Chebyshev Polynomials on Multi-component Sets

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Chebyshev Polynomials

In this talk $\mathsf{E} \subset \mathbb{C}$ will be a compact set consisting of infinitely many points and $\|\cdot\|_{\mathsf{E}}$ will denote the supremum norm on E,

$$\|f\|_{\mathsf{E}} = \sup\{|f(z)| \,:\, z \in \mathsf{E}\}.$$

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The n-th Chebyshev polynomial on E is the polynomial $T_n(z)$ which minimizes the supremum norm on E among all monic polynomials of degree n.

Alternatively, $T_n(z) = z^n - P_{n-1}(z)$, where $P_{n-1}(z)$ is the best uniform approximation on E of the monomial z^n by polynomials of smaller degree. This implies existence of P_{n-1}, and hence of T_n, since the subspace of polynomials of degree less than n is finite dimensional and hence closed.

Classical Chebyshev Polynomials

The classical Chebyshev polynomials correspond to the set $\mathsf{E} = [-1, 1]$. In this case $T_n(z) = 2^{-n+1} Q_n(z)$, where $Q_n(z)$ is the n-th Chebyshev polynomial of the first kind given by

$$Q_n(z) = \cos\left(n \cos^{-1}(z)\right) \quad \text{or} \quad Q_n(\cos z) = \cos(nz)$$

De Moivre's formula $\cos(nx) = \mathrm{Re}\big(\cos(x) + i \sin(x)\big)^n$ shows that $\cos(nx)$ can be expressed as a linear combination of terms of the form $\cos^{n-2k}(x) \sin^{2k}(x)$ and hence $\cos(nx)$ is a polynomial in $\cos(x)$.

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The Chebyshev polynomials have important applications in the approximation theory (e.g., series expansions in the Chebyshev polynomials) and numerical analysis (e.g., zeros of the Chebyshev polynomials are used for interpolation and numerical integration).

Extremal Points and Uniqueness

$z_j \in \mathsf{E}$ such that $|P(z_j)| = \|P\|_{\mathsf{E}}$ is called an extremal point of P on E.

If P_n is a monic polynomial of degree n that minimizes $\|\cdot\|_{\mathsf{E}}$, then P_n has at least $n+1$ extremal points on E.

Proof: If P_n has $k \leq n$ extremal points, one can construct a degree $k-1$ polynomial Q such that $Q(z_j) = P_n(z_j)$ at each extremal point z_j of P_n. Then for sufficiently small $\varepsilon > 0$, the monic polynomial $\tilde{P}_n = P_n - \varepsilon Q$ of degree n will satisfy $\|\tilde{P}_n\|_{\mathsf{E}} < \|P_n\|_{\mathsf{E}}$. ■

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The Chebyshev polynomials $T_n(z)$ are unique.

Proof: Suppose T_n and $\tilde{T}_n$ are monic polynomials of degree n that minimize $\|\cdot\|_{\mathsf{E}}$, then so is $\frac{1}{2}(T_n + \tilde{T}_n)$ and at each of its extremal points $T_n(z_j) = \tilde{T}_n(z_j)$. This implies that $T_n - \tilde{T}_n$ is a polynomial of degree less than n with at least $n+1$ zeros, hence $T_n \equiv \tilde{T}_n$. ∎

The Alternation Theorem

In the case $\mathsf{E} \subset \mathbb{R}$, the Chebyshev polynomials are real (i.e., have real coefficients), since on $\mathbb{R}$, $|\mathrm{Re}(T_n)|$ is no greater than $|T_n|$.

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Let $\mathsf{E} \subset \mathbb{R}$ and P_n be a real monic polynomial of dengree n. Then P_n is the Chebyshev polynomial if and only if P_n has n sign changes on the ordered set $X = \{x \in \mathsf{E} : |P_n(x)| = \|P_n\|_{\mathsf{E}}\}$.

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Proof: $(\Leftarrow)$ If P_n has n sign changes on X but $\|P_n\|_{\mathsf{E}} > \|T_n\|_{\mathsf{E}}$, then $Q = P_n - T_n$ has at least n sign changes and hence at least n zeros which is impossible since the degree of Q is less than n.

