Space-Time Discontinuous Galerkin Methods for Compressible Flows

Part I Conservation Laws

Jaap van der Vegt

Numerical Analysis and Computational Mechanics Group
Department of Applied Mathematics
University of Twente
The Netherlands

Montreal Scientific Computing Days

April 30 - May 2, 2008

Introduction

Challenges for Compressible Flow Simulations

- Efficient capturing of local flow phenomena
  - Shocks
  - Interior and boundary layers
  - Vortical structures
- Time dependent boundaries
  - Fluid-structure interaction
- Robustness and computational efficiency
- Complex geometries

Improving CFD Algorithms

Options to improve CFD algorithms

- Higher order accuracy on unstructured meshes
- hp-Adaptive methods to capture (non smooth) local structures.
- Space-time approach to account for time-dependent boundaries
- Efficient algorithms for massively parallel computers

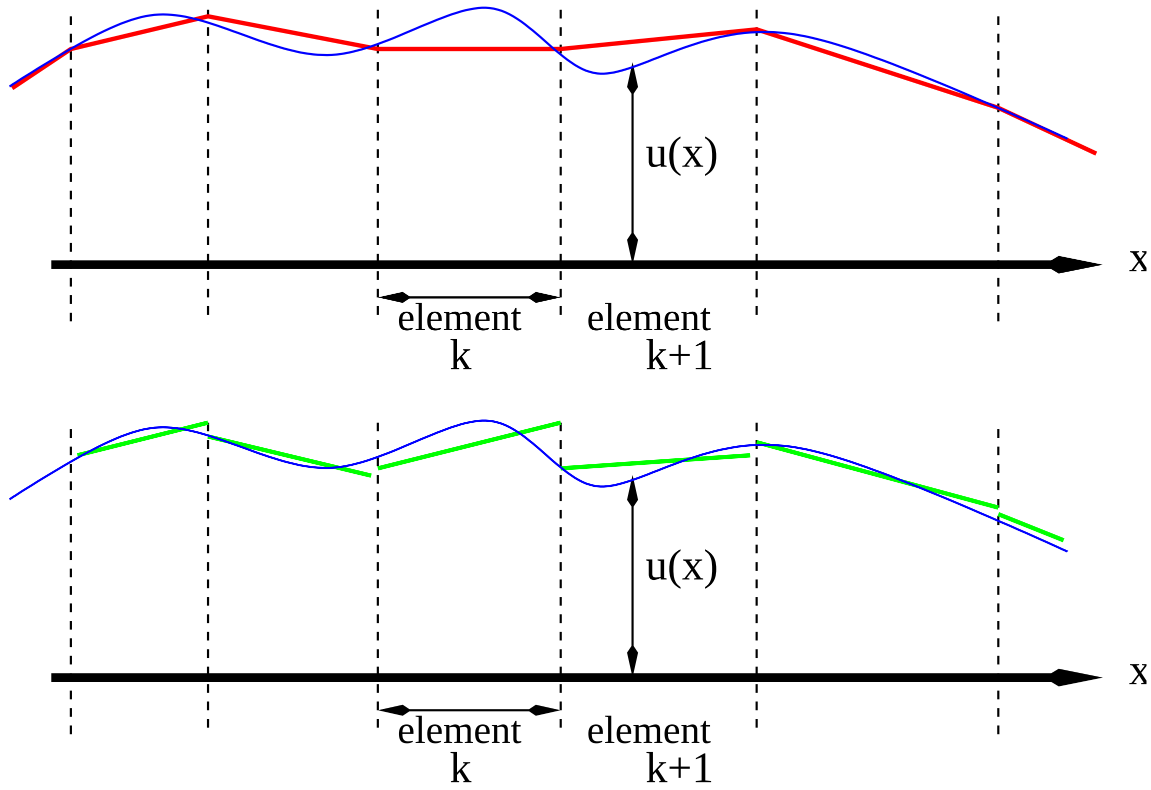
These requirements have motivated the development of

Space-Time Discontinuous Galerkin Finite Element Methods

Key Features of Space-Time Discontinuous Galerkin Methods

- Simultaneous discretization in space and time: time is considered as a fourth dimension.
- Discontinuous basis functions, both in space and time, with only a weak coupling across element faces resulting in an extremely local, element based discretization.
- The space-time DG method is closely related to the Arbitrary Lagrangian Eulerian (ALE) method.

Continuous and Discontinuous Galerkin Approximations



Continuous and discontinuous Galerkin approximation.

Benefits of Discontinuous Galerkin Methods

- Due to the extremely local discretization DG methods provide optimal flexibility for
  - achieving higher order accuracy on unstructured meshes
  - hp-mesh adaptation
  - unstructured meshes containing different types of elements, such as tetrahedra, hexahedra and prisms
  - parallel computing

Benefits of Space-Time Discontinuous Galerkin Methods

- A conservative discretization is obtained on moving and deforming meshes.
- No data interpolation or extrapolation is necessary on dynamic meshes, at free boundaries and after mesh adaptation.

Disadvantages of Space-(Time) Discontinuous Galerkin Methods

- Algorithms are generally rather complicated, in particular for elliptic and parabolic partial differential equations
- On structured meshes DG methods are computationally more expensive than finite difference and finite volume methods.

Overview of Lectures

- Lecture 1
  - One dimensional example
  - DG discretization for conservation laws
  - Extension to space-time DG discretizations
- Lecture 2
  - Space-time DG discretization of the Euler and Navier-Stokes equations
  - Examples of applications
- Lecture 3
  - Pseudo-time and multigrid techniques to solve nonlinear algebraic equations
  - Examples of applications

One-Dimensional Example

Advection equation

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = 0 \qquad \text{in } (0,1)\times(0,T),$$

$$u(x,0) = u_0(x) \qquad \forall x \in (0,1),$$

periodic boundary conditions

with $a \in \mathbb{R}$ a given constant.

DG Discretization for Advection Equation

Basic steps in the derivation of a 1D DG discretization

- Introduce a partition $\{x_{j+\frac{1}{2}}\}_{j=0}^{N}$ of the interval $(0, 1)$.
- Define elements $K_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, with $j = 1, \cdots, N$.
- Introduce the finite element space

$$V_h^k := \Big\{ v \in L^2(\Omega) \mid v|_{K_j} \in P^k(K_j), j = 1, \cdots, N \Big\}$$

with $P^k(K_j)$ polynomials of degree at most k on element K_j.

- **Note, the basis functions are discontinuous at element boundaries.**

- Define the Galerkin approximation:

  Find a $u_h(t) \in V_h^k$, such that for all $v \in V_h^k$,

$$\sum_{j=1}^{N} \int_{K_j} v(x)\left(\frac{\partial u_h(x,t)}{\partial t} + a\frac{\partial u_h(x,t)}{\partial x}\right)dx = 0$$

- Integrate by parts

$$\sum_{j=1}^{N} \frac{d}{dt}\int_{K_j} v(x)u_h(x,t)dx - \int_{K_j} au_h(x,t)\frac{\partial v(x)}{\partial x}dx$$
$$+ v(x^-_{j+\frac{1}{2}})au_h(x_{j+\frac{1}{2}},t) - v(x^+_{j-\frac{1}{2}})au_h(x_{j-\frac{1}{2}},t) = 0$$

- Note, the trace $u_h(x_{j+\frac{1}{2}}, t)$ at element boundaries is multivalued due to the discontinuous basis functions.

