$$\tfrac{1}{2}\langle\Sigma_3\rangle+(\sqrt{3}/2)\langle\Sigma_1\Xi_1\rangle=1,\quad \tfrac{1}{2}\langle\Xi_3\rangle-(\sqrt{3}/2)\langle\Sigma_2\Xi_2\rangle=1, \tag{75}$$
$$\tfrac{1}{2}\langle\Sigma_3\rangle-(\sqrt{3}/2)\langle\Sigma_2\Xi_2\rangle=1,\quad \tfrac{1}{2}\langle\Xi_3\rangle+(\sqrt{3}/2)\langle\Sigma_1\Xi_1\rangle=1.$$

From the solution of these equations we find that $\langle\Xi_3\rangle$ is $\frac{1}{2}$. Therefore

$$\langle DE\rangle=\frac{3}{40}=\frac{7.5}{100}. \tag{76}$$

There is a 7.5% probability that d and e are both 1.

[1]Einstein's letter to Born, 4 December 1926, in *The Born–Einstein Letters*, translated by Irene Born (Walker, New York, 1971).

[2]N. Bohr, "Discussion with Einstein on epistemological problems in atomic physics," in *Albert Einstein: Philosopher-Scientist*, edited by P. A. Schilpp (The Library of Living Philosophers, Evanston, Illinois, 1949), p. 199.

[3]A. Einstein, B. Podolsky, and N. Rosen, "Can quantum mechanical description of physical reality be considered complete?" Phys. Rev. **47**, 777–780 (1935).

[4]Einstein's letter to Born, 3 March 1947, in *The Born–Einstein Letters*, Ref. 1.

[5]D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, 1951), pp. 614–623.

[6]J. S. Bell, "On the Einstein–Podolsky–Rosen paradox," Physics **1**, 195–200 (1964), reprinted in J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University, Cambridge, 1987); J. F. Clauser and A. Shimony, "Bell's theorem: Experimental tests and implications," Rep. Prog. Phys. **41**, 1881–1927 (1978).

[7]S. J. Freedman and J. S. Clauser, "Experimental test of local hidden-variable theories," Phys. Rev. Lett. **28**, 938–941 (1972); M. Lamehi-Rachti and W. Mittig, "Quantum mechanics and hidden variables: A test of Bell's inequality by the measurement of the spin correlation in low-energy proton–proton scattering," Phys. Rev. D **14**, 2543–2555 (1976); the early experiments are reviewed by J. F. Clauser and A. Shimony, Ref. 6; A. Aspect, P. Grangier, and G. Roger, "Experimental realization of Einstein–Podolsky–Rosen–Bohm Gedankenexperiment: A new violation of Bell's inequalities," Phys. Rev. Lett. **49**, 91–94 (1982); A. Aspect, J. Dalibard, and G. Roger, "Experimental tests of Bell's inequalities using time-varying analyzers," *ibid.* **49**, 1804–1807 (1982); W. Perrie, A. J. Duncan, H. J. Beyer, and H. Kleinpoppen, "Polarization correlation of the two photons emitted by metastable atomic deuterium: A test of Bell's inequality," *ibid.* **54**, 1790–1793 (1985); Z. Y. Ou and L. Mandel, "Violation of Bell's inequality and classical probability in a two-photon correlation experiment," *ibid.* **61**, 50–53 (1988); T. E. Kiess, Y. H. Shih, A. V. Sergienko, and C. O. Alley, "Einstein–Podolsky–Rosen–Bohm experiment using pairs of light quanta produced by type-II parametric down-conversion," *ibid.* **71**, 3893–3897 (1993).

[8]D. M. Greenberger, M. Horne, and A. Zeilinger, "Going beyond Bell's theorem," in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, 1989), pp. 69–72; D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, "Bell's theorem without inequalities," Am. J. Phys. **58**, 1131–1143 (1990).

[9]P. Heywood and M. L. G. Redhead, "Nonlocality and the Kochen–Specker paradox," Found. Phys. **13**, 481–499 (1983); H. R. Brown and G. Svetlichny, "Nonlocality and Gleason's lemma. Part I. Deterministic theories," *ibid.* **20**, 1379–1387 (1990); L. Hardy, "Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories," Phys. Rev. Lett. **68**, 2981–2984 (1992); "A quantum optical experiment to test local realism," Phys. Lett. A **167**, 17–23 (1992); B. Yurke and D. Stoler, "Using the Pauli exclusion principle to exhibit local-realism violations in overlapping interferometers," Phys. Rev. A **47**, 1704–1707 (1993); R. Clifton and P. Niemann, "Locality, Lorentz invariance, and linear algebra: Hardy's theorem for two entangled spin-s particles," Phys. Lett. A. **166**, 177–184 (1992).

[10]L. Hardy, "Nonlocality for two particles without inequalities for almost all entangled states," Phys. Rev. Lett. **71**, 1665–1668 (1993).

[11]N. D. Mermin, "Bringing home the atomic world: Quantum mysteries for anybody," Am. J. Phys. **49**, 940–943 (1981).

[12]N. D. Mermin, "Quantum mysteries revisited," Am. J. Phys. **58**, 731–734 (1990); "What's wrong with these elements of reality?" Phys. Today **43** (6), 9–11 (1990).

[13]T. F. Jordan, *Quantum Mechanics in Simple Matrix Form* (Wiley, New York, 1986).

[14]T. F. Jordan, "Testing Einstein–Podolsky–Rosen assumptions without inequalities with two photons or particles with spin 1/2," Phys. Rev. A (submitted).

Quantum mysteries refined

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A gedanken experiment discovered by Lucien Hardy is translated into a very direct black-box gedanken demonstration of quantum nonlocality with red and green lights, using only two far apart detectors each of which operates in only two modes. The quantum mechanical underpinnings of the gedanken demonstration are quite simple. (This paper provides a text for the Klopsteg Memorial Lecture to the American Association of Physics Teachers at Notre Dame University, August 11, 1994.)