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$(\Rightarrow)$ If P_n has fewer than n sign changes on the set of its extremal points then similar to the previous result P_n is not the minimizer for $\|\cdot\|_{\mathsf{E}}$ and hence is not the Chebyshev polynomial. ∎

Harmonic Measure, Green's Function, and Capacity

In the following we'll assume that $\mathsf{E} \subset \mathbb{R}$ is regular (for potential theory), that is, for any $f \in \mathcal{C}(\mathsf{E})$ there exists a unique harmonic function $u(z)$ on $\mathbb{C} \cup \{\infty\} \backslash \mathsf{E}$ with boundary values on E given by f.

The harmonic/equilibrium measure $d\rho_{\mathsf{E}}$ on E is the unique probability measure such that $u(\infty) = \int_{\mathsf{E}} f(x) d\rho_{\mathsf{E}}(x)$, where f, u are as above.

The Green's function G_{E} associated with E is the unique non-negative harmonic function on $\mathbb{C} \backslash \mathsf{E}$ with boundary value 0 on E and such that $G_{\mathsf{E}}(z) - \log |z|$ is harmonic at ∞. Then one has

$$\exp(G_{\mathsf{E}}(z)) = \frac{|z|}{\mathrm{C}(\mathsf{E})} + O(1) \text{ as } z \to \infty.$$

The constant $\mathrm{C}(\mathsf{E})$ above is called the (logarithmic) capacity of E.

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$$\mathsf{E}_n = \{x \in \mathbb{R} : |T_n(x)| \le \|T_n\|_{\mathsf{E}}\} = T_n^{-1}\Big(\big[-\|T_n\|_{\mathsf{E}}, \|T_n\|_{\mathsf{E}}\big]\Big).$$

Polynomial preimages of intervals are convenient because potential theory is rather explicit for them. For example we have:

$$\mathrm{C}(\mathsf{E}_n) = \big[\|T_n\|_{\mathsf{E}}/2\big]^{1/n} \quad \text{or} \quad \|T_n\|_{\mathsf{E}} = 2\mathrm{C}(\mathsf{E}_n)^n.$$

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Corollary of the Alternation Theorem

- $\mathsf{E} \subset \mathsf{E}_n \subset \mathrm{cvh}(\mathsf{E})$ and $\mathsf{E}_n \backslash \mathsf{E}$ consists of intervals with at most one interval in every gap of E.
- Each of these intervals has $d\rho_{\mathsf{E}_n}$-harmonic measure at most $\frac{1}{n}$.

Lower Bounds on the Norm

Bernstein–Walsh Lemma [1912]

If $\mathsf{E} \subset \mathbb{C}$ is compact and P_n is a polynomial of degree n, then

$$|P_n(z)| \leq \|P_n\|_{\mathsf{E}} \exp(nG_{\mathsf{E}}(z)), \quad z \in \mathbb{C} \backslash \mathsf{E}.$$

Bernstein–Walsh Lemma applied to T_n and $z \to \infty$ yields

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New Proof [CSZ 2017]: $\|T_n\|_{\mathsf{E}} = 2\mathrm{C}(\mathsf{E}_n)^n \geq 2\mathrm{C}(\mathsf{E})^n$. ∎

Root Asymptotics

Faber–Fekete–Szegő [1924]

For any compact subset $\mathsf{E} \subset \mathbb{C}$, $\lim_{n\to\infty} \|T_n\|_{\mathsf{E}}^{1/n} = \mathrm{C}(\mathsf{E})$.

Saff–Totik [1997]

For any compact set $\mathsf{E} \subset \mathbb{C}$, $|T_n(z)|^{1/n} \to \mathrm{C}(\mathsf{E}) \exp(G_{\mathsf{E}}(z))$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{C} \backslash \mathrm{cvh}(\mathsf{E})$.