- Introduce a numerical flux to account for the multivalued trace

$$H(u_h)_{j+\frac{1}{2}}(t) := H(u_h(x^-_{j+\frac{1}{2}}, t), u_h(x^+_{j+\frac{1}{2}}, t))$$

- The numerical flux is related to the solution of a Riemann problem with left state $u_h(x^-_{j+\frac{1}{2}}, t)$ and right state $u_h(x^+_{j+\frac{1}{2}}, t)$.

- The Riemann problem introduces upwinding into the DG formulation.

- Numerical flux

$$H(u_h)_{j+\frac{1}{2}}(t) = \frac{1}{2}\left(au_h^- + au_h^+ - |a|(u_h^+ - u_h^-)\right).$$

with $u_h^\pm := u_h(x^\pm_{j+\frac{1}{2}}, t)$.

- Weak formulation: Find a $u_h(t) \in V_h^k$, such that for all $v \in V_h^k$,

$$\sum_{j=1}^{N} \frac{d}{dt}\int_{K_j} v(x)u_h(x,t)dx - \int_{K_j} au_h(x,t)\frac{\partial v(x)}{\partial x}dx$$
$$+ v(x^-_{j+\frac{1}{2}})H(u_h)_{j+\frac{1}{2}}(t) - v(x^+_{j-\frac{1}{2}})H(u_h)_{j-\frac{1}{2}}(t) = 0$$

- Introduce the polynomial expansions for u_h and v

$$u_h(t,x)|_{K_j} = \sum_{m=0}^{k} \hat{U}_m(t)\phi_m(x)$$

$$v(x)|_{K_j} = \phi_i(x), \quad \text{and zero elsewhere}$$

with basis functions $\phi_i \in P^k(K_j)$ into the weak formulation.

- Then we obtain for each element K_j a system of ordinary differential equations

$$\begin{aligned}\sum_{m=0}^{k} \frac{d\hat{U}_m}{dt} \int_{K_j} \phi_i(x)\phi_m(x)dx = \int_{K_j} a u_h(x,t)\frac{\partial \phi_i(x)}{\partial x}dx \\ - \phi_i(x^-_{j+\frac{1}{2}})H(u_h)_{j+\frac{1}{2}}(t) + \phi_i(x^+_{j-\frac{1}{2}})H(u_h)_{j-\frac{1}{2}}(t), \\ j = 1,\cdots,N;\ i = 0,\cdots,k\end{aligned}$$

- or symbolically

$$M\frac{d\hat{\mathbf{U}}}{dt} = \mathbf{R}_h(\hat{\mathbf{U}}).$$

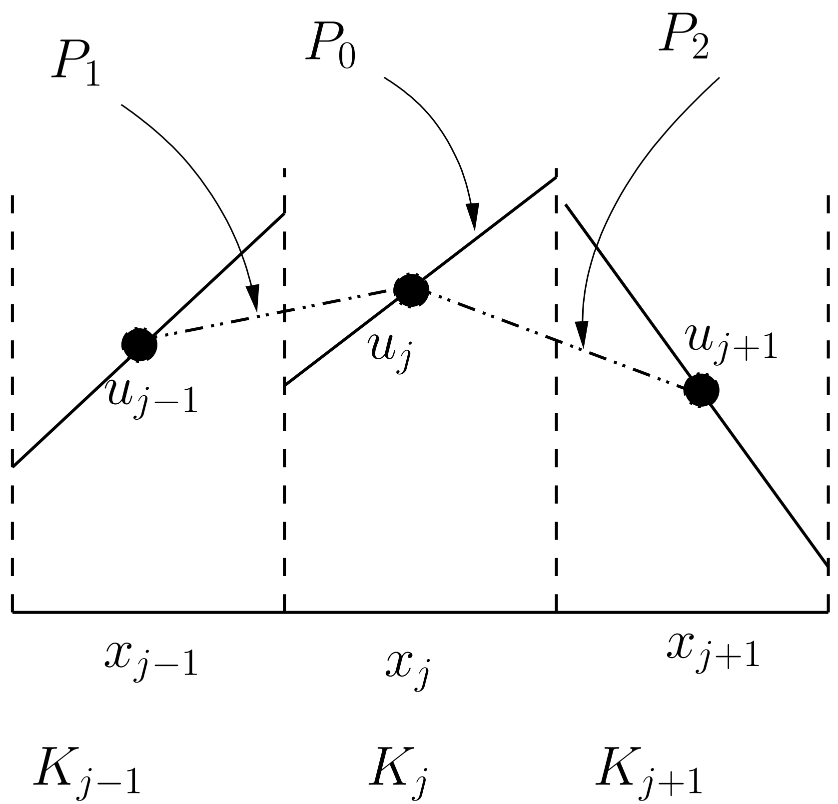
- Integrate in time using the third order Runge-Kutta scheme of Shu and Osher

$$\begin{aligned}
\hat{\mathbf{U}}^{(1)} &= \hat{\mathbf{U}}^n + \triangle t M^{-1}\mathbf{R}_h(\hat{\mathbf{U}}^n) \\
\hat{\mathbf{U}}^{(2)} &= \frac{1}{4}\left[3\hat{\mathbf{U}}^n + \hat{\mathbf{U}}^{(1)} + \triangle t M^{-1}\mathbf{R}_h(\hat{\mathbf{U}}^{(1)})\right] \\
\hat{\mathbf{U}}^{n+1} &= \frac{1}{3}\left[\hat{\mathbf{U}}^n + 2\hat{\mathbf{U}}^{(2)} + 2\triangle t M^{-1}\mathbf{R}_h(\hat{\mathbf{U}}^{(2)})\right]
\end{aligned}$$

- The time integration is stable for the CFL condition $CFL \leq 1$ with

$$CFL = \frac{|a|\triangle t}{\min_j |K_j|}$$

- For non-smooth initial data the solution will be oscillatory.
- To reduce numerical oscillations a slope limiter is used.
- The basic idea is to replace the original polynomial approximation u_h with a less oscillatory polynomial $\tilde{u}_h$ using a reconstruction from data at the midpoints of the element and its neighbors.



Construction of a slope limiter

- For $u_h \in V_h^1$, construct two linear polynomials P_1 and P_2:

$$P_1 = \frac{u_j(x - x_{j-1}) - u_{j-1}(x - x_j)}{x_j - x_{j-1}}, \quad P_2 = \frac{u_j(x - x_{j+1}) - u_{j+1}(x - x_j)}{x_j - x_{j+1}}$$

- Project the polynomials P_i, $i = 0, 1, 2$, with $P_0 = u_h$, onto the DG-space V_h^1 and solve for $(\hat{U}_0)_i$ and $(\hat{U}_1)_i$:

$$\begin{bmatrix} \int_{K_j} \phi_0 \phi_0 \, dK & \int_{K_j} \phi_0 \phi_1 \, dK \\ \int_{K_j} \phi_1 \phi_0 \, dK & \int_{K_j} \phi_1 \phi_1 \, dK \end{bmatrix} \begin{bmatrix} (\hat{U}_0)_i \\ (\hat{U}_1)_i \end{bmatrix} = \begin{bmatrix} \int_{K_j} \phi_0 P_i \, dK \\ \int_{K_j} \phi_1 P_i \, dK \end{bmatrix}$$

- Use an oscillation indicator $o_i = \partial P_i / \partial x$, $i = 0, 1, 2$, to assess the smoothness of the polynomials.

- The polynomial coefficients $\tilde{U}_m$ of the limited solution $\tilde{u}_h$ are constructed as a weighted sum of all polynomials

$$\tilde{U}_m = \sum_{i=0}^{2} w_i (\hat{U}_m)_i, \qquad m = 0, 1$$

- The weights are

$$w_i = \frac{(\epsilon + o_i(P_i))^{-\gamma}}{\sum_{j=0}^{2} (\epsilon + o_i(P_j))^{-\gamma}}$$

- Take $\gamma = 1$ and $\epsilon \ll 1$. For more smoothing increase γ.

- The limited solution then is equal to

$$\tilde{u}_h = \sum_{m=0}^{1} \tilde{U}_m(t)\phi_m(x)$$

- **A serious problem with limiters is that the limited solution does not satisfy the DG discretization. This prevents convergence to steady state.**

- An alternative for limiters are stabilization operators.

General Conservation Laws

- Consider the general conservation law on $\Omega \subset \mathbb{R}^d$

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \operatorname{div} f(u) = 0 \qquad && \forall (x,t) \in \Omega \times (0,T), \\
&u(0,x) = u_0(x) && \forall x \in \Omega, \\
&u(t,x) = \mathcal{B}(u, u_w) && \forall (x,t) \in \partial\Omega \times (0,T).
\end{aligned}
$$

with $u : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ the conserved variable and $f : \mathbb{R} \longrightarrow \mathbb{R}^d$ the flux vector.

Computational Mesh and Basis Functions

- Introduce a tessellation $\mathcal{T}_h$ of Ω

$$\mathcal{T}_h := \left\{ K \mid \cup \bar{K} = \bar{\Omega} \text{ and } K \cap K' = \emptyset \text{ if } K \neq K' \right\} .$$

- Define reference element(s), e.g. a reference cube $\hat{K} = [-1, 1]^d$.
- Define polynomial basis functions $P^k(\hat{K})$ of maximum (or total) degree k on the reference element.

- The element K is related to the reference element $\hat{K}$ using an isoparametric mapping

$$F_K : \hat{K} \longrightarrow K; \xi \mapsto x = \sum_{i=1}^{m} x_i(K)\hat{\phi}_i(\xi).$$

with $x_i(K)$ the nodal points of element K and $\hat{\phi}_i \in P^k(\hat{K})$ the basis functions.

- Use the element mapping $F_K : \hat{K} \longrightarrow K$ to define the basis functions on element K

$$\phi_m(x) = \hat{\phi}_m \circ F_K^{-1}(x).$$

- Define the finite element space

$$V_h^k := \left\{ v \in L^2(\Omega) \mid v|_K \circ F_K \in P^k(\hat{K}) \ \forall K \in \mathcal{T}_h \right\}.$$

- Note, the basis functions are **discontinuous** at the element faces.

Weak Formulation

- Multiply the conservation law with arbitrary test functions $v \in V_h^k$, replace u with u_h, integrate over K and sum over all elements

$$\sum_{K \in \mathcal{T}_h} \int_K v \frac{\partial u_h}{\partial t} dK + \int_K v \, \mathrm{div} f(u_h) dK = 0 \qquad \forall v \in V_h^k.$$

- Integrate by parts

$$\sum_{K \in \mathcal{T}_h} \frac{d}{dt} \int_K v u_h dK - \int_K \mathrm{grad} v \cdot f(u_h) dK + \int_{\partial K} v^- n^- \cdot f(u_h^-) dS = 0 \quad \forall v \in V_h^k.$$

with the traces defined as $u_h^{\pm} = \lim_{\epsilon \downarrow 0} u_h(x \pm n)$ and n the unit outward normal vector at ∂K.

Flux Integrals

- Since the basis functions are discontinuous at the element faces we have to account for the multivalued traces.
- We can transform the boundary integrals into:

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} v^- n^- \cdot f(u_h^-)dS = \sum_S \int_S \frac{1}{2}(v^- n^- + v^+ n^+)\cdot(f(u_h^-)+f(u_h^+))$$

$$+\frac{1}{2}(v^- + v^+)(n^- \cdot f(u_h^-) + n^+ \cdot f(u_h^+))dS$$

with n^-, n^+ the normal vectors at each side of face S, $n^+ = -n^-$.

- The formulation must be conservative, which imposes the condition:

$$\int_S v n^- \cdot f(u_h^-) dS = -\int_S v n^+ \cdot f(u_h^+) dS, \qquad \forall v \in V_h^k$$

hence the contribution

$$\sum_S \int_S \frac{1}{2}(v^- + v^+)(n^- \cdot f(u_h^-) + n^+ \cdot f(u_h^+)) dS = 0$$

- Using the relation $n^+ = -n^-$, the boundary integrals then are equal to:

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} v^- n^- \cdot f(u_h^-) dS = \sum_S \int_S (v^- - v^+) \frac{1}{2} n^- \cdot (f(u_h^-) + f(u_h^+)) dS,$$

Numerical Flux

- In order to stabilize the DG FEM formulation the multi-valued trace of the flux at S is replaced with a numerical flux function:

$$\frac{1}{2}n \cdot (f(u_h^-) + f(u_h^+)) \cong H(u_h^-, u_h^+, n)$$

- To ensure convergence the numerical flux must be
  - consistent: $H(u, u, n) = n \cdot f(u)$;
  - conservative: $H(u_h^-, u_h^+, n^-) = -H(u_h^+, u_h^-, n^+)$;
  - locally Lipschitz continuous:

$$|H(u_h^-, u_h^+, n) - H(v_h^-, v_h^+, n)| \leq C(|u_h^- - v_h^-| + |u_h^+ - v_h^+|)$$

