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CRM
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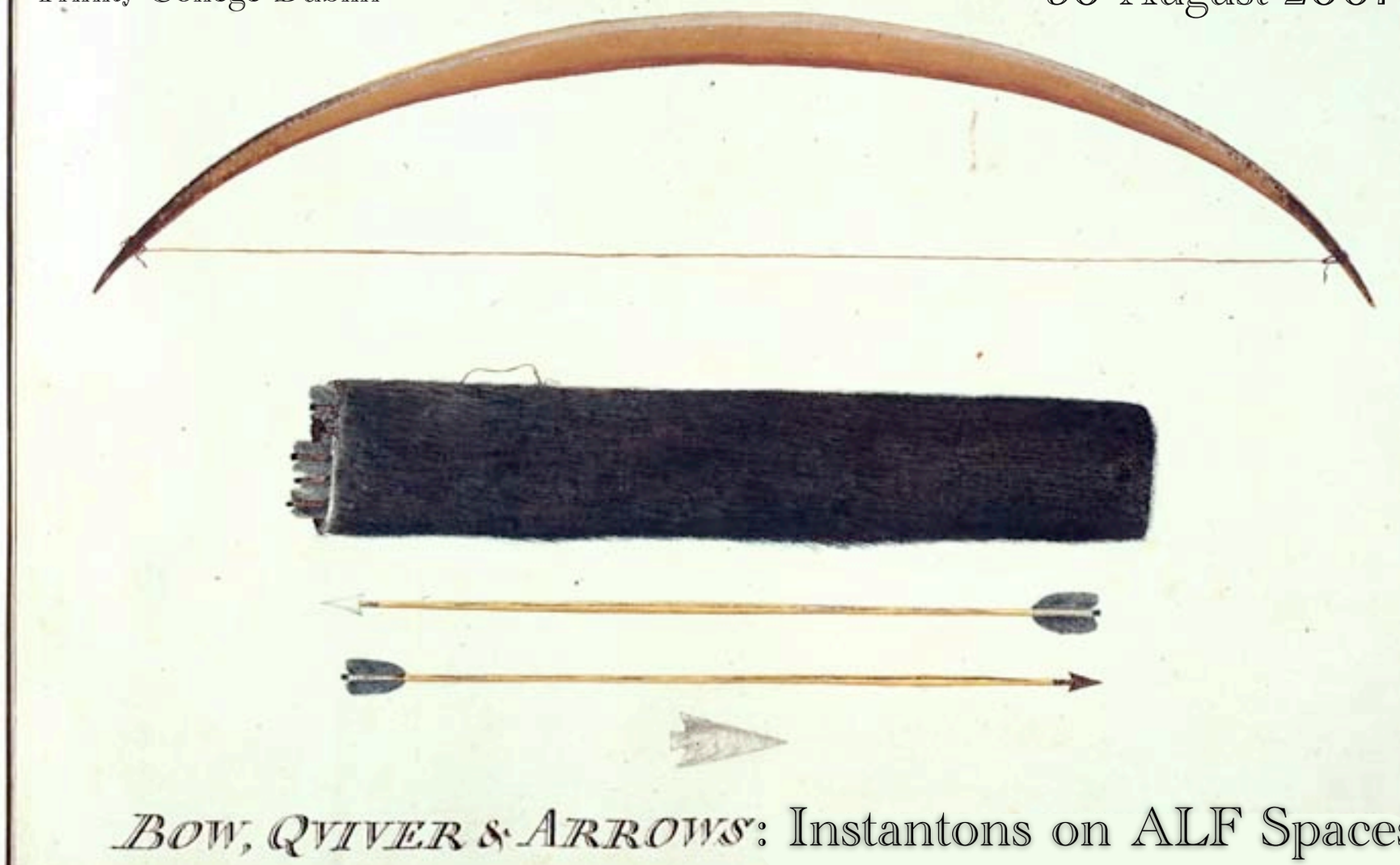


Instantons on ALF Spaces

From: http://www.captcook-ne.co.uk/ccne/themes/objects.htm

Self-dual Gravitational Instantons

are complete four-dimensional Riemannian manifolds that satisfy one of the following equivalent conditions:

i. hyperkähler
ii. admits covariantly constant spinors
iii. Calabi-Yau two-fold
iv. preserves 1/2 Supersymmetry
v. self-dual curvature form

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}R^{\mu\nu}{}_{\gamma\delta}$$

Distinguished by

- Asymptotic behaviour: K3
- Topology: A, D, E, etc.

<u>Questions</u>:

Classification, metrics, <u>Yang-Mills Instantons</u>

Conjecture:

Any gravitational instanton metric with finite Pontrjagin number asymptotically approaches a metric with a local triholomorphic isometry.

$$\int_M R \wedge R < \infty \qquad \Longrightarrow \qquad ds^2 \xrightarrow[|\vec{x}| \to \infty]{} V^{-1}(d\theta + \omega)^2 + V d\vec{x}^2$$

ALE	$V = \dfrac{1}{\lvert\vec{x}\rvert}$
ALF A_k and D_k	$V = C + \dfrac{1}{\lvert\vec{x}\rvert}$
ALG	$V = C + \dfrac{N}{2}\log(x_1^2 + x_2^2)$
ALH	$V = C_1 + C_2 x_1$

The Taub-NUT Space

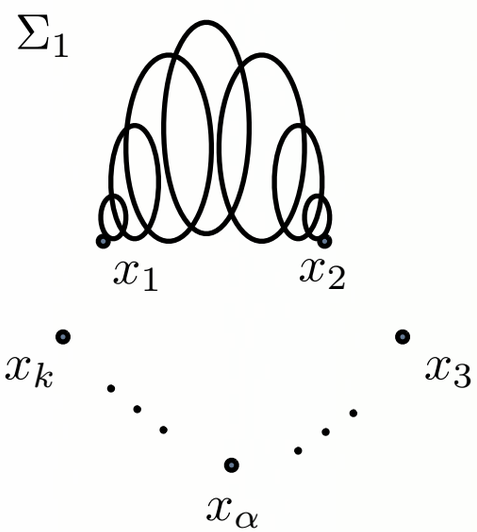
$$ds^2 = V^{-1}\left(d\theta + \omega\right)^2 + V d\vec{x}^2,$$

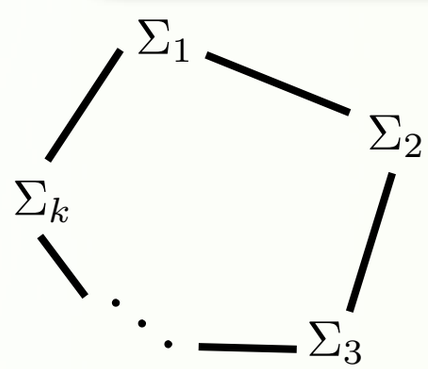
$$d\omega = *_3 dV, \theta \sim \theta + 4\pi, \qquad V = l + \frac{1}{|\vec{x}|}$$

Self-dual Abelian connection:

$$a_0 = \frac{s}{4\pi}\frac{d\theta + \omega}{V}$$

Multi-Taub-NUT (Ak ALF)





$$V = l + \sum_{\alpha=1}^{k+1} \frac{1}{|\vec{x} - \vec{x}_\alpha|}$$

Self-dual Abelian connections:

$$a_\alpha = \frac{1}{4\pi}\left((V_\alpha - V_{\alpha+1})\frac{d\theta + \omega}{V} + \omega_\alpha - \omega_{\alpha+1}\right) \qquad V_\alpha = \frac{1}{|\vec{x} - \vec{x}_\alpha|}$$

$$a_0 = \frac{s}{4\pi}\frac{d\theta + \omega}{V} \qquad d\omega_\alpha = *dV_\alpha$$

Instantons on ALF Spaces

$$F = *F$$

Action $S = \int F \wedge *F$ is finite

Monodromy at infinity $\left(\frac{\partial}{\partial\theta} - iA_\theta\right) W(\vec{x},\theta) = 0,\ W(\vec{x},0) = 1$ $\quad W = \lim_{x\to\infty} W(\vec{x}, 4\pi)$

•Maximal Symmetry Breaking:

EigenValues of W are distinct $-\frac{\pi}{l} < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \frac{\pi}{l}$

EigenBundles of W are line bundles $\mathcal{L}_i \to S^2_\infty$ with Chern classes j_i

Monopole Charges: if $M=\min(j_1, j_1+j_2,\ldots, j_1+j_2+\ldots+j_n)$

then the monopole charges are $(m_1,m_2,\ldots,m_n)=(j_1-M, j_1+j_2-M,\ldots, j_1+j_2+\ldots+j_n-M)$

Instanton Number:

$$n = \frac{1}{32\pi^2}\int \mathrm{Tr}\ F\wedge F - (m_1(l\lambda_1+\pi) + m_2 l(\lambda_2-\lambda_1) + \ldots m_n(\pi - l\lambda_n))$$

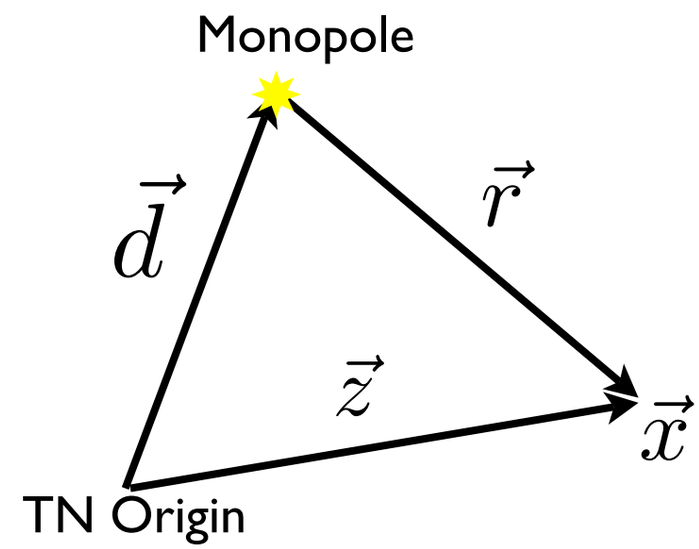
Question: Find explicit SD connections on ALF spaces

Explicit Solution on TN: m=1, n=0

SCh & Brian Durcan

$$V = l + \frac{1}{2z},$$

$$\begin{aligned} a &= z + d \\ \mathcal{D} &= a^2 - r^2 = 2zd + 2\vec{z}\cdot\vec{d}. \end{aligned}$$