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Faber–Fekete–Szegő theorem is sharp for subsets of $\mathbb{R}$:

Goncharov–Hatinoglu [2015]

For any sequence $D_n \geq 1$ of sub-exponential growth, i.e., $\frac{1}{n} \log D_n \to 0$ there exists a compact set $\mathsf{E} \subset \mathbb{R}$ (a non-autonomous Julia set of zero Lebesgue measure) such that the Chebyshev polynomials satisfy

$$\|T_n\|_{\mathsf{E}} \geq D_n \mathrm{C}(\mathsf{E})^n, \quad n \geq 1.$$

Upper Bounds on the Norm

Widom [1969], Totik [2009]

If $\mathsf{E} \subset \mathbb{R}$ is a finite gap set, then for some constant D one has

$$\|T_n\|_{\mathsf{E}} \leq D\,\mathrm{C}(\mathsf{E})^n, \quad n \geq 1.$$

Note: The size of the constant D is not explicit in these works. Though Widom gives a sharp upper bound on $\limsup_{n\to\infty} \|T_n\|_{\mathsf{E}}/\mathrm{C}(\mathsf{E})^n$.

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Widom [1969], Andrievskii [2017]

If $\mathsf{E} \subset \mathbb{C}$ is a finite disjoint union of quasiconformal arcs and/or regions bounded by quasiconformal curves, then Totik–Widom bound holds.

Note: Koch snowflake is an example of a quasiconformal curve.

Conjecture: $\|T_n\|_{\mathsf{E}}/\mathrm{C}(\mathsf{E})^n$ are unbounded for some rough arc/region.

Parreau–Widom and Homogeneous Sets

A regular compact set $\mathsf{E} \subset \mathbb{C}$ is called a Parreau–Widom set if

$$PW(\mathsf{E}) = \sum_j G_{\mathsf{E}}(c_j) < \infty,$$

where $\{c_j\} = \{z \in \mathbb{C} \backslash \mathsf{E} : \nabla G_{\mathsf{E}}(z) = 0\}$. If $\mathsf{E} \subset \mathbb{R}$, the critical points $\{c_j\}$ lie in the gaps of E with exactly one point in each bounded gap.

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It is known that Parreau–Widom sets are of positive Lebesgue measure and the class of Parreau–Widom sets contains compact homogeneous sets $\mathsf{E} \subset \mathbb{R}$, i.e., sets with uniform lower bound on the Lebesgue density:

$$\exists \delta > 0 \ \text{ s.t. } \ |\mathsf{E} \cap (x - \varepsilon, x + \varepsilon)| > \delta\varepsilon \ \text{ for all } \ x \in \mathsf{E},\ 0 < \varepsilon < 1.$$

A canonical example of a homogeneous set is a positive measure middle Cantor set, e.g., $[0, 1]$ with the middle ε_j-th portion removed at step j, where $\sum_{j=1}^{\infty} \varepsilon_j < \infty$.

Sharp Totik–Widom Bound

Christiansen–Simon–Z [2017]

If $\mathsf{E} \subset \mathbb{R}$ is a Parreau–Widom set, then

$$\|T_n\|_{\mathsf{E}} \leq 2\exp[PW(\mathsf{E})]\mathrm{C}(\mathsf{E})^n, \quad n \geq 1.$$

Note: Our constant $D = 2\exp[PW(\mathsf{E})]$ is (asymptotically) sharp!

Open Problem: Does TW bound hold for Parreau–Widom subsets of $\mathbb{C}$?

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We say that a compact set $\mathsf{E} \subset \mathbb{C}$ has rationally independent harmonic measures if for every decomposition $\mathsf{E} = \mathsf{E}_1 \cup \cdots \cup \mathsf{E}_\ell$ into closed disjoint sets, harmonic measures $\rho_{\mathsf{E}}(\mathsf{E}_1), \ldots, \rho_{\mathsf{E}}(\mathsf{E}_\ell)$ are rationally independent.

Christiansen–Simon–Yuditskii–Z [2017]

For regular compact sets $\mathsf{E} \subset \mathbb{C}$ with rationally independent harmonic measures, Totik–Widom bound holds only if the set is Parreau–Widom.

Open Problem: Is there a zero measure set for which TW bound holds?