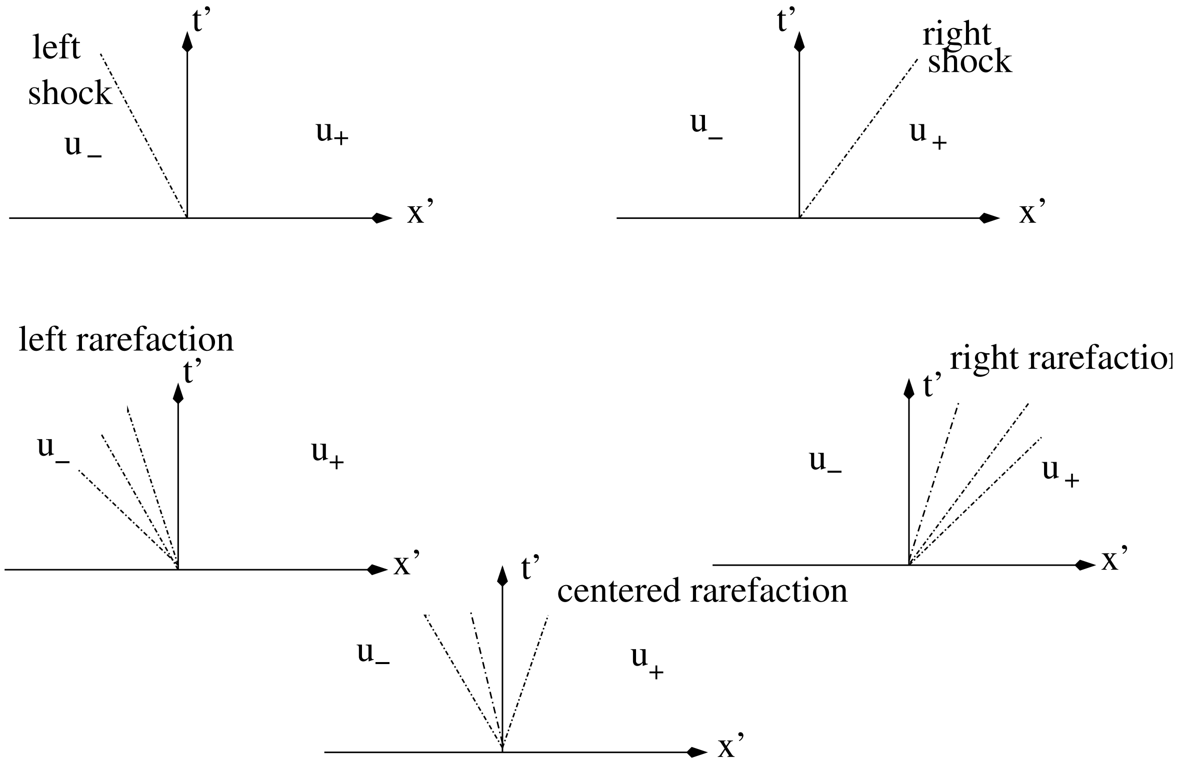
- To ensure monotonicity the numerical flux must also be
  - a nondecreasing function of its first argument, and
  - a nonincreasing function of its second argument

Riemann Problem

- A monotone Lipschitz flux $H(u_h^-, u_h^+, n)$ is obtained by (approximately) solving the Riemann problem with initial states u_h^- and u_h^+ at the element faces ∂K.

- This procedure introduces upwinding into the discontinuous Galerkin finite element method.

Riemann problem for Burgers Equation



Solutions of the Riemann problem for the Burgers equation $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{1}{2}u^2) = 0.$

Upwind Fluxes

Consistent, monotone Lipschitz fluxes are:

- Godunov flux

$$H^G(u_h^-, u_h^+, n) = \begin{cases} \min\limits_{u \in [u_h^-, u_h^+]} \hat{f}(u), & \text{if } u_h^- \leq u_h^+ \\ \max\limits_{u \in [u_h^+, u_h^-]} \hat{f}(u), & \text{otherwise,} \end{cases}$$

with $\hat{f}(u) = f(u) \cdot n$.

Upwind Fluxes

- Local Lax-Friedrichs flux

$$
\begin{aligned}
H^{LLF}(u_h^-, u_h^+, n) &= \frac{1}{2}(\hat{f}(u_h^-) + \hat{f}(u_h^+) - C(u_h^+ - u_h^-)), \\
C &= \max_{\min(u_h^-, u_h^+) \le s \le \max(u_h^-, u_h^+)} |\hat{f}'(s)|,
\end{aligned}
$$

- Roe flux with entropy fix

$$H^{Roe}(u_h^-, u_h^+, n) = \begin{cases} \hat{f}(u_h^-), & \text{if } \hat{f}'(u) \geq 0 \text{ for } u \in \mathcal{U} \\ \hat{f}(u_h^+), & \text{if } \hat{f}'(u) \leq 0 \text{ for } u \in \mathcal{U} \\ H^{LLF}(u_h^-, u_h^+, n) & \text{otherwise} \end{cases}$$

with $\mathcal{U} = [\min(u_h^-, u_h^+), \max(u_h^-, u_h^+)]$

- The choice which numerical flux should be used depends on many aspects, e.g. accuracy, robustness, computational complexity, and personal preference.

DG Discretization

- Introducing the numerical flux into the face integrals then results in

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h}\int_{\partial K} v^- n^- \cdot f(u_h^-)dS &= \sum_S \int_S (v^- - v^+)\frac{1}{2}n^- \cdot (f(u_h^-) + f(u_h^+))dS \\
&\cong \sum_S \int_S (v^- - v^+)H(u_h^-, u_h^+, n^-)d\mathcal{S} \\
&= \sum_{K\in\mathcal{T}_h}\int_{\partial K} v^- H(u_h^-, u_h^+, n^-)dS,
\end{aligned}$$

using the relation $H(u_h^-, u_h^+, n^-) = -H(u_h^+, u_h^-, n^+)$.

- The weak formulation then is equal to:

$$\sum_{K \in \mathcal{T}_h} \frac{d}{dt} \int_K v u_h dK - \int_K \mathrm{grad} v \cdot f(u_h) dK$$
$$+ \int_{\partial K} v^- H(u_h^-, u_h^+, n^-) dS = 0 \quad \forall v \in V_h^k.$$

- The DG discretization is obtained after introducing the basis functions

$$u_h(x,t)|_K = \sum_{j=0}^{M} \hat{U}_j(t) \phi_j(x)$$
$$v(x)|_K = \phi_i(x) \quad \text{and zero elsewhere}$$

- For each element $K \in \mathcal{T}_h$ the DG discretization becomes a system of ordinary differential equations:

$$\sum_{j=0}^{M} \frac{d\hat{U}_j(t)}{dt} \int_K \phi_i \phi_j dK = \int_K \mathrm{grad}\phi_i \cdot f(u_h) dK$$

$$- \int_{\partial K} \phi_i H(u_h^-, u_h^+, n^-) dS = 0, \quad i = 0, \cdots, M.$$

