# Space-Time Discontinuous Galerkin Methods for Compressible Flows

## Part II Euler and Navier-Stokes Equations

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# Space-Time Discontinuous Galerkin Finite Element Methods

## Motivation:

- In many applications one encounters moving and deforming flow domains:
  - Aerodynamics: helicopters, manoeuvering aircraft, wing control surfaces
  - Fluid structure interaction
  - Two-phase and chemically reacting flows with free surfaces
  - Water waves, including wetting and drying of beaches and sand banks
- A key requirement for these applications is to obtain an accurate and conservative discretization on moving and deforming meshes

# Motivation of Research

## Other requirements

- Improved capturing of vortical structures and flow discontinuities, such as shocks and interfaces, using $hp$-adaptation.
- Capability to deal with complex geometries.
- Excellent computational efficiency for unsteady flow simulations.

These requirements have been the main motivation to develop a space-time discontinuous Galerkin method.

# Overview of Lecture

- Space-time discontinuous Galerkin finite element discretization for compressible Navier-Stokes equations
  - main aspects of discretization
  - efficient quadrature
  - choice of variables
- Applications in aerodynamics

# Geometry of Space-Time Domain for Three-Dimensional Time-Dependent Problems

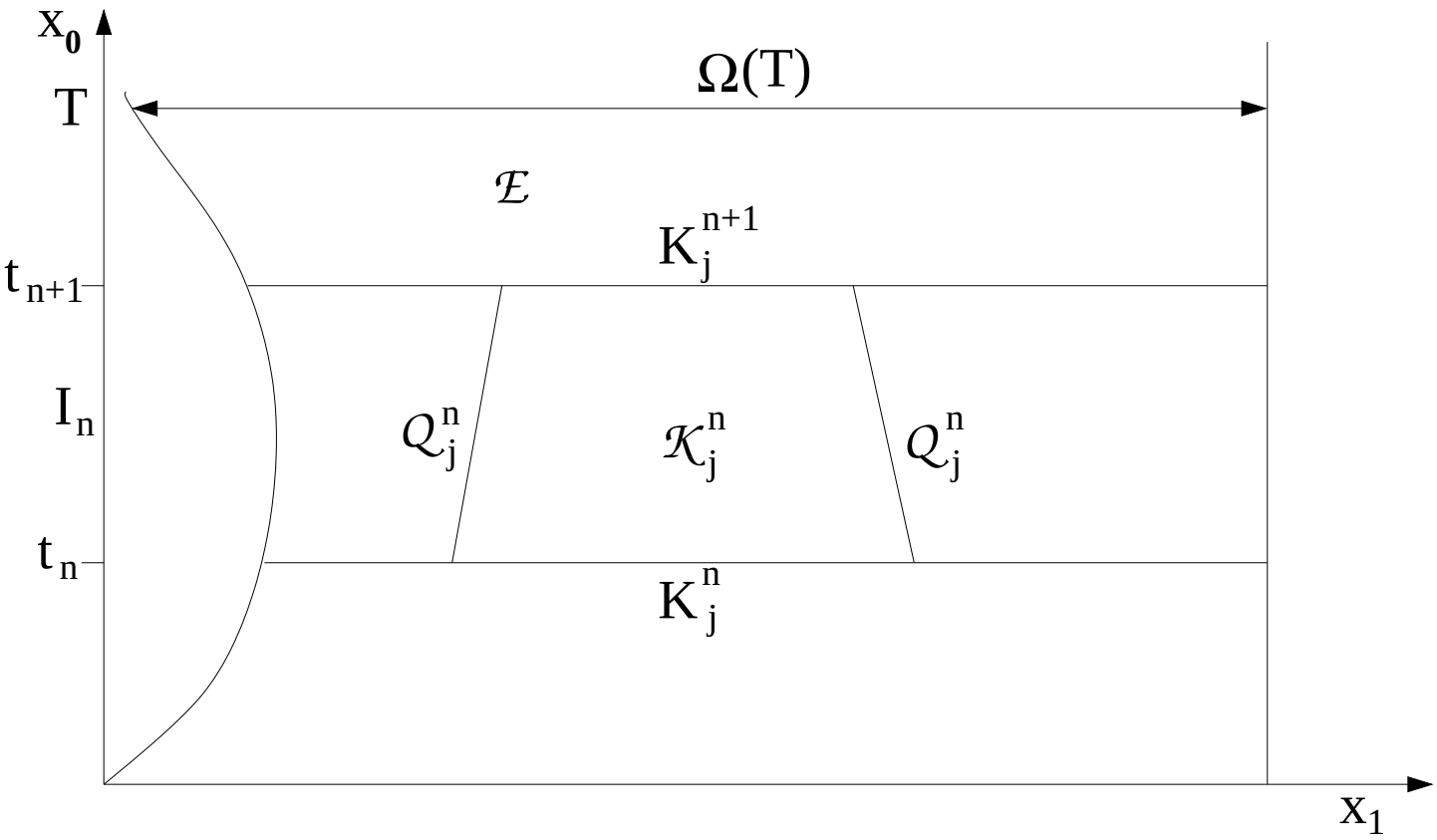
- Consider an open domain: $\mathcal{E} \subset \mathbb{R}^4$.
- The flow domain $\Omega(t)$ at time $t$ is defined as:

$$\Omega(t) := \{x \in \mathcal{E} \mid x_0 = t,\ t_0 < t < T\}$$

- The space-time domain boundary $\partial\mathcal{E}$ consists of the hypersurfaces:

$$\begin{aligned}\Omega(t_0) :=& \{x \in \partial\mathcal{E} \mid x_0 = t_0\},\\ \Omega(T) :=& \{x \in \partial\mathcal{E} \mid x_0 = T\},\\ \mathcal{Q} :=& \{x \in \partial\mathcal{E} \mid t_0 < x_0 < T\}.\end{aligned}$$

# Space-Time Slab




Space-time slab in space-time domain $\mathcal{E}$.

