Special preparatory course

"Random matrices and exactly solvable models of statistical mechanics"

Pavel Bleher

Indiana University-Purdue University Indianapolis

Purpose: This is a special course for PhD students, postdocs, research visitors, and participants of the CRM 2008-9 Thematic Program on "Probabilistic Methods in Mathematical Physics" . The course will give an introduction to the modern theory of random matrix models and exactly solvable models of statistical mechanics.

Prerequisites: real analysis, measure theory, complex analysis, ordinary differential equations.

Outline of the course

I. Introduction to classical spin systems of statistical mechanics

I.1. Gibbs distribution for classical spin systems. Heisenberg and Ising models. Exact solution of one-dimensional models. The method of transfer-matrix.

I.2. Phase transitions in the Ising model on the Bethe lattice.

II. Introduction to random matrix models

II.1. Ensembles of random matrices. Topological expansion.

II.2. Weyl's integration formula and ensemble of eigenvalues.

II.3. Limiting distribution of eigenvalues and equilibrium measure.

II.4. Exact finite *n* solution for joint distribution of eigenvalues in terms of orthogonal polynomials. The Riemann-Hilbert problem for

orthogonal polynomials.

III. Large *n* asymptotics in random matrix models

III.1. The Riemann-Hilbert approach and the Deift-Zhou nonlinear steepest descent method.

III.2. Construction of the parametrix for the RH problem in the one-cut case.

III.3. Construction of the parametrix for the RH problem in the multi-cut case.

III.4. Universality of the local correlations between eigenvalues.

IV. The Pfaffian approach to the dimer model

IV.1. The Pffafian solution of the classical dimer model for a finite *n*.

IV.2. Exact solution of the classical dimer model in the limit as *n* tends to infinity.

IV.3. Introduction to the six-vertex model.

IV.4. The dimer representation of the six-vertex model on the free fermion line. Exact solution of the six-vertex model with the periodic boundary conditions on the free fermion line.

V. The random matrix model approach to the six-vertex model with domain wall boundary conditions

V.1. The Izergin-Korepin solution of the six-vertex model with DWBC for a finite *n*.

V.2. Large *n* asymptotics of the six-vertex model with DWBC in the

disordered phase. Applications to alternating sign matrices.

V.3. Large n asymptotics of the six-vertex model with DWBC in the ferroelectric phase.

V.4. Unsolved problems.

Sources

Random matrix models and orthogonal polynomials

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P. Deift, *Orthogonal Polynomials and Random Matrices:A Riemann-Hilbert Approach*. Courant lecture notes, **3** New York: Courant Institute of Mathematical sciences, New York University, 2000.

A. Kuijlaars, Riemann-Hilbert analysis for orthogonal polynomials. *Orthogonal polynomials and special functions (Leuven, 2002),* 167--210, Lecture Notes in Math., **1817**, *Springer, Berlin,* 2003.

P. Bleher, *Lectures on random matrix models. The Riemann-Hilbert approach.* arXiv:0801.1858 (to appear in the CRM volume on "Random Matrices and Their Application", Springer, 2008)

Exactly solvable models of statistical mechanics

R. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, San Diego, CA.

Kasteleyn, P. W. Dimer statistics and phase transitions. *J. Mathematical Phys.* **4** 1963 287--293.

E. H. Lieb and F. Y. Wu, Two Dimensional Ferroelectric Models, in *Phase Transitions and Critical Phenomena*, C. Domb and M. Green eds., vol. 1, Academic Press (1972) 331-490.

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