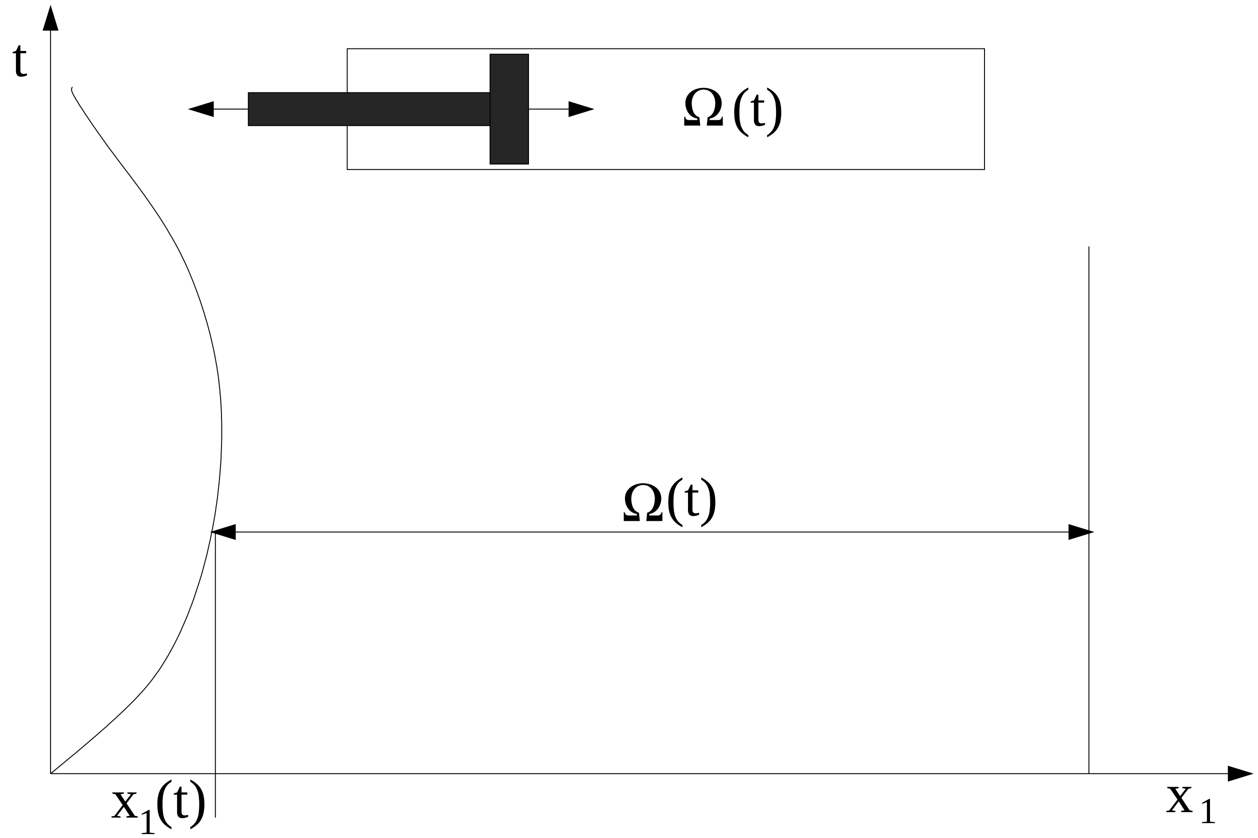
- Evaluate the integrals using quadrature rules. In particular Gaussian quadrature rules are very efficient.

- The resulting discretization can be summarized as

$$M\frac{\partial \hat{\mathbf{U}}}{\partial t} = \mathbf{R}_h(\hat{\mathbf{U}})$$

which can be integrated in time with e.g. a (TVD) Runge-Kutta method.

Time-Dependent Flow Domains



Example of a time dependent flow domain $\Omega(t)$.

Conservation Laws on Time Dependent Flow Domains

- Consider the scalar conservation law on a time dependent flow domain $\Omega(t) \subset \mathbb{R}^d$:

$$\frac{\partial u}{\partial t} + \mathrm{div} f(u) = 0, \qquad \text{on } \Omega(t),\ t \in (t_0, T),$$

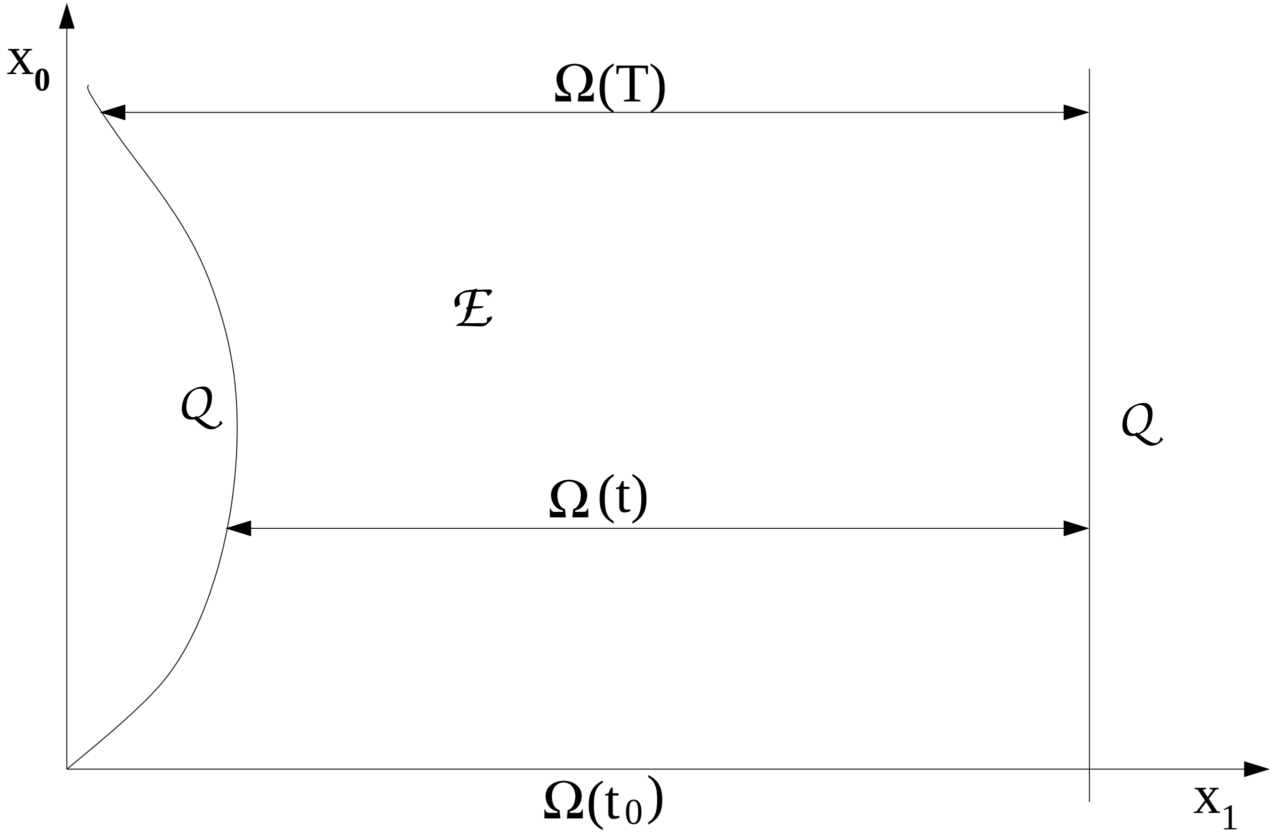
with $u : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ the conserved variable and $f : \mathbb{R} \longrightarrow \mathbb{R}^d$ the flux vector.

- The boundary and initial conditions are

$$\begin{aligned} u(x,t) &= \mathcal{B}(u, u_w), \qquad && \text{at } \partial\Omega(t),\ t \in (t_0, T), \\ u(x,0) &= u_0(x), && \text{in } \Omega(t_0). \end{aligned}$$

- We can also consider this problem in a **space-time framework**

Space-Time Domain



Example of a space-time domain $\mathcal{E}$.