# Definition of Space-Time Slab

- Consider a partitioning of the time interval $(t_0, T)$: $\{t_n\}_{n=0}^{N}$, and set $I_n = (t_n, t_{n+1})$.
- Define a space-time slab as: $\mathcal{I}_n := \{x \in \mathcal{E} \mid x_0 \in I_n\}$
- Split the space-time slab into non-overlapping elements: $\mathcal{K}_j^n$.
- We will also use the notation: $K_j^n = \mathcal{K}_j^n \cap \{t_n\}$ and $K_j^{n+1} = \mathcal{K}_j^n \cap \{t_{n+1}\}$

# Compressible Navier-Stokes Equations

- Compressible Navier-Stokes equations in the space-time domain $\mathcal{E}$:

$$\frac{\partial U_i}{\partial x_0} + \frac{\partial F_k^e(U)}{\partial x_k} - \frac{\partial F_k^v(U, \nabla U)}{\partial x_k} = 0$$

- Conservative variables $U \in \mathbb{R}^5$ and inviscid fluxes $F^e \in \mathbb{R}^{5\times 3}$

$$U = \begin{bmatrix} \rho \\ \rho u_j \\ \rho E \end{bmatrix}, \qquad F_k^e = \begin{bmatrix} \rho u_k \\ \rho u_j u_k + p\delta_{jk} \\ \rho h u_k \end{bmatrix}$$

# Compressible Navier-Stokes Equations

- Viscous flux $F^v \in \mathbb{R}^{5\times 3}$

$$F_k^v = \begin{bmatrix} 0 \\ \tau_{jk} \\ \tau_{kj}u_j - q_k \end{bmatrix}$$

with the total stress tensor $\tau$ defined as:

$$\tau_{jk} = \lambda \frac{\partial u_i}{\partial x_i}\delta_{jk} + \mu(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j})$$

and the heat flux vector $\vec{q}$ as:

$$q_k = -\kappa \frac{\partial T}{\partial x_k}$$

# Compressible Navier-Stokes Equations

- The viscous flux $F^v$ is homogeneous with respect to the gradient of the conservative variables $\nabla U$:

$$F^v_{ik}(U, \nabla U) = A_{ikrs}(U)\frac{\partial U_r}{\partial x_s}$$

with the homogeneity tensor $A \in \mathbb{R}^{5\times3\times5\times3}$ defined as:

$$A_{ikrs}(U) := \frac{\partial F^v_{ik}(U, \nabla U)}{\partial(\nabla U)}$$

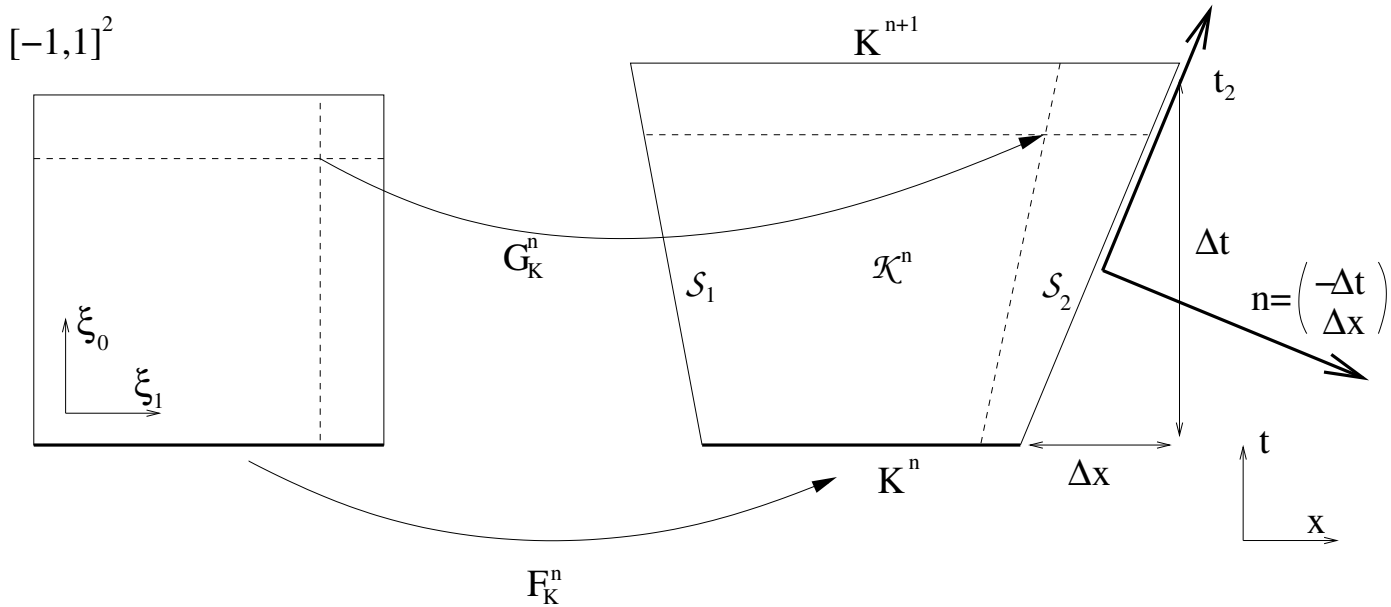
- The system is closed using the equations of state for an ideal gas

$$p = (\gamma - 1)(\rho E - \frac{1}{2}\rho u_i u_i)$$

$$T = \frac{1}{c_v}(E - \frac{1}{2}u_i u_i)$$

with $\gamma = c_p/c_v$ the ratio of specific heats.

# Geometry of Space-Time Element




Geometry of 2D space-time element in both computational and physical space.

# Space-Time Element Definition

- Definition of the mapping $G_K^n$ which the connects the space-time element $\mathcal{K}^n$ to the reference element $\hat{\mathcal{K}} = [-1,1]^4$:

$$G_K^n : [-1,1]^4 \longrightarrow \mathcal{K}^n : \xi \longmapsto x,$$

with:

$$(x_0, \bar{x}) = \Big( \tfrac{1}{2}(1-\xi_0) F_K^n(\bar{\xi}) + \tfrac{1}{2}(1+\xi_0) F_K^{n+1}(\bar{\xi}),$$

$$\tfrac{1}{2}(t_n + t_{n+1}) - \tfrac{1}{2}(t_n - t_{n+1})\xi_0 \Big),$$

and $F_K^n : [-1,1]^3 \longrightarrow K^n$, $F_K^{n+1} : [-1,1]^3 \longrightarrow K^{n+1}$ the mappings for the space elements.

- The space-time tessellation is now defined as:

$$\mathcal{T}_h^n := \{\mathcal{K} = G_k^n(\hat{\mathcal{K}}) \mid K \in \bar{\mathcal{T}}_h^n\}.$$

# Approximation Spaces

- The finite element space associated with the tessellation $\mathcal{T}_h$ is given by:

$$W_h := \left\{ W \in (L^2(\mathcal{E}_h))^5 \ : \ W|_{\mathcal{K}} \circ G_{\mathcal{K}} \in (P^k(\hat{\mathcal{K}}))^5, \quad \forall \mathcal{K} \in \mathcal{T}_h \right\}$$

- We will also use the space:

$$V_h := \left\{ V \in (L^2(\mathcal{E}_h))^{5\times 3} \ : \ V|_{\mathcal{K}} \circ G_{\mathcal{K}} \in (P^k(\hat{\mathcal{K}}))^{5\times 3}, \quad \forall \mathcal{K} \in \mathcal{T}_h \right\}.$$

- Note the fact that $\nabla_h W_h \subset V_h$ is essential for the discretization.

# Trace Operators

- The jump of $f$ in the Cartesian coordinate direction $k$ is defined at internal faces as:
$$[\![f]\!]_k = f^L n_k^L + f^R n_k^R.$$
- The average of $f$ is defined at internal faces as:
$$\{\!\{f\}\!\} = \tfrac{1}{2}(f^L + f^R).$$
- The jump operator satisfies the following product rule at internal faces:
$$[\![g_i f_{ik}]\!]_k = \{\!\{g_i\}\!\}[\![f_{ik}]\!]_k + [\![g_i]\!]_k\{\!\{f_{ik}\}\!\},$$

# First Order System

- Rewrite the compressible Navier-Stokes equations as a first-order system using the auxiliary variable $\Theta$:

$$\frac{\partial U_i}{\partial x_0} + \frac{\partial F^e_{ik}(U)}{\partial x_k} - \frac{\partial \Theta_{ik}(U)}{\partial x_k} = 0,$$

$$\Theta_{ik}(U) - A_{ikrs}(U)\frac{\partial U_r}{\partial x_s} = 0.$$

# Weak Formulation

- Weak formulation for the compressible Navier-Stokes equations

  Find a $U \in W_h$, $\Theta \in V_h$, such that for all $W \in W_h$ and $V \in V_h$, the following holds:

$$-\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\left(\frac{\partial W_i}{\partial x_0}U_i+\frac{\partial W_i}{\partial x_k}(F^e_{ik}-\Theta_{ik})\right)\,d\mathcal{K}$$

$$+\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}W^L_i(\widehat{U}_i+\widehat{F}^e_{ik}-\widehat{\Theta}_{ik})n^L_k\,d(\partial\mathcal{K})=0,$$

$$\sum_{\mathcal{K}\in\mathcal{T}^n_h}\int_{\mathcal{K}}V_{ik}\Theta_{ik}\,d\mathcal{K}=\sum_{\mathcal{K}\in\mathcal{T}^n_h}\int_{\mathcal{K}}V_{ik}A_{ikrs}\frac{\partial U_r}{\partial x_s}\,d\mathcal{K}$$

$$+\sum_{\mathcal{K}\in\mathcal{T}^n_h}\int_{\mathcal{Q}}V^L_{ik}A^L_{ikrs}(\widehat{U}_r-U^L_r)\bar{n}^L_s\,d\mathcal{Q}$$

# Transformation to Arbitrary Lagrangian Eulerian form

- The space-time normal vector on a grid moving with velocity $\vec{v}$ is:

$$n = \begin{cases} (1,0,0,0)^T & \text{at } K(t_{n+1}^-), \\ (-1,0,0,0)^T & \text{at } K(t_n^+), \\ (-v_k \bar{n}_k, \bar{n})^T & \text{at } Q^n. \end{cases}$$

- The boundary integral then transforms into:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} W_i^L (\widehat{U}_i + \widehat{F}_{ik}^e - \widehat{\Theta}_{ik}) n_k^L \, d(\partial\mathcal{K})$$

$$= \sum_{K\in\mathcal{T}_h} \Big( \int_{K(t_{n+1}^-)} W_i^L \widehat{U}_i \, dK + \int_{K(t_n^+)} W_i^L \widehat{U}_i \, dK \Big)$$

$$+ \sum_{K\in\mathcal{T}_h} \int_{\mathcal{Q}} W_i^L (\widehat{F}_{ik}^e - \widehat{U}_i v_k - \widehat{\Theta}_{ik}) \bar{n}_k^L \, d\mathcal{Q}$$

# Numerical Fluxes

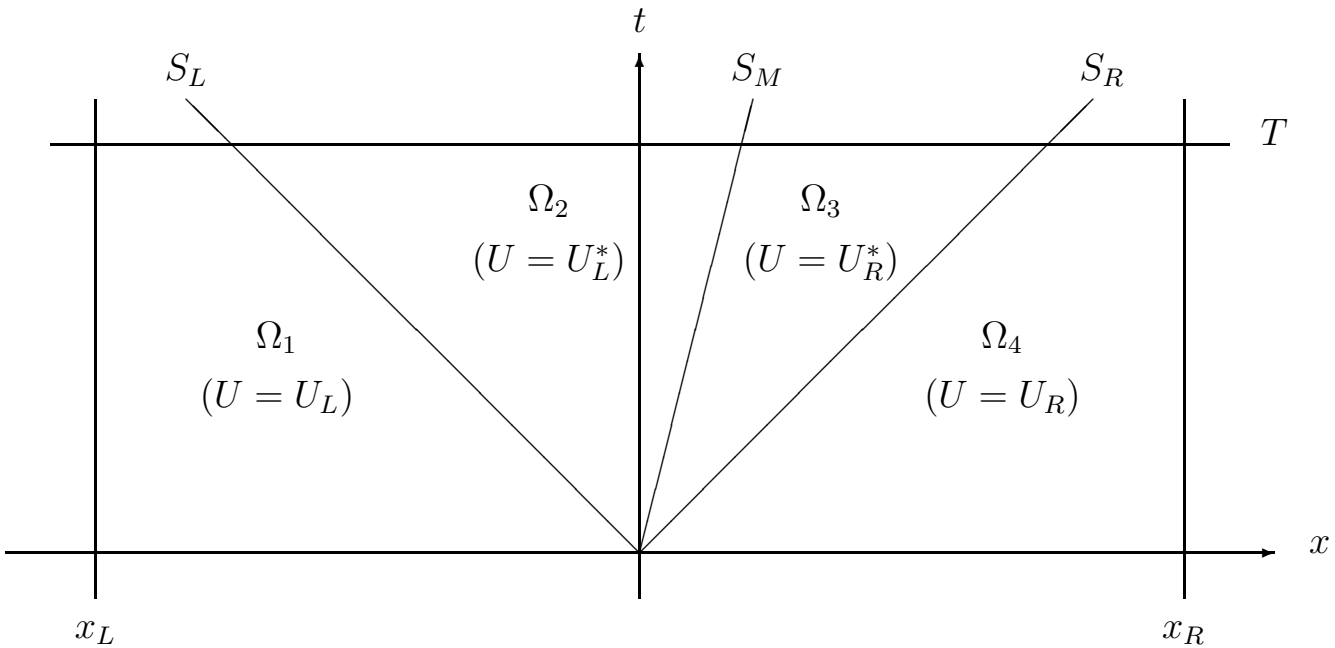
- The numerical flux $\widehat{U}$ at $K(t_{n+1}^-)$ and $K(t_n^+)$ is defined as an upwind flux to ensure causality in time:

$$\widehat{U} = \begin{cases} U^L & \text{at } K(t_{n+1}^-), \\ U^R & \text{at } K(t_n^+), \end{cases}$$

- At the space-time faces $\mathcal{Q}$ we introduce the HLLC approximate Riemann solver as numerical flux:

$$\bar{n}_k(\widehat{F}_{ik}^e - \widehat{U}_i v_k)(U^L, U^R) = H_i^{\mathrm{HLLC}}(U^L, U^R, v, \bar{n})$$

# Wave Pattern in HLLC-Flux




Wave pattern used in the definition of the HLLC flux function. $S_L$ and $S_R$ are the fastest left and right moving signal velocities. The solution in the star region $U^*$ is divided by a wave with velocity $S_M$.

# HLLC Flux

- The HLLC flux, introduced by Toro et al., is used as a numerical flux because it provides an accurate and efficient approximation to the Riemann problem.
- The HLLC flux at a moving interface is defined as:

$$H^{\mathrm{HLLC}}(U_h^-, U_R^+, v, n) = \tfrac{1}{2}\Big( \hat{F}(U_L) + \hat{F}(U_R) - (|S_L - v| - |S_M - v|)U_L^* + (|S_R - v| - |S_M - v|)U_R^* + |S_L - v|U_L - |S_R - v|U_R - v(U_L + U_R))\Big).$$

with $\hat{F} = \bar{n}_{\mathcal{K}}^T \bar{\mathcal{F}}$. In order to completely define the HLLC flux we still need to define the star states $U_L^*$ and $U_R^*$, and the wave speeds $S_L$, $S_R$ and $S_M$.

# ALE Weak Formulation

- The ALE flux formulation of the compressible Navier-Stokes equations transforms now into:

  Find a $U \in W_h$, such that for all $W \in W_h$, the following holds:

$$
\begin{aligned}
&- \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{K}} \Big( \frac{\partial W_i}{\partial x_0} U_i + \frac{\partial W_i}{\partial x_k} (F_{ik}^e - \Theta_{ik}) \Big) \, d\mathcal{K} \\
&+ \sum_{K \in \mathcal{T}_h^n} \Big( \int_{K(t_{n+1}^-)} W_i^L U_i^L \, dK - \int_{K(t_n^+)} W_i^L U_i^R \, dK \Big) \\
&+ \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{Q}} W_i^L (H_i^{\mathrm{HLLC}}(U^L, U^R, v, \bar{n}) - \widehat{\Theta}_{ik} \bar{n}_k^L) \, d\mathcal{Q} = 0.
\end{aligned}
$$

# Auxiliary Equation for $\Theta$

- Recall the auxiliary equation for $\Theta$:

  Find a $\Theta \in V_h$, such that for all $V \in V_h$ the following holds:

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}\Theta_{ik}\, d\mathcal{K} = \sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik} A_{ikrs} \frac{\partial U_r}{\partial x_s}\, d\mathcal{K}$$

$$+ \sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{Q}} V_{ik}^L A_{ikrs}^L (\widehat{U}_r - U_r^L)\bar{n}_s^L\, d\mathcal{Q}$$

- The following relation holds for the element boundary integrals:

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n}\int_{\mathcal{Q}} g_i^L f_{ik}^L \bar{n}_k^L \, d\mathcal{Q} = \sum_{\mathcal{S}\in\mathcal{S}_I^n}\int_{\mathcal{S}} [\![ g_i f_{ik} ]\!]_k \, d\mathcal{S} + \sum_{\mathcal{S}\in\mathcal{S}_B^n}\int_{\mathcal{S}} g_i^L f_{ik}^L \bar{n}_k^L \, d\mathcal{S}.$$

- Transform the element boundary integrals into face integrals in the auxiliary equation:

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n}\int_{\mathcal{Q}} V_{ik}^L A_{ikrs}^L (\widehat{U}_r - U_r^L)\bar{n}_s^L \, d\mathcal{Q} = \sum_{\mathcal{S}\in\mathcal{S}_I^n}\int_{\mathcal{S}} [\![ V_{ik} A_{ikrs} (\widehat{U}_r - U_r) ]\!]_s \, d\mathcal{S}$$
$$+ \sum_{\mathcal{S}\in\mathcal{S}_B^n}\int_{\mathcal{S}} V_{ik}^L A_{ikrs}^L (\widehat{U}_r - U_r^L)\bar{n}_s^L \, d\mathcal{S}.$$

# Numerical Fluxes in Auxiliary Equation

- Introduce the numerical flux proposed by Bassi and Rebay:

$$\widehat{U} = \begin{cases} \{\!\{U\}\!\} & \text{at internal faces,} \\ U^b & \text{at boundary faces.} \end{cases}$$

- Use the relation $[\![V_{ik}A_{ikrs}(\widehat{U}_r - U_r)]\!]_s = -\{\!\{V_{ik}A_{ikrs}\}\!\}[\![U_r]\!]_s$, then the weak formulation for the auxiliary variable $\Theta$ is:

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}\Theta_{ik}\, d\mathcal{K} = \sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}A_{ikrs}\frac{\partial U_r}{\partial x_s}\, d\mathcal{K} - \sum_{\mathcal{S}\in\mathcal{S}_I^n} \int_{\mathcal{S}} \{\!\{V_{ik}A_{ikrs}\}\!\}[\![U_r]\!]_s\, d\mathcal{S}$$
$$- \sum_{\mathcal{S}\in\mathcal{S}_B^n} \int_{\mathcal{S}} V_{ik}^L A_{ikrs}^L (U_r^L - U_r^b)\bar{n}_s^L\, d\mathcal{S}.$$

# Lifting Operator

- Introduce the global lifting operator $\mathcal{R} \in \mathbb{R}^{5\times 3}$, defined in a weak sense as: Find an $\mathcal{R} \in V_h$, such that for all $V \in V_h$:

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}\mathcal{R}_{ik}\, d\mathcal{K} = \sum_{\mathcal{S}\in\mathcal{S}_I^n} \int_{\mathcal{S}} \{\!\{V_{ik}A_{ikrs}\}\!\}[\![U_r]\!]_s\, d\mathcal{S}$$
$$+ \sum_{\mathcal{S}\in\mathcal{S}_B^n} \int_{\mathcal{S}} V_{ik}^L A_{ikrs}^L (U_r^L - U_r^b)\bar{n}_s^L\, d\mathcal{S}.$$

- The weak formulation for the auxiliary variable is now transformed into

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}\Theta_{ik}\, d\mathcal{K} = \sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}(A_{ikrs}\frac{\partial U_r}{\partial x_s} - \mathcal{R}_{ik})\, d\mathcal{K}, \quad \forall V \in V_h.$$

# $\Theta$ Equation

- The primal formulation can be obtained by eliminating the auxiliary variable $\Theta$ using

$$\Theta_{ik} = A_{ikrs}\frac{\partial U_r}{\partial x_s} - \mathcal{R}_{ik}, \qquad \text{a.e. in } \mathcal{E}_h^n$$

- Note, this is possible since $\nabla_h W_h \subset V_h$

# ALE Weak Formulation for Primal Variables

- Recall the ALE flux formulation of the compressible Navier-Stokes equations:

  Find a $U \in W_h$, such that for all $W \in W_h$, the following holds:

$$
\begin{aligned}
&- \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{K}} \Big( \frac{\partial W_i}{\partial x_0} U_i + \frac{\partial W_i}{\partial x_k} (F_{ik}^e - \Theta_{ik}) \Big) \, d\mathcal{K} \\
&+ \sum_{K \in \mathcal{T}_h^n} \Big( \int_{K(t_{n+1}^-)} W_i^L U_i^L \, dK - \int_{K(t_n^+)} W_i^L U_i^R \, dK \Big) \\
&+ \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{Q}} W_i^L (H_i^{\mathrm{HLLC}}(U^L, U^R, v, \bar{n}) - \widehat{\Theta}_{ik} \bar{n}_k^L) \, d\mathcal{Q} = 0.
\end{aligned}
$$

# Numerical Fluxes for $\Theta$

- The numerical flux $\widehat{\Theta}$ in the primary equation is defined following Brezzi as a central flux $\widehat{\Theta} = \{\!\{\Theta\}\!\}$:

$$\widehat{\Theta}_{ik}(U^L, U^R) = \begin{cases} \{\!\{A_{ikrs}\frac{\partial U_r}{\partial x_s} - \eta\mathcal{R}^{\mathcal{S}}_{ik}\}\!\} & \text{for internal faces,} \\ A^b_{ikrs}\frac{\partial U^b_r}{\partial x_s} - \eta\mathcal{R}^{\mathcal{S}}_{ik} & \text{for boundary faces,} \end{cases}$$

