On the Lagrangian Description of the Discrete Isospectral and Isomonodromic Transformations

Anton Dzhamay

School of Mathematical Sciences
University of Northern Colorado

8th International Conference on Symmetries and Integrability of Difference Equations (SIDE 8)
Hôtel Mont-Gabriel, SainteAdéle, Canada, June 2228, 2008

Motivation

Question (I. Krichever):

Can the equations of the isomonodromic transformations of the systems of linear difference equations be written in the (discrete) Lagrangian form?

Answer: Yes (at least in the basic case)

- We have an explicit formula for the Lagrangian, but only in the quadratic case. Still, this suggests that a similar result should hold in the general case as well.
- **Bonus** Under some additional reductions this quadratic case corresponds to the dPV equation.

Isomonodromy Transformations

Linear matrix difference equations

$$\mathbf{\Psi}(z+1)=\mathbf{L}(z)\mathbf{\Psi}(z),$$

where $\mathbf{L}(z)$ is a rational matrix operator on the Riemann sphere.

Isomonodromy Transformations

$$\mathbf{L}(z)\mapsto\tilde{\mathbf{L}}(z)=\mathbf{R}(z+1)\mathbf{L}(z)\mathbf{R}^{-1}(z)$$

where $\mathbf{R}(z)$ is again a certain rational matrix function.

Some References

Classical Theory

- **G. Birkhoff** (1911), General analytic theory of matrix difference equations, the notion of monodromy.

Recent developmets

- **A. Borodin** (2004(2002)), General theory of isomonodromic transformations for polynomial $\mathbf{L}(z)$, relationship with difference Painlevé equations.
- **I. Krichever** (2004), The notion of local monodromy for rational matrices $\mathbf{L}(z)$ with simple poles.
- **D. Arinkin**, **A. Borodin** (2005,2007) Geometric description of isomonodromy transformations using the language of D-connections, the notion of the τ-function.

The Setup

We consider rational matrix functions $\mathbf{L}(z)$ of rank m such that:

- $\mathbf{L}(z)$ is regular, diagonalizable (and diagonalized) at $z = \infty$,

$$\mathbf{L}_0 = \lim_{z\to\infty} \mathbf{L}(z) = \operatorname{diag}\{\rho_1, \ldots, \rho_m\}.$$

- $\mathbf{L}(z)$ has simple poles at the points $z_1, \ldots z_n$.
- The divisor $(\mathbf{L}(z)) = (\det \mathbf{L}(z))$ *is simple as well*,

$$(\det \mathbf{L}(z)) = \sum_{k=1}^{n} (z_k - \zeta_k), \qquad \det \mathbf{L}(z) = \rho_1 \cdots \rho_m \prod_{k=1}^{n} \frac{z - \zeta_k}{z - z_k},$$

  which means that the residues $\mathbf{L}_k$ of $\mathbf{L}(z)$ at the points z_k are matrices of rank one,

$$\mathbf{L}_k = \operatorname{res}_{z_k} \mathbf{L}(z) = \mathbf{a}_k \mathbf{b}_k^\dagger = \alpha_k [\mathbf{a}_k][\mathbf{b}_k^\dagger].$$

- Most of the time we only consider the two-pole case.

Additive and Multiplicative Representations

Such matrices $\mathbf{L}(z)$ can be represented in two different ways:

- Additive representation:

$$\mathbf{L}(z) = \mathbf{L}_0 + \sum_k \frac{\mathbf{L}_k}{z - z_k}$$

- Multiplicative representation:

$$\mathbf{L}(z) = \mathbf{L}_0 \prod_k \left(\mathbf{I} + \frac{\mathbf{G}_k}{z - z_k} \right), \qquad \text{where } \mathbf{G}_k = \mathbf{p}_k \mathbf{q}_k^\dagger = \lambda_k [\mathbf{p}_k][\mathbf{q}_k^\dagger]$$

is again a rank-one matrix.

Definition

We call the matrices $\mathbf{B}_k(z) = \left(\mathbf{I} + \frac{\mathbf{G}_k}{z - z_k} \right)$ appearing in the multiplicative representations the elementary divisors of $\mathbf{L}(z)$.