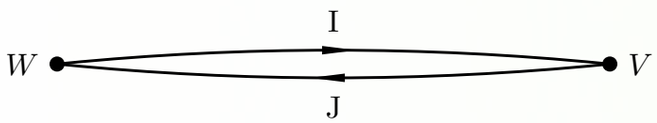


$$\begin{aligned} \mathcal{K} &= \mathcal{K}(a,a) = (a^2+r^2)\cosh(2\lambda r) + 2ra\sinh(2\lambda r) \\ \mathcal{L} &= \mathcal{L}(a,a) = (a^2+r^2)\sinh(2\lambda r) + 2ra\cosh(2\lambda r) \end{aligned}$$

$$\begin{aligned} \mathcal{A} &= \frac{1}{\mathcal{L}}\Big\{\vec{dx}\cdot(\vec{\sigma}\times\vec{r})\Big(\frac{(2\lambda r - \sinh(2\lambda r))\mathcal{D}}{2r^2} - \sinh(2\lambda r)\left(1 + \frac{a}{r}\tanh(\lambda r)\right)\Big) \\ &+ \frac{\vec{dx}\cdot(\vec{r}\times\vec{z})}{z}\frac{\vec{\sigma}\cdot\vec{r}}{r}\left(1 - \frac{\mathcal{K}}{\mathcal{D}}\right) - \frac{r}{z}\vec{dx}\cdot(\vec{\sigma}\times\vec{z}) \\ &+ \frac{d\theta - \omega}{V}\Big(\frac{\vec{\sigma}\cdot\vec{r}}{r}\left((\lambda + \frac{1}{2z})\mathcal{K} - \frac{\mathcal{L}}{2r}\right) - \frac{r}{z}\vec{\sigma}\cdot\vec{d}_\perp\Big)\Big\} \end{aligned}$$

Next Question: Find explicit m=0, n=1 SD connections on TN

Ingredients 1: Arrows and Limbs



$V = \mathbb{C}^v$ and $W = \mathbb{C}^w$

$g_v : (I, J) \mapsto (g_v^{-1} I, J g_v)$

$g_w : (I, J) \mapsto (I g_w, g_w^{-1} J)$

Moment maps:

$$\mu_V^{\mathbb{C}} = \mu_V^1 + i\mu_V^2 = IJ, \quad \mu_V^{\mathbb{R}} = \mu_V^3 = \frac{1}{2}(J^\dagger J - II^\dagger),$$

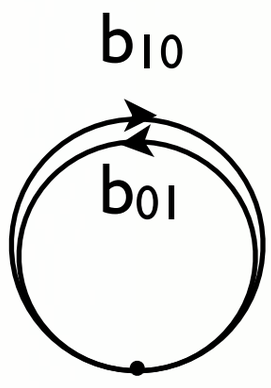
$$\mu_W^{\mathbb{C}} = \mu_W^1 + i\mu_W^2 = -JI, \quad \mu_W^{\mathbb{R}} = \mu_W^3 = \frac{1}{2}(I^\dagger I - JJ^\dagger).$$

A convenient way of writing the moment maps is

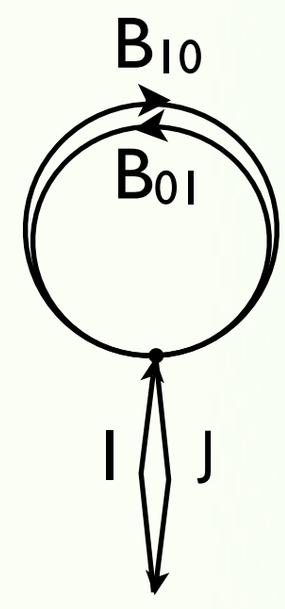
$$Q_V = \begin{pmatrix} J^\dagger \\ I \end{pmatrix} \qquad \not{\mu}_V = \mu_V^i \sigma_i = \mathrm{Vec}(Q_V Q_V^\dagger) \qquad \mathrm{Vec}(M^0 + M^j \sigma_j) = M^j \sigma_j$$

Example: ADHM

$\mathbb{R}^4$

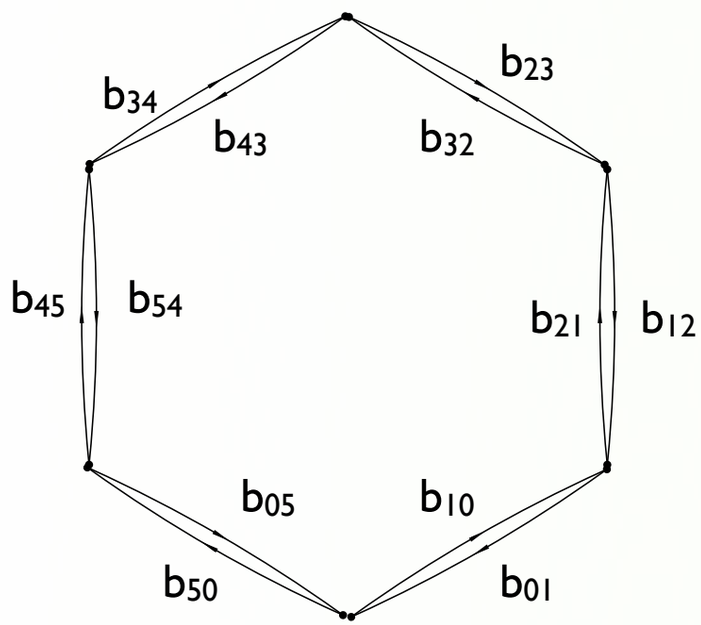


Instanton Data:



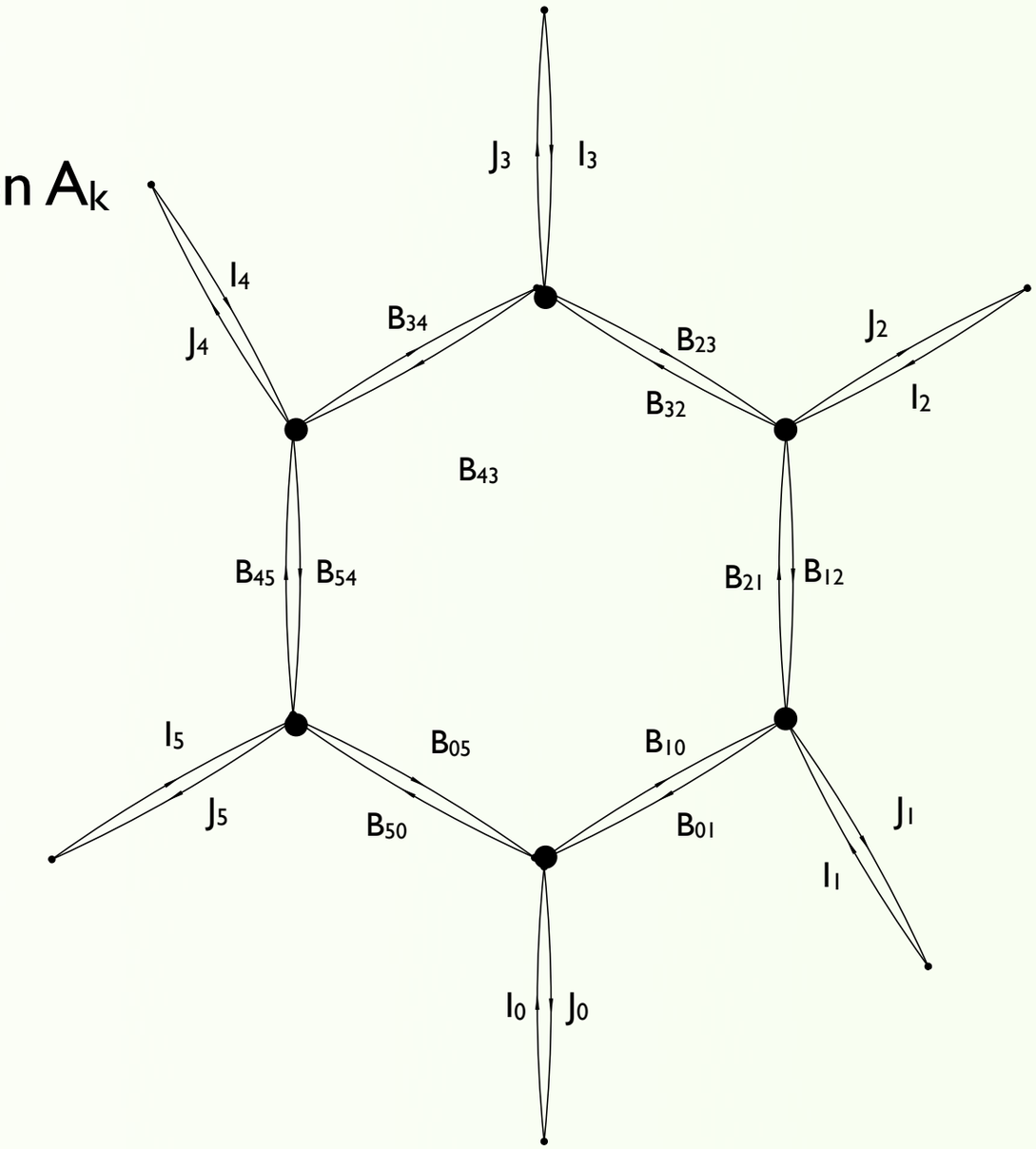
Example: Kronheimer & Nakajima (Instantons on $\widetilde{\mathbb{R}^4/\Gamma}$)

A_k ALE: $\widetilde{\mathbb{R}^4/\mathbb{Z}_{k+1}}$



Affine Dynkin diagram

Instantons on A_k ALE



Moment maps at V_1

$$\mu^C = B_{10}\, B_{01} - B_{12} B_{21} + I_1 J_1$$

$$\mu^R = B^+_{01}\, B_{01} - B_{10} B^+_{10} + B_{12}\, B^+_{12} - B^+_{21} B_{21} + I_1 I^+_1 - J^+_1 J_1$$

Ingredients 2: Strings

$$(T_0(s), T_1(s), T_2(s), T_3(s)).$$

$$g(s) : \begin{pmatrix} T_0(s) \\ T_1(s) \\ T_2(s) \\ T_3(s) \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}T_0 g + i g^{-1}\frac{d}{ds} g \\ g^{-1}T_1 g \\ g^{-1}T_2 g \\ g^{-1}T_3 g \end{pmatrix}$$

$$\begin{aligned} \mu^1 &= \frac{d}{ds}T_1 - i[T_0, T_1] + i[T_2, T_3], \\ \mu^2 &= \frac{d}{ds}T_2 - i[T_0, T_2] + i[T_3, T_1], \\ \mu^3 &= \frac{d}{ds}T_3 - i[T_0, T_3] + i[T_1, T_2]. \end{aligned}$$