Definition of Space-Time Domain

- Let $\mathcal{E} \subset \mathbb{R}^{d+1}$ be an open domain.
- A point $x \in \mathbb{R}^{d+1}$ has coordinates $(x_0, \bar{x})$, where x_0 represents time and $\bar{x} := (x_1, \cdots, x_d)$ the spatial coordinates.
- Define the flow domain Ω at time t as:

$$\Omega(t) := \{\bar{x} \in \mathbb{R} \,|\, (t, \bar{x}) \in \mathcal{E}\}$$

- Define the boundary $\mathcal{Q}$ as:

$$\mathcal{Q} := \{x \in \partial\mathcal{E} \,|\, t_0 < x_0 < T\}$$

- **Note :** The space-time domain boundary $\partial\mathcal{E}$ is equal to:

$$\partial\mathcal{E} = \Omega(t_0) \cup \mathcal{Q} \cup \Omega(T).$$

Space-Time Formulation of Conservation Laws

- Define the space-time flux vector: $\mathcal{F}(u) := (u, f(u))^T$, then scalar conservation laws can be written as:

$$\operatorname{div} \mathcal{F}(u(x)) = 0, \qquad x \in \mathcal{E}$$

with boundary conditions:

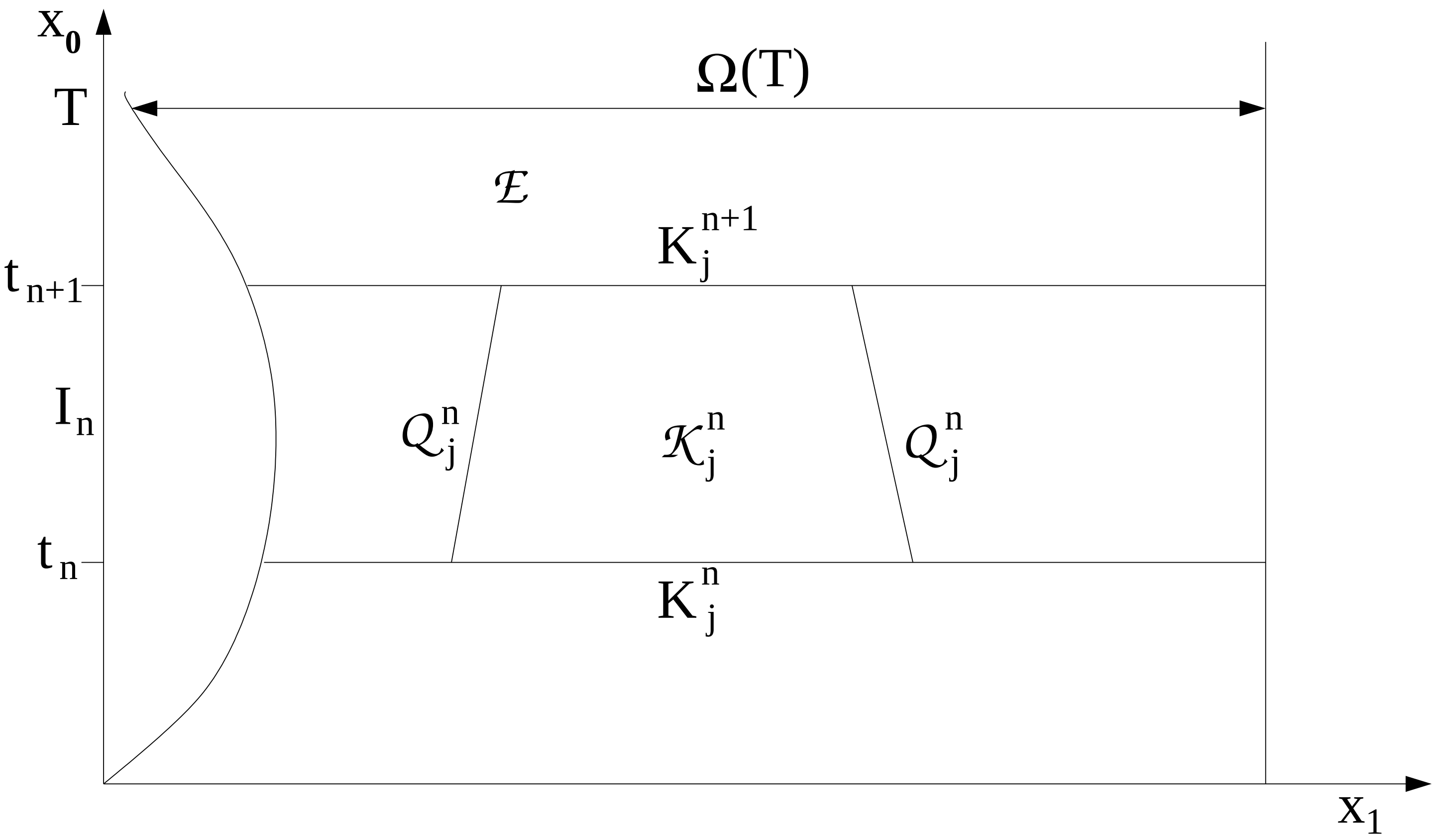
$$u(x) = \mathcal{B}(u, u_w), \qquad x \in \mathcal{Q},$$

and initial condition:

$$u(x) = u_0(x), \qquad x \in \Omega(t_0).$$

- The div operator is defined as: $\operatorname{div} \mathcal{F} = \frac{\partial \mathcal{F}_i}{\partial x_i}$.

Space-Time Slab

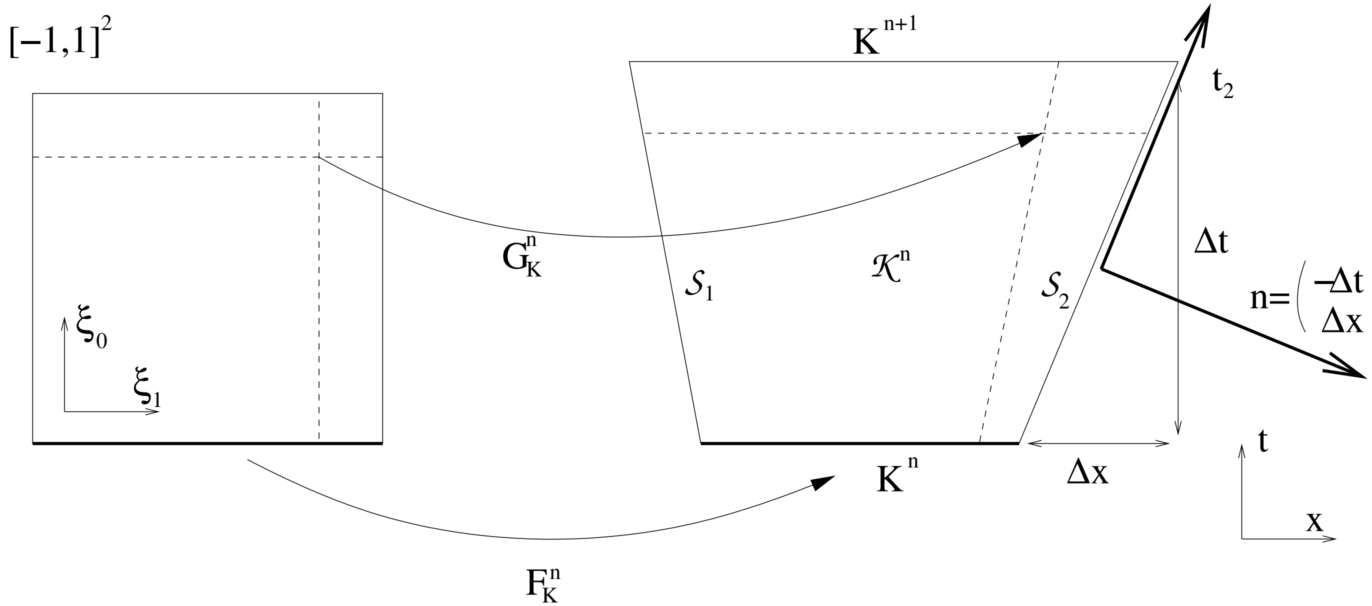


Space-time slab in space-time domain $\mathcal{E}$.

