Multiple orthogonal polynomials recurrence relations and the corresponding Toda lattice

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orthogonal polynomials on the real line

We will use monic polynomials

$$P_n(x) = x^n + \cdots ,$$

with orthogonality conditions

$$\int_{-\infty}^{\infty} P_n(x)x^k w(x)\, dx = 0, \qquad k = 0, 1, \ldots, n-1.$$

These will always exist and

$$\int_{-\infty}^{\infty} P_n(x)x^n w(x)\, dx = \int_{-\infty}^{\infty} P_n^2(x)w(x)\, dx \neq 0.$$

These orthogonal polynomials always satisfy a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x)$$

with $P_0 = 1$ and $P_{-1} = 0$.

type II multiple orthogonal polynomials

$P_{n,m}$ is a monic polynomial

$$P_{n,m}(x) = x^{n+m} + \cdots ;$$

with orthogonality conditions

$$\int_{\Delta_1} P_{n,m}(x) x^k w_1(x)\, dx = 0, \qquad k = 0, 1, \ldots, n-1,$$

$$\int_{\Delta_2} P_{n,m}(x) x^k w_2(x)\, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

These polynomials may not exist. They exist if the (mixed) matrix of moments is not singular.

type I multiple orthogonal polynomials

$(A_{n,m}, B_{n,m})$ is a vector of two polynomials, with

$$\deg(A_{n,m}) \leq n-1 \quad \deg(B_{n,m}) \leq m-1$$

and orthogonality conditions

$$\int_\Delta \Big(A_{n,m}(x)w_1(x) + B_{n,m}(x)w_2(x)\Big)x^k\,dx = 0, \qquad k = 0,1,\ldots,n+m-2,$$

and

$$\int_\Delta \Big(A_{n,m}(x)w_1(x) + B_{n,m}(x)w_2(x)\Big)x^{n+m-1}\,dx = 1.$$

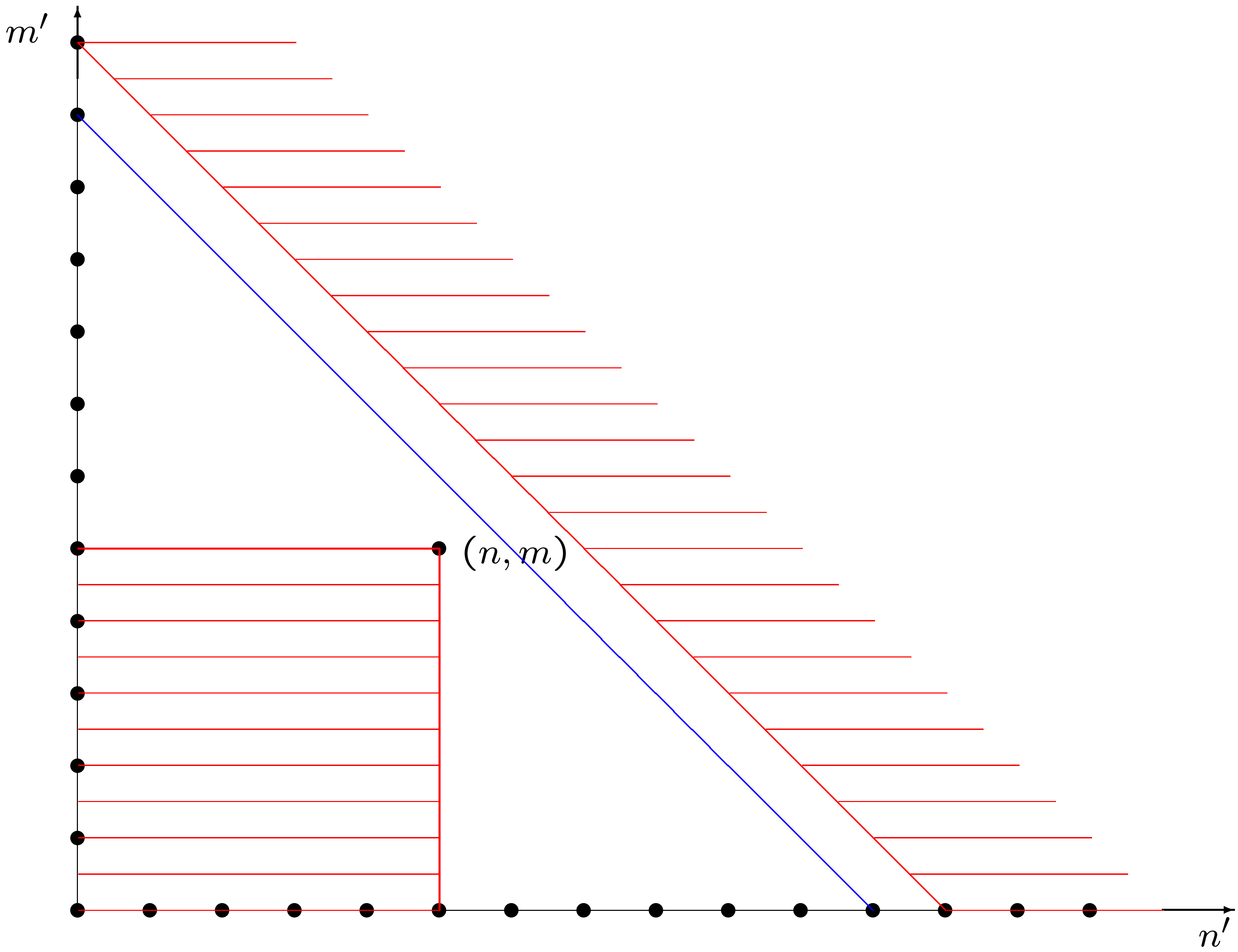
These polynomials may not exist. They exist if the same (mixed) matrix of moments is not singular.