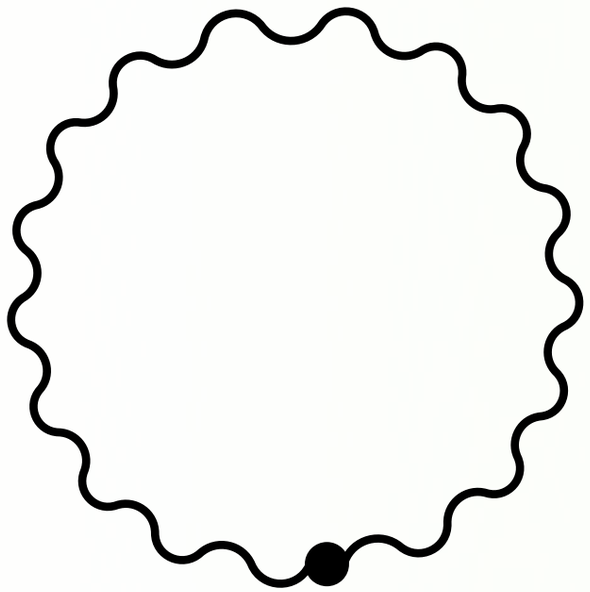
Convenient Notation:

$$\not{T} = \sigma_1 \otimes T_1 + \sigma_2 \otimes T_2 + \sigma_3 \otimes T_3$$

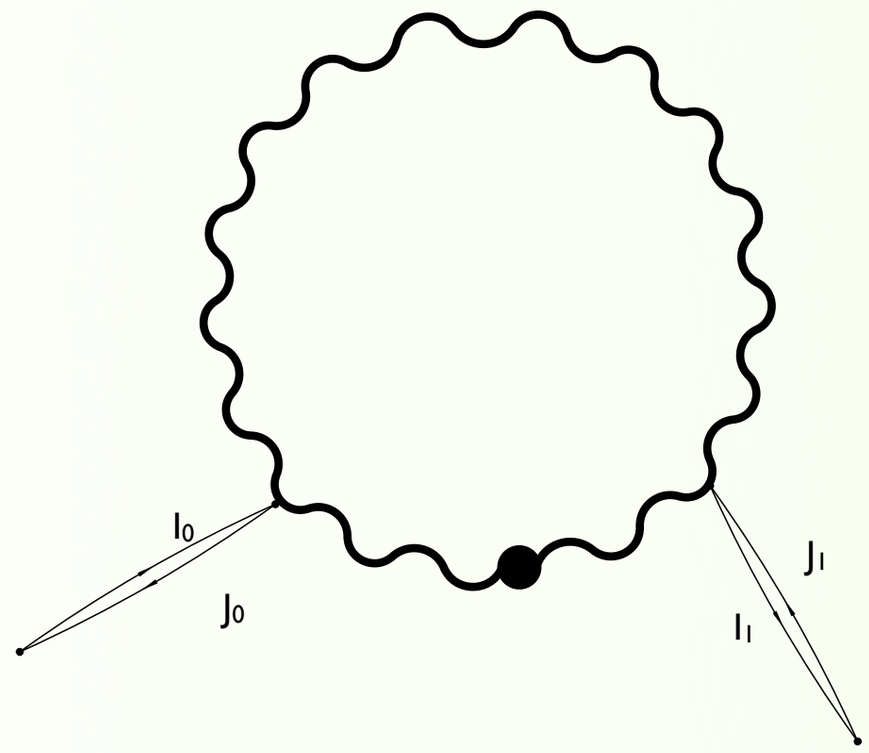
$$\not{\mu} = [\tfrac{d}{ds} - iT_0, \not{T}] + \mathrm{Vec}(\not{T}, \not{T}).$$

Example: Calorons

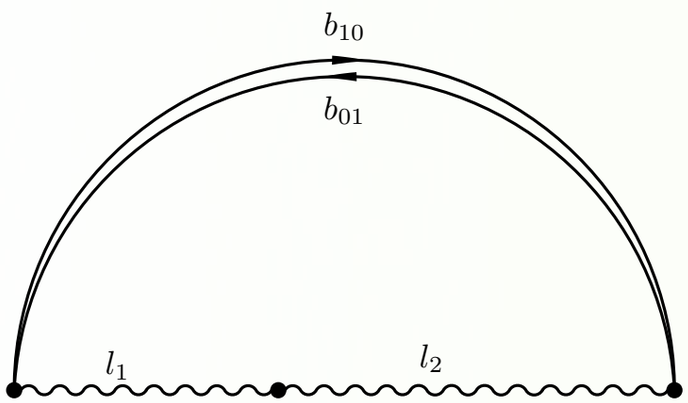
$\mathbb{R}^3 \times \mathbf{S}^1$



Instantons on $\mathbb{R}^3 \times \mathbf{S}^1$



Taub-NUT Bow Diagram



$$\begin{pmatrix} t_0 \\ t_j \\ b_{01} \\ b_{10} \end{pmatrix} \mapsto \begin{pmatrix} h^{-1}t_0 h + ih^{-1}\frac{d}{ds}h \\ h^{-1}t_j h \\ h^{-1}(-\frac{l}{2})b_{01}h(\frac{l}{2}) \\ h^{-1}(\frac{l}{2})b_{10}h(-\frac{l}{2}) \end{pmatrix}$$

$$t = t_1 + it_2 \text{ and } \mathbf{D} = d/ds - it_0 - t_3$$

Moment maps:

$$[\mathbf{D}, t] - \delta_{(s+\frac{l}{2})} b_{01} b_{10} + \delta_{(s-\frac{l}{2})} b_{10} b_{01} = 0,$$

$$[\mathbf{D}^\dagger, \mathbf{D}] + [t^\dagger, t] + \delta_{(s+\frac{l}{2})}(b_{10}^\dagger b_{10} - b_{01} b_{01}^\dagger) + \delta_{(s-\frac{l}{2})}(b_{01}^\dagger b_{01} - b_{10} b_{10}^\dagger) = 0.$$

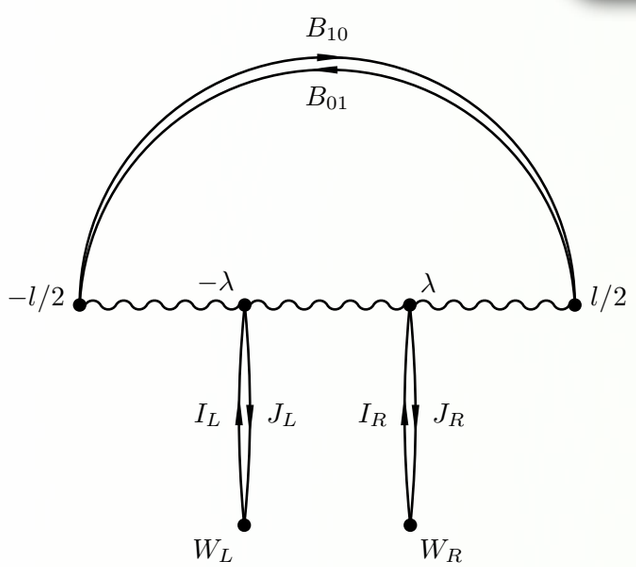
$$ds^2 = \frac{1}{4}\left[(l + \frac{1}{r})d\vec{r}^2 + \frac{1}{l + 1/r}(d\tau + \omega)^2\right] + l_1 \frac{l + 1/r}{l_1 + 1/r}\left[dt_0 + \frac{d\tau + \omega}{2(l + 1/r)}\right]^2.$$

Metric

$$\phi = -l_1 t_0.$$

Natural Connection: $a_0 = \frac{l_1}{2}(d\tau + \omega)/(l + 1/r).$

k_0 Instantons on Taub-NUT



- a rank k_0 vector bundle $E \to [-l/2, l/2]$ with the Nahm data $(T_0, \vec{T})$ on the intervals $[-l/2, -\lambda], [-\lambda, \lambda]$, and $[\lambda, l/2]$ (we do not presume continuity at $s = \pm\lambda$),
- linear maps $B_{10} : E_{-l/2} \to E_{l/2}$ and $B_{01} : E_{l/2} \to E_{-l/2}$,
- linear maps $I_L : W_L \to E_{-\lambda}$, $J_L : E_{-\lambda} \to W_L$, $I_R : W_R \to E_\lambda$, and $J_R : E_\lambda \to W_R$.

$$\begin{pmatrix} T_0 \\ T_j \\ B_{01} \\ B_{10} \\ I_\alpha \\ J_\alpha \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}(s) T_0 g(s) + i g^{-1}(s) \frac{d}{ds} g(s) \\ g^{-1}(s) T_j g(s) \\ g^{-1}(-\frac{l}{2}) B_{01} g(\frac{l}{2}) \\ g^{-1}(\frac{l}{2}) B_{10} g(-\frac{l}{2}) \\ g^{-1}(\lambda_\alpha) I_\alpha \\ J_\alpha g(\lambda_\alpha) \end{pmatrix}$$

Let $D = \frac{d}{ds} - iT_0 - T_3$ and $T = T_1 + iT_2$,

Moment map conditions:

$$[D, T] - \delta(s + \tfrac{l}{2}) B_{01} B_{10} + \delta(s - \tfrac{l}{2}) B_{10} B_{01} + \sum_{\alpha \in \{L,R\}} \delta(s - \lambda_\alpha) I_\alpha J_\alpha = 0$$

$$[D^\dagger, D] + [T^\dagger, T] + \delta(s + \tfrac{l}{2})(B_{10}^\dagger B_{10} - B_{01} B_{01}^\dagger) + \delta(s - \tfrac{l}{2})(B_{01}^\dagger B_{01} - B_{10} B_{10}^\dagger) + \\ + \sum_{\alpha \in \{L,R\}} \delta(s - \lambda_\alpha)(J_\alpha^\dagger J_\alpha - I_\alpha I_\alpha^\dagger) = 0.$$

Nahm Transform

Weyl Operator:

$$\mathfrak{D} : f \mapsto \begin{pmatrix} (-\frac{d}{ds} + iT_0 + \mathbb{T})f \\ (J_L, I_L^\dagger)f(-\lambda) \\ (J_R, I_R^\dagger)f(\lambda) \\ (B_{01}, B_{10}^\dagger)f(l/2) \\ (-B_{10}, B_{01}^\dagger)f(-l/2) \end{pmatrix}$$

