Anatomy of a Number
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On smooth twins

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Abstract

To derive additive properties of a set of integers defined by multiplicative constrains is one of the most difficult problems in number theory. The prime twin conjecture, for example, asserts that p and $p+2$ are simultaneously prime infinitely often, and a proof is still an open challenge. The corresponding result for smooth numbers is long known, however. The aim of this talk is to point out the characteristic property of smooth numbers which provide this result.

A set $\mathcal{A}$ of positive integers is called "stable" if for every fixed positive integer d the relation

$$n \in \mathcal{A} \iff dn \in \mathcal{A} \tag{1}$$

holds for almost all positive integers n, i.e. the set of positive integers n violating (1) has zero (asymptotic) density. In other words, a stable set is one that is almost invariant with respect to multiplication and division by any fixed d. Note that the sets $\mathcal{Q}_{\alpha,\beta} = \{n \colon n^{\alpha} < P(n) \le n^{\beta}\}$, where $P(n)$ denotes the greatest prime factor of n and $0 \le \alpha < \beta \le 1$ are "stable" sets having positive asymptotic density.

Theorem. *Let $a > 0$, $b > 0$ and $c \ne 0$ be integers satisfying $(a,b) \mid c$. If $\mathcal{A}$ is a "stable" set and $\bar{d}(\mathcal{A}) > 0$ then the linear equation $am - bn + c = 0$ has infinitely many solutions in $m \in \mathcal{A}$, $n \in \mathcal{A}$. Moreover $\bar{d}\big(b\mathcal{A} \cap (a\mathcal{A} + c)\big) > 0$.*