Contagion in Inhomogeneous Financial Networks

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joint with Rama Cont and Andreea Minca

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Network structures of banking systems

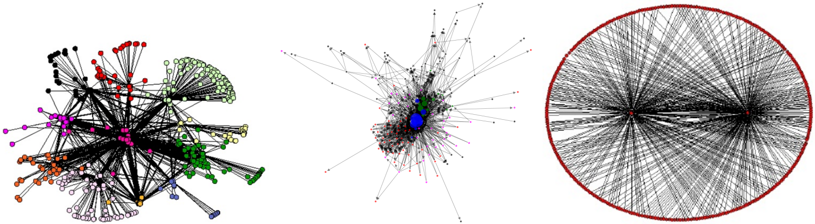

Figure : Structure of interbank networks. From left to right : Austria : scale-free structure (Boss et al. 2004), UK : sparse inhomogeneous structure (Birch et al. 2015) Switzerland : sparse and centralized structure (Müller 2006)

Fundamental questions

- How does the default of a bank transmit in the financial network ?
- Can the default of one or few institutions generate a macro-cascade ?
- How do the answers to the above depend on network structure ?

Outline

1. Network Model of Default Contagion
2. Asymptotic Analysis and Limit Theorems
3. Numerical Results
4. Conclusion

1 Network Model of Default Contagion

2 Asymptotic Analysis and Limit Theorems

3 Numerical Results

4 Conclusion

Financial network

Can be modeled as a weighted directed graph $(V, \mathbf{e}, \mathbf{c})$ on the vertex set $V = \{1, \ldots, n\}$:

- n nodes represent financial market participants
- $e(i,j)$ is the exposure of i to j
- $c(i)$ is the capital buffer of institution i which absorbs market losses

Suppose a loss ε in the assets of institution i : $c(i) \to (c(i) - \varepsilon)_+$

Solvency condition : $c(i) > 0$

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Balance sheet

Assets	Liabilities
Interbank assets $\sum_j e(i,j)$	Interbank liabilities $\sum_j e(j,i)$ Deposits $D(i)$
Other assets $x(i)$	Equity $c(i)$

TABLE – Stylized balance sheet of bank i

Balance sheet equation :

$$c(i) = x(i) + \sum_{j \neq i} e(i,j) - \sum_{j \neq i} e(j,i) - D(i)$$

Insolvency cascades

- The set of initially insolvent institutions is

$$\mathcal{D}_0(\mathbf{e},\mathbf{c}) = \{i \in V \mid \; c(i) \leq 0\}$$

- The default of a market participant j affects its counterparties :
  - Creditors lose a fraction $(1-R)$ of their exposure
  - This leads to default of i if

$$c(i) < (1-R)e(i,j)$$

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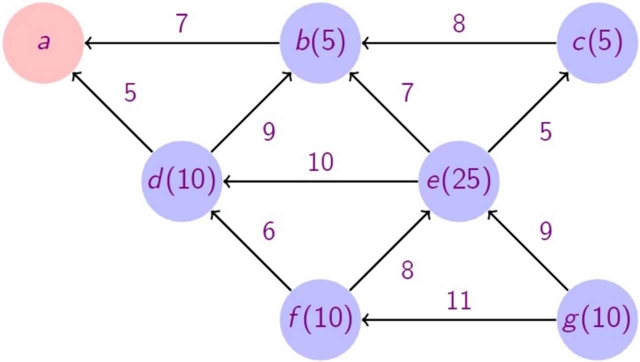
Example



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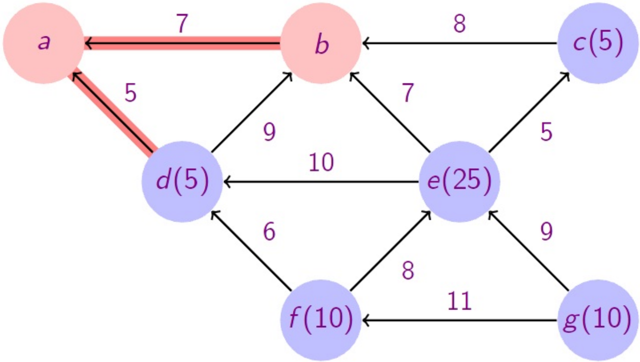