$\chi_\alpha \in E_{\lambda_\alpha}$, $v_- \in E_{-l/2}$ and $v_+ \in E_{l/2}$

cokernel of $\mathfrak{D}$ is given by $(\psi(s), \chi_L, \chi_R, v_-, v_+)$

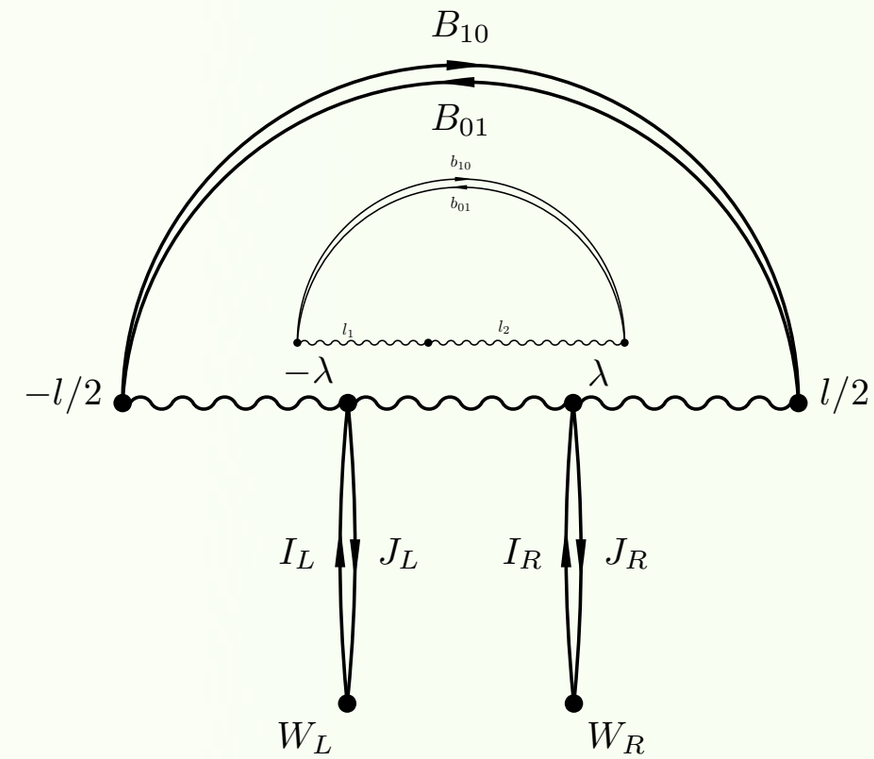
$$\boxed{\begin{aligned} &\left(\frac{d}{ds} - iT_0 + \mathbb{T}\right)\psi = 0, \text{ on } \mathcal{I}\backslash\{\alpha_L, \alpha_R\}, \\ &\psi(\lambda_\alpha+) - \psi(\lambda_\alpha-) = -Q_\alpha \chi_\alpha, \\ &\psi(\frac{l}{2}) \quad = \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix} v_-, \\ &\psi(-\frac{l}{2}) = -\begin{pmatrix} -B_{10}^\dagger \\ B_{01} \end{pmatrix} v_+. \end{aligned}}$$

$$\begin{aligned} \mathfrak{D}^\dagger \;=\; & \begin{pmatrix} -D^\dagger & T^\dagger \\ T & D \end{pmatrix} \oplus \left(\bigoplus_{\alpha \in \{L,R\}} \delta_{(s-\lambda_\alpha)} \begin{pmatrix} J_\alpha^\dagger \\ I_\alpha \end{pmatrix} \right) \\ & \oplus \left(\delta_{(s+\frac{l}{2})} \begin{pmatrix} B_{10}^\dagger \\ -B_{01} \end{pmatrix}, \; \delta_{(s-\frac{l}{2})} \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix} \right). \end{aligned}$$

Moment map conditions are equivalent to $\mathrm{Vec}(\mathfrak{D}^\dagger \mathfrak{D}) = 0.$

Twisted dual Weyl Operator:

given point of the Taub-NUT space $(t_0, \vec{t}, b_{10}, b_{01})$



$$\mathfrak{D}_t^\dagger = \begin{pmatrix} -D^\dagger - t_3 & T^\dagger - t^\dagger \\ T - t & D + t_3 \end{pmatrix} \oplus \left(\bigoplus_{\alpha \in \{L,R\}} \delta(s - \lambda_\alpha) \begin{pmatrix} J_\alpha^\dagger \\ I_\alpha \end{pmatrix} \right)$$
$$\oplus \left(\delta_{(s+\frac{l}{2})} \begin{pmatrix} B_{10}^\dagger & -b_{10}^\dagger \\ -B_{01} & -b_{01} \end{pmatrix} + \delta_{(s-\frac{l}{2})} \begin{pmatrix} -b_{01}^\dagger & B_{01}^\dagger \\ b_{10} & B_{10} \end{pmatrix} \right).$$

ψ a section of $E \otimes e \otimes \mathbb{C}^2 \to \mathcal{I} \backslash \{-\lambda, \lambda\}$.

$v_- \in E_{-l/2} \otimes e_{l/2}$ and $v_+ \in E_{l/2} \otimes e_{-l/2}$.

For

$$\boldsymbol{\psi}_1 = (\psi_1(s), \chi_{L1}, \chi_{R1}^+, v_1)$$
$$\boldsymbol{\psi}_2 = (\psi_2(s), \chi_{L2}, \chi_{R2}, v_2)$$

there is a natural Hermitian product

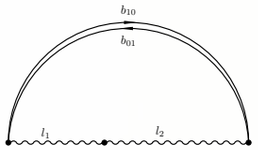
$$(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) = v_1^\dagger v_2 + (\chi_{L1})^\dagger \chi_{L2} + (\chi_{R1})^\dagger \chi_{R2} + \int_{-l/2}^{l/2} \psi_1^\dagger(s) \psi_2(s) ds.$$

the operator $\mathbf{s}$ acting on $\boldsymbol{\psi}$ as follows

$$\mathbf{s} : (\psi(s), \chi_L, \chi_R, v) \mapsto \left(s\psi(s), -\lambda\chi_L, \lambda\chi_R, \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} v \right).$$

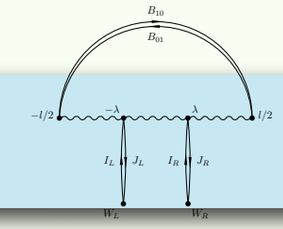
the self-dual connection on TN is

$$A = \left(\Psi, \left(\frac{\partial}{\partial \tau} + \frac{\mathbf{s}}{V} \right) \Psi \right) d\tau + \left(\Psi, \left(\frac{\partial}{\partial x_j} + \omega_j \frac{\mathbf{s}}{V} \right) \Psi \right) dx_j$$



$$\mathfrak{D}^\dagger \Psi = 0$$

Solution of



Taub-NUT data

$$b_- = \begin{pmatrix} -b_{01}^\dagger \\ b_{10} \end{pmatrix}, \; b_+ = \begin{pmatrix} -b_{10}^\dagger \\ -b_{01} \end{pmatrix}$$

$$b_\pm b_\pm^\dagger = |\vec{t}| \pm t$$

Instanton data is simple:

$$\vec{T}(s) = \begin{cases} \vec{T}_1 & \text{for} - l/2 < s < -\lambda \text{ or } \lambda > s > l/2 \\ \vec{T}_2 & \text{for} - \lambda < s < \lambda \end{cases}$$

$$\vec{y} = \vec{T}_2 - \vec{T}_1 = \vec{z}_1 - \vec{z}_2$$

$$B_- = \begin{pmatrix} B_{10}^\dagger \\ -B_{01} \end{pmatrix}, \; B_+ = \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix} \qquad Q_R = Q_+ \text{ and } Q_L = Q_-.$$

$$B_\pm B_\pm^\dagger = |\vec{T}_1| \pm T_1 \qquad Q_\pm Q_\pm^\dagger = y \pm y.$$

Solution:

$$\mathcal{D} = F\bar{F} = (T_1 + t)^2 - z_1^2.$$

$$\mu_\pm = \sqrt{\frac{T_1 + t + \sqrt{\mathcal{D}}}{2}} \pm \sqrt{\frac{T_1 + t - \sqrt{\mathcal{D}}}{2}} \frac{z_1}{z_1},$$

$$\Pi = \frac{1}{2g}\left(e^{-i\theta} e^{\lambda z_2}(y - y) e^{-(l/2-\lambda) z_1} \mu_- + e^{i\theta} e^{-\lambda z_2}(y + y) e^{(l/2-\lambda) z_1} \mu_+\right)$$

$$v = \frac{1}{\sqrt{\mathcal{D}}} \begin{pmatrix} e^{i\tau/2} B_-^\dagger \mu_+ \\ e^{-i\tau/2} B_+^\dagger \mu_- \end{pmatrix}$$

$$\begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} Q_+^\dagger e^{-\lambda z_2} \\ Q_-^\dagger e^{\lambda z_2} \end{pmatrix} \frac{e^{-i\theta} e^{-\lambda z_2} e^{-(\frac{l}{2}-\lambda) z_1} \mu_- - e^{i\theta} e^{\lambda z_2} e^{(\frac{l}{2}-\lambda) z_1} \mu_+}{2g}.$$

$$\psi(s) = \begin{cases} e^{z_1 (s+l/2} e^{i\theta} \mu_+ & \text{for} - l/2 < s < -\lambda \\ e^{z_2 s} \Pi & \text{for} - \lambda < s < \lambda \\ e^{z_1 (s-l/2} e^{-i\theta} \mu_- & \text{for} \lambda < s < l/2 \end{cases}.$$

$$g = y \cosh 2 z_2 \lambda - \frac{\vec{z}_2 \cdot \vec{y}}{z_2} \sinh 2 z_2 \lambda$$

For this Ψ $\quad (\Psi, \Psi) = m\mathbb{I}$

$$
\begin{aligned}
m =& 2 + \frac{1}{g}\Big\{ -\sqrt{\mathcal{D}}\cos 2\theta \\
&+ \frac{(T_1+t)\sinh z_1 d - z_1 \cosh z_1 d}{z_1}\left(y\cosh 2\lambda z_2 + \frac{\vec{z}_1\cdot\vec{z}_2 - y^2}{z_2}\sinh 2\lambda z_2 \right) \\
&+ \frac{(T_1+t)\cosh z_1 d - z_1 \sinh z_1 d}{z_2}(z_2\cosh 2\lambda z_2 + y\sinh 2\lambda z_2)\Big\}. \qquad (50)
\end{aligned}
$$

$$
v = \frac{1}{\sqrt{\mathcal{D}}}\begin{pmatrix} e^{i\tau/2}B_-^\dagger\mu_+ \\ e^{-i\tau/2}B_+^\dagger\mu_- \end{pmatrix}
$$

$$
\psi(s) = \begin{cases} e^{\not{z}_1(s+l/2)}e^{i\theta}\psi_L\mu_+ & \text{for} -l/2 < s < -\lambda \\ e^{\not{z}_2 s}\Pi & \text{for} -\lambda < s < \lambda \\ e^{\not{z}_1(s-l/2)}e^{-i\theta}\psi_R\mu_- & \text{for} \lambda < s < l/2 \end{cases}
$$

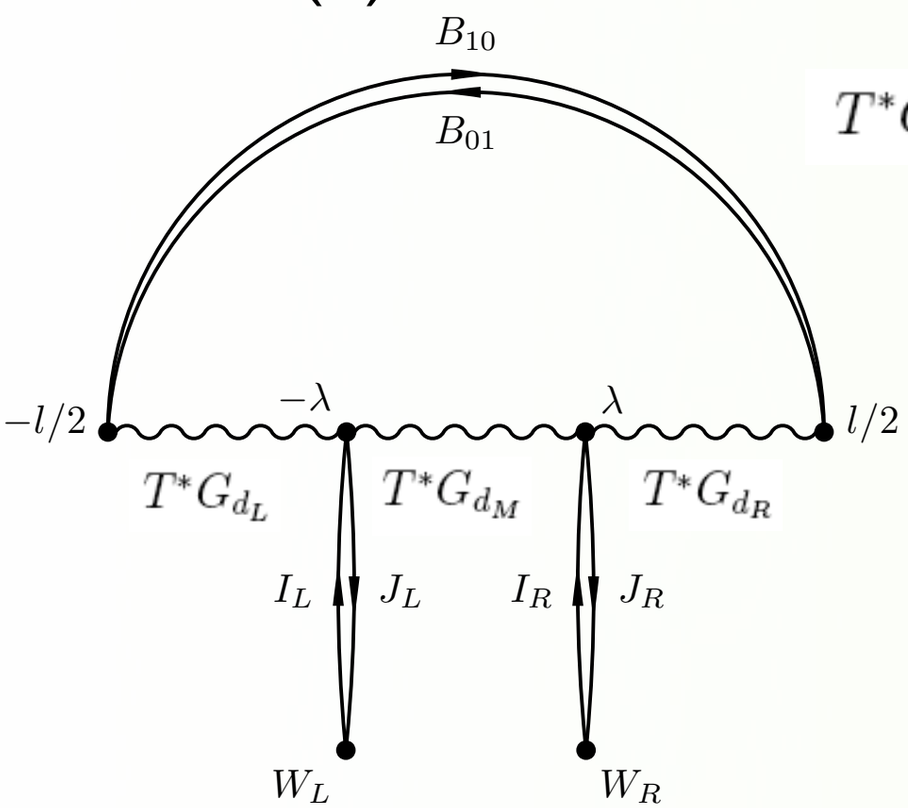
$$
\begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} Q_+^\dagger e^{-\lambda\not{z}_2} \\ Q_-^\dagger e^{\lambda\not{z}_2} \end{pmatrix} \frac{e^{-i\theta}e^{-\lambda\not{z}_2}e^{-(\frac{l}{2}-\lambda)\not{z}_1}\mu_- - e^{i\theta}e^{\lambda\not{z}_2}e^{(\frac{l}{2}-\lambda)\not{z}_1}\mu_+}{2g}.
$$

$$
\boxed{A = \left(\Psi, \left(\frac{\partial}{\partial\theta} + \frac{\mathbf{s}}{V}\right)\Psi\right) d\tau + \left(\Psi, \left(\frac{\partial}{\partial x_j} + \omega_j\frac{\mathbf{s}}{V}\right)\Psi\right) dx_j}
$$

Moduli Space of N SU(2) Instantons on Taub-NUT

One SU(2) instanton on TN

G is $U(N)$



$$T^*G_{d_L} \times \mathbb{H}^N \times T^*G_{d_M} \times \mathbb{H}^N \times T^*G_{d_R} \times \mathbb{H}^{N^2} /\!\!/\!\!/ G_{-l/2} \times G_{-\lambda} \times G_{\lambda} \times G_{l/2}$$

To have algebraic description of this space introduce monodromy H on each interval:

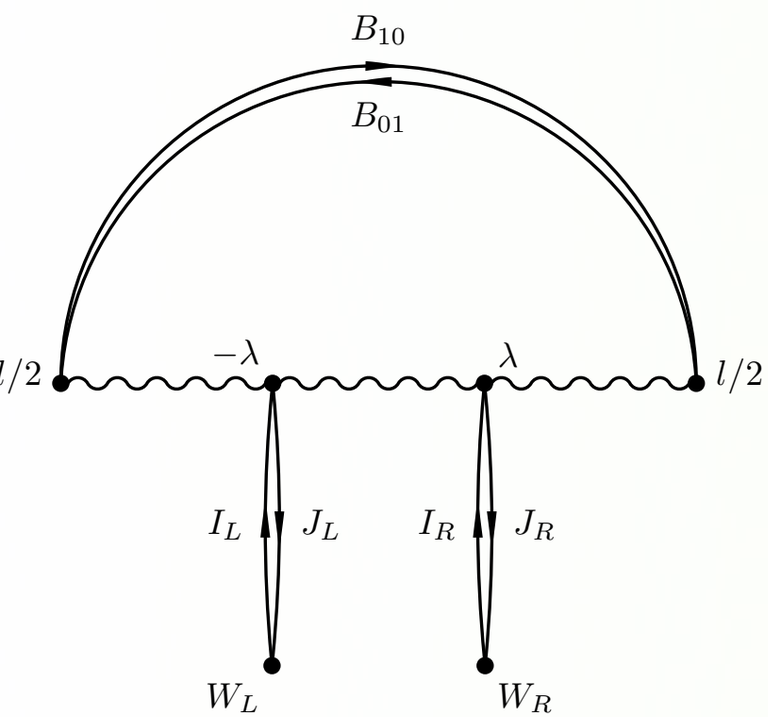
$D_M H_M(s)=0$, $H_M(-\lambda)=1$, $H_M = H_M(\lambda)$

Moment maps can be written as:

$$T_R - H^{-1}_M T_M H_M = I_R J_R, \quad H^{-1}_M T_M H_M = B_{10} B_{01}$$
$$T_M - H^{-1}_L T_L H_L = I_L J_L, \quad T_L = B_{01} B_{10}$$

Up to the gauge equivalence

$$\begin{pmatrix} T_L, H_L \\ T_M, H_M \\ T_R, H_R \\ B_{01}, B_{10} \end{pmatrix} \quad \begin{pmatrix} g^{-1}_{-l/2} T_L\, g_{-l/2}, g^{-1}_{-l/2} H_L\, g_{-\lambda} \\ g^{-1}_{-\lambda} T_M\, g_{-\lambda}, g^{-1}_{-\lambda} H_M\, g_{\lambda} \\ g^{-1}_{\lambda} T_R\, g_{\lambda}, g^{-1}_{\lambda} H_R\, g_{l/2} \\ g^{-1}_{l/2} B_{01}\, g_{-l/2}, g^{-1}_{-l/2} B_{10}\, g_{l/2} \end{pmatrix}$$



$$ds^2 = \left(l + \frac{1}{2r_1}\right) d\vec{r}_1^{\,2} - 4\lambda d\vec{r}_1 d\vec{q} + \left(2\lambda + \frac{1}{q}\right) d\vec{q}^{\,2}$$

$$+ \frac{\left(d\theta - \frac{1}{4}\omega_r\right)^2}{l - 2\lambda + 1/q + 1/(2r)} + \frac{\left(d\alpha + \frac{1}{2}\omega_q\right)^2}{2\lambda + 1/q}$$

