Zero dimensional spectrum of the Holstein model

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The Holstein model (Intro in 4 slides)

1. The Hilbert space is
$$\mathcal{H} = \ell^2(\Lambda) \otimes \mathcal{F}_\Lambda$$
where $\mathcal{F}_\Lambda$ is the symmetric Fock space
$$\mathcal{F}_\Lambda = \oplus_{n=0}^{\infty} \left[\ell^2(\Lambda)\right]^{\bigcirc n}.$$

2. The Hamiltonian for the Boson field acting on $\mathcal{F}_\Lambda$ is
$$H_b = \omega \sum_x b_x^\dagger b_x \qquad\qquad [b_x, b_y^\dagger] = \delta_{x-y}; [b_x, b_y] = [b_x^\dagger, b_y^\dagger] = 0$$

3. The field for the tracer particle is simply
$$V = \sum_x v_x \delta_x$$
where $\text{supp}(v_x) \subset [0, V_\infty]$

The Holstein model II

1. A third term couples the field and tracer particle:

$$H_f = (I \otimes H_b + V \otimes I) + \alpha \sum_x \delta_x \otimes (b_x^\dagger + b_x) + \frac{\alpha^2}{\omega}$$

2. Combining terms gives

$$\begin{aligned} H_f &= V \otimes I + \omega \sum_x (I \otimes b_x^\dagger + \beta \delta_x \otimes I)(I \otimes b_x + \beta \delta_x \otimes I) \\ &= V \otimes I + \omega \sum_x a_x^\dagger a_x \end{aligned}$$

where $\beta = \frac{\alpha}{\omega}$.

The Holstein model III

1. The eigenstates of H_f are

$$|x, m\rangle = e^{\beta(a^\dagger - a)}|x\rangle \otimes |m\rangle$$

2. Where $m : \Lambda \to \mathbb{N}$ so that $|m| = \sum_u m(u) < \infty$

$$|x\rangle \otimes |m\rangle = \left(I \otimes \prod_x \frac{1}{\sqrt{m(x)!}} (b_x^\dagger)^{m(x)} \right) |x\rangle \otimes |0\rangle$$

3. Thus:

$$H_f|x, m\rangle = (\omega|m| + v_x)|x, m\rangle$$

$$|x, m\rangle = \left(\prod_x \frac{1}{\sqrt{m(x)!}} (a_x^\dagger)^{m(x)} \right) |x, 0\rangle$$

The Holstein model IV

1. Now we perturb by a small kinetic term:

$$H = H_f + \gamma\Delta \otimes I$$

2. The spectrum of H is

$$\sigma(H) = \bigcup_{k=0}^{\infty} (k\omega + [0, V_\infty] + \gamma\sigma(\Delta))$$

   fix $\omega > V_\infty$ to separate the spectral bands.

3. The bound on the hopping term is:

$$|\langle y, \xi|\Delta \otimes I|x, m\rangle| \leq C_\mu |\langle x|\Delta|y\rangle| \cdot e^{-\mu\left|\sqrt{|m|}-\sqrt{|\xi|}\right|}$$

Hausdorff dimension

Gauge function:

An increasing function ρ on $(0,1)$ so that $\lim_{s\to 0}\rho(s)\to 0$.

The ρ**-Hausdorff measure** of a set S is defined as

$$\mu^{\rho}(S)=\lim_{\delta\to 0}\inf\left\{\sum_{i}\rho(|U_i|):S\subset\cup_i U_i:|U_i|<\delta\right\}$$

Theorem (Rogers and Taylor, 1959 and 1963)

For any Borel measure μ, there are Borel sets T^{ρ}_{∞} and $T^{\rho}_{<\infty}$, so that

1. $\mu^{\rho}(T^{\rho}_{\infty})=0$
2. $\mu(E\cap T^{\rho}_{<\infty})=0$ *if E is μ^{ρ} σ - finite,*
   and, for any Borel set A,
3. $\mu(A)=\mu(A\cap T^{\rho}_{<\infty})+\mu(A\cap T^{\rho}_{\infty})=\mu_{\rho c}(A)+\mu_{\rho s}(A)$

The measure μ is purely ρ-singular if

$$\mu(A)=\mu(A\cap T^{\rho}_{\infty})=\mu_{\rho s}(A)$$

Spectral dimension bound

Let Q_N project to the space spanned by

$$\{|x, m\rangle : x \in \mathbb{Z}^d; |m| \leq N\}$$

and define

$$H_N = Q_N H Q_N.$$

Theorem (M.)