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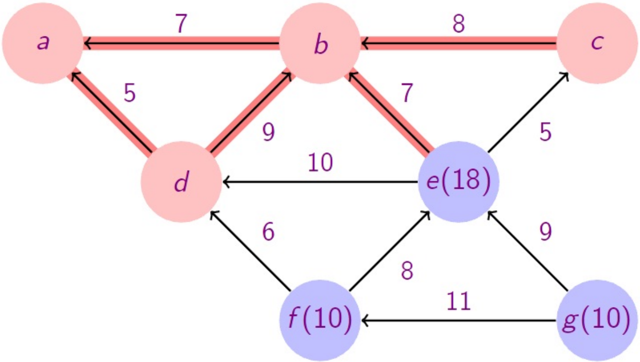

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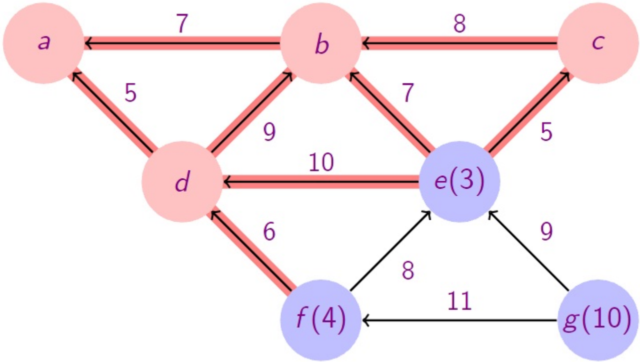
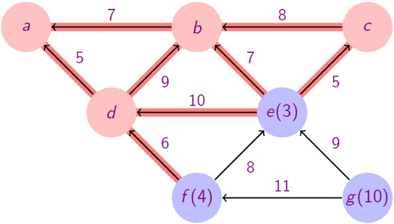


Example

Contagion lasts 3 rounds

- **Fundamental defaults** : $\{a\}$
- **Default cluster** : $\mathcal{D}_f(\mathbf{e}, \mathbf{c}) = \{a, b, c, d\}$
- **Final number of defaults** : $|\mathcal{D}_f(\mathbf{e}, \mathbf{c})| = 4$



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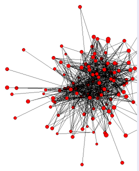
Observation

Real financial networks are large (thousands of nodes) : complex networks

The Brazilian interbank network



The structure of interbank network

- **Number of nodes** $n \simeq 2500$
- **Heterogeneity in number of debtors/creditors**
- **Heterogeneous exposures sizes**

Source : Cont et al. (2010)

Previous works : many simulation studies + average cascade size on homogeneous networks using mean field approximations (Allen, Gale (2000), Watts (2002), Nier et al. (2007), Cont, Moussa (2010), Gai, Kapadia (2011), Battiston et al. (2012), . . .)

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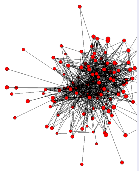

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Degree sequences

- **Financial system :** $(\mathbf{e}_n, \mathbf{c}_n)$ with the vertex set $[n] := \{1, \ldots, n\}$
- The out-degree of node i is given by its number of debtors

$$d_n^+(i) = \#\{j \mid e_n(i,j) > 0\}$$

  and its in-degree is given by its number of creditors

$$d_n^-(i) = \#\{j \mid e_n(j,i) > 0\}$$

- The **empirical distribution of the degrees** :

$$\mu_n(j,k) := \frac{1}{n}\#\{i : d_n^+(i) = j, d_n^-(i) = k\}$$

- The **final fraction of defaults** :

$$\alpha_n(\mathbf{e}_n, \mathbf{c}_n) = \frac{|\mathcal{D}_f(\mathbf{e}_n, \mathbf{c}_n)|}{n}$$

Random financial network

- $\mathcal{G}_n(\mathbf{e}_n)$: the set of all weighted directed graphs with degree sequence $\mathbf{d}_n^+, \mathbf{d}_n^-$ s.t. for all i, the set of exposures is given by the non-zero elements of line i in the exposure matrix $\mathbf{e}_n$
- $\mathbf{E}_n$: random financial network uniformly distributed on $\mathcal{G}_n(\mathbf{e}_n)$
- Capital buffers $\mathbf{c}_n$

$$\#\{j \in [n],\ \mathbf{E}_n(j,i) > 0\} = d_n^-(j),$$
$$\#\{j \in [n],\ \mathbf{E}_n(i,j) > 0\} = d_n^+(i),$$
$$\{\mathbf{E}_n(i,j),\quad \mathbf{E}_n(i,j) \neq 0\} = \{\mathbf{e}_n(i,j),\quad \mathbf{e}_n(i,j) \neq 0\} \quad \mathbb{P} - a.s.,$$