Re-factorization (or Darboux) transformations

The main dynamic that we want to consider is

- Isomonodromic Transformations:

$$\mathbf{L}(z) = \mathbf{L}_0\mathbf{B}_1(z)\mathbf{B}_2(z) \mapsto \tilde{\mathbf{L}}(z) = \mathbf{B}_2(z+1)\mathbf{L}_0\mathbf{B}_1(z) = \mathbf{L}_0\tilde{\mathbf{B}}_1(z)\tilde{\mathbf{B}}_2(z)$$

But in addition we can also look at a simpler case of

- Isospectral Transformations:

$$\mathbf{L}(z) = \mathbf{L}_0\mathbf{B}_1(z)\mathbf{B}_2(z) \mapsto \tilde{\mathbf{L}}(z) = \mathbf{B}_2(z)\mathbf{L}_0\mathbf{B}_1(z) = \mathbf{L}_0\tilde{\mathbf{B}}_1(z)\tilde{\mathbf{B}}_2(z)$$

Goal

Interpret these equations of motion as the discrete Euler-Lagrange equations for some Lagrangian function.

Discrete Euler-Lagrange equations

Let Q be the configuration space of our system.

Continuous Case		Discrete Case

- The Lagrangian

$$\mathcal{L} = \mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}}) \in \mathcal{F}(T\mathcal{Q}) \quad \Longrightarrow \quad \mathcal{L} = \mathcal{L}(\mathbf{Q}, \widetilde{\mathbf{Q}}) \in \mathcal{F}(Q \times Q)$$

- Action

$$S(\gamma) = \int_\gamma \mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}})\, dt \quad \Longrightarrow \quad S(\{\mathbf{Q}_k\}) = \sum_k \mathcal{L}(\mathbf{Q}_k, \mathbf{Q}_{k+1})$$

- Euler-Lagrange Equations (from $\delta S = 0$)

$$\frac{\partial \mathcal{L}}{\partial \mathbf{Q}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{Q}}} = 0 \quad \Longrightarrow \quad \frac{\partial \mathcal{L}}{\partial \mathbf{Y}}(\underset{\sim}{\mathbf{Q}}, \mathbf{Q}) + \frac{\partial \mathcal{L}}{\partial \mathbf{X}}(\mathbf{Q}, \widetilde{\mathbf{Q}}) = 0$$

Discrete Lagrangian Integrable Systems

Re-factorization approach (Veselov, Moser)

- Find a map $\eta : \mathcal{Q} \times \mathcal{Q} \to \mathcal{P}$, where $\mathcal{P}$ is a space of matrix polynomials in a spectral variable z, such that we have the following diagram:

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{\Phi} & (\utilde{\mathbf{Q}}, \mathbf{Q}) & \xrightarrow{\Phi} & (\mathbf{Q}, \tilde{\mathbf{Q}}) & \xrightarrow{\Phi} & \cdots \\
 & & \downarrow \eta & & \downarrow \eta & & \\
\cdots & \xrightarrow{R} & \mathbf{L}(z) = \utilde{\mathbf{L}}_2(z)\utilde{\mathbf{L}}_1(z) = \mathbf{L}_1(z)\mathbf{L}_2(z) & \xrightarrow{R} & \widetilde{\mathbf{L}}(z) = \mathbf{L}_2(z)\mathbf{L}_1(z) = \widetilde{\mathbf{L}}_1(z)\widetilde{\mathbf{L}}_2(z) & \xrightarrow{R} & \cdots
\end{array}
$$

- the isospectral re-factorization map

$$
R : \mathbf{L}(z) \to \tilde{\mathbf{L}}(z) = \mathbf{L}_2(z)\mathbf{L}(z)\mathbf{L}_2^{-1}(z)
$$

  is the discrete analogue of the Lax-pair representation.

This is exactly the type of transformations we have, except that in place of matrix polynomials we have rational functions. The ordering of the poles determines the order of the factors.

