

A Extended Two Dimensional Toda Lattice Hierarchy and Its Solutions

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Introduction

Two dimensional Toda lattice hierarchy (2dTLH) (Ueno & Takasaki, 1982 [1])

- ▶ includes two dimensional Toda lattice (2dTL)

$$\partial_x \partial_y q(n) = e^{q(n)-q(n-1)} - e^{q(n+1)-q(n)}$$

- ▶ differential operator (∂) \rightarrow *difference operator* (Λ)

multi-component generalization (U-T, [1])

- ▶ scalar-difference operator \rightarrow *matrix-difference operator*
- ▶ include non-abelian Toda hierarchy

Introduction

We present a new extended two-dimensional Toda lattice hierarchy (ex2dTLH)

- ▶ introduce new vector field for 2dTLH
- ▶ include two dimensional Toda lattice with self-consistent sources (2dTLSCS, Hu X.B, Wang H.Y. 2006 [2])

We solve 2dTLSCS:

- ▶ construct Darboux transformation (DT) for 2dTLSCS
- ▶ apply method of variation of constants (MVC) to give non-auto-Bäcklund DT ($N \rightarrow N + 1$ SCSs)

Two dimensional Toda Lattice Hierarchy

2dTLH [1]

Let $\mathbf{x} = (x_1, x_2, \dots)$, $\mathbf{y} = (y_1, y_2, \dots)$, $n \in \mathbb{Z}$. Let

$$L = \Lambda + \sum_{i \geq 0} u_i \Lambda^{-i} \quad M = \sum_{i \geq -1} v_i \Lambda^i$$

Λ is *shift operator* such that $\Lambda f(n) = f(n+1)\Lambda$. Then 2dTLH

$$\begin{aligned} \frac{\partial}{\partial x_m} L &= [B_m, L] & \frac{\partial}{\partial y_m} L &= [C_m, L] \\ \frac{\partial}{\partial x_m} M &= [B_m, M] & \frac{\partial}{\partial y_m} M &= [C_m, M] \end{aligned}$$

$$B_m = (L^m)_{\geq 0}, \quad C_m = (M^m)_{< 0}$$

Two dimensional Toda Lattice Hierarchy

Equivalent zero-curvature form

$$B_{k,x_m} - B_{m,x_k} + [B_k, B_m] = 0 \quad (2a)$$

$$C_{k,y_m} - C_{m,y_k} + [C_k, C_m] = 0 \quad (2b)$$

$$B_{k,y_m} - C_{m,x_k} + [B_k, C_m] = 0 \quad (2c)$$

Example

$m = k = 1$, $v_{-1} = e^{q(n)-q(n-1)}$, $u_0 = q_x \Rightarrow$ 2dTL

$$\partial_{x_1} \partial_{y_1} q(n) = e^{q(n)-q(n-1)} - e^{q(n+1)-q(n)} \quad n \in \mathbb{Z}$$

Two dimensional Toda Lattice Hierarchy

Wave operators

$$\hat{W}^{(\infty)} = 1 + b_1 \Lambda^{-1} + b_2 \Lambda^{-2} + \dots$$

$$\hat{W}^{(0)} = c_0 + c_1 \Lambda + c_2 \Lambda^2 + \dots$$

Proposition (U-T)

If L, M satisfy 2dTLH (1), then exists wave operators such that

$$L = \hat{W}^{(\infty)} \Lambda \hat{W}^{(\infty)-1} \quad M = \hat{W}^{(0)} \Lambda^{-1} \hat{W}^{(0)-1}$$

$$\partial_{x_m} \hat{W}^{(\infty)} = -L_{<0}^m \hat{W}^{(\infty)} \quad \partial_{x_m} \hat{W}^{(0)} = L_{\geq 0}^m \hat{W}^{(0)}$$

$$\partial_{y_m} \hat{W}^{(\infty)} = M_{<0}^m \hat{W}^{(\infty)} \quad \partial_{y_m} \hat{W}^{(0)} = -M_{\geq 0}^m \hat{W}^{(0)}$$