Capacity Estimate

Our Totik–Widom bound for Parreau–Widom sets is a consequence of $\|T_n\|_{\mathsf{E}} = 2\mathrm{C}(\mathsf{E}_n)^n$ combined with:

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For $\mathsf{E} \subset \mathbb{R}$ let as before $\mathsf{E}_n = \{x \in \mathbb{R} : |T_n(x)| \leq \|T_n\|_{\mathsf{E}}\}$, then

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$$\mathrm{C}(\mathsf{E}_n)^n \leq \mathrm{C}(\mathsf{E})^n \exp[PW(\mathsf{E})].$$

Proof: Since $G_{\mathsf{E}}(z) - G_{\mathsf{E}_n}(z)$ is harmonic on $\mathbb{C} \cup \{\infty\} \backslash \mathsf{E}_n$, we have

$$\lim_{z\to\infty} [G_{\mathsf{E}}(z) - G_{\mathsf{E}_n}(z)] = \int_{\mathsf{E}_n} [G_{\mathsf{E}}(x) - G_{\mathsf{E}_n}(x)] d\rho_{\mathsf{E}_n}(x).$$

Let (α_j, β_j) be the gaps of E then $\rho_{\mathsf{E}_n}((\alpha_j, \beta_j)) \leq 1/n$ and hence

$$-\log \mathrm{C}(\mathsf{E}) + \log \mathrm{C}(\mathsf{E}_n) = \int_{\mathsf{E}_n \backslash \mathsf{E}} G_{\mathsf{E}}(x) d\rho_{\mathsf{E}_n}(x)$$

$$\leq \sum_j \int_{\alpha_j}^{\beta_j} G_{\mathsf{E}}(x) d\rho_{\mathsf{E}_n}(x) \leq \sum_j G_{\mathsf{E}}(c_j) \rho_{\mathsf{E}_n}(\alpha_j, \beta_j) \leq \frac{PW(\mathsf{E})}{n}. \qquad \blacksquare$$

Exact Asymptotics

For a single Jordan curve $\mathsf{E} \subset \mathbb{C}$ let Ω be the unbounded component of $\mathbb{C} \cup \{\infty\} \backslash \mathsf{E}$. Then Ω is simply connected hence $G_{\mathsf{E}}(z)$ has a harmonic conjugate, so there is a sing-valued analytic "Blaschke" function on Ω,

$$B(z) = e^{G_{\mathsf{E}}(z) + i * G_{\mathsf{E}}(z)}$$

with $|B_{\mathsf{E}}(z)| > 1$ on Ω, $|B_{\mathsf{E}}(z)| = 1$ on E, and

$$\mathrm{C}(\mathsf{E}) B_{\mathsf{E}}(z) = z + \dots \text{ as } z \to \infty.$$

Faber [1919]

If $\mathsf{E} \subset \mathbb{C}$ is a single analytic Jordan curve, then $\dfrac{\|T_n\|_{\mathsf{E}}}{\mathrm{C}(\mathsf{E})^n} \to 1$ and

$$\frac{T_n(z)}{[\mathrm{C}(\mathsf{E}) B_{\mathsf{E}}(z)]^n} \to 1 \text{ as } n \to \infty \text{ uniformly on } \Omega.$$

Widom's Work

In 1969, Widom published a seminal work of over 100 pages long on asymptotics of Chebyshev and orthogonal polynomials. In his set up, $\mathsf{E} \subset \mathbb{C}$ is a finite union of smooth (closed) Jordan curves and/or (open) Jordan arcs. The cases with $\mathsf{E} \subset \mathbb{R}$ are exactly the finite gap sets.

As in the work of Faber, it is natural to look for an analytic function, $B_{\mathsf{E}}(z)$, with $|B_{\mathsf{E}}(z)| = \exp(G_{\mathsf{E}}(z))$ on Ω, the unbounded component of $\mathbb{C} \cup \{\infty\} \backslash \mathsf{E}$. The problem is that Ω is no longer simply connected so the magnitude of $B_{\mathsf{E}}(z)$ is single valued but its phase is multivalued.

Put differently, analytic continuations of $B_{\mathsf{E}}(z)$ along curves in Ω lead to a map from the fundamental group $\pi_1(\Omega)$ to $\partial\mathbb{D}$, which is a character (i.e., group homomorphism), so that after continuation around a closed curve, $B_{\mathsf{E}}(z)$ is multiplied by the character applied to that curve.