biorthogonality

The type I and type II multiple orthogonal polynomials are in many cases biorthogonal. They satisfy

$$\int_\Delta P_{n,m}(x)\Big(A_{n',m'}(x)w_1(x) + B_{n',m'}(x)w_2(x)\Big)\,dx$$

$$= \begin{cases} 0 & \text{if } m' \leq m \text{ and } n' \leq n, \\ 0 & \text{if } n+m \leq n'+m'-2, \\ 1 & \text{if } n'+m' = n+m+1. \end{cases}$$


m'
(n, m)
n'

Riemann-Hilbert problem

Find 3×3 matrix Y such that

- Y is analytic in $\mathbb{C} \setminus \mathbb{R}$,

- Jump condition on $\mathbb{R}$

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R},$$

- Condition near infinity

$$Y(z) = \left(I + \mathcal{O}(1/z)\right) \begin{pmatrix} z^{n+m} & 0 & 0 \\ 0 & z^{-n} & 0 \\ 0 & 0 & z^{-m} \end{pmatrix}, \qquad z \to \infty.$$

Solution:

$$Y(z) = \begin{pmatrix} P_{n,m}(z) & C(P_{n,m}w_1) & C(P_{n,m}w_2) \\ c_1(n,m)P_{n-1,m}(z) & c_1C(P_{n-1,m}w_1) & c_1C(P_{n-1,m}w_2) \\ c_2(n,m)P_{n,m-1}(z) & c_2C(P_{n,m-1}w_1) & c_2C(P_{n,m-1}w_2) \end{pmatrix}$$

with

$$C(Pw) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{P(x)w(x)}{x-z}\,dx,$$

and

$$\frac{2\pi i}{c_1(n,m)} = -\int_{-\infty}^{\infty} P_{n-1,m}(x)x^{n-1}w_1(x)\,dx,$$

$$\frac{2\pi i}{c_2(n,m)} = -\int_{-\infty}^{\infty} P_{n,m-1}(x)x^{m-1}w_2(x)\,dx.$$

A simple observation

Let $F(z) = \det Y(z)$ then

- F is analytic on $\mathbb{C} \setminus \mathbb{R}$,
- $F_+(x) = F_-(x)$ for $x \in \mathbb{R}$,
- $F(z) = 1 + \mathcal{O}(1/z)$ when $z \to \infty$.

Hence F is a bounded and entire function, and Liouville's theorem implies that F is constant.

Conclusion

$$\det Y(z) = 1.$$

Type I multiple orthogonal polynomials

$$(Y^{-1})^t(z) = \begin{pmatrix} 2\pi i C(A_{n,m}w_1 + B_{n,m}w_2) & 2\pi i A_{n,m}(z) & 2\pi i B_{n,m}(z) \\ d_1 C(A_{n+1,m}w_1 + B_{n+1,m}w_2) & d_1 A_{n+1,m}(z) & d_1 B_{n+1,m}(z) \\ d_2 C(A_{n,m+1}w_1 + B_{n,m+1}w_2) & d_2 A_{n,m+1}(z) & d_2 B_{n,m+1}(z) \end{pmatrix}$$

with d_1 and d_2 such that

$$d_1 A_{n+1,m}(z) = z^n + \cdots, \qquad d_2 B_{n,m+1}(z) = z^m + \cdots.$$

recurrence relations

Let $R_1(z) = Y_{n+1,m} Y_{n,m}^{-1}$ then

- R_1 is analytic on $\mathbb{C} \setminus \mathbb{R}$,
- $R_{1+}(x) = R_{1-}(x)$ for $x \in \mathbb{R}$,
- for $z \to \infty$

$$R_1(z) = \begin{pmatrix} z + \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1/z) & \mathcal{O}(1/z) \\ \mathcal{O}(1) & \mathcal{O}(1/z) & 1 + \mathcal{O}(1/z) \end{pmatrix}.$$