Isospectral Case

Fix a divisor $\mathcal{D} = z_1 - \zeta_1 + z_2 - \zeta_2$ and let

$$\mathcal{M}^r_{\mathcal{D}} = \{\mathbf{L}(z) = \mathbf{L}_0\mathbf{B}_1(z)\mathbf{B}_2(z) | (\det \mathbf{L}(z)) = \mathcal{D}\},$$

which in particular means that

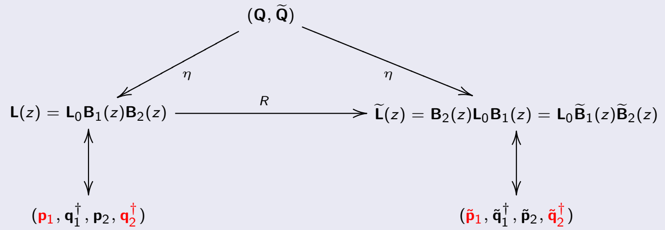
$$\det \mathbf{B}_i(z) = \det \left(\mathbf{I} + \frac{\mathbf{p}_i\mathbf{q}_i^\dagger}{z - z_i} \right) = \frac{z - \zeta_i}{z - z_i}, \qquad \text{where } \mathbf{q}_i^\dagger\mathbf{p}_i = z_i - \zeta_i.$$

We need to do the following:

- Identify the configuration space $\mathcal{Q}$.
- Find the parametrization map $\eta : \mathcal{Q} \times \mathcal{Q} \to \mathcal{M}^r_{\mathcal{D}}$.
- Find the equations of motion in terms of these coordinates.
- Find the Lagrangian function $\mathcal{L} = \mathcal{L}(\mathbf{X}, \mathbf{Y}) \in \mathcal{F}(\mathcal{Q} \times \mathcal{Q})$.

The Coordinates

Consider the diagram



Let $\mathbf{Q} = (\mathbf{p}_1, \mathbf{q}_2^\dagger) \in \mathcal{Q} = \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$

Then we want:

- $\mathbf{p}_2 = \mathbf{p}_2(\mathbf{p}_1, \mathbf{q}_2^\dagger, \tilde{\mathbf{p}}_1, \tilde{\mathbf{q}}_2^\dagger)$
- $\tilde{\mathbf{p}}_2 = \tilde{\mathbf{p}}_2(\mathbf{p}_1, \mathbf{q}_2^\dagger, \tilde{\mathbf{p}}_1, \tilde{\mathbf{q}}_2^\dagger)$
- $\mathbf{q}_1^\dagger = \mathbf{q}_1^\dagger(\mathbf{p}_1, \mathbf{q}_2^\dagger, \tilde{\mathbf{p}}_1, \tilde{\mathbf{q}}_2^\dagger)$
- $\tilde{\mathbf{q}}_1^\dagger = \tilde{\mathbf{q}}_1^\dagger(\mathbf{p}_1, \mathbf{q}_2^\dagger, \tilde{\mathbf{p}}_1, \tilde{\mathbf{q}}_2^\dagger)$