Widom's Minimizers

If $T_n(z)B_{\mathsf{E}}(z)^{-n}\mathrm{C}(\mathsf{E})^{-n}$ had a "limit", that limit cannot be n independent since the character is n dependent!

Widom had an idea that the limit should be given in terms of functions $F_\chi(z)$, defined for each character χ, which minimize $\|F_\chi\|_\Omega$ among all character automorphic functions on Ω with character χ. Widom proved existence and uniqueness of such minimizers and showed that $F_\chi(z)$ and $\|F_\chi\|_\Omega$ are continuous in χ.

Taking into account the character, the limit must be F_χ whose character is the same as the character of $B_{\mathsf{E}}(z)^{-n}$, call it F_n. As functions of n, the limit $F_n(z)$ and its norm $\|F_n\|_\Omega$ are then almost periodic.

Widom's Theorems and Conjectures

Widom's Theorem for Closed Curves [1969]

Suppose E is a finite union of disjoint smooth Jordan curves. Let $F_n(z)$ be Widom's minimizer associated with the character of $B_{\mathsf{E}}(z)^{-n}$ and Ω be the unbounded component of $\mathbb{C} \cup \{\infty\} \backslash \mathsf{E}$. Then

$$\lim_{n\to\infty} \left[\frac{\|T_n\|_{\mathsf{E}}}{\mathrm{C}(\mathsf{E})^n} - \|F_n\|_{\Omega} \right] = 0,$$

and

$$\lim_{n\to\infty} \left[\frac{T_n(z)}{\mathrm{C}(\mathsf{E})^n B_{\mathsf{E}}(z)^n} - F_n(z) \right] = 0,$$

where the limit is uniform on compact subsets of Ω.

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$$\lim_{n\to\infty} \left[\frac{\|T_n\|_{\mathsf{E}}}{\mathrm{C}(\mathsf{E})^n} - 2\|F_n\|_\Omega \right] = 0.$$

Note: Here the norm $\|T_n\|_{\mathsf{E}}$ is twice as large as one might expect!

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Christiansen–Simon–Z [2017]

Widom's conjecture holds for finite gap subsets of $\mathbb{R}$.

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Thiran–Detaille [1991], Eichinger [2017]

If E is a subarc on the unit circle of angular opening $\alpha \in (0, 2\pi)$, then

$$\lim_{n\to\infty} \frac{\|T_n\|_{\mathsf{E}}}{\mathrm{C}(\mathsf{E})^n} = 2\cos^2(\alpha/8) \in (1, 2)$$

$$\lim_{n\to\infty} \frac{T_n(z)}{\mathrm{C}(\mathsf{E})^n B_{\mathsf{E}}(z)^n} = R(z), \quad z \in \Omega$$

for a certain rather explicit nonconstant function $R(z)$.

Asymptotics for the case of finitely many arcs remains an open problem.

Infinite Gap Setting

Christiansen–Simon–Yuditskii–Z [2017]

Let $\mathsf{E} \subset \mathbb{R}$ be a Parreau–Widom set with DCT. Then Widom's minimizer F_χ exists and is unique and continuous. Let $F_n(z)$ be Widom's minimizer associated with the character of $B_{\mathsf{E}}(z)^{-n}$ and $\Omega = \mathbb{C} \cup \{\infty\} \backslash \mathsf{E}$, then

$$\lim_{n\to\infty} \left[\frac{\|T_n\|_{\mathsf{E}}}{\mathrm{C}(\mathsf{E})^n} - 2\|F_n\|_\Omega \right] = 0 \qquad (*)$$

and

$$\lim_{n\to\infty} \left[\frac{T_n(z)}{\mathrm{C}(\mathsf{E})^n B_{\mathsf{E}}(z)^n} - F_n(z) \right] = 0 \ \text{ loc. unif. on } \Omega.$$

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Christiansen–Simon–Yuditskii–Z [2017]

For regular compact sets $\mathsf{E} \subset \mathbb{C}$ with rationally independent harmonic measures, (*) holds and $\|T_n\|_{\mathsf{E}}/\mathrm{C}(\mathsf{E})^n$ is asymptotically almost periodic and bounded only if E is a Parreau–Widom set with DCT.

Finally. . .

Thank you for your attention!