Hence R_1 is an entire matrix function of polynomial growth, and Liouville's theorem implies that R_1 is a matrix polynomial

$$R_1(z) = \begin{pmatrix} z - c_{n,m} & A_{n,m} & B_{n,m} \\ A'_{n,m} & 0 & 0 \\ B'_{n,m} & 0 & 1 \end{pmatrix}.$$

This means that

$$Y_{n+1,m}(z) = \begin{pmatrix} z - c_{n,m} & A_{n,m} & B_{n,m} \\ A'_{n,m} & 0 & 0 \\ B'_{n,m} & 0 & 1 \end{pmatrix} Y_{n,m}(z)$$

In a similar way

$$Y_{n,m+1}(z) = \begin{pmatrix} z - d_{n,m} & A_{n,m} & B_{n,m} \\ A^*_{n,m} & 1 & 0 \\ B^*_{n,m} & 0 & 0 \end{pmatrix} Y_{n,m}(z)$$

The $(1,1)$ entries give

$$P_{n+1,m}(z) = (z - c_{n,m})P_{n,m}(z) - a_{n,m}P_{n-1,m}(z) - b_{n,m}P_{n,m-1}(z),$$

$$P_{n,m+1}(z) = (z - d_{n,m})P_{n,m}(z) - a_{n,m}P_{n-1,m}(z) - b_{n,m}P_{n,m-1}(z).$$

Multiple Hermite polynomials

Let $c_1 \neq c_2$, then

$$\int_{-\infty}^{\infty} H_{n,m}(x) x^k e^{-x^2+c_1 x}\, dx = 0, \qquad k = 0, 1, \ldots, n-1,$$

$$\int_{-\infty}^{\infty} H_{n,m}(x) x^k e^{-x^2+c_2 x}\, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

The Pearson equation: $w_1(x) = e^{-x^2+c_1 x}, w_2(x) = e^{-x^2+c_2 x}$

$$\begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}' = \begin{pmatrix} -2x + c_1 & 0 \\ 0 & -2x + c_2 \end{pmatrix} \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}$$

Gives a system differential equations

$$\begin{aligned} H'_{n,m}(x) &= A_{n,m} H_{n-1,m}(x) + B_{n,m} H_{n,m-1}(x) \\ \left(e^{-x^2+c_1 x} H_{n-1,m}(x)\right)' &= -2 e^{-x^2+c_1 x} H_{n,m}(x) \\ \left(e^{-x^2+c_2 x} H_{n,m-1}(x)\right)' &= -2 e^{-x^2+c_2 x} H_{n,m}(x) \end{aligned}$$

Compatibility

Recurrence relations

$$H_{n+1,m}(x) = (x - c_{n,m})H_{n,m}(x) - a_{n,m}H_{n-1,m}(x) - b_{n,m}H_{n,m-1}(x),$$

$$H_{n,m+1}(x) = (x - d_{n,m})H_{n,m}(x) - a_{n,m}H_{n-1,m}(x) - b_{n,m}H_{n,m-1}(x).$$

Differential equations

$$\begin{aligned} H'_{n,m}(x) &= A_{n,m}H_{n-1,m}(x) + B_{n,m}H_{n,m-1}(x) \\ \left(e^{-x^2+c_1x}H_{n-1,m}(x)\right)' &= -2e^{-x^2+c_1x}H_{n,m}(x) \\ \left(e^{-x^2+c_2x}H_{n,m-1}(x)\right)' &= -2e^{-x^2+c_2x}H_{n,m}(x) \end{aligned}$$

$$\begin{aligned} a_{n,m} &= a_{n-1,m} + \frac{1}{2} \\ b_{n,m} &= b_{n-1,m} \\ c_{n,m} &= c_{n-1,m} \\ d_{n,m} &= d_{n-1,m} \end{aligned}$$

