Anatomy of a Number
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Sieving by large moduli

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Abstract

Let C be a finite set of ordered pairs of positive integers (n, r) which we interpret as a collection of residue classes $r(\bmod n)$. We say such a set is a residue system. We assume that for any n there is at most one r with $(n, r) \in C$. By $\delta(C)$ we denote the density of the set of integers not congruent to $r(\bmod n)$ for any $(n, r) \in C$. Let

$$S(C) = \{n\colon (n, r) \in C\}, \quad \alpha(C) = \prod_{(n,r)\in C} \left(1 - \frac{1}{n}\right).$$

Some problems of Erdös, Graham, and Selfridge are settled. We prove that if $0 < \varepsilon < 1/3$, $N \geq N(\varepsilon)$, C is a residue system with $S(C)$ consisting of

integers $> N$, and such that either

$$\sum_{n \in S(C)} \frac{1}{n} \leq (tfrac13 - \varepsilon) \log N \log\log\log N / \log\log N,$$

or

$$S(C) \subset (N, N \exp\big((1/(1-\log 2) - \varepsilon\big) \log N \log\log\log N / \log\log N)],$$

then $\delta(C) > 0$. For fixed $\varepsilon \in (0, tfrac12)$,

$$S(C) \subset (N, N \exp(\big(\tfrac{1}{2} - \varepsilon\big) \log N \log\log\log N / \log\log N)],$$

we have $\delta(C) = \alpha(C)\big(1+o(1)\big)$ as $N \to \infty$. However, if K is a large integer then for any N there is some residue system C with $S(C) \subset (N, KN]$ such that $\delta(C) \leq (1/K) \exp\big(-\log K/(3N)\big)$. Also, we prove that if for any finite $A \subset \{N+1, N+2, \ldots,\}$ we choose C randomly so that $S(C) = A$, then $\delta(C) = \alpha(C)\big(1 + o(1)\big)$ with probability tending to 1 as $N \to \infty$. Some of the results can be generalized to the case that moduli may be repeated a certain number of times.

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