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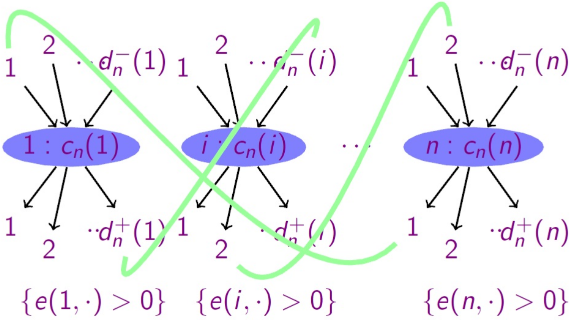
Configuration model



Assumption : $\sum_i (d_n^+(i) + d_n^-(i))^2 = O(n)$

$\Longrightarrow \liminf_{n\to\infty} \mathbb{P}(CM(e_n) \text{ is simple}) > 0$ Janson (2009)

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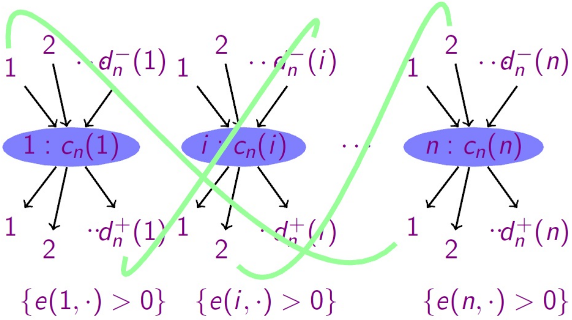

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Default threshold

$\Theta(i,\tau)$: measures how many counterparty defaults i can tolerate before it becomes insolvent, if its counterparties default in the order specified by τ :

$$\Theta(i,\tau) := \min\{k \geq 0, c(i) < \sum_{j=1}^{k} (1-R)e(i,\tau(j))\}$$

$$p_n(j,k,\theta) := \frac{\#\{(i,\tau) \mid d_n^+(i) = j, d_n^-(i) = k, \Theta(i,\tau) = \theta\}}{n\mu_n(j,k)j!}$$

Natural assumptions

There exists a

1. probability distribution μ with average $\lambda \in (0, \infty)$ s.t.

$$\mu_n(j,k) \to \mu(j,k) \text{ as } n \to \infty$$

2. function p s.t.

$$p_n(j,k,\theta) \to p(j,k,\theta), \text{ as } n \to \infty$$

for all $j,k,\theta \in \mathbb{N}$.

Ex : exposures and capitals i.i.d. or exchangeable arrays

Asymptotic size of contagion

We can completely describe the asymptotic behaviour of contagion :

Theorem

There exists $I : [0,1] \to [0,1]$ s.t. if π^* is the smallest fixed point of I, we have

1. *if $\pi^* = 1$, then asymptotically almost all nodes default*
2. *if $\pi^* < 1$ and furthermore π^* is a stable fixed point of I, then*

$$\alpha_n(\mathbf{E}_n, \mathbf{c}_n) \xrightarrow{p} J(\pi^*) := \sum_{j,k} \mu(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\mathbb{P}(\mathrm{Bin}(j,\pi^*) \geq \theta)$$

$$I(\pi) := \sum_{j,k} \frac{k\mu(j,k)}{\lambda} \sum_{\theta=0}^{j} p(j,k,\theta)\mathbb{P}(\mathrm{Bin}(j,\pi) \geq \theta)$$

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Phase transition

Stress scenario : apply a common macro-shock Z, measured in % loss in capital to all balance sheets in network

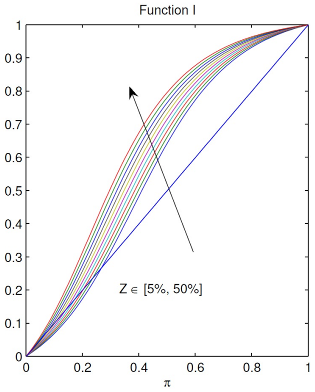

Figure : Function I for increasing magnitude of the common macro-shock Z

Contagion described by a Markov chain

The state variables of the Markov chain are :

- $S_n^{j,k,\theta,l}(t)$, $l < \theta \leq j$, the number of solvent banks with degree (j,k), default threshold θ and l defaulted debtors before time t.

We introduce the additional variables of interest in determining the size and evolution of contagion :

- $D_n^{j,k}(t)$, the number of defaulted banks at time t with degree (j,k)

$$D_n^{j,k}(t) = n\mu_n(j,k) - \sum_{\theta,l,0\leq l<\theta\leq j} S_n^{j,k,\theta,l}(t),$$

- $D_n(t)$, the number of defaulted banks at time t :

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- $D_n(t)$, the number of defaulted banks at time t :

$$D_n(t) = \sum_{j,k} D_n^{j,k}(t),$$

- $D_n^-(t)$, the number of remaining in-coming half-edges belonging to defaulted banks :

$$D_n^-(t) = \sum_{j,k} k D_n^{j,k}(t) - t.$$

The process will finish at the stopping time T_n which is the first time $t \in \mathbb{N}$ when $D_n^-(t) = 0$. The final number of defaulted banks will be $D_n(T_n)$.

We then show that, as the network size increases, the rescaled Markov chain converges in probability to a limit described by a system of ordinary differential equations, which can be solved in closed form !

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Resilience condition

Corollary

If

$$\sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j,k,1) < 1,$$

then w.h.p. (with probability $\to 1$ *as* $n \to \infty$*), the default of a finite set of nodes cannot trigger the default of a positive fraction of the financial network.*

$i \to j$ is a contagious link if the default of j generates the default of i

$p(j,k,1)$: proportion of contagious exposures belonging to nodes with degree (j,k)

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Skeleton of contagious links

Theorem

If the resilience condition fails :

$$\sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j,k,1) > 1,$$

then w.h.p. there exists a strongly connected set of nodes representing a positive fraction of the financial system s.t. any node belonging to this set can trigger the default of all nodes in the set.

- A decentralized recipe for regulating systemic risk
- No need to monitor/know the entire network of counterparty exposures but simply the skeleton of contagious links

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Simulation-free stress testing of banking systems

- These analytical results may be used for stress-test the resilience of a banking system, without the need for large scale simulation
- Stress scenario : apply a common macro-shock Z, measured in % loss in asset value, to all balance sheets in network
- Network remains resilient (no macro-cascade) as long as

$$\sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p_Z(j,k,1) < 1 \iff Z < Z^*$$

Amplification of default by contagion when $\sum_{j,k} jk\mu(j,k) < \infty$

Assume that the resilience condition is satisfied :

- When a fraction ε of all nodes represent fundamental defaults, i.e. $p(j,k,0) = \varepsilon$ for all j,k :

$$\alpha_n(\mathbf{E}_n, \mathbf{c}_n) \xrightarrow{p} \varepsilon \underbrace{\left(1 + \frac{\sum_{j,k} j\mu(j,k) p(j,k,1)}{1 - \sum_{j,k} \frac{\mu(j,k)jk}{\lambda} p(j,k,1)}\right)}_{\textit{Amplification}} + o(\varepsilon)$$

- When $p(d^+, d^-, 0) = \varepsilon$ and $p(j,k,0) = 0$ for all $(j,k) \neq (d^+, d^-)$:

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Amplification of default by contagion when $\sum_{j,k} jk\mu(j,k) = \infty$

Assume that for some $\Theta \in \mathbb{N}, \ , \gamma \in \mathbb{R}^+$ and $\beta \in (2,3)$ and for all $j \in \mathbb{N}$:

$$\sum_k \sum_{\theta=1}^{\Theta} k\mu(j,k)p(j,k,\theta) \geq \gamma j^{-\beta+1}$$

Theorem

There exists $\hat{\pi} > 0$ the smallest positive solution of

$$\pi = \sum_{j,k} \sum_{\theta \geq 1} \frac{k\mu(j,k)}{\lambda} p(j,k,\theta) \mathbb{P}(\mathrm{Bin}(j,\pi) \geq \theta)$$

such that, for ε small enough and n large enough, the final fraction of defaults is given by $J(\hat{\pi}) > 0$.

Numerical results

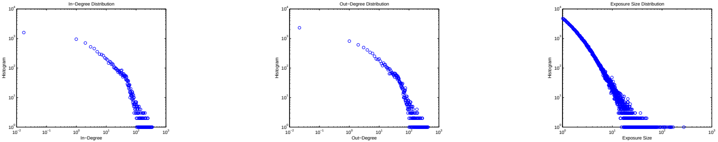


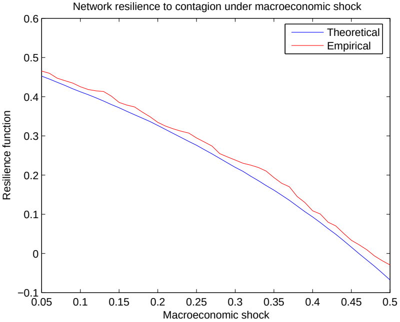
FIGURE – (a) The distribution of in-degree has a Pareto tail with exponent 2.19, (b) The distribution of the out-degree has a Pareto tail with exponent 1.98, (c) The distribution of the exposures (tail-exponent 2.61).

The finite sample

In a finite network the resilience condition becomes

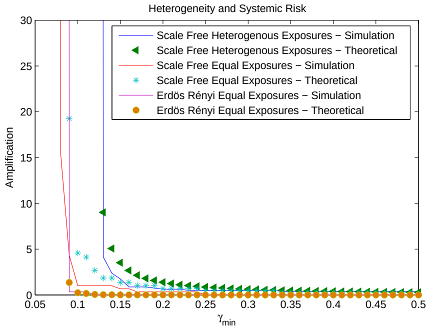
$$\frac{1}{m_n}\sum_i d_n^-(i)q_n^{(Z)}(i) < 1,$$

with m_n the total number of links in the network and $q_n^{(Z)}(i)$: the number of 'contagious' links of bank i.



The impact of heterogeneity

Amplification of the number of defaults in a Scale-Free Network (in and out-degree of the scale-free network are Pareto distributed with tail coefficients 2.19 and 1.98 respectively, the exposures are Pareto distributed with tail coefficient 2.61), the same network with equal weights and an Erdös Rényi Network with equal exposures $n = 10000$.



Related to bootstrap percolation

Bootstrap percolation process with *activation threshold* an integer $\theta \geq 2$:

- $\mathcal{A}_0$: initially infected nodes, selected deterministically or randomly
- An uninfected node with (at least) θ infected neighbours becomes infected
- $\mathcal{A}_f$: final infected set

Aizenman, Lebowitz (1988) - Cerf, Manzo (2002) - Holroyd (2003) (grids)
Balogh, Bollobàs, Duminil-Copin and Morris (2012) (higher dimensions)
Balogh, Bollobás (2006) (hypercube)
Balogh, Peres and Pete (2006) - Fontes, Schonmann (2008) (infinite trees)
Balogh, Pittel (2007) (random regular graphs)
Janson et al. (2011) (Erdős-Rényi random graph)
A. (2010), Lelarge (2011) (random graphs with given vertex degrees)
A., Fountoulakis and Panagiotou (2014) (inhomogeneous random graphs)
...

Bootstrap percolation in power-law random graphs

In power-law random graphs with parameter $2 < \beta < 3$ and maximum degree $d_{max} = \Theta(n^{\zeta})$:

$$a_c(n) = n^{\frac{\theta(1-\zeta)+\zeta(\beta-1)-1}{\theta}} = o(n)$$

Theorem (A., Fountoulakis 2014)

We have w.h.p.

- *if* $|\mathcal{A}_0| \ll a_c(n)$, *then* $\mathcal{A}_f = \mathcal{A}_0$
- *if* $|\mathcal{A}_0| \gg a_c(n)$, *then* $|\mathcal{A}_f| > \varepsilon n$ *for some* $\varepsilon > 0$

Random recovery rates and contagion probability matrix

First order cascade : the contagion carried through the contagious links

$$\mathcal{C}_1 = \{ j \mid \exists\, k,\ (j_0 \in \mathcal{D}_0, j_1, \ldots, j_k = j),\ j_{\ell-1} \to j_\ell \text{ is contagious } \forall \ell = 1, \ldots, k \}$$

Let β_{ij} denote the probability that a link from i to j becomes contagious. Given the exposure matrix and the realized shocks (in stress scenarios),

$$\beta_{ij} := \mathbb{P}\left(R_{ji} < \frac{e(i,j) - C_i}{e(i,j)} \right)$$

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Bounds in general networks

Let $B := (\beta_{ij})_{i,j\in[n]}$ and $\lambda_{\max}(B) = ||B||_2$ the largest singular value of B :

Proposition

If $\lambda_{\max}(B) < 1$, then

$$\mathbb{E}\left[|\mathcal{C}_1|\right] \leq \frac{1}{1-\lambda_{\max}(B)}\sqrt{n|\mathcal{D}_0|},$$

which in particular implies that w.h.p. $|\mathcal{C}_1|/|\mathcal{D}_0| = O\left(\sqrt{n}\right)$.

Ex : This upper bound is asymptotically tight in the case of star networks

Inhomogeneous random financial networks

- Financial institutions have different types which are in a certain type space $\mathcal{S} = \{s_1, s_2, \ldots, s_r\}$ and we only have information about their types. **Ex.** Core-periphery structure.
- Let n_i denote the number of vertices of type s_i, i.e.,

$$n_i := \#\{v \in [n] \mid s(v) = s_i\},$$

  so that $n_1 + n_2 + \cdots + n_r = n$.
- We shall assume that

$$\lim_{n \to \infty} n_i / n = \mu_i,$$

  so that we have $\mu_i > 0$ and $n_i - \mu_i n = o(n)$.
- We then assume that the probability β_{ij} of having a contagious link from bank i to bank j depends only on the types of i and j.

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Consider now the $r \times r$ matrix $\bar{B} = \left(\bar{b}_{ij}\right)_{1 \le i,j \le r}$ where $\bar{b}_{ij}$ is the average number of contagious links from an institution of type s_i to the institutions of type s_j.

Theorem

- *If* $\lambda_{\max}(\bar{B}) < 1$ *then w.h.p.* $|\mathcal{C}_1|/|\mathcal{D}_0| = O(\log n)$ *;*
- *If* $\lambda_{\max}(\bar{B}) > 1$ *then w.h.p. there exists a strongly connected set of nodes representing a positive fraction of the financial system such that the default of any node belonging to this set can trigger the default of all nodes in the set.*

1. Network Model of Default Contagion
2. Asymptotic Analysis and Limit Theorems
3. Numerical Results
4. Conclusion

Conclusions

- We have proposed different frameworks for testing the possibility of large cascades in financial networks.
- These results hold for a model flexible enough to accommodate interpretations as insolvency cascade of illiquidity cascade.
- The regulator can efficiently contain insolvency contagion by focusing on fragile nodes, especially those with high connectivity and over-exposed.
- In particular, higher capital requirements could be imposed on them to reduce their number of contagious links.

THANK YOU !

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Independent threshold model with security investment

- We study the independent threshold model with security investment, in which a node's threshold is drawn independently from a distribution which depends on the node's degrees and its security investment. Consider a finite ensemble of security classes $\mathcal{S}$.
- A node i with degrees (j,k) can obtain a security level $s \in \mathcal{S}$ for a cost $C_i(s)$.
- It is natural to assume that for all nodes the threshold is stochastically increasing with the security investment i.e., for a node i with degrees (j,k) and $s < s'$ we have for all $\ell \in \mathbb{N}$:

$$\sum_{\theta \geq \ell} p^{(s)}(j,k,\theta) \leq \sum_{\theta \geq \ell} p^{(s')}(j,k,\theta).$$

- On the other hand, investing in a higher security class increases the cost, i.e., $C_i(s)$ is strictly increasing in s. If a node $i \in V$ is not among the final defaulted nodes, it earns a profit u_i ; otherwise it earns 0. So we have

$$U_i(\mathbf{s}) = u_i \mathbf{1}(i \in D_f^{(\mathbf{s})}).$$

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We define the function $\Phi : [0,1] \to [0,1]$ as

$$\Phi(x) := \sum_{j,k} \sum_{s \in \mathcal{S}} \frac{k\mu^{(s)}(j,k)}{\lambda} \sum_{\theta=0}^{j} p^{(s)}(j,k,\theta)\mathbb{P}(\mathrm{Bin}(j,x) \geq \theta),$$

and let x^* be the smallest fixed point of Φ in $[0,1]$.

Theorem

We have

$$\frac{|D_f^{(s)}(j,k)|}{\mu^{(s)}(j,k)n} \xrightarrow{p} \alpha_{j,k}^{(s)}(x^*) := \sum_{\theta=0}^{j} p^{(s)}(j,k,\theta)\mathbb{P}(\mathrm{Bin}(j,x^*) \geq \theta).$$

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Security investment Nash Equilibrium

We say that a security investment across agents $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a Nash equilibrium if

$$s_i^* \in \arg\max_{s \in \mathcal{S}} \mathbb{E}[U_i(s, \mathbf{s}_{-i}^*) - C_i(s)],$$

for all $i \in [n]$.