$$\begin{aligned} a_{n,m} &= a_{n,m-1} \\ b_{n,m} &= b_{n,m-1} + \frac{1}{2} \\ c_{n,m} &= c_{n,m-1} \\ d_{n,m} &= d_{n,m-1} \end{aligned}$$

$$\begin{aligned} a_{n,m} &= a_{n-1,m} + \frac{1}{2} \\ b_{n,m} &= b_{n-1,m} \\ c_{n,m} &= c_{n-1,m} \\ d_{n,m} &= d_{n-1,m} \end{aligned}$$

$$\begin{aligned} a_{n,m} &= a_{n,m-1} \\ b_{n,m} &= b_{n,m-1} + \frac{1}{2} \\ c_{n,m} &= c_{n,m-1} \\ d_{n,m} &= d_{n,m-1} \end{aligned}$$

$$a_{n,m} = \frac{n}{2}, \quad b_{n,m} = \frac{m}{2}, \quad c_{n,m} = \frac{c_1}{2}, \quad d_{n,m} = \frac{c_2}{2}$$

$$H_{n+1,m}(x) = \left(x - \frac{c_1}{2}\right) H_{n,m}(x) - \frac{n}{2} H_{n-1,m}(x) - \frac{m}{2} H_{n,m-1}(x),$$

$$H_{n,m+1}(x) = \left(x - \frac{c_2}{2}\right) H_{n,m}(x) - \frac{n}{2} H_{n-1,m}(x) - \frac{m}{2} H_{n,m-1}(x).$$

$$H'_{n,m}(x) = nH_{n-1,m}(x) + mH_{n,m-1}(x)$$

multiple Laguerre polynomials, first kind

Let $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, then

$$\int_0^\infty L_{n,m}(x) x^k x^{\alpha_1} e^{-x}\, dx = 0, \qquad k = 0, 1, \ldots, n-1,$$

$$\int_0^\infty L_{n,m}(x) x^k x^{\alpha_2} e^{-x}\, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

Pearson equation

$$\left(x \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix} \right)' = \begin{pmatrix} \alpha_1 + 1 - x & 0 \\ 0 & \alpha_2 + 1 - x \end{pmatrix} \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}$$

$$L_{n+1,m}(x) = (x - c_{n,m}) L_{n,m}(x) - a_{n,m} L_{n-1,m}(x) - b_{n,m} L_{n,m-1}(x),$$

$$L_{n,m+1}(x) = (x - d_{n,m}) L_{n,m}(x) - a_{n,m} L_{n-1,m}(x) - b_{n,m} L_{n,m-1}(x).$$

$$a_{n,m} = \frac{n + \alpha_1 - \alpha_2}{n + \alpha_1 - \alpha_2 - m}\, n(n + \alpha_1), \qquad b_{n,m} = \frac{m + \alpha_2 - \alpha_1}{m + \alpha_2 - \alpha_1 - n}\, m(m + \alpha_2),$$

$$c_{n,m} = 2n + m + \alpha_1 + 1, \qquad d_{n,m} = 2m + n + \alpha_2 + 1$$

multiple Laguerre polynomials, second kind

Let $0 < c_1 \neq c_2$, then

$$\int_0^\infty L_{n,m}(x) x^k x^\alpha e^{-c_1 x}\, dx = 0, \qquad k = 0, 1, \ldots, n-1,$$

$$\int_0^\infty L_{n,m}(x) x^k x^\alpha e^{-c_2 x}\, dx = 0, \qquad k = 0, 1, \ldots, m-1.$$

Pearson equation

$$\left(x \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix} \right)' = \begin{pmatrix} \alpha + 1 - c_1 x & 0 \\ 0 & \alpha + 1 - c_2 x \end{pmatrix} \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}$$

$$L_{n+1,m}(x) = (x - c_{n,m}) L_{n,m}(x) - a_{n,m} L_{n-1,m}(x) - b_{n,m} L_{n,m-1}(x),$$

$$L_{n,m+1}(x) = (x - d_{n,m}) L_{n,m}(x) - a_{n,m} L_{n-1,m}(x) - b_{n,m} L_{n,m-1}(x).$$

$$a_{n,m} = \frac{n(n+m+\alpha)}{c_1^2}, \qquad b_{n,m} = \frac{m(n+m+\alpha)}{c_2^2},$$

$$c_{n,m} = \frac{n}{c_1} + \frac{m}{c_2} + \frac{n+m+\alpha+1}{c_1}, \qquad d_{n,m} = \frac{n}{c_1} + \frac{m}{c_2} + \frac{n+m+\alpha+1}{c_2}$$

multiple Charlier polynomials

Let $0 < a_1 \neq a_2$

$$\sum_{j=0}^{\infty} C_{n,m}(j) j^k \, \frac{a_1^j}{j!} = 0, \qquad k = 0, 1, \ldots, n-1,$$

$$\sum_{j=0}^{\infty} C_{n,m}(j) j^k \, \frac{a_2^j}{j!} = 0, \qquad k = 0, 1, \ldots, m-1,$$