Equations of Motion

- $\mathbf{p}_2 = (z_2 - z_1)\frac{\mathbf{L}_0\mathbf{p}_1}{\mathbf{q}_2^\dagger\mathbf{L}_0\mathbf{p}_1} + (z_1 - \zeta_2)\frac{\mathbf{L}_0\tilde{\mathbf{p}}_1}{\mathbf{q}_2^\dagger\mathbf{L}_0\tilde{\mathbf{p}}_1} = \frac{\partial\mathcal{L}}{\partial\mathbf{x}_2^\dagger}(\mathbf{Q},\widetilde{\mathbf{Q}})$
- $\tilde{\mathbf{p}}_2 = (z_2 - \zeta_1)\frac{\mathbf{p}_1}{\tilde{\mathbf{q}}_2^\dagger\mathbf{p}_1} + (\zeta_1 - \zeta_2)\frac{\tilde{\mathbf{p}}_1}{\tilde{\mathbf{q}}_2^\dagger\mathbf{L}_0\tilde{\mathbf{p}}_1} = -\frac{\partial\mathcal{L}}{\partial\mathbf{y}_2^\dagger}(\mathbf{Q},\widetilde{\mathbf{Q}})$
- $\mathbf{q}_1^\dagger = (z_1 - z_2)\frac{\mathbf{q}_2^\dagger\mathbf{L}_0}{\mathbf{q}_2^\dagger\mathbf{L}_0\mathbf{p}_1} + (z_2 - \zeta_1)\frac{\tilde{\mathbf{q}}_2^\dagger}{\tilde{\mathbf{q}}_2^\dagger\mathbf{p}_1} = -\frac{\partial\mathcal{L}}{\partial\mathbf{x}_1}(\mathbf{Q},\widetilde{\mathbf{Q}})$
- $\tilde{\mathbf{q}}_1^\dagger = (z_1 - \zeta_2)\frac{\mathbf{q}_2^\dagger\mathbf{L}_0}{\mathbf{q}_2^\dagger\mathbf{L}_0\tilde{\mathbf{p}}_1} + (\zeta_2 - \zeta_1)\frac{\tilde{\mathbf{q}}_2^\dagger}{\tilde{\mathbf{q}}_2^\dagger\tilde{\mathbf{p}}_1} = \frac{\partial\mathcal{L}}{\partial\mathbf{y}_1}(\mathbf{Q},\widetilde{\mathbf{Q}})$

Discrete Euler-Lagrange Equations

$$\frac{\partial\mathcal{L}}{\partial\mathbf{x}_2^\dagger}(\mathbf{Q},\widetilde{\mathbf{Q}}) = \mathbf{p}_2 = \underset{\sim}{\mathbf{p}}_2 = -\frac{\partial\mathcal{L}}{\partial\mathbf{y}_2^\dagger}(\underset{\sim}{\mathbf{Q}},\mathbf{Q})$$

$$-\frac{\partial\mathcal{L}}{\partial\mathbf{x}_1}(\mathbf{Q},\widetilde{\mathbf{Q}}) = \mathbf{q}_1^\dagger = \underset{\sim}{\widetilde{\mathbf{q}}}{}_1^\dagger = \frac{\partial\mathcal{L}}{\partial\mathbf{y}_1}(\underset{\sim}{\mathbf{Q}},\mathbf{Q})$$

The Lagrangian

Thus, we have the following

Theorem

The equations of the isospectral dynamic can be written in the Lagrangian form with

$$\begin{aligned}\mathcal{L}(\mathbf{X},\mathbf{Y}) = (z_2 - z_1)\log(\mathbf{x}_2^\dagger \mathbf{L}_0 \mathbf{x}_1) + (z_1 - \zeta_2)\log(\mathbf{x}_2^\dagger \mathbf{L}_0 \mathbf{y}_1) \\ + (\zeta_2 - \zeta_1)\log(\mathbf{y}_2^\dagger \mathbf{y}_1) + (\zeta_1 - z_2)\log(\mathbf{y}_2^\dagger \mathbf{x}_1)\end{aligned}$$

where $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2^\dagger)$ *and* $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2^\dagger)$.

The main idea of the proof is to take residues of the matrix function $\widetilde{\mathbf{L}}(z) = \mathbf{B}_2(z)\mathbf{L}_0\mathbf{B}_1(z) = \mathbf{L}_0\widetilde{\mathbf{B}}_1(z)\widetilde{\mathbf{B}}_2(z)$ at its poles. They will be matrices of rank-one, and we compare their row and column spaces. To do that we need some simple properties of elementary divisors.

Properties of Elementary Divisors

Lemma

Let $\mathbf{B}_i(z) = \mathbf{I} + \frac{\mathbf{p}_i\mathbf{q}_i^\dagger}{z-z_i}$. *Then*

- $\mathbf{B}_i^{-1}(z) = \mathbf{I} - \frac{\mathbf{p}_i\mathbf{q}_i^\dagger}{z-\zeta_i}, \qquad \zeta_i = z_i - \mathbf{q}_i^\dagger\mathbf{p}_i;$
- *if* $\mathbf{B}_i(z)\mathbf{w} = \mathbf{v}$, *then* $\mathbf{p}_i = \mathbf{p}_i(\mathbf{w},\mathbf{v},\mathbf{q}_i^\dagger) = (z_i - z)\frac{\mathbf{w}}{\mathbf{q}_i^\dagger\mathbf{w}} + (z-\zeta_i)\frac{\mathbf{v}}{\mathbf{q}_i^\dagger\mathbf{v}};$
- *if* $\mathbf{w}^\dagger\mathbf{B}_i(z) = \mathbf{v}^\dagger$, *then* $\mathbf{q}_i^\dagger = \mathbf{q}_i^\dagger(\mathbf{w}^\dagger,\mathbf{v}^\dagger,\mathbf{p}) = (z_i - z)\frac{\mathbf{w}^\dagger}{\mathbf{w}^\dagger\mathbf{p}_i} + (z-\zeta_i)\frac{\mathbf{v}^\dagger}{\mathbf{v}^\dagger\mathbf{p}_i}.$

Proof.

Let $\left(\mathbf{I} + \frac{\mathbf{p}_i\mathbf{q}_i^\dagger}{z-z_i}\right)\mathbf{w} = \mathbf{w} + \frac{\mathbf{q}_i^\dagger\mathbf{w}}{z-z_i}\mathbf{p}_i = \mathbf{v}$. Then $\mathbf{p}_i = \frac{z_i-z}{\mathbf{q}_i^\dagger\mathbf{w}}\mathbf{w} + \frac{z-z_i}{\mathbf{q}_i^\dagger\mathbf{w}}\mathbf{v}$. But since $\mathbf{q}_i^\dagger\mathbf{B}_i(z)\mathbf{w} = \frac{z-\zeta_i}{z-z_i}\mathbf{q}_i^\dagger\mathbf{w} = \mathbf{q}_i^\dagger\mathbf{v}$, we get $\mathbf{p}_i = \frac{z_i-z}{\mathbf{q}_i^\dagger\mathbf{w}}\mathbf{w} + \frac{z-\zeta_i}{\mathbf{q}_i^\dagger\mathbf{v}}\mathbf{v}$. □

Proof (Theorem)

Consider $\widetilde{\mathbf{L}}(z) = \mathbf{B}_2(z)\mathbf{L}_0\mathbf{B}_1(z) = \mathbf{L}_0\widetilde{\mathbf{B}}_1(z)\widetilde{\mathbf{B}}_2(z)$.

- res_{z_1} : $\mathbf{B}_2(z_1)\mathbf{L}_0\mathbf{p}_1\mathbf{q}_1^\dagger = \mathbf{L}_0\tilde{\mathbf{p}}_1\tilde{\mathbf{q}}_1^\dagger\widetilde{\mathbf{B}}_2(z_1)$

$$\mathbf{p}_2 = (z_2 - z_1)\frac{\mathbf{L}_0\mathbf{p}_1}{\mathbf{q}_2^\dagger\mathbf{L}_0\mathbf{p}_1} + (z_1 - \zeta_2)\frac{\mathbf{L}_0\tilde{\mathbf{p}}_1}{\mathbf{q}_2^\dagger\mathbf{L}_0\tilde{\mathbf{p}}_1}$$

- res_{z_2} : $\mathbf{p}_2\mathbf{q}_2^\dagger\mathbf{L}_0\mathbf{B}_1(z_2) = \mathbf{L}_0\widetilde{\mathbf{B}}_1(z_2)\tilde{\mathbf{p}}_2\tilde{\mathbf{q}}_2^\dagger$

$$\mathbf{q}_1^\dagger = (z_1 - z_2)\frac{\mathbf{q}_2^\dagger\mathbf{L}_0}{\mathbf{q}_2^\dagger\mathbf{L}_0\mathbf{p}_1} + (z_2 - \zeta_1)\frac{\tilde{\mathbf{q}}_2^\dagger}{\tilde{\mathbf{q}}_2^\dagger\mathbf{p}_1}.$$

This gives expressions for $\mathbf{p}_2$ and $\mathbf{q}_1^\dagger$.