Some time ago I described in these pages a nontechnical gedanken demonstration of Bell's theorem using two far apart black boxes, each with a switch that could be set in one of three positions.[1] The boxes acted as detectors for a pair of particles emanating from a distant source. Each box responded to its particle by flashing a red or green light in a manner that a little analysis revealed to be entirely mysterious in view of the absence of connections between the boxes or between the boxes and the source. More recently I described a modification of the gedanken demonstration that worked on the same nontechnical level, embodying a variation on the Greenberger–Horne–Zeilinger (GHZ) version of Bell's theorem to make the point more directly and dramatically.[2] The new gedanken demonstration used three (one more) far-apart black boxes with red and green lights, each of which had only two (one less) switch settings.

Now Lucien Hardy[3] has discovered a form of Bell's theorem that leads to a gedanken demonstration that works with a more economical apparatus than either of these, and is even easier to describe and analyze. Only *two* boxes with red and green lights are required (as in the first gedanken demonstration), each box has only *two* switch settings (as in the second), and the mysterious character of the data is virtually self-evident.

Hardy and subsequently Henry Stapp,[4] Sheldon Goldstein,[5] and Thomas Jordan[6] have stated this new argument so lucidly that its conversion into the boxes of my gedanken demonstration is quite straightforward, as Jordan has explicitly noted. I carry out the exercise here[7] because the rhetoric that accompanies the Hardy version of the gedanken demonstration—the song and dance that leads you irresistibly down the garden path to the wrong conclusion—is interestingly and instructively different from the patter common to the Bell or GHZ versions. This is because the Einstein–Podolsky–Rosen (EPR) argument, in the form which plays so central a role in the Bell or GHZ versions of my device, cannot be made in the Hardy version. I believe it is the absence of a full blown EPR argument that explains why nobody had noticed so beautiful a simplification in the three decades since Bell's theorem first appeared.

The entire gedanken demonstration, with the complete and surprising analysis of the gedanken data, is contained in my tiny Sec. I. Section II, only slightly longer, gives the full quantum mechanical derivation of the gedanken data, at a level suitable for the early part of an (appropriately designed) introductory course. Section III expands on how the Hardy gedanken demonstration, in contrast to the earlier ones, neither implies nor requires the existence of EPR "elements of reality." And Sec. IV casts doubt on the claim that a real Hardy experiment would provide a better test of quantum nonlocality than existing experiments based on generalizations of Bell's original correlation inequality. Appendix A gives a numerically simple example of a Hardy state and uses it to make an important point about the "spookiness" of the nonlocality Hardy states exhibit. Appendix B gives a more elaborate discussion of the Bell inequality used in Sec. IV.

It is not necessary to be acquainted with Refs. 1 and 2 to follow the exposition of the Hardy gedanken demonstration and its quantum mechanical explanation in Secs. I and II[8] Sections III and IV address slightly more technical issues, but they can also be read without a knowledge of Refs. 1 and 2, though one of my purposes in Sec. III is to contrast the Hardy demonstration with the Bell and GHZ versions. Section IV does not refer to the earlier demonstrations.

I. TWO BLACK BOXES WITH TWO SWITCHES

As promised, my new device (shown in Fig. 1) has only two detectors (instead of the three in the GHZ device of Ref. 2) and each operates in only two modes (instead of the three in the Bell device of Ref. 1). Just as in the earlier gedanken demonstrations, there are no connections between the detectors, which are far apart and far from the source. There are also no connections between source and detectors beyond those mediated by a pair of particles that originate at the source and fly apart to the detectors, the arrival at each detector being signaled by a flash of red or green.

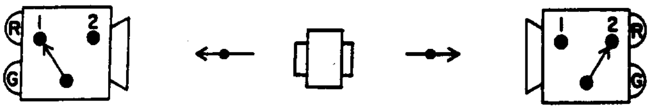

Fig. 1. The Hardy gedanken demonstration. Two particles emerge from a common source heading for two far apart detectors, each of which can operate in two modes, specified by the setting of a switch. When a particle reaches a detector the detector flashes a red or green light. Aside from the passage of the particles from the source to the detectors, there are no connections between the source and the detector or between the detectors.

In each run of the gedanken demonstration one presses a button at the source to send the particles off to the detectors. After the particles have left the source but before they have arrived at and triggered the detectors, one randomly and independently sets the switch on each detector to either of its two positions (labeled 1 and 2) by tossing a coin at each detector. One then waits for the lights to flash and records the color flashed at each detector. As with the earlier Bell device, one summarizes the result of a run by writing down the settings determined by the tosses of the coins, followed by two letters indicating the colors subsequently flashed. Thus $21GR$ describes a run in which the switch on the left detector was set to 2, the switch on the right detector was set to 1, the left detector flashed green, and the right detector flashed red.

The data exhibit the following important features:

(a) In runs in which the detectors end up with different settings, they never both flash green: $21GG$ and $12GG$ never occur.
(b) In runs in which both detectors end up set to 2, one occasionally[9] finds both flashing green: $22GG$ sometimes occurs.
(c) In runs in which both detectors end up set to 1, they never both flash red: $11RR$ never occurs.

As in earlier versions of the device, because there are no direct connections between detectors the explanation for their coordinated behavior can only come from the fact that both are triggered by particles coming from a single source. Something in the common origin of the particles must be responsible for the correlations. Since the switches on the detectors are not set until after the particles have left their source, whatever features of the particles produce these correlations and however they might vary from one run to the next, in any particular run those features cannot be affected by the setting of the switches. Furthermore, since each detector is triggered by only one of the two particles, whatever features it responds to can reside only in that particle and not in the particle that went off to the other faraway detector. In view of this it is very hard to resist the following line of thought, which constitutes the simplest version of Bell's theorem I can imagine:

Since any run might end up as a 12 or a 21 run, whenever one of the particles is of a variety that allows a type 2 detector to flash green, the other particle must be of a variety that requires a type 1 detector to flash red, for otherwise we would see instances of $12GG$ or $21GG$, which never occur. It follows that in any of those occasional 22 runs in which both detectors flash green, both particles must be of a variety that requires a type 1 detector to flash red. So if the random setting of the detectors that resulted in any such 22 run had instead resulted in a 11 run, both detectors would have flashed red. But this is impossible, since $11RR$ is never observed.