- The local lifting operator $\mathcal{R}^{\mathcal{S}} \in \mathbb{R}^{5\times 3}$ is defined as follows:
  Find an $\mathcal{R}^{\mathcal{S}} \in V_h$, such that for all $V \in V_h$:

$$\sum_{\mathcal{K}\in\mathcal{T}^n_h} \int_{\mathcal{K}} V_{ik}\mathcal{R}^{\mathcal{S}}_{ik}\, d\mathcal{K} = \begin{cases} \displaystyle\int_{\mathcal{S}} \{\!\{V_{ik}A_{ikrs}\}\!\}[\![U_r]\!]_s\, d\mathcal{S} & \text{for internal faces,} \\ \displaystyle\int_{\mathcal{S}} V^L_{ik}A^L_{ikrs}(U^L_r - U^b_r)\bar{n}_s\, d\mathcal{S} & \text{for external faces.} \end{cases}$$

# Space-Time Formulation for NS

Find a $U \in W_h$, such that for all $W \in W_h$:

$$
\begin{aligned}
&- \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{K}} \left( W_{i,0} U_i + W_{i,k} (F_{ik}^e - A_{ikrs} U_{r,s} + \mathcal{R}_{ik}) \right) d\mathcal{K} \\
&+ \sum_{K \in \mathcal{T}_h^n} \left( \int_{K(t_{n+1}^-)} W_i U_i^L \, dK - \int_{K(t_n^+)} W_i U_i^R \, dK \right) \\
&+ \sum_{\mathcal{S} \in \mathcal{S}_{IB}^n} \int_{\mathcal{S}} (W_i^L - W_i^R) H_i(U^L, U^R, \bar{n}^L) \, d\mathcal{S} \\
&- \sum_{\mathcal{S} \in \mathcal{S}_I^n} \int_{\mathcal{S}} [\![ W_i ]\!]_k \{\!\{ A_{ikrs} U_{r,s} - \eta \mathcal{R}_{ik}^{\mathcal{S}} \}\!\} \, d\mathcal{S} \\
&- \sum_{\mathcal{S} \in \mathcal{S}_B^n} \int_{\mathcal{S}} W_i^L \left( A_{ikrs}^b U_{r,s}^b - \eta \mathcal{R}_{ik}^{\mathcal{S}} \right) \bar{n}_k^L \, d\mathcal{S} = 0,
\end{aligned}
$$

# Ensuring Monotonicity of Second and Higher Order DG Discretizations

Second and higher-order DG discretizations do not preserve the monotonicity of the solution.

- Using a slope limiter in DG discretizations results in an inconsistent discretization with limit cycle behavior, which hampers implicit time integration methods and prevents convergence to steady state.
- The use of a slope limiter also seriously degrades the accuracy of a DG discretization.

# Stabilization Operator

- Stabilization operator for flow discontinuities added to the weak formulation

$$\sum_{j=1}^{N_n} \int_{\mathcal{K}_j^n} (\nabla\, W_h)^T \cdot \mathfrak{D}(U_h) : \nabla\, U_h \, d\mathcal{K}$$

  The dyadic product is defined as $A : B = A_{ij} B_{ij}$ for $A, B \in R^{n \times m}$.

- A stabilization operator results in a numerical scheme which can converge to steady state and has improved accuracy, but requires additional research to ensure monotonicity.

# Stabilization Operator

- The effectiveness of the stabilization operator $\mathcal{D}$ strongly depends on the artificial viscosity matrix $\mathfrak{D} \in \mathbb{R}^{4\times 4}$. The definition of the artificial viscosity matrix is more straightforward if the stabilization operator acts independently in all computational coordinate directions. This is achieved by introducing the artificial viscosity matrix $\tilde{\mathfrak{D}} \in \mathbb{R}^{4\times 4}$ in computational space using the relation:

$$\mathfrak{D}(U_h|_{\mathcal{K}^n}, U_h^*|_{\mathcal{K}^n}) = R^T\, \tilde{\mathfrak{D}}(U_h|_{\mathcal{K}^n}, U_h^*|_{\mathcal{K}^n})\, R,$$

where the matrix $R \in \mathbb{R}^{4\times 4}$ is defined as:

$$R = 2\, H^{-1}\, \mathrm{grad}\, G_K.$$

- The matrix $H \in \mathbb{R}^{4\times 4}$ is used introduced to ensure that both $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ have the same mesh dependence, and is defined as:

$$H = \mathrm{diag}\,(h_0, h_1, h_2, h_3).$$

# Artificial Viscosity Model

- Artificial viscosity model proposed and analyzed by Jaffre, Johnson and Szepessy, *Math. Models and Meth. in Appl. Sci*, **158**, pp. 81-116 (1998), is used. In this model both the jumps at the element faces and the element residual are used to define the artificial viscosity:

$$\tilde{\mathfrak{D}}_{qq}(U_h|_{\mathcal{K}^n}, U_h^*|_{\mathcal{K}^n}) = \max\big(C_2 h_{\mathcal{K}}^{2-\beta} R_q(U_h|_{\mathcal{K}^n}, U_h^*|_{\mathcal{K}^n})\,,\; C_1 h_{\mathcal{K}}^{\frac{3}{2}}\big), \quad q = 1,2,3,$$
$$= 0, \qquad \text{otherwise,}$$

with

$$R(U_h|_{\mathcal{K}^n}, U_h^*|_{\mathcal{K}^n}) = \Big|\sum_{k=0}^{3} \frac{\partial \mathcal{F}(U_h)}{\partial U_{h,i}} \frac{\partial U_{h,i}(G_K(0))}{\partial x_k}\Big| + \frac{C_0}{h_{\mathcal{K}}}\big|U_h^+(x_{(7)}) - U_h^-(x_{(7)})\big| +$$
$$\sum_{m=1}^{6} \frac{1}{h_{\mathcal{K}}}\big|\bar{n}_{\mathcal{K}}^T \mathcal{F}(U_h^+(x_{(m)})) - \bar{n}_{\mathcal{K}}^T \mathcal{F}(U_h^-(x_{(m)}))\big|.$$

