

Theorem (Berman 1986). For any graph G whose edges are partitioned into two spanning trees, one with degrees $h(v)$ and the other with degrees $K(v)$, there is another such partition of the edges.

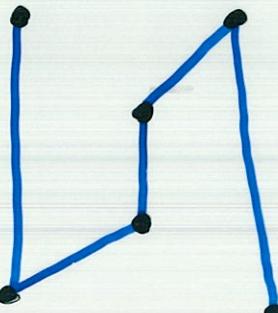
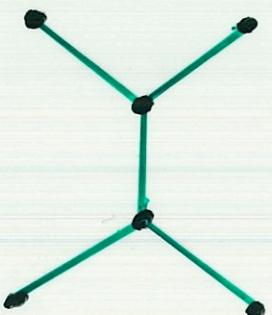
- Cameron and Edmonds found an algorithm to get from one partition to another
- complexity is unknown

The Intermediate Spanning Tree Problem (ISTP):

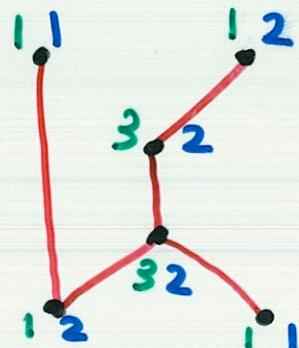
Let G be a graph. Let H and K be spanning trees of G with degrees $h(v)$ and $K(v)$.

Is there a spanning tree T (besides H or K) such that the degree of v in T is between $h(v)$ and $K(v)$ for all vertices v ?

If such a T exists, it is called an **intermediate tree** of H and K .



have
intermediate
tree :



If H and K are edge disjoint and $h(v) \neq K(v)$ for some v , it follows from theorem above that H and K have an intermediate tree.

ISTP is NP-complete since the following problem is:

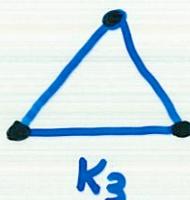
Given a graph containing two distinct Hamiltonian paths with ends u and v , is there another such Hamiltonian path?

Proof is based on a proof by Papadimitriou and Steiglitz (1977) that the following is NP-complete:

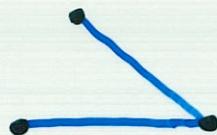
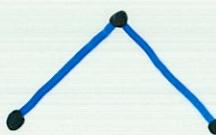
Given a graph G containing a Hamiltonian path, does G contain a Hamiltonian circuit?

Are there any graphs G in which no pair of spanning trees has an intermediate tree?

Yes :



has 3 spanning trees :



No two spanning trees of K_3 have an intermediate tree.

In fact, K_3 is essentially the only such graph:

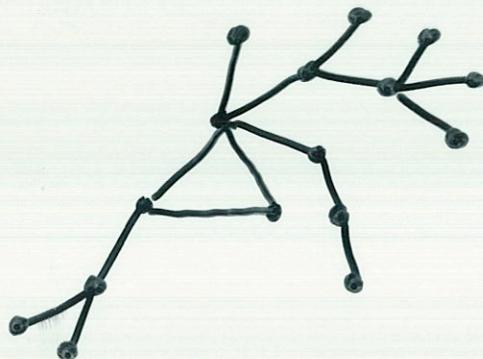
Theorem Let G be a connected graph with no loops and at least three spanning trees.

No two spanning trees of G have an intermediate tree

if and only if

the graph G' obtained from G by repeatedly removing vertices of degree 1 is K_3 .

i.e. G looks like :



Proof:

(\Rightarrow) Suppose G' has m vertices. Then $m \geq 3$. It is easy to show that G' must be simple. If $m = 3$, we're done.

Suppose $m \geq 4$. G' must contain a cycle C of length $3 \leq k \leq m$. Suppose $k \geq 4$.

$$C = (v_1, v_2, \dots, v_k, v_1).$$

Some spanning tree H of G must contain:

$$v_1 \leftarrow \dots \rightarrow v_4$$

In H :

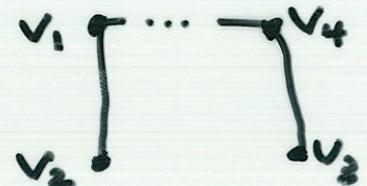
$$v_2 \leftarrow v_3$$

$$\text{Let } K = H + v_1 v_2 - v_3 v_4 \quad \text{and} \quad T = H + v_1 v_2 - v_2 v_3.$$

$$\text{In } K: v_1 \leftarrow \dots \rightarrow v_4$$



$$\text{In } T: v_1 \leftarrow \dots \rightarrow v_4$$



Then T is an intermediate tree of H and K ~~\otimes~~ . Thus $K=3$, $\Rightarrow m \geq 5$.