Two dimensional Toda Lattice Hierarchy

Wave function & adjoint wave function

$$\begin{aligned}w^{(\infty)} &= \hat{W}^{(\infty)}(\lambda^n) e^{\xi(\mathbf{x}, \lambda)} & w^{(\infty)*} &= \hat{W}^{(\infty)*-1}(\lambda^{-n}) e^{-\xi(\mathbf{x}, \lambda)} \\w^{(0)} &= \hat{W}^{(0)}(\lambda^n) e^{\xi(\mathbf{y}, \lambda^{-1})} & w^{(0)*} &= \hat{W}^{(0)*-1}(\lambda^{-n}) e^{-\xi(\mathbf{y}, \lambda^{-1})}\end{aligned}$$

$$\xi(\mathbf{x}, \lambda) = \sum x_i \lambda^i, \quad \xi(\mathbf{y}, \lambda^{-1}) = \sum y_i \lambda^{-i}.$$

Wave functions satisfy

$$\begin{aligned}L(w^{(\infty)}) &= \lambda w^{(\infty)} & M(w^{(0)}) &= \lambda^{-1} w^{(0)} \\ \partial_{x_m} w^{(\infty)} &= B_m(w^{(\infty)}) & \partial_{x_m} w^{(0)} &= B_m(w^{(0)}) \\ \partial_{y_m} w^{(\infty)} &= C_m(w^{(\infty)}) & \partial_{y_m} w^{(0)} &= C_m(w^{(0)})\end{aligned}$$

Adjoint wave functions have similar expressions

Two dimensional Toda Lattice Hierarchy

Proposition

$$\sum_{k \geq 0} L_{\geq 0}^k \lambda^{-k} = -w^{(\infty)} \Delta_+^{-1} w^{(\infty)*}$$

$$\sum_{k \in \mathbb{Z}} M_{\geq 0}^k \lambda^k = -w^{(0)} \Delta_+^{-1} w^{(0)*}$$

$$\sum_{k \in \mathbb{Z}} L_{< 0}^k \lambda^{-k} = w^{(\infty)} \Delta_-^{-1} w^{(\infty)*}$$

$$\sum_{k > 0} M_{< 0}^k \lambda^k = w^{(0)} \Delta_-^{-1} w^{(0)*}$$

where

$$\Delta_+^{-1} = - \sum_{i \geq 0} \Lambda^i \quad \Delta_-^{-1} = \sum_{i \leq -1} \Lambda^i$$

Proof. Need

$$\text{Res}_\Lambda PQ^* = \text{Res}_\lambda \lambda^{-1} P(\lambda^n) Q(\lambda^{-n})$$

New extended 2D Toda Lattice Hierarchy

Define \bar{y}_k such that

$$\partial_{\bar{y}_k} = \partial_{y_k} + \sum_{i=1}^N \sum_{j \geq 1} \lambda_i^j \partial_{y_j}$$

for distinct non-zero parameters λ_i . Then

$$\frac{\partial}{\partial \bar{y}_k} L = [\bar{C}_k, L] \quad \frac{\partial}{\partial \bar{y}_k} M = [\bar{C}_k, M] \quad (3a)$$

where

$$\bar{C}_k = C_k + \sum_{i=1}^N \sum_{j \geq 1} \lambda_i^j C_j$$

New extended 2D Toda Lattice Hierarchy

Define \bar{y}_k such that

$$\partial_{\bar{y}_k} = \partial_{y_k} + \sum_{i=1}^N \sum_{j \geq 1} \lambda_i^j \partial_{y_j}$$

for distinct non-zero parameters λ_i . Then

$$\frac{\partial}{\partial \bar{y}_k} L = [\bar{C}_k, L] \quad \frac{\partial}{\partial \bar{y}_k} M = [\bar{C}_k, M] \quad (3a)$$

where (according to proposition)

$$\bar{C}_k = C_k + \sum_{i=1}^N \sum_{j \geq 1} \lambda_i^j C_j = C_k + \sum_{i=1}^N w_i^{(0)} \Delta_-^{-1} w_i^{(0)*}.$$