Definition of a Space-Time Slab

- Consider a partitioning of the time interval (t_0, T): $\{t_n\}_{n=0}^N$, and set $I_n = (t_n, t_{n+1})$.
- Define a space-time slab as: $\mathcal{I}_n := \{x \in \mathcal{E} \mid x_0 \in I_n\}$
- Split the space-time slab into non-overlapping elements: $\mathcal{K}_j^n$.
- We will also use the notation: $K_j^n = \mathcal{K}_j^n \cap \{t_n\}$ and $K_j^{n+1} = \mathcal{K}_j^n \cap \{t_{n+1}\}$

Geometry of Space-Time Element



Geometry of 2D space-time element in both computational and physical space.

Element Mappings

Definition of the mapping $G^n_{\mathcal{K}}$ which the connects the space-time element $\mathcal{K}^n$ to the reference element $\hat{\mathcal{K}} = (-1, 1)^d$:

- Define a smooth, orientation preserving and invertible mapping Φ^n_t in the interval I_n as:

$$\Phi^n_t : \Omega(t_n) \to \Omega(t) : \bar{x} \mapsto \Phi^n_t(\bar{x}), \quad t \in I_n.$$

- Split $\Omega(t_n)$ into the tessellation $\bar{\mathcal{T}}^n_h$ with non-overlapping spatial elements K_j.
- Define $\phi_i(\bar{\xi}), \bar{\xi} \in (-1, 1)^d$ as the standard Lagrangian finite element shape functions.

Element Mappings

- The mapping F_K^n is defined as:

$$F_K^n : (-1,1)^d \to K^n : \bar{\xi} \longmapsto \sum_{i=1}^{N_n} x_i(K^n)\phi_i(\bar{\xi}),$$

with $x_i(K^n)$ the spatial coordinates of the nodal points of the space-time element at time $t = t_n$.

- Similarly we define the mapping F_K^{n+1}:

$$F_K^{n+1} : (-1,1)^d \to K^{n+1} : \bar{\xi} \longmapsto \sum_{i=1}^{N_n} \Phi_{t_{n+1}}^n(x_i(K^n))\phi_i(\bar{\xi}).$$

Element Mappings

- The space-time element is defined by linear interpolation in time:

$$G_{\mathcal{K}}^n : (-1,1)^d \to \mathcal{K}^n : (\xi_0, \bar{\xi}) \longmapsto (x_0, \bar{x}),$$

with:

$$(x_0, \bar{x}) = \Big(\tfrac{1}{2}(t_n + t_{n+1}) - \tfrac{1}{2}(t_n - t_{n+1})\xi_0,$$
$$\tfrac{1}{2}(1-\xi_0)F_K^n(\bar{\xi}) + \tfrac{1}{2}(1+\xi_0)F_K^{n+1}(\bar{\xi}) \Big).$$

- The space-time tessellation is now defined as:

$$\mathcal{T}_h^n := \{\mathcal{K} = G_{\mathcal{K}}^n(\hat{\mathcal{K}}) \mid K \in \bar{\mathcal{T}}_h^n\}.$$

Basis Functions

- Define the basis functions ϕ_m in a space-time element $\mathcal{K}$ as:

$$\phi_m(x) = \hat{\phi}_m \circ (G^n_{\mathcal{K}}(x))^{-1}.$$

with $\hat{\phi}_m \in P^k(\hat{\mathcal{K}})$ polynomial basis functions of maximum (or total) degree k on the reference element.

- Introduce the basis functions $\psi_m : \mathcal{K} \longrightarrow \mathbb{R}$ and split the test and trial functions into an element mean at time t_{n+1} and a fluctuating part:

$$\begin{aligned}\psi_m(x) &= 1, & m &= 0,\\ &= \phi_m(x) - \frac{1}{|K(t_{n+1})|}\int_{K(t_{n+1})} \phi_m(x)\,dK, & m &\geq 1.\end{aligned}$$

- The splitting is beneficial for the definition of the stabilization operator and multigrid convergence acceleration.

Finite Element Space

- Define the finite element space $V_h^k(\mathcal{T}_h^n)$ as:

$$V_h^k(\mathcal{T}_h^n) := \left\{ v_h \,\middle|\, v_h|_{\mathcal{K}} \circ G_{\mathcal{K}}^n \in P^k(\hat{\mathcal{K}}),\ \forall \mathcal{K} \in \mathcal{T}_h^n \right\}$$

- The trial functions $u_h : \mathcal{E} \longrightarrow \mathbb{R}$ are defined in each element $\mathcal{K} \in \mathcal{T}_h^n$ as:

$$u_h(x) = \sum_{m=0}^{M} \hat{U}_m(\mathcal{K})\psi_m(x), \quad x \in \mathcal{K},$$

with $\hat{U}_m$ the expansion coefficients.

Finite Element Space

- **Note :** Since $\int_{K(t_{n+1})} \psi_m(x)dK = 0$ for $m \geq 1$, we have the relation:

$$\bar{u}_h(K(t_{n+1})) := \frac{1}{|K(t_{n+1})|} \int_{K(t_{n+1})} u_h dK = \hat{U}_0,$$

  and we can write:

$$u_h(x) = \bar{u}_h(K(t_{n+1})) + \tilde{u}_h(x),$$

  with $\int_{K(t_{n+1})} \tilde{u}_h(x)dK = 0$.

- One of the main benefits of this splitting is that the equation for $\hat{U}_0$ is very similar to a first order finite volume discretization and is only weakly coupled to the equations for $\tilde{u}_h$.

Weak Formulation for STDG Method

The scalar conservation law can now be transformed into a weak formulation:

- Find a $u_h \in V_h^k(\mathcal{T}_h^n)$, such that for all $w_h \in V_h^k(\mathcal{T}_h^n)$, we have:

$$\sum_{n=0}^{N_T} \sum_{j=1}^{N_e} \Big(\int_{\mathcal{K}_j^n} w_h \, \mathrm{div}\mathcal{F}(u_h) d\mathcal{K} + \int_{\mathcal{K}_j^n} (\mathrm{grad}\, w_h)^T \, \mathfrak{D}(u_h) \, \mathrm{grad}\, u_h d\mathcal{K} \Big) = 0.$$

- The second integral with $\mathfrak{D}(u_h) \in \mathbb{R}^{d+1}$ is the stabilization operator necessary to obtain monotone solutions near discontinuities.