Thus the regularities described in (a)–(c) are inconsistent with a very common-sense—indeed, an apparently unavoidable—explanation for them, leaving those correlations profoundly mysterious.

II. THE QUANTUM MECHANICS BEHIND THE DEVICE

So how does this magic trick work? You can pick any two one-particle observables you like for the detectors to measure when their switches are set to 1.[10] Divide the spectrum of each observable into any two sets you desire and rig the detectors so that a flash of red or green indicates which set the measured value was found in. The source produces a pair of particles in a two-particle state $|\Psi\rangle$ constructed as follows.

Pick for the left particle any two states that are superpositions of eigenstates of the observable measured in mode 1 on the left, taken entirely from the red and entirely from the green part of its spectrum. Call the states $|1R\rangle_l$ and $|1G\rangle_l$. Pick two states $|1R\rangle_r$ and $|1G\rangle_r$ for the particle on the right, similarly defined in terms of its own mode-1 observable.[11] Take the two particle state $|\Psi\rangle$ to be a superposition of the three two-particle states:

$$
\begin{aligned}
&|1R,1G\rangle=|1R\rangle_l|1G\rangle_r;\\
&|1G,1R\rangle=|1G\rangle_l|1R\rangle_r;\\
&|1G,1G\rangle=|1G\rangle_l|1G\rangle_r.
\end{aligned}
\tag{1}
$$

Feature (c) of the data is guaranteed by the absence of $|1R,1R\rangle$ from the superposition:

$$|\Psi\rangle=\alpha|1R,1G\rangle+\beta|1G,1R\rangle+\gamma|1G,1G\rangle. \tag{2}$$

The choices of the two observables measured by the detectors in modes 2 are almost as flexible. All we require is that the observable measured on the left have an eigenstate $|2G\rangle_l$ with nonzero components along both $|1G\rangle_l$ and $|1R\rangle_l$. A green flash on the left indicates that the measurement in mode 2 did indeed find the particle in the state $|2G\rangle_l$. Anything else results in a red flash. A similar arrangement is made on the right with a state $|2G\rangle_r$.

Feature (a) requires $|\Psi\rangle$ to be orthogonal to $|1G,2G\rangle$ and $|2G,1G\rangle$:

$$0=\langle 1G,2G|\Psi\rangle=\beta\langle 2G|1R\rangle_r+\gamma\langle 2G|1G\rangle_r, \tag{3}$$

and

$$0=\langle 2G,1G|\Psi\rangle=\alpha\langle 2G|1R\rangle_l+\gamma\langle 2G|1G\rangle_l. \tag{4}$$

Feature (b), apparently incompatible with features (a) and (c), merely requires $|\Psi\rangle$ *not* to be orthogonal to $|2G,2G\rangle$:

$$
\begin{aligned}
0\neq p=|\langle 2G,2G|\Psi\rangle|^2= \,|\, &\alpha\langle 2G|1R\rangle_l\langle 2G|1G\rangle_r\\
&+\beta\langle 2G|1G\rangle_l\langle 2G|1R\rangle_r\\
&+\gamma\langle 2G|1G\rangle_l\langle 2G|1G\rangle_r|^2,
\end{aligned}
\tag{5}
$$

which Eqs. (3) and (4) reduce simply to

$$0\neq p=|\gamma|^2|\langle 2G|1G\rangle_l|^2|\langle 2G|1G\rangle_r|^2. \tag{6}$$

This tells us that the amplitude γ must be nonzero, and Eqs. (3) and (4) then tell us that α and β cannot be zero either. So we can do the trick with any two choices of nontrivial local one-particle observables to be measured in mode 1 on the right and on the left, and any state $|\Psi\rangle$ of the form (2) with three nonzero amplitudes[12] α, β, and γ.

How big can we make the probability p for the "impossible" $22GG$ events? The coefficient $|\gamma|^2$ appearing in Eq. (6) is determined by the normalization $1=\langle\Psi|\Psi\rangle$ together with Eqs. (3) and (4):

$$1=|\gamma|^2\left(1+\frac{|\langle 2G|1G\rangle_l|^2}{|\langle 2G|1R\rangle_l|^2}+\frac{|\langle 2G|1G\rangle_r|^2}{|\langle 2G|1R\rangle_r|^2}\right). \tag{7}$$

Evidently for given $\langle 2G|1G\rangle_l$ and $\langle 2G|1G\rangle_r$, p is biggest when γ is largest, and γ is biggest when the magnitudes of $\langle 2G|1R\rangle_l$ and $\langle 2G|1R\rangle_r$ are as large as possible. So to maximize p we should take the states $|2G\rangle$ on each side to be linear combinations of just the two states $|1G\rangle$ and $|1R\rangle$ on that side. If we define

$$p_l=|\langle 2G|1G\rangle_l|^2,\quad p_r=|\langle 2G|1G\rangle_r|^2, \tag{8}$$

then

$$|\langle 2G|1R\rangle_l|^2=1-p_l,\quad |\langle 2G|1R\rangle_r|^2=1-p_r, \tag{9}$$

and Eqs. (6) and (7) reduce to

$$p=\frac{p_l(1-p_l)p_r(1-p_r)}{1-p_lp_r}. \tag{10}$$

Maximizing Eq. (10) gives uniquely $p_l=p_r=1/\tau$, where τ is the golden mean, $\frac{1}{2}(\sqrt{5}+1)$, and this gives p the maximum value $1/\tau^5=0.090\,17$.