For any N, let γ be sufficiently small. For any ψ, the spectral measure μ_ψ of H_N is purely ρ-singular for

$$\rho(s) = \frac{1}{|\log s|^\alpha}$$

for α sufficiently large

In particular spectral measures are all $\widetilde{\rho}$ singular for $\widetilde{\rho}(s) = s^p$ for $0 < p < 1$.

Reference

Let $\widetilde{\rho}(s) = s^p$ for $0 < p < 1$,

Theorem (Last 1996)

If H is SA on $\ell^2(Z^d)$ and $P_{\widetilde{\rho}c}\psi \neq 0$, then, for any $m > 0$ every $T > 0$,

$$\langle\langle\psi|e^{itH}|X|^m e^{-itH}|\psi\rangle\rangle_T > CT^{pm/d}$$

Let $H_{AMO} = \Delta + \lambda\cos(2\pi\beta X + \theta)$

Theorem (Last 1996)

If β is a Liouville number and $\lambda > 2$, then for all θ, H_{AMO} is supported on a set of zero Hausdorff dimension.

Dynamical localization bound

Theorem (M. + Schenker (CMP Dec 2018))

For any E, there is γ_E so that for $0 < \gamma < \gamma_E$, any $0 < s < 1$, and $|x, m\rangle$, and $|y, \xi\rangle$,

$$\mathbb{E}\left(\left|\langle x, m|\frac{1}{H-E}|y, \xi\rangle\right|^s\right) < C_s e^{-\mu\left(L_{x,m;y,\xi}+\left|\sqrt{|m|}-\sqrt{|\xi|}\right|\right)}$$

Theorem (M. + Schenker 2018)

For any $|x, m\rangle$, and $|y, \xi\rangle$, for sufficiently small γ,

$$\mathbb{E}\sup_{|f|<1}\left(|\langle x, m|f(H)|y, \xi\rangle|\right) < Ce^{-\mu\|x-y\|}$$

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Theorem (M. + Schenker 2018)

For any $|x, m\rangle$, and $|y, \xi\rangle$, for sufficiently small γ,

$$\mathbb{E}\sup_{|f|<1}\left(|\langle x, m|f(H)|y, \xi\rangle|\right) < Ce^{-\mu\|x-y\|}$$

Similarly: $|x, m\rangle, |y, \xi\rangle \in$ *Range* Q_N

$$\mathbb{E}\sup_t\left(|\langle x, m|e^{-itH_N}|y, \xi\rangle|\right) < Ce^{-\mu\|x-y\|}$$

Reference

Multiparticle Anderson model:

$$H_{MP}^{(N)} = \sum_{j=1}^{N}(-\Delta_j + \lambda V(x_j)) + \sum_{j=2}^{p}\alpha_j \sum_{\substack{A:|A|=N \\ diam(A)<\ell}} U_A(N_A(\mathbf{x}))$$

Theorem (Aizenman + Warzel 2008)

For a multiparticle Anderson model $H_{MP}^{(N)}$, for sufficiently large λ depending on $\alpha_2, .., \alpha_p$,

$$\mathbb{E}\left[\sup_{\|f\|_\infty \leq 1} \left|\langle \delta_{\mathbf{x}} | f(H_{MP}^{(N)}))|\delta_{\mathbf{y}}\rangle\right|\right] \leq Ce^{-\mu d(\mathbf{x},\mathbf{y})}$$

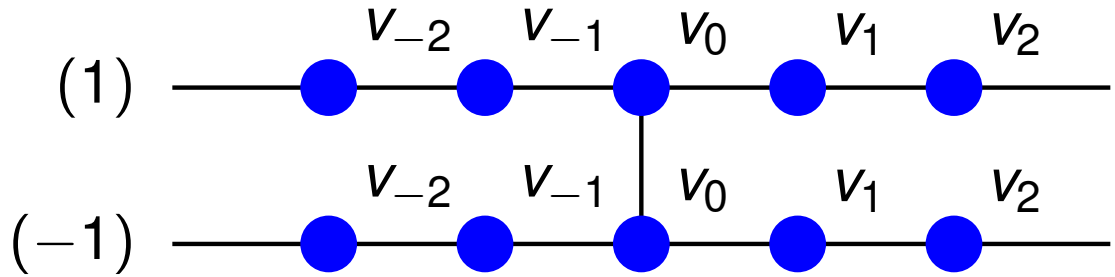
where

$$d(\mathbf{x},\mathbf{y}) = \max\left\{\max_i \text{dist}\,(x_i, Y), \max_i \text{dist}\,(X, y_i)\right\}$$

Digression: Anderson with spin on Z^d

$$H_{spin} = (\gamma\Delta + V) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + g|0\rangle\langle 0| \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

naturally embedded into graph



Theorem (M. + Schenker, arXiv 2017)

1. $\mathbb{E}|\langle x,i|(H-E)|y,j\rangle|^s < C_s e^{-\mu(|x-y|(1+jk)+(|x|+|y|)(1-jk))}$
2. $\mathbb{E}[\sup_{|f|<1} |\langle x,j|(f(H_{spin}))|y,k\rangle|] \leq Ce^{-\mu|x-y|}$
3. $\exists x_\ell \to \infty$, *and* $\tau_\ell > e^{\nu|x|}$ *so that,* $\forall n$,

$$\|e^{-in\tau_\ell H_{spin}}|\psi_{x_\ell}\rangle \otimes |j\rangle - |\psi_{x_\ell}\rangle \otimes |(-1)^n j\rangle\| = O(e^{-\nu|x_\ell|})$$

Holstein: Small time localization of excitations

Let $L_{|x,m\rangle;|y,\xi\rangle}$ be diameter of the set

$$\{x, y\} \cup \{u : m(u) \neq \xi(u)\}$$

Theorem (M. + Schenker)

For sufficiently small γ, for any $|x, m\rangle$, and $|y, \xi\rangle \in$ Range Q_N and all t,

$$\mathbb{E}\left(\frac{1}{t}|\langle x, m|e^{-itH_N}|y, \xi\rangle|\right) < Ce^{-\mu L_{|x,m\rangle;|y,\xi\rangle}}$$

Define: $\langle A_t\rangle_T = \frac{1}{T}\int_0^T A_t dt$ and $\psi_t = e^{-tH_N}\psi$

Corollary

For sufficiently small γ, and any $\epsilon > 0$ there is some $\nu > 0$ so that, for sufficiently large L

$$\langle\|(1 - P_{0,L})\psi_t\|^2\rangle_T < \epsilon\|\psi\|^2$$

For $T = e^{\nu L}$

Hilbert space decomposition

Given any gauge function ρ and a Hamiltonian H on Hilbert space $\mathcal{H}$, and any $\psi \in \mathcal{H}$, the spectral measure is decomposed as

$$\mu_\psi = \mu_{\psi\rho s} + \mu_{\psi\rho c}$$

moreover, there is a decomposition

$$H = H_{\rho s} \oplus H_{\rho c}$$

where

$$H_{\rho s} = \{\psi | \mu_\psi = \mu_{\psi,\rho s}\}$$

$$H_{\rho c} = \{\psi | \mu_\psi = \mu_{\psi,\rho c}\}$$

Spectral decomposition

1. The measure μ is UρH if, for all $\epsilon < 1$,

$$\sup_s \mu_\psi(s - \epsilon/2, s + \epsilon/2) \leq C_\epsilon \rho(\epsilon)$$

Theorem (Rogers and Taylor, 1959 and 1963)

For any $\epsilon > 0$, there is decomposition

$$d\mu_{\rho c} = d\mu_{\rho c,1} + d\mu_{\rho c,2}$$

so that $\mu_{\rho c,1}$ is UρH and $\mu_{\rho c,2}(\mathbb{R}) < \epsilon$

Results of uniform continuity

Lemma

For $\psi \in$ Range Q_N such that μ_ψ is UρH

$$\langle |\langle \phi, \psi_t \rangle| \rangle_T \leq C_\psi \rho\left(\frac{1}{T}\right)$$

Let P_{xL} project to $\{|y, m\rangle : |y - x| \leq L; m(u) = 0 \ \forall \ |u - x| > L\}$

Corollary

If $\psi \in$ Range Q_N is such that μ_ψ is UρH then

$$\langle |\langle \psi_t | P_{xL} | \psi_t \rangle| \rangle_T \leq C_{\psi,N} L^{d(N+1)} \left[\rho\left(\frac{1}{T}\right)\right]$$

Application to Holstein Hamiltonian H_N

Let ρ be a gauge function

$$\rho(s) = |\ln(s)|^{-\alpha}$$

Theorem (M.)

Suppose $\psi \in$ Range Q_N and $P_{\rho c}\psi \neq 0$, then, for sufficiently large $L > 0$,

$$\langle \|(1 - P_{0,L})\psi_t\|^2 \rangle_T > \frac{1}{2}\|\psi_{\rho c}\|^2$$

For $T = e^{\beta L}$ for $\beta > d(N+1)/\alpha$

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Extension of spectral localization to H, requires dynamical bounds on excitations or bounds on

$$[P_E, Q_N]$$

for P_E the spectral projection to $(-\infty, E)$