And $w_i^{(0)(*)} = w^{(0)(*)}(\lambda_i)$ are (adj-)wave functions s.t.

$$\partial_{x_m} w_i^{(0)} = B_m(w_i^{(0)}) \quad \partial_{x_m} w_i^{(0)*} = -B_m^*(w_i^{(0)*}) \quad (3b)$$

$$\partial_{y_m} w_i^{(0)} = C_m(w_i^{(0)}) \quad \partial_{y_m} w_i^{(0)*} = -C_m^*(w_i^{(0)*}) \quad (3c)$$

New extended 2D Toda Lattice Hierarchy

ex2dTLH: Use $w_i^{(*)}$ instead of $w_i^{(0)(*)}$, (1) and (3) give rise to: When $m \neq k$:

$$B_{m,x_k} - B_{k,x_m} + [B_m, B_k] = 0 \quad (4a)$$

$$C_{m,\bar{y}_k} - \bar{C}_{k,y_m} + [C_m, \bar{C}_k] = 0 \quad (4b)$$

$$B_{m,\bar{y}_k} - \bar{C}_{k,x_m} + [B_m, \bar{C}_k] = 0 \quad (4c)$$

$$B_{k,y_m} - C_{m,x_k} + [B_k, C_m] = 0 \quad (4d)$$

$$w_{i,x_m} = B_m(w_i) \quad w_{i,y_m} = C_m(w_i) \quad (i = 1, \dots, N) \quad (4e)$$

$$w_{i,x_m}^* = -B_m^*(w_i^*) \quad w_{i,y_m}^* = -C_m^*(w_i^*) \quad (4f)$$

When $m = k$:

$$B_{k,\bar{y}_k} - \bar{C}_{k,x_k} + [B_k, \bar{C}_k] = 0 \quad (5a)$$

$$\partial_{x_k} w_i = B_k(w_i), \quad \partial_{x_k} w_i^* = -B_k^*(w_i^*) \quad (i = 1, \dots, N) \quad (5b)$$

where $\bar{C}_k = C_k + \sum_{i=1}^N w_i \Delta_{-1}^{-1} w_i^*$.

New extended 2D Toda Lattice Hierarchy

When $m \neq k$, Lax representation for (4a)-(4d)

$$\begin{aligned}\psi_{x_m} &= B_m(\psi) & \psi_{x_k} &= B_k(\psi) \\ \psi_{y_m} &= C_m(\psi) & \psi_{\bar{y}_k} &= \bar{C}_k(\psi)\end{aligned}$$

When $m = k$, Lax representation for (5a)

$$\psi_{x_k} = B_k(\psi) \quad \psi_{\bar{y}_k} = \bar{C}_k(\psi)$$

New extended 2D Toda Lattice Hierarchy

Example (2dTLSCS)

When $m = k = 1$, let $u = u_0$, $v = v_{-1}$, $x = x_1$, $y = \bar{y}_1$

$$B_1 = \Lambda + u, \quad C_1 = v\Lambda^{-1}.$$

Denote $f(n+k) = f^{(k)}$, then

$$u_y = -\Delta(v + \sum w_i w_i^{*(-1)}), \quad v_x = v(u - u^{(-1)}), \quad (6a)$$

$$w_{i,x} = B_1(w_i), \quad w_{i,x}^* = -B_1^*(w_i^*), \quad i = 1, \dots, N \quad (6b)$$