If you are willing to settle for a p that is just a hair less (9% exactly), then by taking $p_l=p_r=\frac{3}{5}$ you can get very simple numerical relations between the relevant eigenstates of observables 1 and 2—helpful in explaining the gedanken demonstration to a freshman or sophomore physics class or even a class of suitably informed nonscientists. These relations are given in Appendix A and used to illustrate the important fact that no data collected entirely at one end of the laboratory can reveal how the switch is set on the detector at the other end.

III. THE ABSENCE OF AN EPR ARGUMENT

A philosopher[13] once asked why I had given the two boxes in Ref. 1 three switches rather than just two, as in many of the experimental tests for violations of the form of Bell's inequality given by Clauser and Horne.[14] My reason was that the gedanken demonstration of Ref. 1 relies on the perfect correlations, for appropriately paired switch settings, of a pair of particles in the singlet state. From these one concludes in the manner of Einstein, Podolsky, and Rosen[15] that the particles must carry to their detectors identical instruction sets, telling them what color to flash for every switch setting. If there are only two settings at each detector, and the colors are always the same when the switches have the same setting, then it is easy to distribute identical instruction sets to the particles that reproduce the singlet state statistics regardless of how the switches end up. When one has a third perfectly correlated pair of switch settings, however, it is possible to find singlet-state statistics that instruction sets cannot reproduce.

But the actual laboratory experiments that explore these peculiar correlations do get away with just two settings at each detector. This is because they do not rely on the EPR argument to make their point. Real experiments cannot rely on the EPR argument because perfect correlations are an idealization beyond experimental reach. Instead, laboratory experiments test certain other assumptions about the joint distributions of colors revealed by the various pairs of switch settings. Although those assumptions, which I discuss in Sec. IV, are certainly plausible enough to be worth testing, they lack the simplicity and intuitive power of the EPR assumptions. And that is why my first gedanken demonstration had three switch settings at each detector, rather than just two.

Yet Hardy has managed to come up with an argument using only two settings at each detector that makes a gedanken demonstration every bit as compellingly mysterious as my three-setting version, and a lot simpler to analyze. How did he do it?

He did it by working with data that are, surprisingly, just as powerful as the EPR data for purposes of the gedanken demonstration even though they are not as strongly correlated. In particular, Hardy's correlations do not permit one to invoke the EPR reality criterion.

According to EPR an "element of reality"—i.e., a preexisting value for an observable or, in the language of my earlier gedanken demonstrations, an entry in an "instruction set" specifying the color to be flashed by a detector—must exist if one can predict in advance the result of a localized measurement of that observable by other localized experiments done far away. In the Hardy experiment the condition of predictability is not invariably met. Sometimes you can predict with certainty the result of a distant experiment but you can only do it in an uncontrollable fraction of the runs. If, for example, you try to learn what the particle on the left will do at a type 1 detector by measurements on the faraway particle on the right, you run into a problem. If you subject the particle on the right to a type 1 detector, then if that detector flashes red you can indeed predict with certainty that the particle on the left will cause its detector to flash green, since $11RR$ is never observed. But if the detector on the right flashes green when both switches are set to 1, you are in no position to predict what will happen on the left. Since the outcome on the right is not under your control, your ability to predict with certainty the behavior on the left is a matter of chance. If you're lucky you can do it; otherwise you cannot. The same difficulty arises if instead you subject the particle on the right to a type 2 detector. If it flashes green then you can predict red on the left with certainty, since $12GG$ never occurs, but if it flashes red then you are stumped. Similar problems arise if you want to predict what happens on the left at a type 2 detector, or what happens at either type of detector on the right.

In contrast to the two earlier versions of my gedanken demonstration, the data no longer demand that every particle must always carry a set of instructions telling its detector what color to flash for each setting of its switch. In keeping with this, the brief argument in the final paragraph of Sec. I makes no explicit reference to instruction sets, talking instead about particles that "allow" and particles that "require." While particles that "require" presumably carry instructions, particles that "allow" need not.[16]

To emphasize the difference between the Hardy version of the device and its predecessors we can make the argument clumsier by trying to reformulate it in an appropriately generalized language of instruction sets. In addition to instructions R, "flash red," and G, "flash green," we must now allow a third designation, N, signifying that the particle carries no instruction to a detector specifying what color it must flash. We can then examine when the data require a particle to carry to its detector instructions R or G and when they allow it to carry the noncommittal noninstruction N.

To account for the correlations for all four pairs of switch settings we suppose that each particle carries a generalized instruction set XY, where X and Y, which can each be either R, G, or N, specify the behavior at type 1 (X) and type 2 (Y) detectors. We must admit the additional noninstruction N because, as noted, in the Hardy experiment we do not have the kinds of correlations that in conjunction with the EPR argument require every particle to specify the color to be flashed at either type of detector. In the EPR situations of my earlier gedanken demonstrations N cannot appear in an instruction set because a particle carrying the noninstruction N for a given type of detector cannot manage invariably to flash the same color as its partner flashes at a detector of the same type.

But although the Hardy experiment allows noninstructions N, it does not permit very many. Suppose, for example, a particle has the noninstruction N for a type 1 detector. Because this allows its type 1 detector to flash red, its partner must have the instruction G for a type 1 detector, since $11RR$ never occurs. This in turn requires the original particle to have the instruction R for a type 2 detector, because GG never occurs when the switches have different settings. For this same reason, since a type 1 detector can flash green in response to a particle carrying the noninstruction N, its partner must have the instruction R for a type 2 detector. We conclude that a particle can carry the noninstruction N for a type 1 detector provided the full instruction set it carries is NR and provided its partner carries the instruction set GR. A similar line of thought establishes that a particle can carry the noninstruction N for a type 2 detector, provided the full instruction set it carries is GN and its partner carries the instruction set RR. In neither case is the flexibility introduced by a noninstruction N enough to permit a $22GG$ run. Whether the N is associated with a type 1 or type 2 detector, at least one of the particles must carry an R for a type 2 detector. Since runs without any noninstructions N are even less flexible, there is never any possibility of a $22GG$ run.

From the point of view of the earlier gedanken demonstrations, the surprising thing about the Hardy argument is that even though the correlations are not strong enough to support an EPR argument, those correlations that continue to be perfect—the strict absence of any $11RR$, $12GG$, or $21GG$ runs—suffice to prohibit $22GG$ runs with the full force of the earlier arguments. And the analysis is now so wonderfully concise that it can be stated in a few short sentences without explicitly invoking instruction sets at all.

IV. INEQUALITIES FOR REAL EXPERIMENTS

Hardy's analysis is an example of what Greenberger, Horne, and Zeilinger call a "Bell's theorem without inequalities." In both the Hardy and GHZ argument one deduces from strong correlations in the outcomes of experiments in distant places that a new correlation experiment cannot have a certain outcome. One then discovers that the new experiment sometimes (and in the case of GHZ always) does have that forbidden outcome. These arguments refute the implicit hypotheses underlying the deduction more directly than the earlier forms of Bell's theorem. By resting the analysis entirely on events of probability 1 or 0, they also reduce the room for interpretive maneuver in assessing what went wrong with the deduction.

Hardy[3] has suggested that real versions of gedanken experiments involving all or nothing probabilities should provide more clear-cut laboratory demonstrations of quantum nonlocality since in the present case, for example, the bizarre behavior can be established by a single observation of $22GG$. This is questionable. The problem is this:

The fact that $11RR$, $21GG$, and $12GG$ are never observed is a deduction from theory—a consequence of the quantum mechanical structure of the state $|\Psi\rangle$. If the theory is correct then a single observation of $22GG$ refutes the apparently unavoidable explanation of those three absolute absences in terms of information carried by a particle to its box. But without prior assurance that $11RR$, $21GG$, and $12GG$ never happen, an observation of $22GG$ no longer has momentous consequences. It would, of course, be revolutionary if quantum mechanics were incorrect in so fundamental and elementary an application as its prohibition of these three outcomes. But if we are willing to accept without laboratory tests the theoretical predictions for the $11RR$, $21GG$, and $12GG$ correlations, then why not accept them for the $22GG$ correlations as well and settle for the gedanken experiment, which makes a powerful conceptual point without the bother, expense, and inevitable ambiguity of any real laboratory investigation?

Are there compelling reasons to take the trouble to perform a real version of the gedanken experiment beyond the pleasure derived from performing a superb magic trick? While it would be startling indeed if quantum mechanics turned out to be wrong in so elementary and straightforward an application, even granting its validity one could harbor worries that some hitherto overlooked interactions were responsible for disrupting the predicted correlations as the particles flew apart to their detectors, preventing them from attaining the strong forms necessary to produce so distressingly unintuitive a body of data. A friend of mine once maintained that anything—a passing cosmic ray in the next room—would suffice to disrupt the delicate correlations of an EPR state. One can demonstrate that cosmic rays are not up to that job, but perhaps something we know nothing about comes into play in such situations, weakening these inexplicable correlations in a perfectly respectable quantum mechanical manner, through hitherto unknown interactions. Perhaps these theoretically predicted correlations should be viewed not as a gedanken demonstration of how strange the world can be, but as establishing constraints on which of the states described by quantum mechanics we are actually able to achieve in real world applications with stuff that the world actually makes available to us.

If the experiment is performed to test whether something we haven't yet been clever enough to notice comes into play and disrupts the correlations between the particles, then it would be ridiculous to assume that as the particles separated such unknown interactions with hitherto undetected entities disrupted only the 22 correlations. Any sensible experimental test of whether nature manages to spare us the intellectual distress of Sec. I ought surely to probe all the ingredients of that distress: both the existence of $22GG$ events *and* the *non*existence of $11RR$, $21GG$, and $12GG$ events. But even with perfect detectors one can only establish that the probability of an event is of the order of or less than the inverse of the number of times it fails to happen.[17] With the imperfect detectors available for real experiments the ambiguities in ruling out $11RR$, $21GG$, and $12GG$ events are greater still. But what becomes of our argument if we can deduce from the data only that $11RR$, $21GG$, and $12GG$ events are highly improbable? One feels that under such circumstances $22GG$ should also be highly improbable. Can we arrive at some relations between these improbabilities as apparently compelling as our conclusion that impossibilities implied impossibility? We might reason something like this:

Let $p(11RR)$, $p(21GG)$, $p(12GG)$, and $p(22GG)$ be the probabilities of getting those particular colors in those particular runs. Consider only the runs in which the tosses of the coins resulted in both detectors operating in mode 2. Since those coin tosses selected an entirely random quarter of all the runs, it is surely safe to assume that had those tosses resulted in other than 22 runs, the frequencies of colors flashed in those runs would have been given by the appropriate probabilities $p(ijXY)$. Now consider that fraction $p(22GG)$ of the 22 runs in which both lights flashed green. What might have happened in any particular such run if the switches had ended up not 22, but 21, 12, or 11? Under the plausible assumption that an alteration of the switch on one side would not have affected the outcome on the other side, we could have had $12XG$, $21GY$, or $11VW$. Under the (perhaps somewhat less compelling) assumption that the result we would have had for a setting 1 on one side also should not depend on the setting on the other side, we can identify V with X and W with Y, replacing $11VW$ with $11XY$. But if we do this, then regardless of whether XY is RR, RG, GR, or GG, at least one of the three possibilities $12XG$, $21GY$, or $11XY$ will be one of the improbable ones $12GG$, $21GG$, or $11RR$. Thus the combined frequencies of GG among 12 runs, GG among 21 runs, and RR among 11 runs must be at least as great as the frequency of GG among 22 runs:

$$p(22GG) \leq p(21GG) + p(11RR) + p(12GG). \tag{11}$$

The relevance of an inequality like this to the case of imperfect detectors was first noted by Clauser and Horne.[14] All subsequent experiments have tested appropriate forms of this "Bell–CH" inequality. The derivation I have given is my cartoon version of an argument that Henry Stapp[18] has been refining over many years, with the aim of extracting nonlocality directly from the theoretical predictions of quantum mechanics without appealing to possible hidden variables or to EPR elements of reality. Jon Jarrett[19] has given an analysis of the assumptions underlying Eq. (11) that many people have found more congenial. In Appendix B I sketch Jarrett's argument in the context of this gedanken demonstration.