Proof (continued).

To get the expressions for $\tilde{\mathbf{p}}_2$ and $\tilde{\mathbf{q}}_1^\dagger$, we need to consider the *inverse* map $\widetilde{\mathbf{L}}^{-1}(z) = \mathbf{B}_1^{-1}(z)\mathbf{L}_0^{-1}\mathbf{B}_2^{-1}(z) = \widetilde{\mathbf{B}}_2^{-1}(z)\widetilde{\mathbf{B}}_1^{-1}(z)\mathbf{L}_0^{-1}$, which has poles at ζ_1 and ζ_2. Thus,

- $\operatorname{res}_{\zeta_2}$: $\quad -\mathbf{p}_1\mathbf{q}_1^\dagger\mathbf{L}_0^{-1}\mathbf{B}_2^{-1}(\zeta_1) = -\widetilde{\mathbf{B}}_2^{-1}(\zeta_1)\tilde{\mathbf{p}}_1\tilde{\mathbf{q}}_1^\dagger\mathbf{L}_0^{-1}$

$$\tilde{\mathbf{p}}_2 = (z_2 - \zeta_1)\frac{\mathbf{p}_1}{\tilde{\mathbf{q}}_2^\dagger\mathbf{p}_1} + (\zeta_1 - \zeta_2)\frac{\tilde{\mathbf{p}}_1}{\tilde{\mathbf{q}}_2^\dagger\tilde{\mathbf{p}}_1}.$$

- $\operatorname{res}_{\zeta_2}$: $\quad -\mathbf{B}_1^{-1}(\zeta_2)\mathbf{L}_0^{-1}\mathbf{p}_2\mathbf{q}_2^\dagger = -\tilde{\mathbf{p}}_2\tilde{\mathbf{q}}_2^\dagger\widetilde{\mathbf{B}}_1^{-1}(\zeta_2)\mathbf{L}_0^{-1}.$

$$\tilde{\mathbf{q}}_1^\dagger = (z_1 - \zeta_2)\frac{\mathbf{q}_2^\dagger\mathbf{L}_0}{\mathbf{q}_2^\dagger\mathbf{L}_0\tilde{\mathbf{p}}_1} + (\zeta_2 - \zeta_1)\frac{\tilde{\mathbf{q}}_2^\dagger}{\tilde{\mathbf{q}}_2^\dagger\tilde{\mathbf{p}}_1}.$$

□

Isomonodromic Case: The Lagrangian

Similarly to the isospectral case, we have

Theorem

The equations of the isomonodromic dynamic are given by the time-dependent Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \mathbf{Y}}(\mathbf{Q}_{k-1}, \mathbf{Q}_k, k-1) + \frac{\partial \mathcal{L}}{\partial \mathbf{X}}(\mathbf{Q}_k, \mathbf{Q}_{k+1}, k) = 0$$

with the Lagrangian function

$$\begin{aligned}\mathcal{L}(\mathbf{X}, \mathbf{Y}, t) = (z_2(t) - z_1)\log(\mathbf{x}_2^\dagger \mathbf{L}_0 \mathbf{x}_1) + (z_1 - \zeta_2(t))\log(\mathbf{x}_2^\dagger \mathbf{L}_0 \mathbf{y}_1)\\ + (\zeta_2(t) - \zeta_1)\log(\mathbf{y}_2^\dagger \mathbf{y}_1) + (\zeta_1 - z_2(t))\log(\mathbf{y}_2^\dagger \mathbf{x}_1)\end{aligned},$$

where $z_2(t) = z_2 - t$ *and* $\zeta_2(t) = \zeta_2 - t$.

Isomonodromic Case: Reduction to dPV

This part closely follows one of the examples considered by **D. Arinkin** and **A. Borodin**.

- Restrict our attention to the rank-two case, let

$$\mathbf{L}_\infty = -\operatorname{res}_\infty \mathbf{L}(z)\,dz = \mathbf{L}_1 + \mathbf{L}_2, \qquad \text{and } k_i = \frac{1}{\rho_i}(\mathbf{L}_\infty)_{ii}.$$

- Define type $\theta(\mathbf{L}(z)) = \{z_1, z_2, \zeta_1, \zeta_2, \rho_1, \rho_2, k_1, k_2\}$ with the single relation $k_1 + k_2 = z_1 - \zeta_1 + z_2 - \zeta_2$, and let $\mathcal{M}_2^\theta$ be the space of matrices of type θ. Factoring out by the action of the diagonal matrices we get a surface $\widehat{\mathcal{M}_2^\theta}$.
- Introduce the following spectral coordinates (γ, π) on this space:

$$\gamma : (\mathbf{L}(\gamma))_{12} = 0 \qquad \text{and} \qquad \pi = \frac{\gamma - z_1}{\gamma - z_2}(\mathbf{L}(\gamma))_{11}.$$

Then

- The isomonodromic transformation that we consider defines a birational map $\psi : \widehat{\mathcal{M}}_2^{\theta} \to \widehat{\mathcal{M}}_2^{\tilde{\theta}}$, where $\tilde{z}_2 = z_2 - 1$, $\tilde{\zeta}_2 = \zeta_2 - 1$ and all other parameters are unchanged.
- In the spectral coordinates (γ, π) on $\widehat{\mathcal{M}}_2^{\theta}$ and $(\tilde{\gamma}, \tilde{\pi})$ on $\widehat{\mathcal{M}}_2^{\tilde{\theta}}$ this map is given by the difference Painlev equation dPV of the Sakais hierarchy,

$$\begin{cases} \gamma + \tilde{\gamma} = z_2 + \tilde{\zeta}_2 - \dfrac{\rho_1(k_2 + \zeta_1 - z_2)}{\pi - \rho_1} + \dfrac{\rho_2(k_2 + \tilde{\zeta}_2 - z_1)}{\pi - \rho_2} \\ \pi\tilde{\pi} = \rho_1\rho_2 \dfrac{(\tilde{\gamma} - \zeta_1)(\tilde{\gamma} - z_1)}{(\tilde{\gamma} - \tilde{\zeta}_2)(\tilde{\gamma} - \tilde{z}_2)} \end{cases}.$$

Proof is a direct calculation that uses the following explicit expressions of our rank-one matrices in the spectral coordinates:

$$\mathbf{L}_1 = \frac{\gamma - z_1}{z_2 - z_1} \begin{bmatrix} 1 \\ \frac{\rho_2}{\pi}\mu(z_2,\zeta_1) + \rho_2 k_2 \end{bmatrix} \begin{bmatrix} -\frac{\gamma - z_2}{\gamma - z_1}\mu(\zeta_2, z_1) + \rho_1 k_1 & 1 \end{bmatrix},$$

$$\mathbf{L}_2 = \frac{\gamma - z_2}{z_1 - z_2} \begin{bmatrix} 1 \\ \frac{\rho_2(\gamma - z_1)}{\pi(\gamma - z_2)}\mu(z_2,\zeta_1) + \rho_2 k_2 \end{bmatrix} \begin{bmatrix} -\mu(\zeta_2, z_1) + \rho_1 k_1 & 1 \end{bmatrix},$$

$$\mathbf{G}_1 = \frac{\pi}{\rho_1(\pi - \rho_1)} \begin{bmatrix} 1 \\ \frac{\rho_1}{\pi}\mu(z_2,\zeta_1) + \rho_1 k_2 \end{bmatrix} \begin{bmatrix} -\frac{\rho_1}{\pi}\mu(\zeta_2, z_1) + \rho_1 k_1 & 1 \end{bmatrix},$$

$$\mathbf{G}_2 = \frac{1}{(\rho_1 - \pi)} \begin{bmatrix} 1 \\ \mu(z_2,\zeta_1) + \rho_1 k_2 \end{bmatrix} \begin{bmatrix} -\mu(\zeta_2, z_1) + \rho_1 k_1 & 1 \end{bmatrix},$$

where $\mu(a,b) = \pi(\gamma - a) - \rho_1(\gamma - b)$.