Under $u = q_x$, $v = \exp(q - q^{(-1)})$, (6) becomes 2dTLSCS

$$q_{xy} = e^{q-q^{(-1)}} - e^{q^{(1)}-q} + \sum (w_i w_i^*)_x,$$

$$w_{i,x} = w_i^{(1)} + q_x w_i, \quad (i = 1, \dots, N)$$

$$w_{i,x}^* = -w_i^{*(-1)} - q_x w_i^*.$$

New extended 2D Toda Lattice Hierarchy

Another ex2dTLH: Introduce \bar{x}_k , such that

$$\partial_{\bar{x}_k} = \partial_{x_k} + \sum_{i=1}^N \sum_{j \geq 1} \lambda_i^{-j} \partial_{x_j}$$

When $m \neq k$

$$\begin{aligned} B_{m, \bar{x}_k} - \bar{B}_{k, x_m} + [B_m, \bar{B}_k] &= 0, & C_{m, \bar{x}_k} - \bar{B}_{k, y_m} + [C_m, \bar{B}_k] &= 0, \\ C_{m, y_k} - C_{k, y_m} + [C_m, C_k] &= 0, & B_{m, y_k} - C_{k, x_m} + [B_m, C_k] &= 0, \\ w_{i, y_m} &= C_m(w_i), & w_{i, x_m} &= B_m(w_i), \quad i = 1, \dots, N, \\ w_{i, y_m}^* &= -C_m^*(w_i^*), & w_{i, x_m}^* &= -B_m^*(w_i^*). \end{aligned}$$

when $m = k$

$$\begin{aligned} C_{k, \bar{x}_k} - \bar{B}_{k, y_k} + [C_k, \bar{B}_k] &= 0, \\ \partial_{y_k} w_i &= C_k(w_i), & \partial_{y_k} w_i^* &= -C_k^*(w_i^*). \quad i = 1, \dots, N, \end{aligned}$$

where

$$\bar{B}_k = B_k - \sum_{i=1}^N w_i \Delta_+^{-1} w_i^*.$$

New extended 2D Toda Lattice Hierarchy

Example (2dTLSCS, Hu X.B., Wang H.Y. 2006 [2])

When $m = k = 1$

$$q_{xy} = e^{q-q^{(-1)}} - e^{q^{(1)}-q} + \sum (w_i w_i^*)_y$$

$$w_{i,y} = e^{q-q^{(-1)}} w_i^{(-1)} \quad (i = 1, \dots, N)$$

$$w_{i,y}^* = -e^{q^{(1)}-q} w_i^{*(1)}$$

This equation is equivalent to the previous 2dTLSCS, under:

$$\begin{aligned} x &\rightarrow -y, & y &\rightarrow -x & q &\rightarrow q \\ w_i &\rightarrow -e^q w_i^*, & w_i^* &\rightarrow e^{-q} w_i \end{aligned}$$

This transformation was discovered by Prof. Hu.

Darboux Transformations (DT) + Method of Variation of Constants (MVC)

Soliton equation with self-consistent sources (SESCS) \rightarrow soliton equation with non-homogeneous terms. This inspired us to use MVC (celebrated in ODE theory) to solve SESCS.

Apply MVC to Binary DT : Grammian det. (Y. Zeng, 2001, [3])

- ▶ KdVSCS, mKdVSCS, NLSSCS
- ▶ Non-auto-Bäcklund BDT: $N \rightarrow N + 1$ SCSs

Apply MVC to Ordinary DT : Wronskian or Casoratian det. etc. (R. Lin, X. Liu 2005, [4])

- ▶ TLSCS, Ablowitz-Ladik with SCS, 2dTLSCS
- ▶ Non-auto-Bäcklund DT: $N \rightarrow N + 1$ SCSs

DT and MVC

General Scheme

- ▶ Ordinary DT (auto-Bäcklund)
- ▶ Eigenfunction $h = f + a \cdot g$
- ▶ MVC: $a \rightarrow a(t)$
- ▶ Prove: Non-auto-Bäcklund DT: $N \rightarrow N + 1$ sources
- ▶ m -iteration of DT (different eigenvalues)

This enables us to construct non-trivial solutions of SESCS from trivial one.