Violating the Bell–CH inequality is all that a real experiment with a finite number of runs and/or imperfect detectors can accomplish. The existing experimental tests of that inequality exploit choices of the observables 1 and 2 on the left and right of Eq. (11) that lead to theoretical probabilities

$$p(22GG) = \tfrac{1}{8}(2+\sqrt{2}) = 0.427, \tag{12}$$

$$p(21GG) + p(11RR) + p(12GG) = \tfrac{3}{8}(2-\sqrt{2}) = 0.220, \tag{13}$$

leaving a comfortable margin for experimental error. The corresponding numbers for the strongest version of Hardy's experiment are

$$p(22GG) = \left(\frac{2}{1+\sqrt{5}}\right)^5 = 0.090, \tag{14}$$

$$p(21GG) + p(11RR) + p(12GG) = 0. \tag{15}$$

Thus any experiment using real detectors will require better instruments to confirm the violation of Eq. (11) in a Hardy state than are needed to establish a violation with the data (12) and (13) predict for the existing tests.

The importance of the Hardy states is not that they provide the basis for a more definitive experimental test. Even if such a test were accurate enough to establish a violation of the Bell–CH inequality in a Hardy state it would, like the other experimental tests, only have implications for the rather subtle assumptions underlying that inequality in either the Stapp or the Jarrett versions of its derivation. It is certainly a matter of great interest to establish that those assumptions are in conflict with experiment. But the additional importance and the great beauty of the Hardy experiment lies in what it tells us, as an extremely simple and direct deduction from elementary quantum theory, about the world that theory describes in the absence of hypothetical and hitherto undetected correlation destroying interactions. The argument leading to the *vanishing* of $p(22GG)$ from the *vanishing* of $p(21GG)$, $p(11RR)$, and $p(12GG)$, as given in Sec. I, is both simpler and more compelling than the arguments that underly the derivations of the Bell–CH inequality (11) when those probabilities are not zero. The brisk gedanken refutation of those hypotheses stands in its pristine simplicity as one of the strangest and most beautiful gems yet to be found in the extraordinary soil of quantum mechanics.

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APPENDIX A: A NUMERICALLY SIMPLE EXAMPLE

As noted at the end of Sec. II, taking $p_l=p_r=\frac{3}{5}$ gives a probability $p(22GG)$ of 9%, only a shade less than the maximum possible. This choice is convenient for illustrating at an elementary level another important and very general feature of the Hardy states (possessed by any such superposition of orthogonal product states). A Hardy state (2) with $p(22GG)=9\%$ is given by

$$|\Psi\rangle=\sqrt{\tfrac{3}{8}}|1R\rangle_l|1G\rangle_r+\sqrt{\tfrac{3}{8}}|1G\rangle_l|1R\rangle_r-\tfrac{1}{2}|1G\rangle_l|1G\rangle_r, \tag{A1}$$

with the eigenstates of observable 2 defined in terms of those of observable 1 (for either particle) by

$$|2G\rangle=\sqrt{\tfrac{3}{5}}|1G\rangle+\sqrt{\tfrac{2}{5}}|1R\rangle,$$

$$|2R\rangle=-\sqrt{\tfrac{2}{5}}|1G\rangle+\sqrt{\tfrac{3}{5}}|1R\rangle. \tag{A2}$$

It is an elementary exercise[20] to calculate that the probabilities of all the outcomes for each of the four possible settings of the detectors are as given in the following table:

	GG	RR	RG	GR
12	0	0.15	0.225	0.625
21	0	0.15	0.625	0.225
22	0.09	0.64	0.135	0.135
11	0.25	0	0.375	0.375

These probabilities do indeed have properties (a)–(c) of Sec. I.

With this table one can make the important point that although the data in a Hardy state strongly suggest Einstein's "spooky actions at a distance," the spookiness only emerges when one tries to explain the correlations between outcomes at both ends of the laboratory. One cannot change the behavior at any given end by altering the only thing that one can actually control at the other end—the setting of the faraway detector. The probability, for example, of getting red on the left in a 21 run is $p(21RR)+p(21RG)=0.15+0.625=0.775$; the probability of red on the left in a 22 run, on the other hand, is $p(22RR)+p(22RG)=0.64+0.135=0.775$—i.e., exactly the same.

APPENDIX B: JARRETT'S JUSTIFICATION OF THE BELL–CH INEQUALITY

I give here a summary of the justification for the inequality (11) that follows from the analysis of Jarrett,[19] which is rather different from the informal justification I offered in Sec. IV. One introduces a parameter λ to stand for everything stemming from the common origin of the two particles that might account for their correlated behavior at the two distant detectors. One expresses the probability distributions $p(ijXY)$ as a suitably weighted average of the distributions $p_\lambda(ijXY)$ associated with the various possible conditions λ prevailing at the source:

$$p(ijXY)=\langle p_\lambda(ijXY)\rangle, \tag{B1}$$

where the angular brackets $\langle\ \rangle$ denote such an average over λ.

It is an elementary probabilistic identity that

$$p_\lambda(ijXY)=p_\lambda^l(ijX|Y)p_\lambda^r(ijY), \tag{B2}$$

where $p_\lambda^l(ijX|Y)$ is the probability (for given λ,i,j) of getting X on the left in just those runs in which the result on the right was Y, and $p_\lambda^r(ijY)$ is the probability (for given λ,i,j) of getting Y on the right regardless of what happens on the left. Jarrett distills everything into two assumptions. The first, which he calls "completeness",[21] is that $p_\lambda^l(ijX|Y)$ does not depend on Y:

$$p_\lambda^l(ijX|Y)=p_\lambda^l(ijX). \tag{B3}$$

This is a formal statement of the intuition that nothing beyond what was available to the particles at their common source can be needed to account for their correlated behavior. The assumption (B3) states that it ought to be possible to specify (through λ) the features the particles acquired at their source in enough detail that no further information about the subsequent behavior of the particle on the right beyond what a knowledge of λ implies can alter our expectations for what happens on the left. I elaborate no further than that since my point here is only that Jarrett's assumptions are more subtle than those behind the argument of Sec. I.

Jarrett's second assumption, which he calls "locality",[22] is that for no value of λ can the distribution of results at one detector depend on the setting of the other:

$$p_\lambda^l(ijX)=p_\lambda^l(iX), \quad p_\lambda^r(ijY)=p_\lambda^r(jY). \tag{B4}$$

While a failure of this assumption for the distributions averaged over λ would imply the ability to signal instantaneously at a distance,[23] imposing it for each value of λ individually is a strong requirement, since instantaneous signalling might still be impossible if λ were inherently uncontrollable.

Using the assumptions (B3) and (B4) of completeness and locality to simplify the identity (B2), the general form (B1) of the distributions $p(ijXY)$ reduces to

$$p(ijXY)=\langle p_\lambda^l(iX)p_\lambda^r(jY)\rangle. \tag{B5}$$

Given this, the Bell–CHSH inequality (11) follows directly:[24]

We have

$$p(22GG)=\langle p_\lambda^l(2G)p_\lambda^r(2G)\rangle$$
$$=\sum_{XY}\langle p_\lambda^l(2G)p_\lambda^l(1X)p_\lambda^r(1Y)p_\lambda^r(2G)\rangle, \tag{B6}$$

since

$$1=\sum_X p_\lambda^l(1X)=\sum_Y p_\lambda^l(1Y). \tag{B7}$$

But because of the normalization conditions

$$1=\sum_X p_\lambda^l(2X)=\sum_Y p_\lambda^r(2Y), \tag{B8}$$

and the non-negativity of p_λ^r and p_λ^l, the terms appearing on the right of Eq. (B6) can be bounded as follows:

$$\langle p_\lambda^l(2G)p_\lambda^l(1R)p_\lambda^r(1R)p_\lambda^r(2G)\rangle\leq\langle p_\lambda^l(1R)p_\lambda^r(1R)\rangle=p(11RR); \tag{B9}$$

$$\langle p_\lambda^l(2G)p_\lambda^l(1G)p_\lambda^r(1R)p_\lambda^r(2G)\rangle\leq\langle p_\lambda^l(1G)p_\lambda^r(2G)\rangle=p(12GG); \tag{B10}$$

and

$$\langle p_\lambda^l(2G)p_\lambda^l(1R)p_\lambda^r(1G)p_\lambda^r(2G)\rangle+\langle p_\lambda^l(2G)p_\lambda^l(1G)p_\lambda^r(1G)p_\lambda^r(2G)\rangle$$
$$=\langle p_\lambda^l(2G)p_\lambda^r(1G)p_\lambda^r(2G)\rangle\leq\langle p_\lambda^l(2G)p_\lambda^r(1G)\rangle=p(21GG). \tag{B11}$$

Equations (B6), (B9), (B10), and (B11) give Eq. (11).

[1]N. David Mermin, "Bringing home the atomic world: Quantum mysteries for anybody," Am. J. Phys. **49**, 940–943 (1981).

[2]N. David Mermin, "Quantum mysteries revisited," Am. J. Phys. **58**, 731–734 (1990).

[3]Lucien Hardy, "Non-locality for two particles without inequalities for almost all entangled states," Phys. Rev. Lett. **71**, 1665–1668 (1993). Hardy gives an earlier version of his gedanken experiment, in "Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories," Phys. Rev. Lett. **68**, 2981–2984 (1992).

[4]Henry P. Stapp, Section VII of "Noise induced reduction of wave packets and faster-than-light influences," Phys. Rev. A **46**, 6860–6868 (1992). Stapp also gives a very elegant formulation of Hardy's gedanken experiment in *Mind, Matter, and Quantum Mechanics* (Springer, New York, 1993), pp. 5–9.

[5]Sheldon Goldstein, "Nonlocality without inequalities for almost all entangled states for two particles," Phys. Rev. Lett. **72**, 1951 (1994).

[6]Thomas F. Jordan, "Testing Einstein–Podolsky–Rosen assumptions without inequalities with two photons or particles with spin $\frac{1}{2}$," Phys. Rev. A (submitted); "Quantum mysteries explored," Am. J. Phys. **62**, 874–880 (1994).

[7]I give a version of journalistic brevity without the red and green lights in "What's wrong with this temptation?," Physics Today, June 1994, pp. 9–11.

[8]I have not, however, reproduced here the general talk in Ref. 1 that applies equally well to the Hardy experiment, elaborating on the lack of connections between the detectors and describing the evidence that they are indeed triggered by two things, conveniently called "particles," that emanate from the source

[9]More precisely a little more than 9% of the time, or, even more precisely, as we shall see in Sec. II, a fraction $1/\tau^5$ of the time where τ is the golden mean, $\frac{1}{2}(\sqrt{5}+1)$.

[10]I exaggerate only slightly. Each observable must of course be measurable by a localized detector, and you cannot pick either to be the trivial observable that has just a single number in its spectrum.

[11]The two one-particle observables you pick to measure in mode 1 on the right and the left can be quite different if you wish.

[12]Note that the singlet state of two spin-$\frac{1}{2}$ particles has the form (2) with only two nonzero coefficients. The failure of the Hardy argument to work in the singlet state is a consequence of the fact, noted in Sec. III, that the singlet state supports an EPR argument that cannot be refuted with detectors that operate in only two modes.

[13]Arthur Fine (private communication).

[14]J. F. Clauser and M. A. Horne, "Experimental consequences of objective local theories," Phys. Rev. D **10**, 526–535 (1974).

[15]A. Einstein, B. Podolsky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?," Phys. Rev. **47**, 777–780 (1935).

[16]That the Hardy gedanken experiment is not of the EPR type becomes evident if you try to present it as an investigation of whether particles can possess certain properties, as I have done with the GHZ experiment in "The (non)world (non)view of quantum mechanics," New Literary History **23**, 855–875 (1992). [Ignore the (unnecessary) caption of Fig. 7, or see the Erratum in **24**, 947 (1993).] You can also present the Bell gedanken demonstration along those lines, but the Hardy demonstration does not work in that vein precisely because one cannot appeal to the kinds of intuitions about physical properties on which the EPR reality criterion rests.

[17]If I leave myself open to Bayesian criticism with this remark, I would be delighted to hear about it.

[18]See, for example, Henry Stapp, "Bell's theorem and the foundations of quantum physics," Am. J. Phys. **53**, 306–317 (1985). My own most determined effort to make Stapp's arguments intuitive can be found in "Can you help your team tonight by watching on TV?," in *Philosophical Consequences of Quantum Theory: Reflections on Bell's Theorem*, edited by James T. Cushing and Ernan McMullin (University of Notre Dame, South Bend, 1989), pp. 38–49; reprinted as Chap. 11 of N. David Mermin, *Boojums all the Way Through* (Cambridge University, New York, 1990), pp. 95–100.

[19]A particularly clear exposition of Jarrett's argument has been given by L. Ballentine and J. Jarrett, "Bell's theorem: Does quantum mechanics contradict relativity?" Am. J. Phys. **55**, 696–701 (1987).

[20]Your students need not even have learned about bra vectors or inner products. They need only solve Eq. (17) to express the single-particle 1 states in terms of the 2 states, make the appropriate substitutions into Eq. (16), and read off the squared coefficients.

[21]Abner Shimony calls it "outcome independence." I prefer Jarrett's terminology.

[22]Shimony calls it "parameter independence." I again prefer Jarrett's terminology.

[23]The last paragraph of Appendix A illustrates the fact that when averaged over λ Eq. (B4) does indeed hold in a Hardy state.

[24]A more graceful but more subtle (though entirely correct) route from Eq. (B5) to Eq. (11) consists of simply noting that $\langle p^l_\lambda(2X')p^l_\lambda(1X)p^r_\lambda(1Y)p^r_\lambda(2Y')\rangle$ can be interpreted as a distribution for an ensemble of pairs of particles in which each member of the pair has a specified outcome (R or G) for each of the switch settings (1 or 2) it might encounter, and in which the marginal distributions that describe each of the four sets of experiments one might actually perform (11, 12, 21, or 22) agree with the experimental distributions. If the experimental distributions can indeed be simulated by such an ensemble, then my derivation of Eq. (11) in Sec. IV in the manner of Stapp is indisputably valid, whether or not that ensemble makes any physical sense.

General relativity before special relativity: An unconventional overview of relativity theory

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It is suggested how Bernhard Riemann might have discovered General Relativity soon after 1854 and how today's undergraduate students can be given a glimpse of this before, or independently of, their study of Special Relativity. At the same time, the whole field of relativity theory is briefly surveyed from the space–time point of view.

I. INTRODUCTION

Historically, Einstein's General Relativity of 1915—the theory of curved spacetime—arose as a generalization of his Special Relativity of 1905—the theory of flat spacetime—much as the geometry of curved surfaces arises as a generalization of the Euclidean geometry of the plane. This historical sequence from the special to the general theory is followed in every presentation of the subject known to me. And for good reason: in this way the required level of mathematical sophistication rises only gradually, whereas the inverse sequence would seem to require some heavy mathematics up front. However, it is amusing and instructive to fantasize how, in the best of all possible worlds, General Relativity might have been developed *ab initio* long before 1905, for example by Bernhard Riemann soon after 1854, and how it could then have led to Special Relativity. At the same time, a mathematically diluted version of such a development can prove to be of interest to bright undergraduate students. It gives them a quick and direct taste of spacetime and of General Relativity, two topics which are often promised them "at the end of Special Relativity," but which only too often are never quite reached. This sequence also well illuminates the inner logic and self-sufficiency of General Relativity.

The following is a sample of such a development, which, with suitable omissions, can be presented to students in an hour's lecture.

II. HOW THEORIES ORIGINATE

New theories are as a rule not developed for sport. Rather, they arise in response to difficulties, paradoxes, or puzzles in the older theories. Thus Special Relativity grew out of difficulties in reconciling Maxwell's theory with Newtonian kinematics, and, in spite of Einstein's well-known disclaimer, it could hardly have come into being without the acute paradox of the Michelson–Morley experiment of 1887. This experiment showed that, no matter how fast you chase a light signal, you can never reduce its speed relative to you. General Relativity, on the other hand, has its roots in the much older mechanics of Newton. But Newton's theory, too, is by no means free of puzzles. Above all, it has long been criticized for its reliance, if not necessarily on absolute space, on the set of global inertial frames whose absoluteness ("they act but cannot be acted on") so offended the scientific sensitivities of Mach and Einstein. And then there is the mystery of the equality of gravitational and inertial mass, appearing simply as a postulate in Newton's theory. Why should a quantity measuring a body's inertia or resistance to acceleration act at the same time as its "gravitational charge?" It would seem that these two puzzles alone (and there were others) could drive a man to search for a new theory, i.e., a new mathematical model, especially when a new and suitable mathematical avenue had just opened up. The man might have been Riemann, and the avenue his newly discovered differential geometry of (irregularly) curved spaces of higher dimensions.

III. GAUSS' GEOMETRY OF SURFACES

The year 1854 was a memorable one in the annals of the famous old German university town of Göttingen. The recently developed railroad had finally reached the town. And also, though unbeknown to most of its good burghers, the