# Choice of Variables

- Conservative variables are not suitable for (nearly) incompressible flows since the density is not an independent variable.
- In the incompressible limit the flux Jacobians $\frac{\partial \hat{\mathbf{F}}^e(\mathbf{U})}{\partial \mathbf{U}}$ contains indefinite $(0/0)$ and infinitely large terms.
- Better alternative variables are:
  - Pressure primitive variables

$$Y = (p, \mathbf{v}, T)^T$$

  - Entropy variables

$$V = \frac{1}{T}(\mu - \frac{1}{2}\mathbf{v} \cdot \mathbf{v}, \mathbf{v}, -1)^T$$

with $\mathbf{v}$ the velocity, $T$ temperature and $\mu$ the chemical potential

- The Navier-Stokes equations have a proper incompressible limit in these variables and they are also suitable for general equations of state.
- The solution is then expanded in terms of these variables

$$\mathbf{U} = \mathbf{U}(\mathbf{Y}) \quad \text{or} \quad \mathbf{U} = \mathbf{U}(\mathbf{V})$$

  but the conservative formulation is maintained.
- A benefit of using the $\mathbf{Y}$ or $\mathbf{V}$ variables is also that residual based stabilization operators remain well defined.

- The entropy variables symmetrize the Navier-Stokes equations.

  The resulting Friedrichs system has a direct relation with the second law of thermodynamics and satisfies the Clausius-Duhem inequality.

- The pressure primitive variables are computationally less expensive, whereas the entropy variables require quite some complicated thermodynamic transformations but have a more solid mathematical foundation.

# Flux Quadrature

- In the discontinuous Galerkin discretization the integration of the (approximate) Riemann solver flux is by far the computationally most expensive part.
- The standard Gauss (product) quadrature rules require too many flux evaluations to be practical. In particular for discretizations in three-dimensions or in space-time (4D).
- A more efficient alternative is provided by the Taylor quadrature rule which also uses the gradient information available in a DG discretization.

# Taylor Flux Quadrature Rule

**Main Features:**

- In order to improve the computational efficiency the flux function in the integrand is replaced with a second or higher order Taylor series expansion evaluated at the face or element center.
- This approximation improves the computational efficiency significantly:
  - since only one flux evaluation per element face is necessary, instead of $2^n$ or $3^n$ for a product Gauss quadrature rule;
  - the locality of the required flow data improves the cache performance.
- The approximate flux integration does not result in a loss of accuracy in the DG discretization when a sufficient number of terms in the Taylor series expansion are used.

# Taylor Quadrature Rule for Space-Time DG

- The integrals of the flux tensor $\mathcal{F}_{ik}(U)$ over a space-time face $\mathcal{S}_m \subset \partial\mathcal{K}$ can be approximated as:

$$\int_{\mathcal{S}_m} \phi_m \mathcal{F}_{ik}(U) n_k dx \cong \mathcal{F}_{ik}(U(\bar{\xi}_m)) \int_{\hat{\mathcal{S}}} \xi_m d\hat{\mathcal{S}}_k^m$$

$$+ \sum_{l \in I(\mathcal{S}_m)} \frac{\partial \mathcal{F}_{ik}}{\partial U^j}(U(\bar{\xi}_m)) \frac{\partial U^j}{\partial \xi_l}(\bar{\xi}_m) \int_{\hat{\mathcal{S}}} \xi_l \xi_m d\hat{\mathcal{S}}_k^m$$

$$+ \text{ higher-order terms,}$$

with $\bar{\xi}_m$ the computational face center of face $\mathcal{S}_m$, defined by $\xi_{m,i} = \pm\delta_{im}$.

# Benefits of Taylor Quadrature Rule

The flow derivatives necessary for the Taylor approximation can be easily computed:

- In computational coordinates the solution vector $U_h$ in cell $\mathcal{K}$, restricted to the face $\mathcal{S}_{m_1}$, can be written as:

$$U_{|\mathcal{S}_{m_1}} = \bar{U}(\bar{\xi}_{m_1}) + \xi_{m_2}\hat{U}_{m_2}(\mathcal{K}) + \xi_{m_3}\hat{U}_{m_3}(\mathcal{K}) + \xi_4\hat{U}_4(\mathcal{K})$$

$$+ \text{ higher-order terms,}$$

  hence, the flow derivatives $D^\alpha U_{|\mathcal{S}_{m_1}}$ can be computed directly using the series representation of $U$.

- The integrals $\int_{\mathcal{S}_m} \hat{\phi}(\xi)\xi_{i_1}^{k_1}\xi_{i_2}^{k_2}\cdots\xi_{i_d}^{k_d} d\hat{\mathcal{S}}_k^m$ can be easily computed analytically.
- For multi-dimensional integrals the number of flux evaluations in the Taylor quadrature rule is nearly independent of the dimension $n$ of the integration domain. A product Gauss quadrature rule would require $2^n$ or $3^n$ flux evaluations.

# Taylor Quadrature Rule for Space-Time DG

**Remarks**

- The gradient contribution is necessary to obtain second order accuracy for the element face flux integrals.
- For the stability of a DG discretization with linear basis functions it is also essential to incorporate the gradient contributions in the approximation of the integral.
- For an upwind flux:

$$\hat{\mathcal{F}}(U_L, U_R) := \tfrac{1}{2}(\mathcal{F}(U_L) + \mathcal{F}(U_R))n - D(U_L, U_R),$$