DT and MVC for 2dTLSCS

Recall 2dTLSCS and its Lax representations

$$q_{xy} = e^{q-q^{(-1)}} - e^{q^{(1)}-q} + \sum (w_i w_i^*)_x, \quad (7a)$$

$$w_{i,x} = w_i^{(1)} + q_x w_i, \quad (i = 1, \dots, N) \quad (7b)$$

$$w_{i,x}^* = -w_i^{*(-1)} - q_x w_i^*. \quad (7c)$$

Under (7b) and (7c), $(u = q_x, v = e^{q-q^{(-1)}})$

$$\psi_x = (\Lambda + u)(\psi), \quad (8a)$$

$$\psi_y = (v\Lambda^{-1} + \sum_{i=1}^N w_i \Delta_-^{-1} w_i^*)(\psi), \quad (8b)$$

Darboux Tf and MVC for 2dTLSCS

Proposition. Let h is solution to Lax representation (8). Define

$$\mathcal{D} = \Lambda + \sigma, \quad \sigma = -h^{(1)}/h.$$

Then Darboux Transformation for 2dTLSCS is

$$\tilde{u} = u^{(1)} + \sigma - \sigma^{(1)}, \quad (9a)$$

$$\tilde{v} = v\sigma/\sigma^{(-1)}, \quad (9b)$$

$$\tilde{\psi} = \mathcal{D}(\psi) = \frac{\text{cas}(h, \psi)}{h}, \quad (9c)$$

$$\tilde{w}_i = \mathcal{D}(w_i) = \frac{\text{cas}(h, w_i)}{h}, \quad (9d)$$

$$\tilde{w}_i^* = \mathcal{D}^{*-1}(w_i^*) = -\frac{S(hw_i^*)}{h^{(1)}}, \quad (9e)$$

where $S = \Lambda\Delta_-^{-1}$.

Darboux Tf and MVC for 2dTLSCS

Proposition. Let $h = f + ag$ is solution to Lax representation (8). Define

$$\mathcal{D} = \Lambda + \sigma, \quad \sigma = -h^{(1)}/h.$$

Then Darboux Transformation for 2dTLSCS is

$$\tilde{u} = u^{(1)} + \sigma - \sigma^{(1)}, \quad (9a)$$

$$\tilde{v} = v\sigma/\sigma^{(-1)}, \quad (9b)$$

$$\tilde{\psi} = \mathcal{D}(\psi) = \frac{\text{cas}(h, \psi)}{h}, \quad (9c)$$

$$\tilde{w}_i = \mathcal{D}(w_i) = \frac{\text{cas}(h, w_i)}{h}, \quad (9d)$$

$$\tilde{w}_i^* = \mathcal{D}^{*-1}(w_i^*) = -\frac{S(hw_i^*)}{h^{(1)}}, \quad (9e)$$

where $S = \Lambda\Delta_-^{-1}$.

Darboux Tf and MVC for 2dTLSCS

Proposition. Let $h = f + a(y)g$. Define

$$\mathcal{D} = \Lambda + \sigma, \quad \sigma = -h^{(1)}/h.$$

Then Darboux Transformation for 2dTLSCS is

$$\tilde{u} = u^{(1)} + \sigma - \sigma^{(1)}, \quad (9a)$$

$$\tilde{v} = v\sigma/\sigma^{(-1)}, \quad (9b)$$

$$\tilde{\psi} = \mathcal{D}(\psi) = \frac{\text{cas}(h, \psi)}{h}, \quad (9c)$$

$$\tilde{w}_i = \mathcal{D}(w_i) = \frac{\text{cas}(h, w_i)}{h}, \quad (9d)$$

$$\tilde{w}_i^* = \mathcal{D}^{*-1}(w_i^*) = -\frac{S(hw_i^*)}{h^{(1)}}, \quad (9e)$$

$$\tilde{w}_{N+1} = a^{-1}\mathcal{D}(f), \quad (9f)$$

$$\tilde{w}_{N+1}^* = \frac{\dot{a}}{h^{(1)}}, \quad (9g)$$

where $S = \Lambda\Delta_-^{-1}$.

m -time DT

Theorem. Let f_j and g_j ($j = 1, 2, \dots, m$) be solutions to (8).

$$h_j := f_j + a_j(y)g_j.$$

Then

$$u[m] = u^{(m)} + \frac{\widetilde{\text{cas}}^{(1)}(h_1, \dots, h_m)}{\text{cas}^{(1)}(h_1, \dots, h_m)} - \frac{\widetilde{\text{cas}}(h_1, \dots, h_m)}{\text{cas}(h_1, \dots, h_m)}, \quad (10a)$$

$$v[m] = v \frac{\text{cas}^{(1)}(h_1, \dots, h_m) \text{cas}^{(-1)}(h_1, \dots, h_m)}{\text{cas}^2(h_1, \dots, h_m)}, \quad (10b)$$

$$w_i[m] = \frac{\text{cas}(h_1, \dots, h_m, w_i)}{\text{cas}(h_1, \dots, h_m)}, \quad i = 1, \dots, N, \quad (10c)$$

$$w_i^*[m] = (-1)^m \frac{\overline{\text{cas}}(h_1, \dots, h_m, w_i^*)}{\text{cas}^{(1)}(h_1, \dots, h_m)}, \quad (10d)$$

$$w_{N+j}[m] = a_j^{-1} \frac{\text{cas}(h_1, \dots, h_m, f_j)}{\text{cas}(h_1, \dots, h_m)}, \quad j = 1, \dots, m \quad (10e)$$

$$w_{N+j}^*[m] = (-1)^{m-j} \dot{a}_j \frac{\text{cas}^{(1)}(h_1, \dots, \hat{h}_j, \dots, h_m)}{\text{cas}^{(1)}(h_1, \dots, h_m)}, \quad (10f)$$

m -time DT

$$\text{cas}(h_1, \dots, h_m) = \begin{vmatrix} h_1 & \dots & h_m \\ \vdots & & \vdots \\ h_1^{(m-1)} & \dots & h_m^{(m-1)} \end{vmatrix},$$

$$\widetilde{\text{cas}}(h_1, \dots, h_m) = \begin{vmatrix} h_1 & \dots & h_m \\ \vdots & & \vdots \\ h_1^{(m-2)} & \dots & h_m^{(m-2)} \\ h_1^{(m)} & \dots & h_m^{(m)} \end{vmatrix},$$

$$\overline{\text{cas}}(h_1, \dots, h_m, f) = \begin{vmatrix} S(h_1 f) & \dots & S(h_m f) \\ h_1^{(1)} & \dots & h_m^{(1)} \\ \vdots & & \vdots \\ h_1^{(m-1)} & \dots & h_m^{(m-1)} \end{vmatrix}$$

Solutions

Starting: $q = 1$, $v = 1$, $u = 0$, $N = 0$. Lax representation for 2dTLSCS:

$$\psi_x = \psi^{(1)}, \quad \psi_y = \psi^{(-1)}. \quad (11)$$

Independent solutions to (11) w.r.t. parameter $z = e^\omega$ are

$$\begin{aligned} f(n, x, y) &= \exp(n\omega + zx + z^{-1}y), \\ g(n, x, y) &= \exp(-n\omega + z^{-1}x + zy), \\ \frac{\partial^k f}{\partial z^k}, \frac{\partial^k g}{\partial z^k}, \quad k &= 1, 2, \dots \end{aligned}$$

Different choices will give different solutions.