$$d\omega_r = *d\frac{1}{r_1}$$

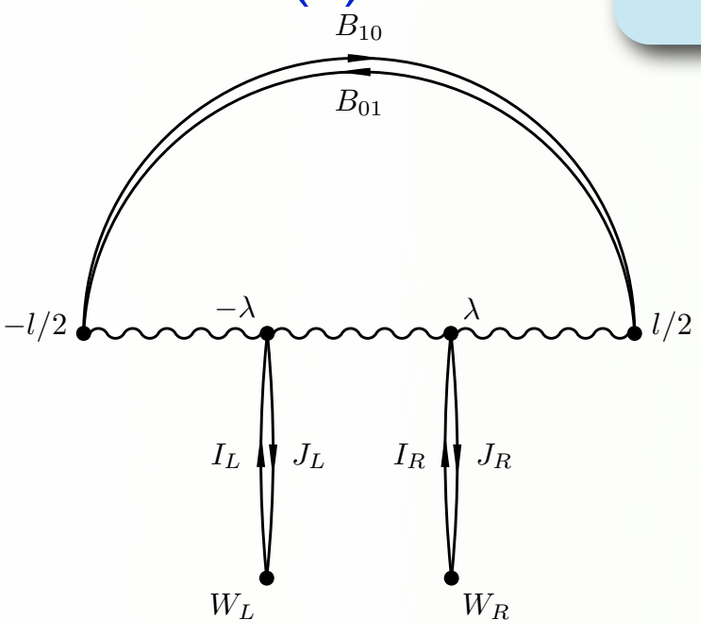
$$d\omega_q = *d\frac{1}{q}$$

$$\theta \sim \theta + 2\pi$$

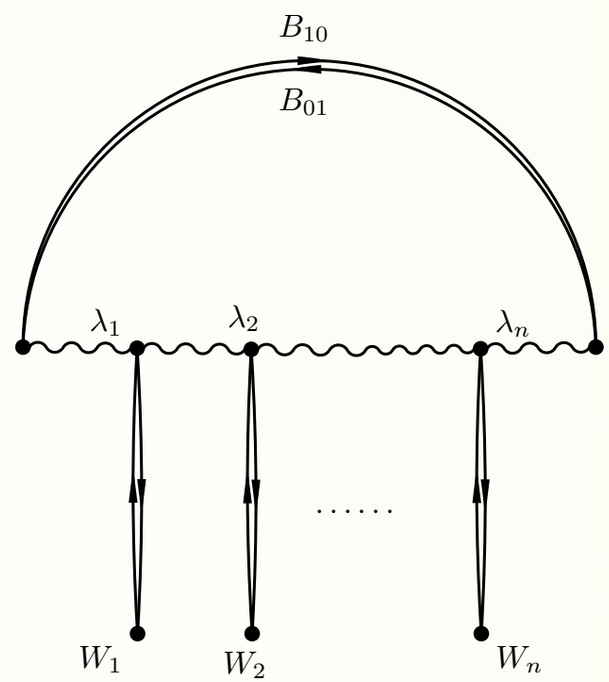
$$\alpha \sim \alpha + 2\pi$$

Data Determining N U(m) Instantons on TN_k Bow Diagrams

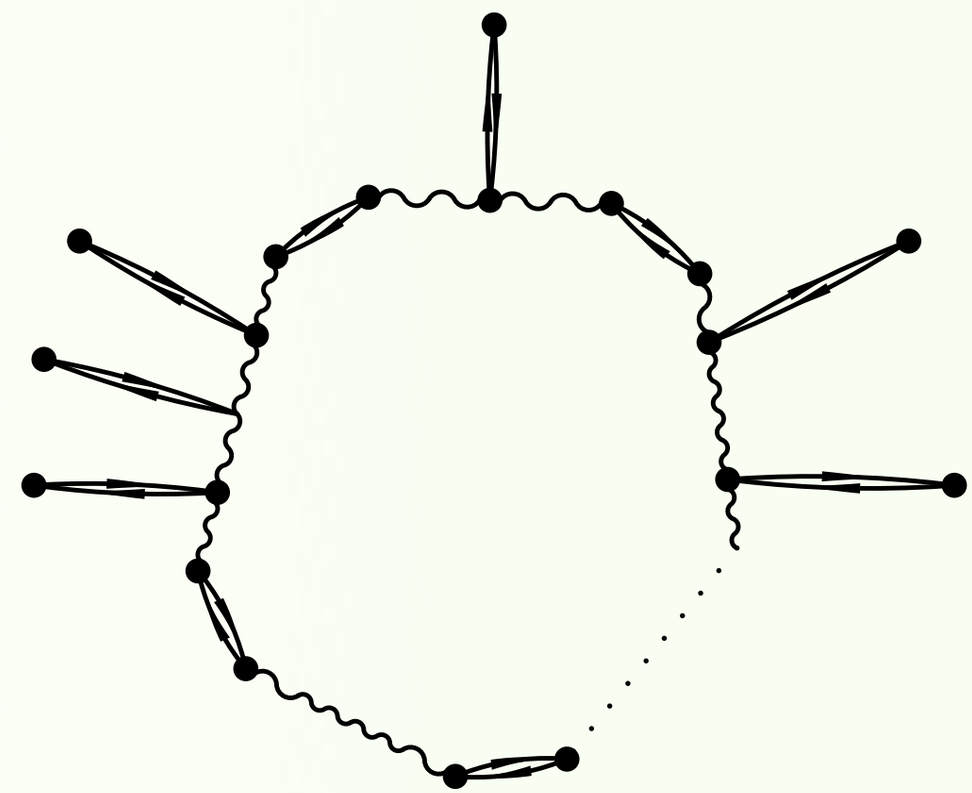
N SU(2) on TN



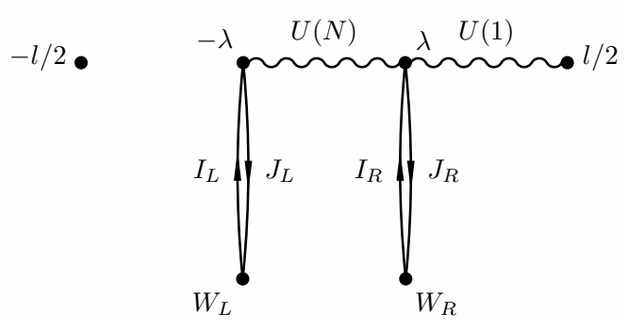
N SU(m) on TN



N SU(m) on multi-TN

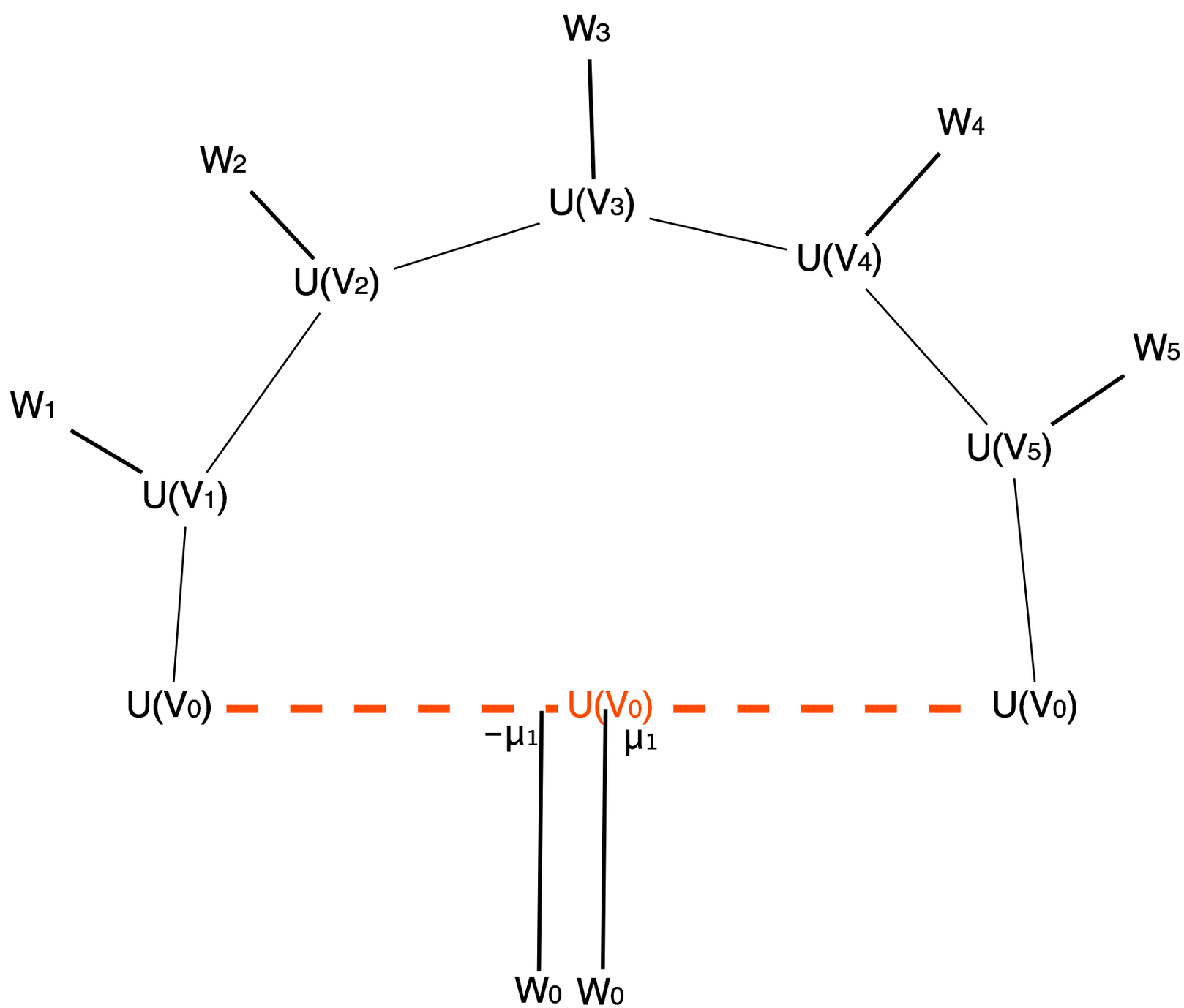


N SU(2) "monopoles" on TN

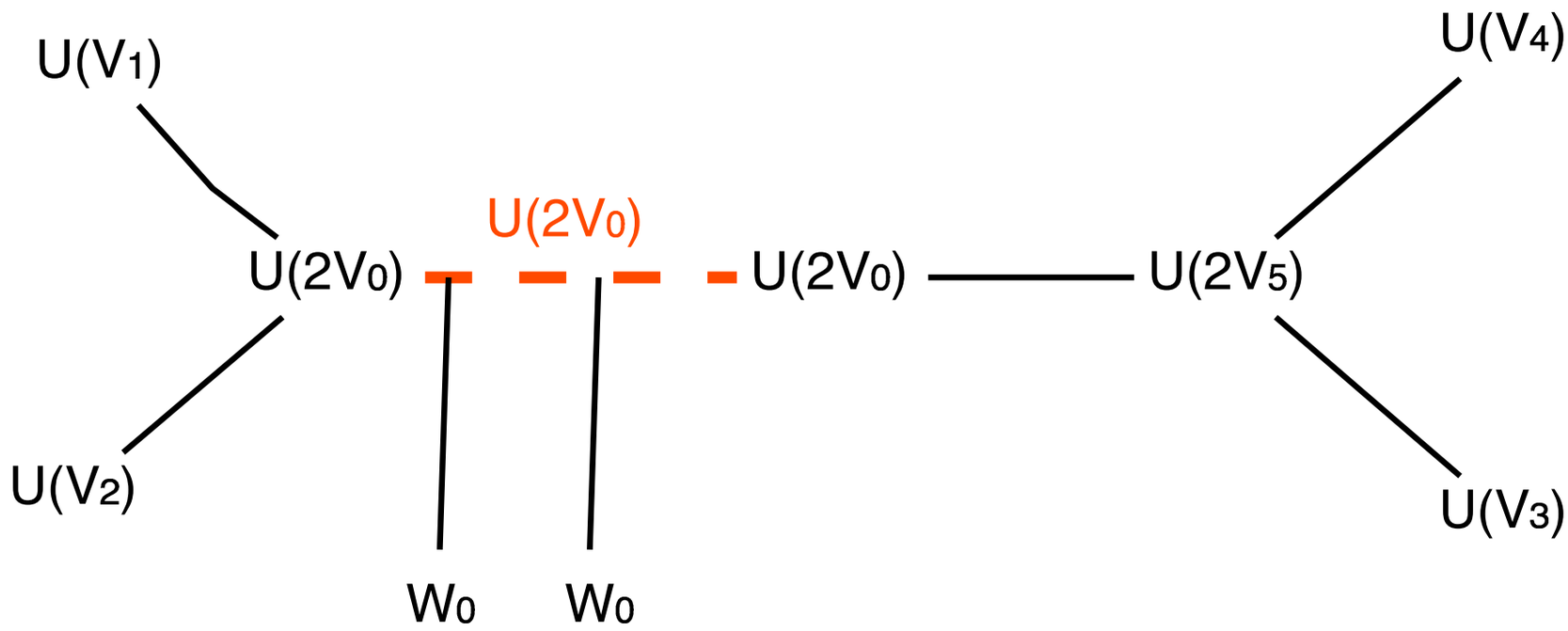


Instantons on ALF Spaces:

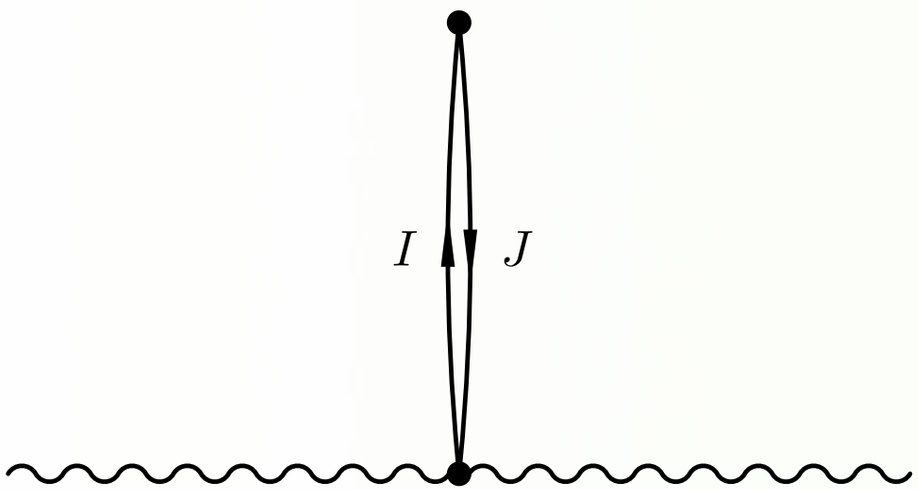
A_5 ALF



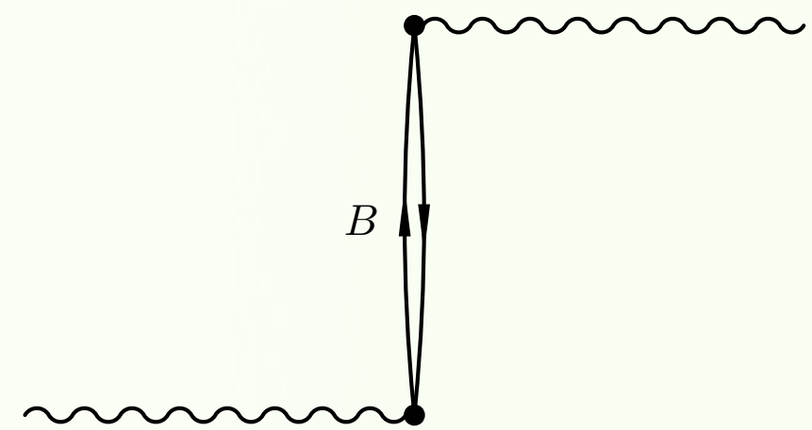
Instanton on D_k ALF



Electric-Magnetic Duality



EM duality

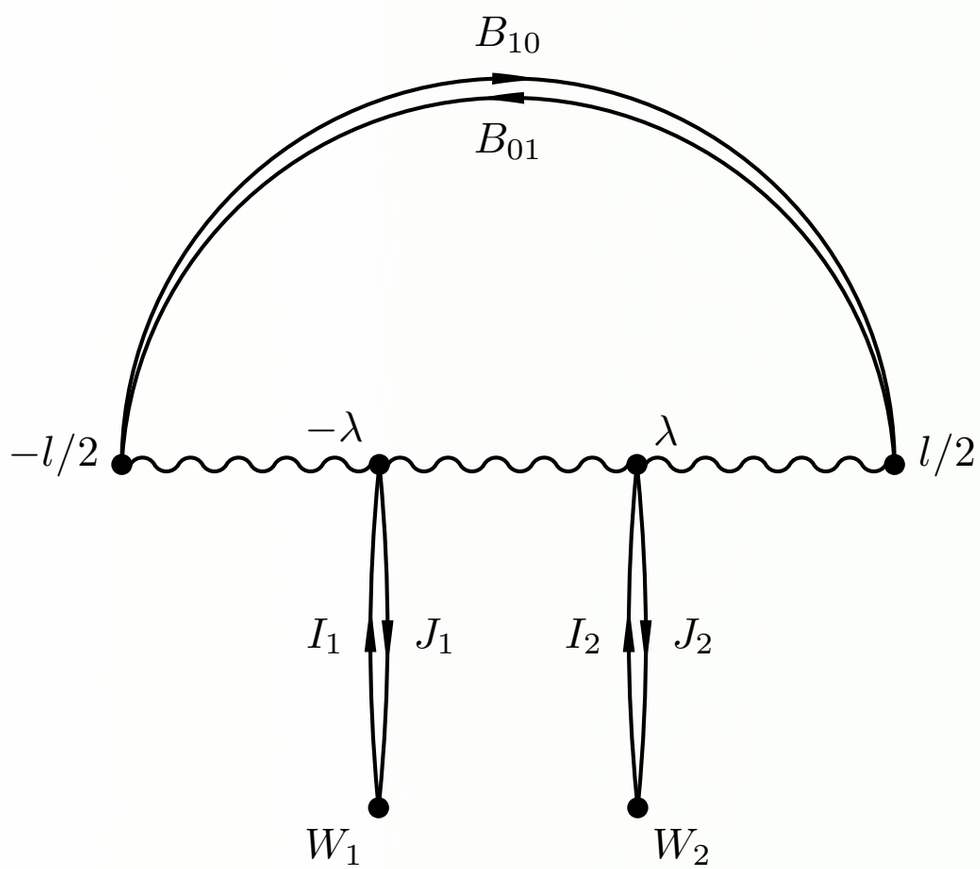


Adjoint

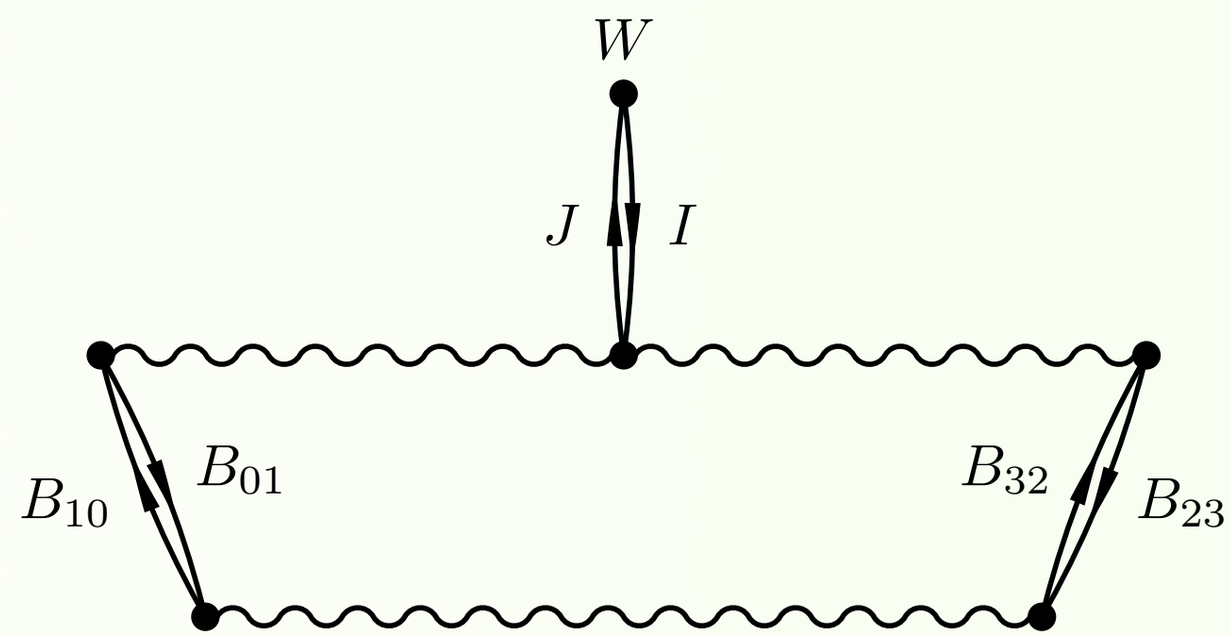
Bifundamental

Bow Doublet

Higgs Branch

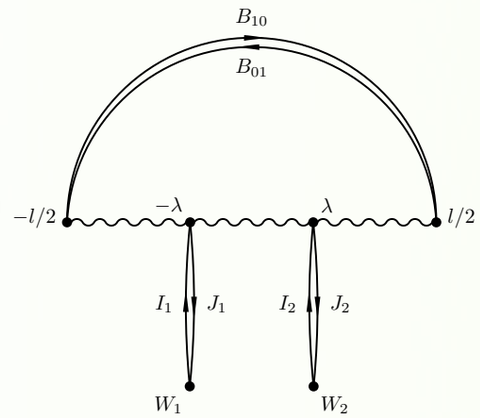


Coulomb Branch



Higgs Branch

$$< H^2 > \neq 0, < Y^2 > = 0$$



N U(m) Inst / TN_k

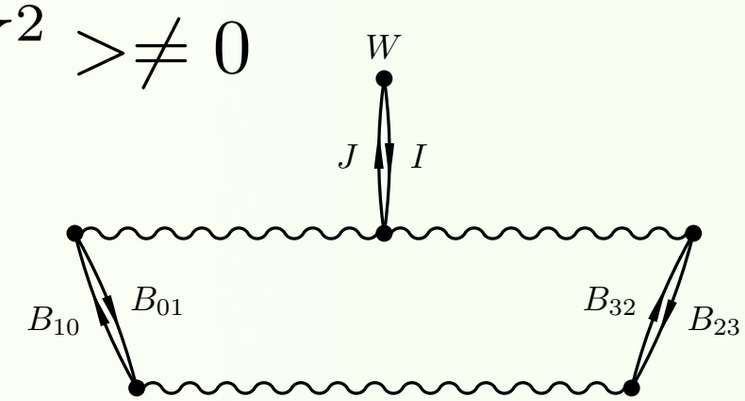
Mixed Branch

$$H_\alpha \sim I, Y_J \sim I$$

Coulomb Branch

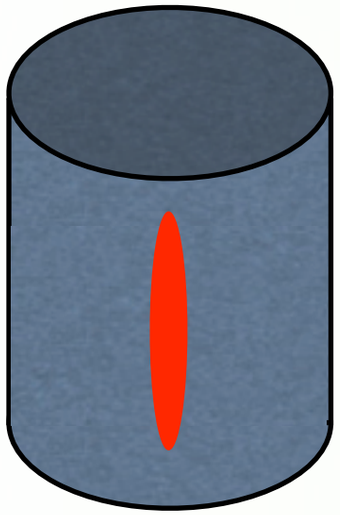
$$< H^2 > = 0, < Y^2 > \neq 0$$

N U(k) Inst / TN_m



String Theory Reasoning

D6



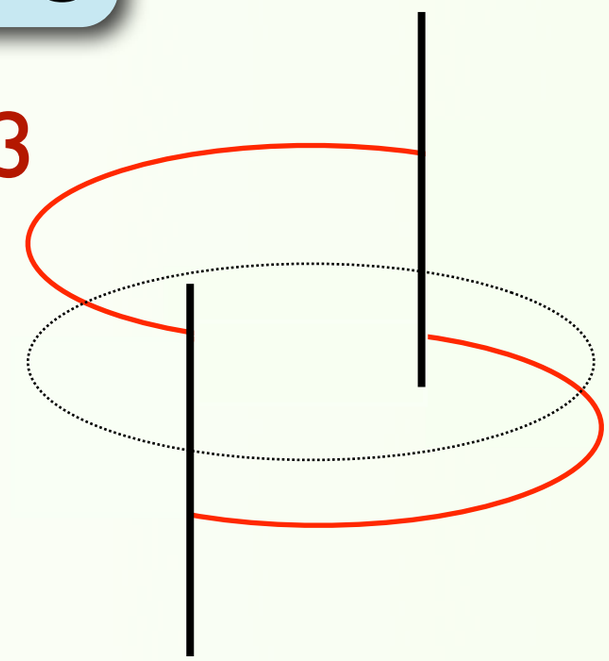
D2