So G' contains at least one of:



or



both of which lead to contradictions, similar to the above.

(\Leftarrow) Clear. □

Note: We can restrict our attention to simple connected graphs which contain no vertices of degree one.

Suppose there exist vertices u and w s.t.

$$h(v) = \begin{cases} h(v) - 1 & \text{if } v = u \\ h(v) + 1 & \text{if } v = w \\ h(v) & \text{else} \end{cases}$$

Then we call H and K a skew pair.

Note: Any intermediate tree of a skew pair has the same degree sequence as H or K .

What kinds of spanning tree pairs have no intermediate tree?

- Required edge missing

- Same degree sequence

e.g. In K_4 :



- Skew trees

e.g. In K_3 :



Suppose H and K have different degree sequences and are **not** skew trees.

There exist vertices u and w s.t.

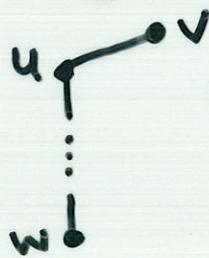
$$h(u) - k(u) > 0$$

and

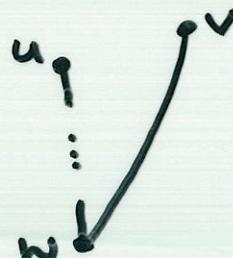
$$h(w) - k(w) < 0$$

Then $h(u) \geq 2 \Rightarrow$ there is vertex v adjacent to u not in unique path in H connecting u and w .

In H :



Form T :



H and T are skew pairs, so $T \neq K$, and

- $k(u) \leq h(u)-1 = t(u) \leq h(u)$
- $h(w) \leq t(w) = h(w)+1 \leq k(w)$
- $t(x) = h(x)$ for all vertices $x \neq u, w$.

Thus, if edge wv is in G , H and K have an intermediate tree.

If edge wv is not in G , then H and K have an intermediate tree in $G \cup \{wv\}$.

In K_n ; $n \geq 5$:

- If H and K are spanning stars:



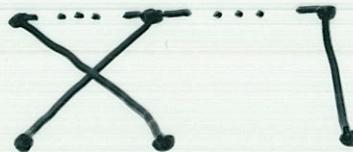
then H and K have an intermediate tree (in fact, there are $2^{n-3} - 2$ of them).

- If H is not a spanning star, then K_n contains at least 2 other spanning trees with the same degree sequence as H :

- If H is a path, there are $(n-2)! > 3$ spanning trees with H 's degree sequence
- If H is not a path, then H contains:



or



We have just proved the following:

Theorem. If $n \geq 5$, then every pair of spanning trees of K_n has an intermediate tree.

* Moreover, if H and K have different degree sequences and are not skew pairs, then they have an intermediate tree whose degree sequence differs from that of H and K .

Theorem. If $n \geq 4$, $m \geq 4$, then * is true for $K_{n,m}$.



And this more general result is true:

Theorem. Let G be the complete k -partite graph for which no set in k -partition contains 2 or 3 vertices. Let n be the number of sets in k -partition containing exactly 1 vertex.

- If
- $n=0$ and $k \geq 2$, or
 - $k \geq n+1 \geq 5$, or
 - $k=n \geq 5$

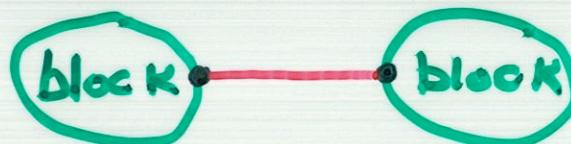
then * is true for G .

* { Building bigger graphs in which every pair of spanning trees has an intermediate tree:

Recall: • cut vertex disconnects graph if deleted
• block is maximal subgraph containing no cut vertices.

Theorem. Let G be a simple connected graph. If $\textcircled{2}$ is true for G , then $\textcircled{1}$ is true for every block of G .

What about the converse?



"trivial block"

Theorem. Let G be a simple connected graph in which every cut vertex is in at most one nontrivial block of G . If $\textcircled{1}$ is true for every block of G , then $\textcircled{2}$ is true for G .

i.e. G looks like:

