# Tau functions, integrable systems, random matrices and random processes

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## Outline

1. **Definition and uses of tau functions**
   - Uses of tau functions
   - Grassmannians, Abelian group actions, determinants
   - Fermionic Fock space VEV's
2. **Classical integrable systems**
   - KP, Toda hierarchies. Baker function. Dispersionless limit
   - Isospectral flows. Finite dimensional reductions
   - Relation to Hamilton's Principle Function
3. **Random two-matrix models**
   - Partition functions, expectation values, correlators
   - Two matrix characteristic polynomial correlators
   - Double Schur function expansions
4. **Random Processes**
   - Determinantal processes on partitions
   - Other relations to integrable systems: Bethe ansatz
   - Other related work

# Uses of tau functions

1. Classical integrable systems
   - Canonical generator for commuting flows
2. Random matrices, quantum integrable systems, solvable lattice models,
   - Partition function, spectral correlation functions
   - Boltzmann weight on statistical ensembles
3. Large $N$ limit $\sim$ dispersionless limit $\sim$ semiclassical limit
   - Free energy $= \lim_{N\to\infty} \frac{1}{N^2}$ log (tau function) $\sim$ electrostatic self-energy of Coulomb gas (eigenvalues in RMM) in continuum limit; ($\{t_i\}_{=1,2\dots}$ = exterior harmonic moments)
4. Random processes
   - Weight on path space
   - Generating function for transition probabilities

## Sato-Segal-Wilson definition of the *KP* $\tau$ function

The $\tau$-function is an infinite Fredholm determinant

$$\tau_g(\mathbf{t}) = \det(\pi_+ : W(\mathbf{t}) \to \mathcal{H}_+), \quad \mathbf{t} = (t_1, t_2, \dots)$$

of the orthogonal projection operator defined on a Hilbert space Grassmannian $Gr_{\mathcal{H}_+}(\mathcal{H})$

$$\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_- = L^2(S^1)$$

from a linearly evolving subspace (commensurable with $\mathcal{H}_+$):

$$W(\mathbf{t}) = \gamma(\mathbf{t})(W) \in Gr_{\mathcal{H}_+}(\mathcal{H}), \quad W := g(\mathcal{H}_+)) \subset \mathcal{H}, \quad g \in GL(\mathcal{H})$$

under an Abelian group action:

$$e^{\sum_{i=1}^{\infty} t_i z^i} =: \gamma(\mathbf{t}) : \mathcal{H} \to \mathcal{H},$$

to the "origin" (reference subspace $\mathcal{H}_+ \in Gr_{\mathcal{H}_+}(\mathcal{H})$).

## Equivalent fermionic definition (via Plücker map)

**Fermionic Fock space** (exterior space)

$$\mathcal{F} := \Lambda\mathcal{H}$$

**Monomial basis for** $\mathcal{H}$, and **dual basis for** $\mathcal{H}^*$

$$\{e_i = z^i\}_{i\in\mathbf{Z}}, \quad \{\tilde{e}_i\}_{i\in\mathbf{Z}}$$

**Vacuum state:** $|0\rangle := e_0 \wedge e_1 \wedge e_2 \cdots$

**Fermi creation and annihilation operators**

$$f_i := i(\tilde{e}^i), \quad \bar{f}_i := e_i\wedge$$

**Free fermion anticommutation relations**

$$[f_n, f_m]_+ = [\bar{f}_n, \bar{f}_m]_+ = 0, \quad [f_n, \bar{f}_m]_+ = \delta_{nm}$$

**"Charged" vacuum vectors**:

$$|N\rangle := f_N f_{N-1} \dots f_0|0\rangle$$

# Fermionic form for KP and $2-D$ Toda Tau functions

## KP $\tau$-function

$$\tau_{N,g}(\mathbf{t},\tilde{\mathbf{t}}) := \langle N|\gamma(\mathbf{t})g|N\rangle$$
$$\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i H_i} \qquad g = e^{\mathcal{A}}$$
$$H_i := \sum_{j\in\mathbf{Z}} f_j \bar{f}_{j+i} \qquad \mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j$$

## 2-D Toda $\tau$-function

$$\tau_{N,g}(\mathbf{t},\tilde{\mathbf{t}}) := \langle N|\gamma(\mathbf{t})g\tilde{\gamma}(\tilde{\mathbf{t}})|N\rangle$$
$$\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i H_i} \qquad \tilde{\gamma}(\tilde{\mathbf{t}}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i H_{-i}} \qquad g = e^{\mathcal{A}}$$
$$H_i := \sum_{j\in\mathbf{Z}} f_j \bar{f}_{j+i} \qquad \mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j$$

## Baker-Akhiezer function. Sato formula

Define the **Baker-Akhiezer function**

$$\psi(z,\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i z^i} \frac{\tau(\mathbf{t} - [z^{-1}])}{\tau(\mathbf{t})}$$

where $\tau$ is a KP tau function, $[z^{-1}] := (\frac{1}{z}, \frac{2}{z^2} \dots)$
Then $\psi(z,\mathbf{t})$ satisfies the infinite set of evolution equations

$$\frac{\partial \psi}{\partial t_i} = \mathcal{D}_i \psi$$

$$\mathcal{D}_i := (\mathcal{L}^i)_+$$

$$\mathcal{L} := \partial + \sum_{i=1}^{\infty} u_i \partial^{-i}, \quad \partial := \frac{\partial}{\partial x} = \frac{\partial}{\partial t_1}$$

## KP hierarchy/ 2-D Toda hierarchy

It follows that $\mathcal{L}$, $\mathcal{D}_i$ satisfy the equations of the **KP hierarchy**

### KP hierarchy

$$\frac{\partial \mathcal{L}}{\partial t_i} = [\mathcal{D}_i,\ \mathcal{L}],$$
$$\frac{\partial \mathcal{D}_i}{\partial t_j} - \frac{\partial \mathcal{D}_j}{\partial t_i} + [\mathcal{D}_i,\ \mathcal{D}_j] = 0$$

A similar construction (replacing $\partial \rightarrow e^{\partial}$) leads to the equations of the $2 - D$ **Toda hierarchy**

### 2-D Toda hierarchy

$$\frac{\partial \mathcal{L}}{\partial t_i} = [\mathcal{D}_i,\ \mathcal{L}], \quad \frac{\partial \tilde{\mathcal{L}}}{\partial t_i} = [\mathcal{D}_i,\ \tilde{\mathcal{L}}], \quad \frac{\partial \mathcal{L}}{\partial \tilde{t}_i} = [\tilde{\mathcal{D}}_i,\ \mathcal{L}], \quad \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{t}_i} = [\tilde{\mathcal{D}}_i,\ \tilde{\mathcal{L}}]$$

## Dispersionless limit

In the dispersionless / semiclassical limit (introducing a small parameter $\hbar$, $\partial \to \hbar\partial$), we have (Takesaki, Takebe (1995))

$$e^{\hbar\partial} \to w \in \mathbf{C}, \quad \mathcal{L}, \tilde{\mathcal{L}}^{-1} \to f(w,t), \tilde{f}(w,t) \quad \text{(complete symbol)}$$
$$[\hat{f},\ \hat{g}] \to \{f,\ g\} := w\frac{\partial f}{\partial w}\frac{\partial g}{\partial t} - w\frac{\partial g}{\partial w}\frac{\partial f}{\partial t}$$

where $f, \tilde{f}$ satisfy the equations:

**Dispersionless 2-D Toda hierarchy**

$$\frac{\partial f}{\partial t_i} = \{(f^i)_+, f\}, \frac{\partial f}{\partial t_i} = \{(f^i)_+, f\}, \frac{\partial f}{\partial t_i} = \{(f^i)_+, f\}\frac{\partial f}{\partial t_i} = \{(f^i)_+, f\}$$

In this setting, the equation of **Laplacian growth** is viewed (Mineev-Zabrodin (1990)) as the compatible constraint equation

$$\{f, \tilde{f}\} = 1$$

# Finite dimensional reductions. Lax matrix, spectral curves

## Isospectral Hamiltonian flows of Lax matrices

Restricting to finite dimensional phase space (or **finite band gap** solutions to KP and Toda hierarchies), we introduce a **Lax matrix** $L(\lambda)$ depending rationally on a spectral parameter $\lambda$ .
For a suitable Poisson structure ($R$-matrix) and spectral invariant Hamiltonians: Hamilton's equation $\rightarrow$ Lax equation

$$\frac{dL(\lambda)}{dt} = [A(L, \lambda), L(\lambda)]$$

Invariants ($\mathbf{P} = \mathbf{C}$) $\leftrightarrow$ coefficients of the **spectral curve**

$$\det(L(\lambda) - z\mathbf{I}) = 0$$

The $\tau$-function is essentially the Riemann $\Theta$ function, (which is an **infinite** determinant ):

$$\tau(\mathbf{t}) = \Theta(\mathbf{Q}(\mathbf{t})), \quad \mathbf{Q}(\mathbf{t}) = \mathbf{Q_0} + (\nabla_{\mathbf{P}} H, \mathbf{t})$$

# Solitons. Relation to Hamilton's Principle Function

**Solitons** arise through degeneration to rational spectral curves with cusp singularities

$$\tau \sim \det(e^{(\Lambda_{ij},\mathbf{t})+\kappa_{ij}})$$

**Hamilton's Principle Function** on Lagrangian leaves:

$$\begin{aligned} S(\mathbf{q}(\mathbf{t}),\mathbf{C}) &= \int_{\mathbf{P}=\mathbf{C}} \mathbf{p}\cdot d\mathbf{q} \\ &= \mathcal{D}(\log\Theta(t_1,t_2,\dots)) \end{aligned}$$

Where $\mathcal{D}$ is a constant coefficient linear differential operator in the flow parameters $(t_1, t_2, \dots)$.

## Random two-matrix models

Most **statistical properties** of the spectrum are expressible as **expectation values**

$$< F >= \frac{1}{\mathbf{Z}_N^{(2)}} \int F(M_1, M_2) d\Omega(M_1, M_2)$$

where the **Partition function** is

$$\mathbf{Z}_N^{(2)} := \int d\Omega(M_1, M_2)$$

For some **conjugation invariant** $F$'s, unitarily diagonalizable matrices $M_1$, $M_2$, and certain matrix measures $d\Omega(M_1, M_2)$ this reduces to integrals over the **eigenvalues**

$$< F > \propto \prod_{i=1}^{N} \int\int_{\kappa\Gamma} d\mu(x_i, y_i) \Delta_N(x) \Delta_N(y) \tilde{F}(x_1, \ldots, x_N, y_1, \ldots y_N)$$

## Random two-matrix models

where the Vandermonde determinants

$$\Delta(\mathbf{x}) := \prod_{i<j}^{N}(x_i - x_j), \qquad \Delta(\mathbf{y}) := \prod_{i<j}^{N}(y_i - y_j)$$

give rise to the 2-Coulomb gas interpretation (Gibbs measure)

$$\Delta(\mathbf{x})\Delta(\mathbf{y}) = e^{\sum_{i<j}\log(x_i - x_j) + \sum_{i<j}\log(y_i - y_j)}$$

**Example: (Itzykson-Zuber (1980))**

$$d\Omega(M_1, M_2) = d\mu_0(M_1)d\mu_0(M_2)e^{\mathrm{Tr}(V_1(M_1)+V_2(M_2)+M_1M_2)}$$

General reduced 2−matrix integrals become

$$\tilde{\mathbf{Z}}_N^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}) := \prod_{i=1}^{N} \int\int_{\kappa\Gamma} d\mu(x_i, y_i)\gamma(x_i, \mathbf{t})\tilde{\gamma}(y_i, \tilde{\mathbf{t}})\Delta_N(\mathbf{x})\Delta_N(\mathbf{y})$$

## General form of two-variable measure

Here $d\mu(x,y)$ is some generalized two-variable measure

$$d\mu(x,y) = d\mu(x)d\tilde{\mu}(y)h(xy)\sum_{a=1}^{k}\sum_{b=1}^{l} z_{ab}\chi_a(x)\tilde{\chi}_b(y)$$

The terms $\gamma(x,\mathbf{t}), \gamma(y,\tilde{\mathbf{t}})$ are viewed as abelian deformation factors, with parameters: $\mathbf{t} = (t_1, t_2, \dots), \quad \tilde{\mathbf{t}} = (\tilde{t}_1, \tilde{t}_2, \dots)$

$$\gamma(x,\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i x^i}, \quad \tilde{\gamma}(y,\tilde{\mathbf{t}}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i y^i}$$

The terms $(\chi_a(x)\tilde{\chi}_b(y))$ give limits of the domains of integration.
$\kappa = \{\kappa_{\alpha\beta}\}\big|_{\substack{1\leq\alpha\leq d_1 \\ 1\leq\beta\leq d_2}}$ give the relative weights of segments

$$\int\int_{\kappa\Gamma} := \sum_{\alpha\beta} \kappa_{\alpha\beta} \int_{\Gamma_\alpha}\int_{\tilde{\Gamma}_\beta}$$

$$\Gamma = \{\Gamma_\alpha \times \tilde{\Gamma}_\beta\}$$

and $(z_{ab})$ are generating function parameters.

## Examples

- Two-matrix **partition function**:

$$k = l = 1, \quad z_{11} = 1, \quad \chi_1 = \tilde{\chi}_1 = 1$$

$$d\mu(x) = e^{V_1(x)}dx, \quad d\tilde{\mu}(y) = e^{V_2(y)}, \quad h(xy) = e^{xy}$$

- Generating function for $(k, l)$ **point correlators** (marginal distributions) of eigenvalues

$$\chi_a = \delta(x - X_a), \quad \chi_b = \delta(y - Y_b)$$

- Generating function for **gap probabilities**

$$\chi_a = \chi_{[\alpha_{2a-1},\alpha_{2a}]}(x) \quad \tilde{\chi}_b = \chi_{[\beta_{2b-1},\beta_{2b}]}(x)$$

- Generating function for **Janossy distributions**. (Combine the above two.)

# Two matrix characteristic polynomial correlators

$$\mathbf{I}_N^{(2)} := \left\langle \prod_{a=1}^{N} \frac{\prod_{\alpha=1}^{L_1} \det(\xi_\alpha \mathbf{I} - M_1) \prod_{\beta=1}^{L_2} \det(\zeta_\beta \mathbf{I} - M_2)}{\prod_{j=1}^{M_1} \det(\eta_j \mathbf{I} - M_1) \prod_{k=1}^{M_2} \det(\mu_k \mathbf{I} - M_2)} \right\rangle$$

For suitable measures, this reduces to **Integrals of rational symmetric functions** in $2N$ variables:

$$\begin{aligned} \mathbf{I}_N^{(2)}(\xi,\zeta,\eta,\mu) := & \frac{1}{\mathbf{Z}_N^{(2)}} \prod_{i=1}^{N} \int\int_{\kappa\Gamma} d\mu(x_i, y_i) \Delta_N(x) \Delta_N(y) \\ & \times \prod_{a=1}^{N} \frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_a) \prod_{\beta=1}^{L_2}(\zeta_\beta - y_a)}{\prod_{j=1}^{M_1}(\eta_j - x_a) \prod_{k=1}^{M_2}(\mu_k - y_a)} \end{aligned}$$

$$\mathbf{Z}_N^{(2)} := \prod_{i=1}^{N} \int\int_{\kappa\Gamma} d\mu(x_i, y_i) \Delta_N(x) \Delta_N(y) \qquad (3.1)$$

# Biorthogonal polynomials

Assuming generic conditions on the matrix of **bimoments**:

$$B_{jk} := \iint_{\kappa\Gamma} d\mu(x,y) x^j y^k < \infty, \quad 0, \quad \forall j,k \in \mathbf{N}$$
$$\det(B_{jk})_{0 \le j,k \le N} \neq 0, \quad \forall N \in \mathbf{N}$$

implies the existence of a unique sequence of **biorthogonal polynomials** $\{P_j(x), S_j(y)\}$

$$\iint_{\kappa\Gamma} d\mu(x,y) P_j(x) S_k(y) = \delta_{jk},$$

normalized to have leading coefficients that are equal:

$$P_j(x) = \frac{x^j}{\sqrt{h_j}} + O(x^{j-1}), \qquad S_j(y) = \frac{y^j}{\sqrt{h_j}} + O(y^{j-1}).$$

## Determinantal expression for correlator

Also, assume existence of their **Hilbert transforms**

$$\begin{aligned}\tilde{P}_j(\mu) &:= \iint_{\kappa\Gamma} d\mu(x,y)\,\frac{P_j(x)}{\mu-y},\\ \tilde{S}_j(\eta) &:= \iint_{\kappa\Gamma} d\mu(x,y)\,\frac{S_j(x)}{\eta-x}\end{aligned}$$

**Determinantal expression for correlator**
(Assume $N+L_2-M_2 \geq N+L_1-M_1 \geq 0$)

$$\begin{aligned}\mathbf{I}_N^{(2)} &= \epsilon(L_1,L_2,M_2,M_2)\,\frac{\prod_{n=0}^{N+L_2-M_2-1}\sqrt{h_n}\ \prod_{n=0}^{N+L_1-M_1-1}\sqrt{h_n}}{\prod_{n=0}^{N-1} h_n}\\ &\quad\times\frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\ \det G\end{aligned}$$

where $\epsilon(L_1,L_2,M_2,M_2) := (-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}(-1)^{L_1M_2}$
and $G$ is the $(L_2+M_1)\times(L_2+M_1)$ matrix:

## Determinantal expression for correlator

$$G = \begin{pmatrix} \overset{N+L_1-M_1}{K_{11}}(\xi_\alpha, \eta_j) & \overset{N+L_1-M_1}{K_{12}}(\xi_\alpha, \zeta_\beta) \\ \overset{N+L_1-M_1}{K_{21}}(\mu_k, \eta_j) & \overset{N+L_1-M_1}{K_{22}}(\mu_k, \zeta_\beta) \\ \tilde{S}_{N+L_1-M_1}(\eta_j) & S_{N+L_1-M_1}(\zeta_\beta) \\ \vdots & \vdots \\ \tilde{S}_{N+L_2-M_2-1}(\eta_j) & S_{N+L_2-M_2-1}(\zeta_\beta) \end{pmatrix}$$

where the kernels $K^J_{11}, K^J_{12}, K^J_{21}, K^J_{22}$ are defined by:

$$\overset{J}{K}_{11}(\xi, \eta) := \sum_{n=0}^{J-1} P_n(\xi)\tilde{S}_n(\eta) + \frac{1}{\xi - \eta}, \quad \overset{J}{K}_{12}(\xi, \zeta) := \sum_{n=0}^{J-1} P_n(\xi) S_n(\zeta),$$

$$\overset{J}{K}_{21}(\mu, \eta) := \sum_{n=0}^{J-1} \tilde{P}_n(\mu)\tilde{S}_n(\eta) - \iint_{\kappa\Gamma} \frac{d\mu(x, y)}{(\eta - x)(\mu - y)},$$

$$\overset{J}{K}_{22}(\mu, \zeta) := \sum_{n=0}^{J-1} \tilde{P}_n(\mu) S_n(\zeta) + \frac{1}{\zeta - \mu}$$

# Two methods of derivation

## Direct method

J. Harnad and A. Yu. Orlov,

"Integrals of rational symmetric functions, two-matrix models and biorthogonal polynomials", *J. Math. Phys.* **48** (in press, Sept. 2007)

## Fermionic vacuum state expectation values

J. Harnad and A.Yu. Orlov,

"Fermionic approach to the evaluation of integrals of rational symmetric functions", arXiv:0704.1150)

# Previous work

## Analogous results for one-matrix models:

- V.B. Uvarov, (1969) (general case)
- E. Brezin and S. Hikami, (2000),(polynomial integrals)
- Y.V. Fyodorov and E. Strahov (2003) ($N \geq M$)
- J. Baik, P. Deift and E. Strahov, (2003) ($N \geq M$)
- A. Borodin and E. Strahov (2006)

## Complex matrix model: polynomial case

- G. Akemann and G. Vernizzi, (2003)

## Complex matrix model: rational case

- M. Bergère hep-th/0404126)

## Relation to integrable systems

Deform the measure

$$d\Omega(M_1, M_2) \to d\Omega(M_1, M_2) e^{\mathrm{tr}(\sum_{j=0}^{\infty}(t_j M_1^j + \tilde{t}_j M_2^j)} \\ := d\Omega(M_1, M_2, \mathbf{t}, \tilde{\mathbf{t}})$$

The deformed partition function

$$\mathbf{Z}_N^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}) := \int d\Omega(M_1, M_2, \mathbf{t}, \tilde{\mathbf{t}})$$

is a **2-Toda $\tau$ function**. The **biorthogonal polynomials** and their Hilbert transforms

$$\{P_j(x, \tilde{\mathbf{t}}, \mathbf{t}), \tilde{P}_j(y, \mathbf{t}, \tilde{\mathbf{t}})\}, \{\tilde{S}_j(x, \mathbf{t}, \tilde{\mathbf{t}}), S_j(y, \mathbf{t}, \tilde{\mathbf{t}})\}_{j \in \mathbf{N}}$$

are **Baker-Akhiezer** and **dual Baker-Akhiezer functions**.

## Double Schur function perturbation expansions

In terms of the **Schur functions** $s_\lambda(\mathbf{t})$, $s_\mu(\tilde{\mathbf{t}})$ (irreducible characters) corresponding to partitions

$$\lambda := (\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0), \quad \mu := (\mu_1 \geq \cdots \geq \mu_{\ell(\mu)} > 0)$$

of lengths $\ell(\lambda), \ell(\mu) \leq N$

$$\mathbf{Z}_N^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}) = N! \sum_{\substack{\lambda, \mu, \\ \ell(\lambda)\ell(\mu) \leq N}} B_{\lambda\mu} s_\lambda(\mathbf{t}) s_\mu(\tilde{\mathbf{t}})$$

where

$$B_{\lambda,\mu} = \det(B_{\lambda_i - i + N, \mu_j - j + N})|_{i,j=1,\ldots,N},$$

is the **Matrix of bimoments**.

# Two methods of derivation

## Direct method


J. Harnad and A.Yu. Orlov
"Scalar products of symmetric functions and matrix integrals", *Theor. Math. Phys.* **137**, 1676–1690 (2003).

J. Harnad and A. Yu. Orlov
"Matrix integrals as Borel sums of Schur function expansions", In: Symmetries and Perturbation theory SPT2002, eds. S. Abenda and G. Gaeta, World Scientific, Singapore, (2003).


## Fermionic vacuum state expectation values


J. Harnad and A. Yu. Orlov
"Fermionic construction of partition functions for two-matrix models and double Schur function expansions", *J. Phys. A* **39**, 8783–8809 (July 2006) math-phys/0512056


## Earlier work


V. A. Kazakov, M. Staudacher and T. Wynter, *Commun. Math. Phys.* **177**, 451-468 (1996)

## Main tools for direct proof of formula for rational integrals

### 1. Multivariable partial fraction expansions

For $N \geq M$:

$$\frac{\Delta_N(x)\Delta_M(\eta)}{\prod_{a=1}^N \prod_{j=1}^M (\eta_j - x_a)} =$$

$$(-1)^{MN} \sum_{\sigma \in S_M} \mathrm{sgn}(\sigma) \sum_{a_1 < \cdots < a_M}^{N} (-1)^{\sum_{j=1}^M a_j} \frac{\Delta_{N-M}(x[\mathbf{a}])}{\prod_{j=1}^M (\eta_{\sigma_j} - x_{a_j})}$$

For $N \leq M$:

$$\frac{\Delta_N(x)\Delta_M(\eta)}{\prod_{a=1}^N \prod_{j=1}^M (\eta_j - x_a)} =$$

$$\frac{(-1)^{\frac{1}{2}N(N-1)}}{(M-N)!} \sum_{\sigma \in S_M} \mathrm{sgn}(\sigma) \frac{\Delta_{M-N}(\eta_{\sigma_{N+1}}, \dots, \eta_{\sigma_M})}{\prod_{a=1}^N (\eta_{\sigma_a} - x_a)}$$

# Main tools for direct proof of formula for rational integrals

## 2. Cauchy-Binet identity

If $V$ is an oriented Euclidean vector space with volume form $\Omega$ and $(P^1, \dots, P^L)$, $(S^1, \dots, S^L)$ are two sets of $L$ vectors, then the scalar product of their exterior products $(\wedge_{\alpha=1}^L P^\alpha, \wedge_{\beta=1}^L S^\beta)$ is equal to the determinant of the matrix formed from the scalar products of the vectors:

$$(\wedge_{\alpha=1}^L P^\alpha, \wedge_{\beta=1}^L S^\beta) = \det G$$
$$G^{\alpha\beta} := (P^\alpha, S^\beta), \quad 1 \leq i, j \leq L \tag{3.2}$$

### Remark

This is just the fermionic form of the **Wick theorem**.

# Main tools for direct proof of the Schur function expansion

## 3. Cauchy Littlewood identity

$$e^{\sum_{i=1}^{\infty} i t_i \tilde{t}_i} = \sum_{\lambda} s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}})$$

## 4. Andreief identity

$$\prod_{a=1}^{N} \int\int d\mu(x_a, y_a) \det \phi_i(x_j) \det \psi_k(y_l)$$
$$= N! \det \left( \int \int d\mu(x,y) \phi_i(x) \psi_j(y) \right)$$
$$(1 \le i, j, k, l \le N)$$

### Remark

Both of these may be derived as consequences of the Cauchy- Binet identity (or Fermionic Wick theorem).

# Fermionic approach: vacuum state expectation values (VEV)

**Two-component fermions.**

$$[f_n^{(\alpha)}, f_m^{(\beta)}]_+ = [\bar{f}_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = 0, \quad [f_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = \delta_{\alpha,\beta}\delta_{nm}, \quad \alpha = 1, 2$$

**Fermionic fields.**

$$f^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^k f_k^{(\alpha)}, \quad \bar{f}^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^{-k-1} \bar{f}_k^{(\alpha)}$$

**Right and left vacuum vectors.** $|0,0\rangle, \ \langle 0,0|$

$$\begin{array}{llll} f_m^{(\alpha)}|0,0\rangle = 0 & (m < 0), & \bar{f}_m^{(\alpha)}|0,0\rangle = 0 & (m \geq 0), \\ \langle 0,0|f_m^{(\alpha)} = 0 & (m \geq 0), & \langle 0,0|\bar{f}_m^{(\alpha)} = 0 & (m < 0) \end{array}$$

**Wick's theorem** implies, for linear elements of the Clifford algebra

$$\langle 0,0|w_1 \cdots w_N \bar{w}_N \cdots \bar{w}_1|0,0\rangle = \det\left(\langle 0,0|w_i \bar{w}_j|0,0\rangle\right)|_{i,j=1,\dots,N}$$

## Charged vacuum states and $gl(\infty)$ operators

**Charged vacuum states** $|n^{(1)}, n^{(2)}\rangle := \bar{C}_{n^{(2)}} \bar{C}_{n^{(1)}} |0,0\rangle$, where

$$\bar{C}_{n^{(\alpha)}} := f^{(\alpha)}_{n^{(\alpha)}-1} \cdots f^{(\alpha)}_0 \quad \text{if} \quad n^{(\alpha)} > 0$$
$$\bar{C}_{n^{(\alpha)}} := \bar{f}^{(\alpha)}_{n^{(\alpha)}} \cdots \bar{f}^{(\alpha)}_{-1} \quad \text{if} \quad n^{(\alpha)} < 0, \quad \bar{C}_0 := 1$$

### $gl(\infty)$ operators

$$\mathcal{A} := \int\int f^{(1)}(x)\bar{f}^{(2)}(y)d\mu(x,y), \qquad H(\mathbf{t},\tilde{\mathbf{t}}) := \sum_{k=1}^{\infty} H^{(1)}_k t_k - \sum_{k=1}^{\infty} H^{(2)}_k \tilde{t}_k$$

where

$$H^{(\alpha)}_k := \sum_{n=-\infty}^{+\infty} f^{(\alpha)}_n \bar{f}^{(\alpha)}_{n+k}, \quad k \neq 0, \quad \alpha = 1, 2.$$

are two sequences of commuting operators.

# Partition function as a 2-Toda $\tau$ function (VEV)

## Theorem. $\mathbf{Z}_N^{(2)}(\mathbf{t},\tilde{\mathbf{t}})$ is a 2-Toda $\tau$ function (VEV)

$$\tau_N(\mathbf{t},\tilde{\mathbf{t}}) := \langle N,-N|e^{H(\mathbf{t},\tilde{\mathbf{t}})}e^{A}|0,0\rangle = \frac{1}{N!}(-1)^{\frac{1}{2}N(N+1)}\mathbf{Z}_N^{(2)}(\mathbf{t},\tilde{\mathbf{t}})$$

## Sketch of proof

$$\langle N,-N|\prod_{i=1}^{N} f^{(1)}(x_i)\overline{f}^{(2)}(y_i)|0,0\rangle = (-1)^{\frac{1}{2}N(N+1)}\Delta_N(x)\Delta_N(y)$$

$$\langle N,-N|\prod_{i=1}^{k} f^{(1)}(x_i)\overline{f}^{(2)}(y_i)|0,0\rangle = 0 \quad \text{if } k \neq N,$$

$$\langle N,-N|A^k|0,0\rangle = 0 \quad \text{if} \quad k\neq N$$

$$e^{H(\mathbf{t})}f^{(1)}(x)\overline{f}^{(2)}(y)e^{-H(\mathbf{t})} = e^{\sum_{j=1}^{\infty}t_j x^j + \sum_{j=1}^{\infty}\tilde{t}_j y^j} f^{(1)}(x)\overline{f}^{(2)}(y),$$

# Double Schur function expansion

## Theorem

$$\mathbf{Z}_N^{(2)}(\mathbf{t},\tilde{\mathbf{t}}) = N! \sum_{\substack{\lambda,\mu,\\ \ell(\lambda)\ell(\mu)\leq N}} B_{\lambda\mu} s_\lambda(\mathbf{t}) s_\mu(\tilde{\mathbf{t}})$$

*where*

$$B_{\lambda,\mu} = \det\left(B_{\lambda_i - i + N, \mu_j - j + N}\right)\big|_{i,j=1,\dots,N},$$

## Fermionic proof. Key element follows from Wick's theorem

$$\langle N, -N| e^{H(\mathbf{t},\tilde{\mathbf{t}})} f^{(1)}_{h_1} \bar{f}^{(2)}_{-h'_1-1} \cdots f^{(1)}_{h_N} \bar{f}^{(2)}_{-h'_N-1} |0,0\rangle$$
$$= (-1)^{\frac{1}{2}N(N+1)} s_\lambda(\mathbf{t}) s_\mu(\tilde{\mathbf{t}})$$
$$h_i := \lambda_i - i + N, \quad h'_j := \mu_j - j + N$$

# Random processes on partitions

## Fermionic construction

J. Harnad and A. Yu. Orlov, "Fermionic construction of tau functions and random processes", *Theor. Math. Phys.* (in press, 2007) arXiv:0704.1157

## Maya diagrams as basis for fermionic Fock space

For each integer $N$, and partition $\lambda$ of length $\ell(\lambda)$

$$\lambda := \lambda_1 \geq \lambda_2 \geq \dots \quad \lambda_\ell(\lambda) > 0 \in \mathbf{N} \qquad \lambda_i := 0 \forall i > \ell(\lambda),$$

define **particle positions** (levels): $\{l_i := \lambda_i - i + N, \}_{i=1,\dots\infty}$ to form a **Maya Diagram**, and a **basis vector**:

$$\begin{aligned} |\lambda, N\rangle &:= (-1)^{\sum_{i=1}^k \beta_i} f_{n+\alpha_k} \bar{f}_{n-1-\beta_k} \cdots f_{n+\alpha_1} \bar{f}_{n-1-\beta_1} |N\rangle \\ |N\rangle &:= |0, N\rangle \quad \text{(charge } N \text{ vacuum)} \end{aligned}$$

where $(\alpha_1, \dots \alpha_k | \beta_1, \dots \beta_k)$ is the **Frobenius notation** for a partition. ($(\alpha_i, \beta_i)$ = no. of blocks to the right, resp. beneath the diagonal block in the $(i, i)$ position.)

# Maya diagrams

| Fig.1 | Fig.2 |
|---|---|
| ○ N+2 | ○ N+2 |
| ○ N+1 | ● N+1 |
| ○ N | ○ N |
| ● N-1 | ○ N-1 |
| ● N-2 | ● N-2 |
| ● N-3 | ● N-3 |

Fig.1 Dirac sea of level $N$. $|0; N\rangle$

Fig.2 Maya diagram for $|(2,1); N\rangle$

## $gl(\infty)$ action on $\mathcal{F}$

$$gl(\infty) : \mathcal{F} \rightarrow \mathcal{F}$$
$$gl(\infty) = \mathrm{span}\{E_{ij} := f_i \bar{f}_j\}_{i,j \in \mathbf{Z}}$$

This determines weighted actions on **Maya diagrams**

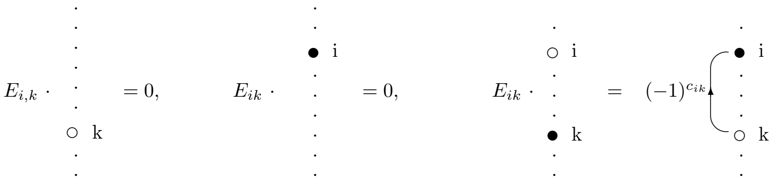
$$\mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j \in gl(\infty)$$
$$\mathcal{A} : |\lambda; N > \rightarrow \sum_{ij} a_{ij} f_i \bar{f}_j |\lambda; N >= \sum_{N',\mu} C_{\mu\lambda}^{N'N} |\mu, N' >$$

For positive coefficients $a_{ij}$, we can view

$$< \lambda, N' | \mathcal{A}^k | \mu, N >$$

as an (unnormalized) **transition weight** after $k$ (discrete) time steps.

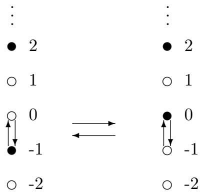
# Action of $E_{i,k}$ on Maya diagrams



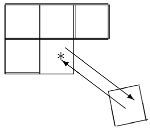

1. Elimination of Maya diagrams

2. Nontrivial action

# Up-down hopping. Adding / removing squares of Young diagram




1. Upward/downward hop
of a particle on a Maya diagram




2. Adding/removing boxes on a Young diagram

# The $\tau$ function as a generating function for transition probabilities

## 2 *D*-Toda tau function

In the 1-component formulation, the 2 *D* Toda tau function is:

$$\tau_{N,g}(\mathbf{t},\tilde{\mathbf{t}}) := < N|\gamma(\mathbf{t})e^{\mathcal{A}}\tilde{\gamma}(\tilde{\mathbf{t}})|N >$$
$$\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i H_i}, \quad \tilde{\gamma}(\tilde{\mathbf{t}}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i H_{-i}}$$
$$H_i := \sum_{j\in\mathbf{Z}} f_i \bar{f}_{j+i}, \quad \mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j$$

Assume $\mathcal{A}$ preserves $N$, and use

$$< N|\tilde{\gamma}(\tilde{\mathbf{t}})|\mu, N >= s_\mu(\tilde{\mathbf{t}}) \qquad < \lambda, N|\gamma(\mathbf{t})|N >= s_\lambda(\mathbf{t})$$

## The $\tau$ function expansion in Schur functions

$$\tau_{N,g}(\mathbf{t},\tilde{\mathbf{t}}) = \sum_{\lambda,\mu}\sum_{k=0}^{\infty}\frac{1}{k!} < \lambda, N|\mathcal{A}^k|\mu, N > s_\mu(\mathbf{t})s_\lambda(\tilde{\mathbf{t}})$$

$$< \lambda, N|\mathcal{A}^k|\mu, N > := \frac{1}{k!} < \lambda, N|\mathcal{A}_+^k + \mathcal{A}_-^k|\mu, N > \quad (\pm \text{ permutations})$$

The **transition probability** in $k$ time steps is

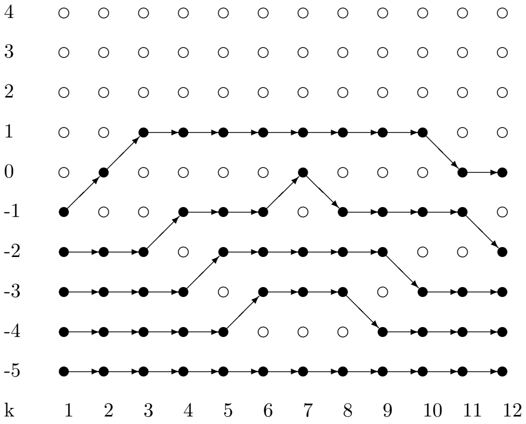
$$P_k((\mu, N)\rightarrow(\lambda, N)) = \frac{W^k_{N,g}(\lambda,\mu)}{\sum_\nu W^k_{N,g}(\nu,\mu)},$$

$$W^k_{N,g}(\nu,\mu) :=< \lambda, N|\mathcal{A}_+^k|\mu, N > - < \lambda, N|\mathcal{A}_-^k|\mu, N >$$

**Example. Random turn non-intersecting walkers**

$$\mathcal{A} := \sum_{i\in\mathbf{Z}}(p_l f_{i-1}\bar{f}_i + p_r f_{i+1}\bar{f}_i), \qquad p_l, p_r \geq 0, \quad p_l + p_r = 1$$

# Random turn non-intersecting walkers




$$\mathcal{A} = \sum_i (p_l f_{i-1} \bar{f}_i + p_r f_{i+1} \bar{f}_i)$$

# Asymmetric Exclusion Process (ASEP)

Other relations to integrable systems: **Bethe ansatz solution of ASEP**, using equivalence with integrable spin models


- K-H Gwa and H. Spohn
  "Six-vertex model, roughened surfaces and an asymmetric spin hamiltonian", *Phys. Rev. Let.* **68**, 725 - 728 (1992); "Bethe solution for the dynamical-scaling exponent of the noisy Burgers equation", *Phys. Rev.* **A 46**, 844-854 (1992)
- G.M. Schütz
  "Exact solution of the master equation for the asymmetric exclusion process", *J. Stat. Physics* **88**, 427-445 (1997)
- C. Tracy and H. Widom
  "Integral formulas for the simple asymmetric exclusion process", ArXiv0704.2633v2 [math.PR] (2007)

## Bethe ansatz solution of ASEP

**Continuous time limit (ASEP)**, for a finite number of particles.
Particle positions: $X = (x_1, \dots x_n), \quad Y = (y_1, \dots y_n)$

Let $u_Y(X;t) :=$ probability of being in state $X$ at time $t$, given they are in state $Y$ at $t = 0$.

**Master equation**:

$$\frac{du_Y}{dt} = \sum_{i=1}^{n} \left(p_r u_Y(X_i^-;t) + p_l u_Y(X_i^+;t) - u(X;t)\right)$$
$$X_i^{\pm} := (x_1, \dots x_{i-1}, x_i \pm 1, \dots, x_n)$$

**Initial** and **boundary** conditions:

$$\begin{aligned} u_Y(X;0) &= \delta_{X,Y} \\ u_Y((x_1, \dots x_i, x_i+1, \dots x_n);t) &= p_r u_Y((x_1, \dots, x_i, x_i, \dots x_n);t) \\ &\quad + p_l u_Y((x_1, \dots, x_i+1, x_i+1, \dots x_n);t) \end{aligned}$$

# Bethe ansatz solution

## Integral formula for transition probabilities

(Tracy, Widom (2007))

$$u_Y(X;t) = \sum_{\sigma \in S_n} \left(\frac{1}{2\pi i}\right)^n \prod_{j=1}^{n} \oint_{\xi_j=0} A_\sigma \xi_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} e^{\sum_{j=1}^{n} \epsilon(\xi_j) t} d\xi_j$$

where

$$A_\sigma := \prod_{\text{inversions}(\alpha,\beta)\subset\sigma} \left(-\frac{p_r + p_l \xi_\alpha \xi_\beta - \xi_\alpha}{p_r + p_l \xi_\alpha \xi_\beta - \xi_\beta}\right)$$

$$\epsilon(\xi) := p\xi^{-1} + q\xi - 1$$

# Further relations of random processes to integrable systems

## Tau function as partition function for Fermion statistical ensembles


- J. Harnad and A. Yu. Orlov, “Fermionic construction of tau functions and random processes”, *Theor. Math. Phys.* (in press, 2007) arXiv:0704.1157


## Tau functions as weights on 2-D partitions (path space weight for 1-D partitions)


- A. Okounkov and N. Reshetikhin, “Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram ”*J. Amer. Math. Soc.* **16** 581-603 (2003); “Random skew plane partitions and the Pearcey process ” *Commun. Math. Phys.* **269**, (2007)


## Asymptotics of random partitions, growth problems, limiting shapes


- A. Borodin and G. Olshanski, “Z-measures on partitions and their scaling limits ” *Eyr. J. Comb.* **26**, 795-834 (2005); “Random partitions and the gamma kernel” *Adv. Math.* **194** , 141-202 (2005)
- R. Kenyon, A. Okounkov and S. Sheffield, “Dimers and amoebae ” *Ann. Math.* **163** , 1019-1056 (2006)