Pearson equation

$$\nabla \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix} = \begin{pmatrix} 1 - x/a_1 & 0 \\ 0 & 1 - x/a_2 \end{pmatrix} \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}, \qquad x \in \mathbb{N}$$

$$C_{n+1,m}(x) = (x - n - m - a_1) C_{n,m}(x) - n a_1 C_{n-1,m}(x) - m a_2 C_{n,m-1}(x),$$

$$C_{n,m+1}(x) = (x - n - m - a_2) C_{n,m}(x) - n a_1 C_{n-1,m}(x) - m a_2 C_{n,m-1}(x).$$

Formulas for the recurrence coefficients

$$
\begin{aligned}
xP_{n,m}(x) &= P_{n+1,m}(x) + c_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x) \\
xP_{n,m}(x) &= P_{n,m+1}(x) + d_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x)
\end{aligned}
$$

Let $Q_{n,m}(x) = A_{n,m}(x)w_1(x) + B_{n,m}(x)w_2(x)$, then

$$c_{n,m} = \int xP_{n,m}(x)Q_{n+1,m}(x)\,dx, \qquad d_{n,m} = \int xP_{n,m}(x)Q_{n,m+1}(x)\,dx,$$

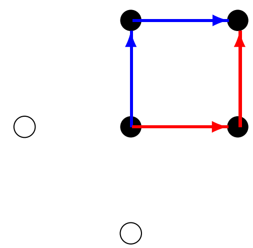
$$a_{n,m} = \frac{\int x^n P_{n,m}(x)w_1(x)\,dx}{\int x^{n-1}P_{n-1,m}(x)w_1(x)\,dx}, \qquad b_{n,m} = \frac{\int x^m P_{n,m}(x)w_2(x)\,dx}{\int x^{m-1}P_{n,m-1}(x)w_2(x)\,dx}$$

$(a_{n,0}, c_{n,0})$ recurrence coefficients of orthogonal polynomials for w_1

$(b_{0,m}, d_{0,m})$ recurrence coefficients of orthogonal polynomials for w_2

$a_{n,m}, b_{n,m}, c_{n,m}, d_{n,m}$ are fully determined by the boundary values $(a_{n,0}, c_{n,0})$ and $(b_{0,m}, d_{0,m})$

$P_{n+1,m+1}(z)$ can be computed from $P_{n,m}(z), P_{n-1,m}, P_{n,m-1}$ in two ways



consistency rules

Theorem 1 *The recurrence coefficients for multiple orthogonal polynomials satisfy*

$$c_{n,m+1} - c_{n,m} = d_{n+1,m} - d_{n,m}$$

$$a_{n+1,m} - a_{n,m+1} + b_{n+1,m} - b_{n,m+1} = \det \begin{pmatrix} c_{n,m} & c_{n,m+1} \\ d_{n,m} & d_{n+1,m} \end{pmatrix}$$

$$\frac{a_{n,m}}{a_{n,m+1}} = \frac{d_{n-1,m} - c_{n-1,m}}{d_{n,m} - c_{n,m}}$$

$$\frac{b_{n,m}}{b_{n+1,m}} = \frac{d_{n,m-1} - c_{n,m-1}}{d_{n,m} - c_{n,m}}$$

A dynamical system

Let us consider weights of the form

$$w_1(x;t) = w_1(x;0)e^{-tx}, \qquad w_2(x;t) = w_2(x;0)e^{-tx}$$

Then the recurrence coefficients will be functions of t

$$a_{n,m}(t), b_{n,m}(t), c_{n,m}(t), d_{n,m}(t).$$

Theorem 2 *The recurrence coefficients satisfy the equations*

$$a'_{n,m} = a_{n,m}(c_{n-1,m} - c_{n,m}), \qquad b'_{n,m} = b_{n,m}(d_{n,m-1} - d_{n,m})$$

$$c'_{n,m} = (a_{n,m} - a_{n+1,m}) + (b_{n,m} - b_{n+1,m}),$$

$$d'_{n,m} = (a_{n,m} - a_{n,m+1}) + (b_{n,m} - b_{n,m+1}).$$