Special Preparatory Course: for CRM 2008-9 Thematic Year on
Probabilistic Methods in Mathematical Physics

Title: Introduction to Tau functions and their applications

Lecturers: This series will be given mainly by John Harnad and Marco Bertola, with further lectures by other members of the CRM Mathematical Physics Group, and visitors.
(Ferenc Balogh, Leonid Chekhov, Dmitri Korotkin, Jacques Hurtubise, Gabor Pusztai)

Times: Fridays 13:30-16:00, starting January 25, 2008

Locale: CRM, UdeM, Pav. André-Aisenstadt, 2920, ch. de la Tour, salle 4336

Purpose: This is a special series of lectures to be offered weekly throughout this semester, intended, in part, as a preparatory course for several of the workshops comprising the **CRM 2008-9 Thematic Program on Probabilistic Methods in Mathematical Physics.**

The topics to which the material covered will have applications include:
Quantum and Classical Integrable Systems; Random Matrices, Random Processes, Random Tilings, Topological Invariants of Riemann surfaces; Group Representation Theory, Combinatorics

Prerequisites: Elementary familiarity with group representations, measure theory and differential equations. (Any further familiarity with the areas of application and the mathematical tools used in them will be helpful, but no specific knowledge will be assumed. These include: elementary notions from probability theory, random processes, symmetric functions, Riemann surfaces, combinatorics, topology, geometry)

Outline: (J.H. = John Harnad M.B. = Marco Bertola)

Chapter 1: (J.H.): **The Sato KP Tau function**
Part A:
- Examples: Schur functions, more general polynomial tau functions, theta functions, matrix integrals
- Applications: integrable systems; random matrices, random processes, combinatorics

Part B:
The KP hierarchy: pseudodifferential operators, formal dressing operator, Baker functions; Sato's formula, Bilinear relations

Chapter 2: (J.H.) **Geometric definition of KP tau functions: the Sato-Segal-Wilson Grassmannian**
Part A:
- Review of finite dimensional Grassmannians, Plücker coordinates, Plücker relations, Schur functions
- The Segal-Wilson Grassmannian; partitions
- The abelian flow group: (simplified) Sato definition of the KP tau function

Part B:
- Fermi Fock space (Bose-Fermi correspondence)
- Interpretation as fermionic expectation values
- The Hirota bilinear equations for KP as Plücker relations
- Plücker coordinates and Schur function expansions

Chapter 3 (J.H.): **Examples of tau functions I: Integrable systems**
Part A:
- Schur functions, general polynomial tau functions
- The Calogero-Moser system: rational solutions of KP hierarchy

- Toda lattice

Part B:

- KP solitons
- Finite gap solutions: algebraic curves
- Multi-KP generalizations

Chapter 4 (J.H.): **Examples of tau functions II: Matrix integrals**

Part A:

- Hermitian matrix models: Partition functions and expectation values as KP tau functions
- Kontsevich-Witten integral

Part B:

- Two -matrix models and multi-matrix chains; 2-Toda, and multi-component KP hierarchy

Chapter 5 (J.H.): **Other applications I: Fermi systems in equilibrium and random processes**

Part A:

- Equilibrium statistical models of fermions

Part B:

- Generating functions for exclusion processes
- Limiting distributions and shapes

Chapter 6 (J.H.): **Other applications II: Random partitions, random tilings, random permutations**

Part A

- Plancherel random partitions (Borodin-Olshansky)

Part B

- Schur processes (Okounkov-Reshetikhin)
- Tau functions as weights on path space:

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References (First three weeks only; others to follow)

1) L.A. Dickey, "Soliton Equations", (World Scientific, 2003, 2nd ed.)
2) G. Segal and G. Wilson "Loop groups and equations of KdV type", *Publ. Math. IHES* **63**, 1-64 (1985)
3) E. Date, M. Jimbo, M. Kashiwara and T. Miwa, "Transformation groups for soliton equations", in *Nonlinear Integrable Systems,* Proc. RIMS Symposium, pgs. 39 -120, eds. T. Miwa and M. Jimbo, (World Scientific, Singapore 1983)
4)M. Sato and Y. Sato, "Soliton Equations as Dynamical Systems on Infinite Grassmann Manifold" in: Nonlinear PDE in Applied Science, U.S.-Japan Seminar. Tokyo. *Lecture Notes in Appl. Anal*. **5**, 259-271 (1982)

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Chapter 7: (M.B.): **Isomonodromic tau functions**

- Isomonodromic deformations
- Definition of isomonodromic tau function; residue formulae
- Spectral curve; Hamiltonian structure

Chapter 8: (M.B.): **Matrix models, biorthogonal polynomials**

- Random Matrix Partitions functions as isomonodromic tau functions
- Generalized orthogonal polynomials
- Dual Spectral curve and deformations systems

- The partition functon for 2-matrix models as an isomondromic tau function
- Other matrix models

Chapter 9: (M.B): **Asymptotics: the Riemann Hilbert Method**
- Asymptotics of tau functions using the Riemann-Hilbert nonlinear steepest descent method

Further possible sessions on:
- **Large N asymptotics of matrix model partitions functions and orthogonal polynomials** (Bertola, Korotkin)
- **Dispersionless limit and equliibrium measures for normal matrix models** (Balogh)
- **Relations to Laplacian growth** ("Tau function of an analytic curve": Wiegman-Zabrodin) (Balogh)
- **Relation of tau functions to Hamiltonian theory of isomonodromic deformations** (Bertola, Pusztai, Hurtubise)
- **The algebraic geometry of isomonodromic deformations** (Hurtubise)
- **Bergmann kernels, Hurwitz spaces d-bar determinants, Frobenius manifolds** (Korotkin)
- **Genus expansions, Witten-Jones generating functions** (Chekhov)