Taub-NUT

T duality

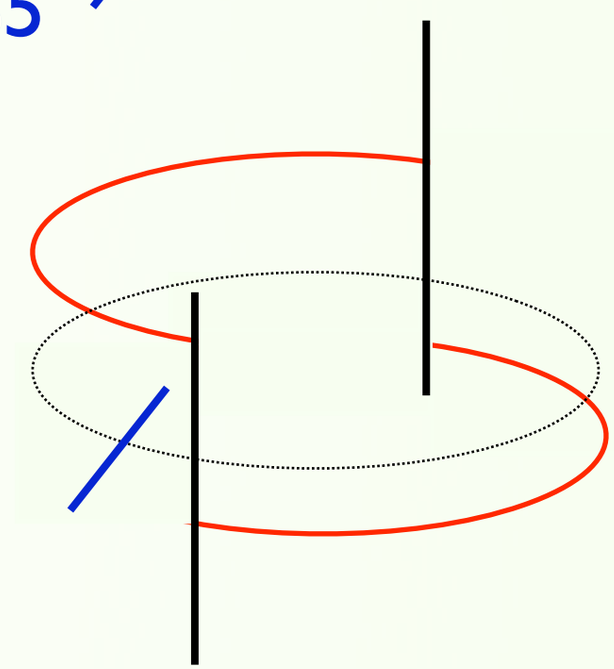
D5

D3



T duality

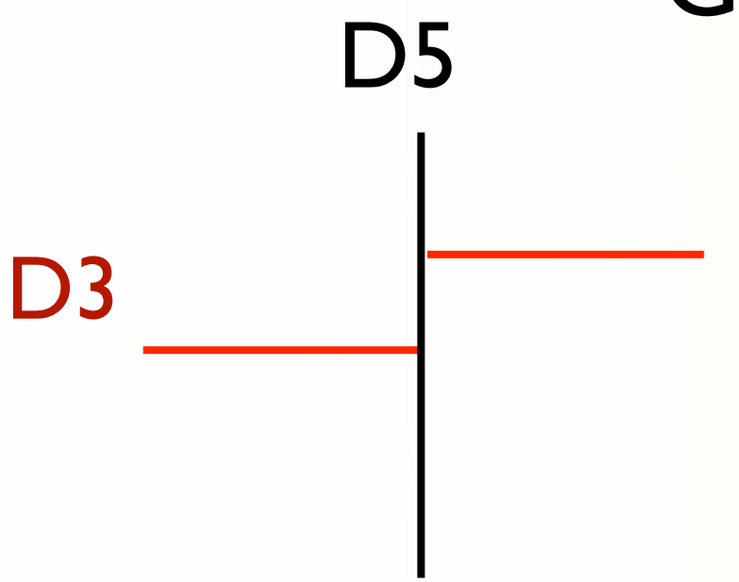
NS5

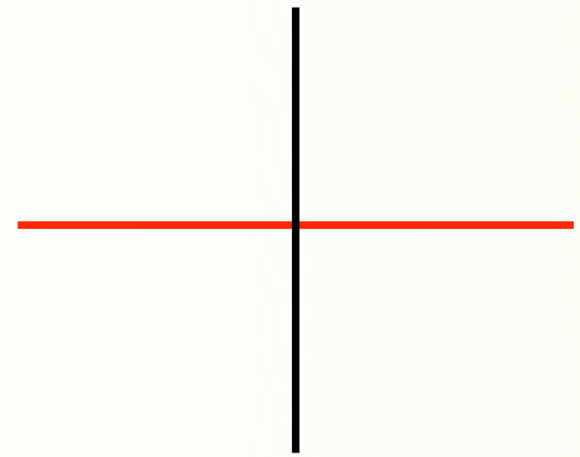


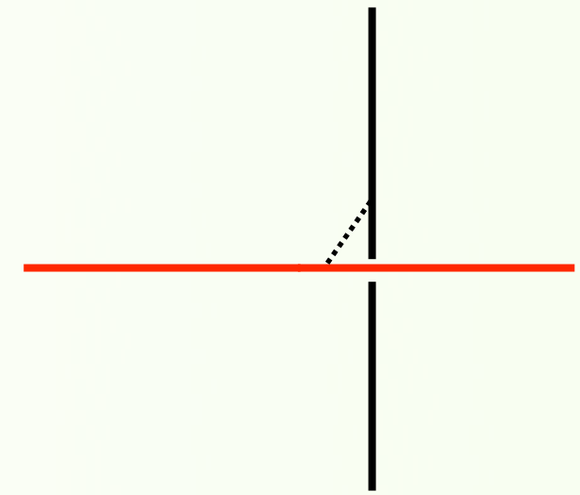
2 D6 and D2
on Taub-NUT

T duality

Gauge Theory on D3-brane

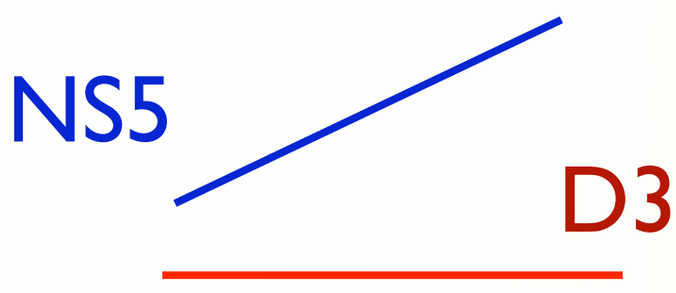


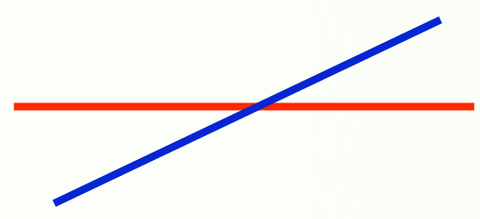


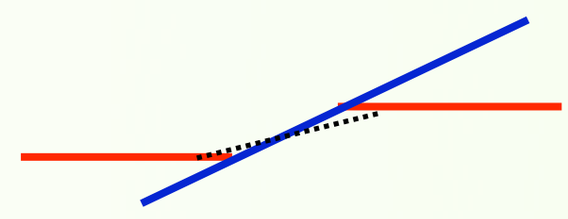


Massless fundamental hypermultiplet: f

Massive fundamental hypermultiplet from D3-D5 open string mode

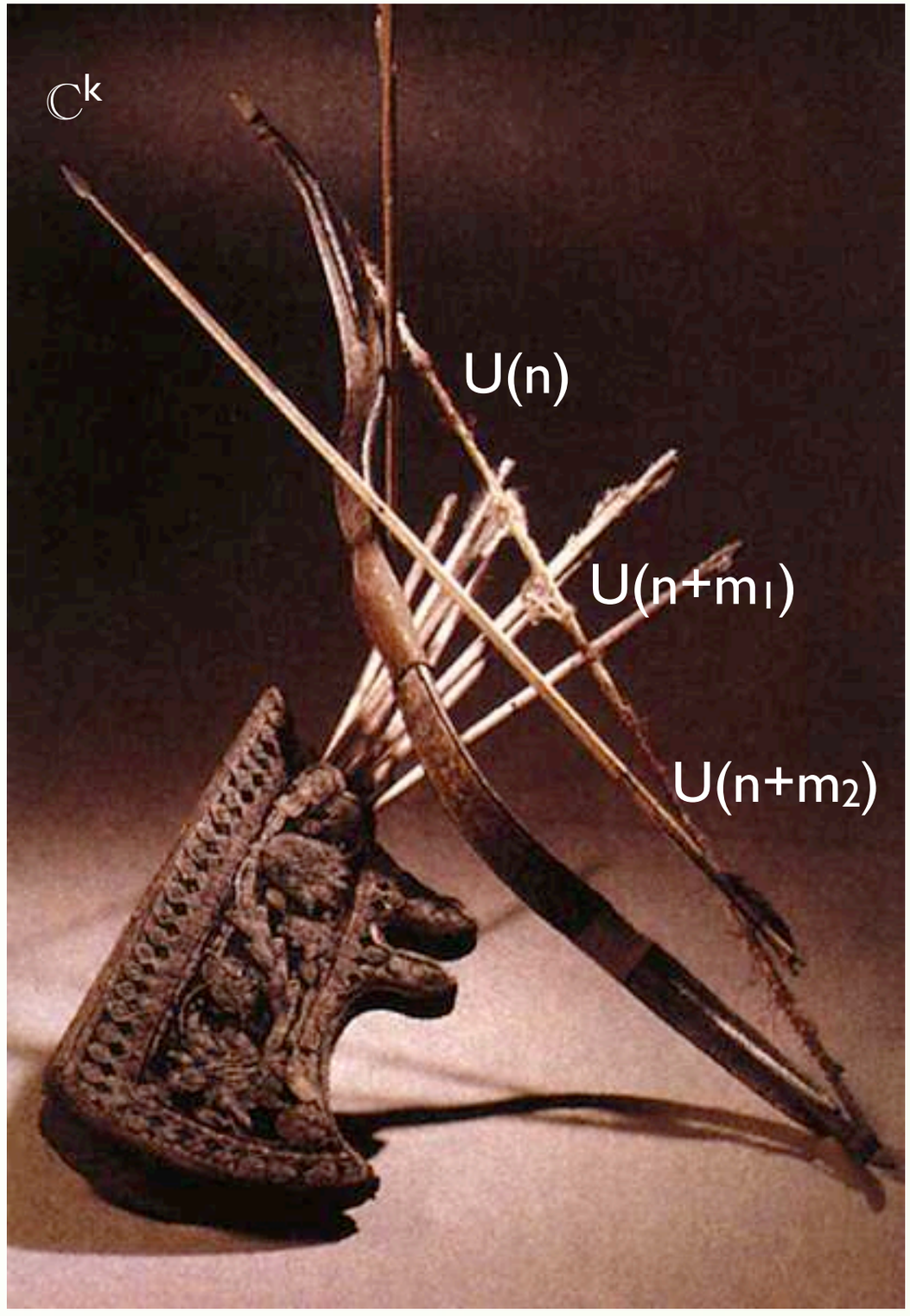






Massless bifundamental hypemultiplet: B

Massive bifundamental hypemultiplet from D3-D3 open string mode


$\mathbb{C}^k$
U(n)
U(n+m$_1$)
U(n+m$_2$)