Solutions

Example (Solitons)

Let $a(y) = e^{\alpha(y)}$, then

$$h = f + a(y)g = 2 \exp \Omega \cdot \cosh Z,$$

$$\Omega = \cosh \omega \cdot x + \cosh \omega \cdot y + \alpha/2,$$

$$Z = n\omega + \sinh \omega \cdot x - \sinh \omega \cdot y - \alpha/2.$$

DT + MVC \Rightarrow 1-soliton

$$u[1] = \frac{\cosh(Z + 2\omega)}{\cosh(Z + \omega)} - \frac{\cosh(Z + \omega)}{\cosh Z},$$

$$v[1] = \frac{\cosh(Z + \omega) \cosh(Z - \omega)}{\cosh^2 Z},$$

$$w[1] = \frac{\sinh \omega \cdot e^{\Omega}}{\cosh Z}, \quad w^*[1] = \frac{\dot{\alpha} e^{-\Omega}}{2 \cosh(Z + \omega)}.$$

Solutions

Example (Solitons)

Let

$$h_i = f_i + a_i(y)g_i = 2 \exp \Omega_i \cdot \cosh Z_i,$$

then m -time DT + MVC with different parameters $z = z_i$
($i = 1, \dots, m$) \Rightarrow

m -soliton

Solutions

Example (Rational Solution)

Let

$$h_k = \frac{\partial^k}{\partial z^k} g + a(y)g.$$

For e.g. $k = 1$, DT+MVC \Rightarrow a rational solution

$$u[1] = -\frac{z}{(\eta + za + 1/2)^2 - 1/4},$$

$$v[1] = 1 - \frac{1}{(\eta + za)^2},$$

$$w[1] = \frac{z^{n+1}e^\xi}{\eta + za},$$

$$w^*[1] = \frac{\dot{a}z^{-n}e^{-\xi}}{\eta + za + 1},$$

where $\eta = n + zx - z^{-1}y$, $\xi = zx + z^{-1}y$.

Solutions

Example

Let $h = f + a(y)g$, $h_z = \partial_z f + a(y)\partial_z g$. Then 2-iteration of DT+MVC
 \Rightarrow a singular solution

$$u[2] = \frac{C_2^{(1)}}{C_1^{(1)}} - \frac{C_2}{C_1}, \quad v[2] = \frac{C_1^{(1)} C_1^{(-1)}}{C_1^2},$$

$$w_1[2] = -2 \frac{\sinh^2 \omega e^\Omega}{z C_1} \cosh(Z + \omega),$$

$$w_2[2] = \frac{e^\Omega \left(D_1^{(1)} \cosh Z + D_1 \cosh(Z + 2\omega) - D_2 \cosh(Z + \omega) \right)}{C_1},$$

$$w_1^*[2] = -\dot{a} \frac{\Omega_z \cosh(Z + \omega) + (Z_z + 1/z) \sinh(Z + \omega)}{2e^\Omega C_1^{(1)}},$$

$$w_2^*[2] = \dot{a} \frac{\cosh(Z + \Omega)}{2e^\Omega C_1^{(1)}}.$$

Solutions

where

$$C_k = \left(Z_z + \frac{k}{2z}\right) \sinh(k\omega) + \frac{k}{2z} \sinh(2Z + k\omega), \quad k = 1, 2$$

$$D_k = -\left(\frac{n}{z} + \frac{k}{2z} + F_z\right) \left(\frac{n}{z} + \frac{k}{2z} - G_z\right) \sinh(k\omega) + \frac{k^2}{4z^2} \sinh(k\omega) + \frac{k\Omega_z}{z} \cosh(k\omega)$$

$$F = zx + z^{-1}y, \quad G = z^{-1}x + zy$$

Conclusions

This extensions applied to many 2+1 systems include

- ▶ KP, BKP,CKP,mKP, 2dTL
- ▶ Dispersionless KP, q-deformed KP etc.
- ▶ all consider symmetry generating vector fields

Apply DT to MVC for solving soliton equations with self-consistent sources

- ▶ available for 1+1, 2+1 systems
- ▶ provide non-auto-Bäcklund DTs.

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