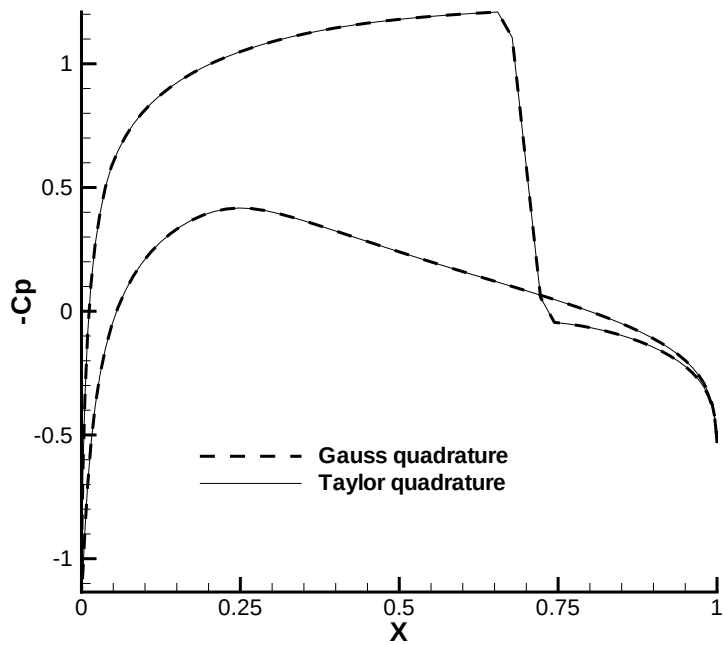
  it is essential to not just expand the central part of the flux, but also the dissipative part $D(U_L, U_R)$.
- It is important to incorporate the dissipative part of the upwind flux in the Taylor approximation which has been one of the reasons to apply the HLLC-flux.

# Theoretical Analysis of Taylor Quadrature

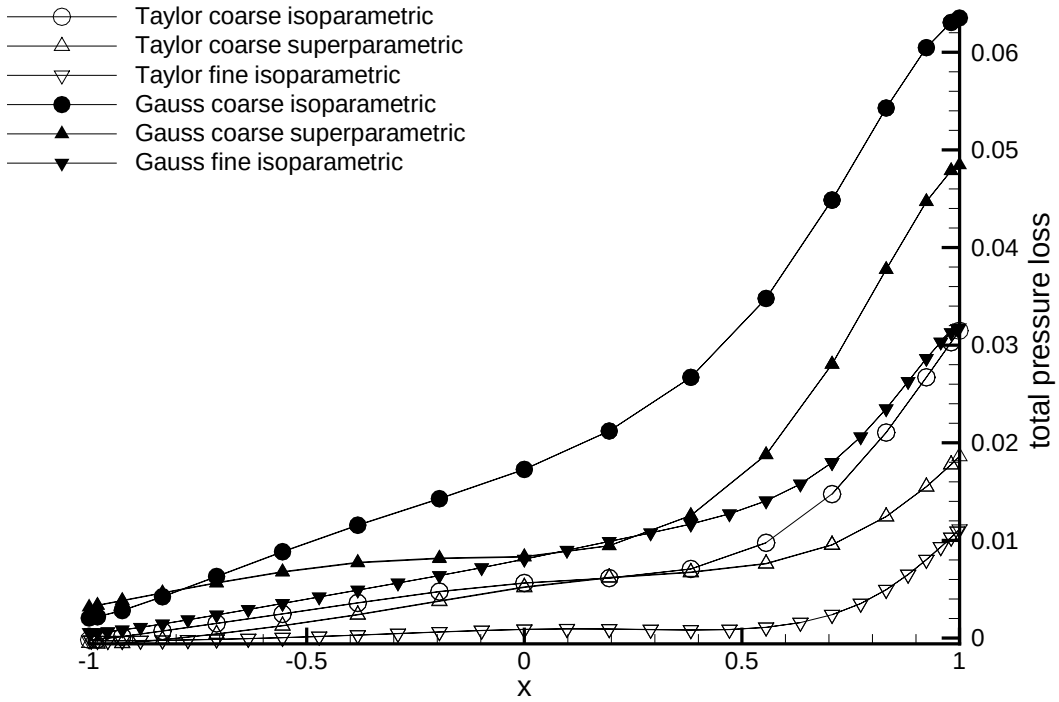
- An extensive theoretical analysis showing that for linear basis functions the Taylor and Gauss quadrature rules provide the same order of accuracy can be found in:

  H. van der Ven and J.J.W. van der Vegt, Space-time discontinuous Galerkin finite element method with dynamic grid motion for inviscid compressible flows. II. Efficient flux quadrature, *Comput. Meth. Appl. Mech. Engrg.* **191**, pp. 4747-4780 (2002).

# Comparison of Taylor and Gauss Quadrature Rules




Pressure distribution for transonic flow over a NACA0012 airfoil computed with Taylor and Gauss quadrature rules for the element and face fluxes ($M_\infty = 0.8$, $\alpha = 2^\circ$).




Total pressure loss at the wall for the flow around a circular cylinder ($M_\infty = 0.38$) using Gauss and Taylor flux quadrature rules on coarse ($32 \times 48$ elements) and fine mesh ($64 \times 96$).

# Conclusions

The space-time discontinuous Galerkin method has the following interesting properties:

- Accurate, unconditionally stable scheme for the compressible Navier-Stokes equations.
- Conservative discretization on moving and deforming meshes which satisfies the geometric conservation law.
- Local, element based discretization suitable for $h$-$(p)$ mesh adaptation.

# Conclusions

- The Taylor quadrature significantly improves the efficiency of the flux integrals without a reduction in accuracy.
- The use of a stabilization operator instead of a slope limiter makes it possible to converge to machine accuracy for steady state problems, whereas a limiter prevents convergence to steady state and reduces accuracy in a large part of the domain.
- Optimal accuracy and stability of space-time DG discretization for advection-diffusion equation proven in.

  J.J. Sudirham, J.J.W. van der Vegt, R.M.J. van Damme, Space-time discontinuous Galerkin method for advection-diffusion problems on time-dependent domains, *Applied Numerical Mathematics*, **56** (2006) 1491-1518.

For more information on space-time (DG) methods:

`wwwhome.math.utwente.nl/~vegtjjw/`