Weak Formulation

After integration by parts we obtain the following weak formulation:

- Find a $u_h \in V_h^k(\mathcal{T}_h^n)$, such that for all $w_h \in V_h^k(\mathcal{T}_h^n)$, we have:

$$\sum_{n=0}^{N_T} \sum_{j=1}^{N_e} \Big(- \int_{\mathcal{K}_j^n} \operatorname{grad} w_h \cdot \mathcal{F}(u_h) d\mathcal{K} + \int_{\partial \mathcal{K}_j^n} w_h^- n^- \cdot \mathcal{F}(u_h^-) d\mathcal{S}$$

$$+ \int_{\mathcal{K}_j^n} (\operatorname{grad} w_h)^T \mathfrak{D}(u_h) \operatorname{grad} u_h d\mathcal{K} \Big) = 0.$$

Flux Integrals

- Due to the summation over all space-time slabs and elements, the boundary integrals can be transformed into:

$$\sum_{\mathcal{K}} \int_{\partial\mathcal{K}} w_h^- n^- \cdot \mathcal{F}(u_h^-) d\mathcal{S} = \sum_{\mathcal{S}} \int_{\mathcal{S}} \frac{1}{2}(w_h^- - w_h^+) n^- \cdot (\mathcal{F}(u_h^-) + \mathcal{F}(u_h^+)) d\mathcal{S}$$

- Before we can introduce a numerical flux on the right hand side we first need to consider the space-time normal vector.

Arbitrary Lagrangian Eulerian Formulation

- At faces $\mathcal{S} \subseteq \Omega(t_{n+1})$ the space-time normal vector is equal to

$$n = (1, 0, \cdots, 0)$$

and at faces $\mathcal{S} \subseteq \Omega(t_n)$ we have

$$n = (-1, 0, \cdots, 0).$$

- At faces $\mathcal{S} \subseteq \mathcal{Q}$ the space-time normal vector can be expressed as:

$$n = (-v_g \cdot \bar{n}, \bar{n}),$$

with v_g the mesh velocity.

- If we introduce this relation into the flux then we obtain at faces $\mathcal{S} \subseteq \mathcal{Q}$

$$\mathcal{F}(u) \cdot n = f(u) \cdot \bar{n} - v_g \cdot \bar{n}\, u,$$

which is exactly the flux in an Arbitrary Lagrangian Eulerian (ALE) formulation.

Numerical Fluxes

- The numerical flux at the boundary faces $K(t_n)$ and $K(t_{n+1})$, which have as normal vectors $n^- = (\mp 1, 0, \cdots, 0)^T$, respectively, is defined as:

$$\begin{aligned} H_\Omega(u_h^-, u_h^+, n^-) &= u_h^+ \qquad \text{at } K(t_n) \\ &= u_h^- \qquad \text{at } K(t_{n+1}) \end{aligned}$$

  which ensures causality in time.

- The numerical flux at the boundary faces $\mathcal{Q}^n$ is a monotone Lipschitz $H(u_h^-, u_h^+, \bar{n}; v_g)$, which is consistent:

$$H(u, u, \bar{n}; v_g) = n \cdot \mathcal{F}(u) = f(u) \cdot \bar{n} - v_g \cdot \bar{n}\, u$$

  and conservative:

$$H(u_h^-, u_h^+, n^-; v_g) = -H(u_h^+, u_h^-, n^+; v_g).$$

Weak Formulation for DG Discretization

After introducing the numerical fluxes we can transform the weak formulation into:

- Find a $u_h \in V_h^k$, such that for all $w_h \in V_h^k$, the following variational equation is satisfied:

$$\sum_{j=1}^{N_n} \Big(- \int_{\mathcal{K}_j^n} (\mathrm{grad}\, w_h) \cdot \mathcal{F}(u_h) d\mathcal{K} + \int_{K_j(t_{n+1})} w_h^- u_h^- dK$$

$$- \int_{K_j(t_n)} w_h^- u_h^+ dK + \int_{\mathcal{Q}_j^n} w_h^- H(u_h^-, u_h^+, n^-; v_g) d\mathcal{S}$$

$$+ \int_{\mathcal{K}_j^n} (\mathrm{grad}\ w_h)^T \mathfrak{D}(u_h)\, \mathrm{grad}\, u_h\, d\mathcal{K} \Big) = 0.$$

- **Note:** Due to the causality of the time-flux the solution in a space-time slab only depends explicitly on the data u_h^+ from the previous space-time slab.

DG-Expansion Coefficient Equations for Element Mean

- Introduce the polynomial expansions for u_h and w_h into the weak formulation then the following set of equations for the element mean $\bar{u}_h(K_j(t_{n+1}))$ is obtained:

$$\big|K_j(t_{n+1})\big|\bar{u}_h(K_j(t_{n+1})) - \big|K_j(t_n)\big|\bar{u}_h(K_j(t_n)) + \int_{\mathcal{Q}_j^n} H(u_h^-, u_h^+, \bar{n}^-; v_g)\,d\mathcal{Q} = 0.$$

- These equations are equivalent to a first order accurate finite volume formulation, except that more accurate data are used at the element faces.

DG-Expansion Coefficient Equations for Element Fluctuations

- The equations for the coefficients $\hat{U}_m(\mathcal{K}_j^n)$, $(m \geq 1)$ for the fluctuating part of the flow field $\tilde{u}_h$ in each space-time element $\mathcal{K}_j^n$ satisfy the algebraic system

$$\sum_{m=1}^{M} \hat{U}_m(\mathcal{K}_j^n)\Big(-\int_{\mathcal{K}_j^n} \frac{\partial \psi_l}{\partial t}\psi_m d\mathcal{K} + \int_{K_j^{n+1}} \psi_l(t_{n+1}^-, \bar{x})\psi_m(t_{n+1}^-, \bar{x})dK$$

$$+ \int_{\mathcal{K}_j^n} \frac{\partial \psi_l}{\partial x_k}\mathfrak{D}_{kp}(u_h)\frac{\partial \psi_m}{\partial x_p}d\mathcal{K}\Big)$$

$$- \int_{K_j^n} u_h(t_n^-, \bar{x})\psi_l(t_n^+, \bar{x})dK - \bar{u}_h(K_j^{n+1}) \int_{\mathcal{K}_j^n} \frac{\partial \psi_l}{\partial t}d\mathcal{K}$$

$$+ \int_{\mathcal{Q}_j^n} \psi_l H(u_h^-, u_h^+, \bar{n}^-; v_g)d\mathcal{S} - \int_{\mathcal{K}_j^n} \frac{\partial \psi_l}{\partial \bar{x}_i}f_i(u_h)d\mathcal{K} = 0, \qquad l = 1, \cdots, M.$$

Solution of DG Expansion Coefficient Equations

- The space-time DG formulation results in an implicit time-integration scheme.
- The equations for the DG expansion coefficients are represented as:

$$\mathcal{L}(\hat{U}^n; \hat{U}^{n-1}) = 0.$$

- The non-linear equations for the expansion coefficients $\hat{U}^n$ are solved by introducing a pseudo-time τ and marching the solution with a Runge-Kutta method to a steady state:

$$\frac{\partial \hat{U}(\mathcal{K}^*)}{\partial \tau} = \frac{1}{\triangle t}\mathcal{L}(\hat{U}^*; \hat{U}^{n-1}).$$

- Convergence to steady state in pseudo-time can be accelerated using a FAS multigrid procedure.