Vanishing of Hyperelliptic L-Functions at the Central Point

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Abstract

We obtain a lower bound on the number of quadratic Dirichlet L-functions over the rational function field which vanish at the central point $s = 1/2$. The approach is based on the observation that vanishing at the central point can be interpreted geometrically, as the existence of a map to a fixed abelian variety from the hyperelliptic curve associated to the character.

Motivation: Chowla's conjecture

Conjecture 1 (Chowla, 1965)**.** *For any quadratic Dirichlet character χ, $L(s,\chi) \neq 0$ for all $s \in (0,1)$.*
In particular, $L(1/2,\chi) \neq 0$.

Theorem 1 (Soundrarajan, 2000)**.** *At least 87.5% of odd squarefree integers $d > 0$ have the property that $L(1/2, \chi_{8d}) \neq 0$ where χ_{8d} denotes the real quadratic character with conductor $8d$.*

Function Field Analogy

Number field	Function field
$\mathbb{Q}$	$\mathbb{F}_q(x)$
$\mathbb{Z}$	$\mathbb{F}_q[x]$
positive primes	monic, irreducible polynomials
$\lvert n\rvert$	$\lvert f\rvert = q^{\deg f}$
quadratic characters	monic, squarefree polynomials

Definiton 2. *Let $\mathbb{F}_q$ be a finite field with odd characteristic. Define*

$$g(N) = \{D \in \mathbb{F}_q[x],\ monic,\ squarefree : |D| < N, L(1/2, \chi_D) = 0\}$$

Question: Is $g(N)$ equal to 0?

Theorem 3 (Bui–Florea, 2016)**.** *With the notation above,*

$$|g(N)| \leq 0.057N + o(N)$$

for any $N = q^{2g+1}$ where $g \in \mathbb{Z}$.

The Main Result

Theorem 4 (L., 2017)**.**
- *When q is a square, for any $\epsilon > 0$, $|g(N)| \geq B_\epsilon N^{1/2-\epsilon}$ with some nonzero constant B_ϵ and $N > N_\epsilon$.*
- *When q is not a square and $q \neq 3$, for any $\epsilon > 0$, $|g(N)| \geq B_\epsilon N^{1/3-\epsilon}$ with some nonzero constant B_ϵ and $N > N_\epsilon$.*
- *When $q = 3$, for any $\epsilon > 0$, $|g(N)| \geq B_\epsilon N^{1/5-\epsilon}$ with some nonzero constant B_ϵ and $N > N_\epsilon$.*

Although Chowla's conjecture does not hold over $\mathbb{F}_q(t)$, it may hold for almost all quadratic characters, i.e. it may be the case that $|g(N)|/N \to 0$ as $N \to \infty$.

Geometric Interpretation

Let D be a monic, squarefree polynomial. Let $P(x) \in \mathbb{Z}[x]$ be the characteristic polynomial of geometric Frobenius acting on the Jacobian of the hyperelliptic curve defined by $y^2 = D$.

$$L(1/2, \chi_D) = 0 \iff P(q^{-1/2}) = 0.$$

$$P(q^{-1/2}) = 0 \iff \alpha_j = \sqrt{q} \text{ for some } \alpha_j$$

when q is a square, there exists an elliptic curve E_0 over $\mathbb{F}_q$ which admits $\sqrt{q}$ as a Frobenius eigenvalue.

$$P(q^{-1/2}) = 0 \iff J(C) \sim E_0 \times A \text{ for some abelian variety A}$$

By composing with a map $C \to J(C)$, we get the existence of a dominant map $C \to E_0$.

Proposition 5 (L., 2017)**.** *Let C_0 be a genus g hyperelliptic curve defined over $\mathbb{F}_q$. There exists a positive constant B_ϵ such that the number of monic squarefree polynomials $D \in \mathbb{F}_q[x]$ satisfying*

1. $|D| < N$

2. $C: y^2 = D$ admits a dominant map to C_0

is at least $B_\epsilon N^{\frac{1}{g+1}-\epsilon}$ for any $\epsilon > 0$.

Application to Ranks of Elliptic Curves

From $E_0 : y^2 = f(x)$ over $\mathbb{F}_q$, we construct the constant elliptic curve over the rational function field $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(x)$. Denote E_D as the quadratic twist of E by $D \in \mathbb{F}_q[x]$. Let C be a hyperelliptic curved defined by $y^2 = D$.

$$rank(E_D) = |\{\phi : C \to E_0, \text{ dominant map}\}| \cdot (rank(End(E_0)))$$

Corollary 6 (L., 2017)**.** *Let $E = E_0 \times \mathbb{F}_q(x)$ be a constant elliptic curve over $\mathbb{F}_q(x)$.*
Let $P(N) = \{D \in \mathbb{F}_q[x] :$ monic, squarefree, $|D| < N\}$.
$R_m(N) = \{D \in P(N) : E_D$ has even rank $\geq m\}$.
Then there exists a nonzero constant B_ϵ such that

$$\lim_{N\to\infty} \frac{|R_2(N)|}{|P(N)|} \geq B_\epsilon N^{1/2-\epsilon}$$

Moreover, if E_0 is supersingular, then the statement holds with $R_2(N)$ replaced by $R_4(N)$.

Data

$\mathbb{F}_9$					
Degree d	$g(9^d)$	$9^d - 9^{d-1}$	$g(9^d)/(9^d - 9^{d-1})$	$1/(9^d)^{1/2}$	$1/(9^d)^{1/4}$
3	6	648	0.9%	3.7%	19.2%
4	18	5832	0.3%	1.2%	11.1%
5	216	52488	0.4%	0.4%	6.4%
6	180	472392	0.038%	0.1%	3.7%
7	8658	4251528	0.2%	0.045%	2.1%
8(sample)	2660	5000000	0.05%	0.015%	1.2%
9(sample)	3262	5000000	0.065%	0.005%	0.7%
10(sample)	532	5000000	0.01%	0.002%	0.4%

References

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