Physics and Mathematics of Link Homology
Proposal for the Séminaire de Mathématiques Supérieures 2013
June 24–July 5, 2013

Organizers:
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1 Executive Summary

A two-week summer school on link homology is proposed to be held in June/July 2013 at Montréal's Centre de Recherches Mathématiques, as part of the series "Séminaire de Mathématiques Supérieures" (SMS). This will be just the right time to introduce a new generation of researchers to this growing field at the interface of mathematics with physics. Participants of the school will obtain a broad and unified view of the subject of link homology, and be prepared and stimulated to pursue original research in the field. Secured funding represents more than half of the $80,000 necessary to support 12–15 lecture sequences for around 60 students. Confirmed lecturers include Mina Aganagic, Dror Bar-Natan, Joel Kamnitzer, Anton Kapustin, Marcos Mariño, Alexei Oblomkov, Peter Ozsvath, Jake Rasmussen, Catharina Stroppel and Ben Webster. Further invited lecturers whose participation will be confirmed closer to the actual event, include Peter Kronheimer, Tomasz Mrowka, Cumrun Vafa, and Edward Witten. The proceedings of the SMS will be published in the proceeding series of the American Mathematical Society. The present request is for partial funding of $10,000 from the *Pacific Institute for the Mathematical Sciences.*

2 Scientific Description

The last decade has witnessed an explosion in the study of theories of knot homology. One of the starting points was the categorification of the Jones polynomial by Khovanov, followed by categorification of the Alexander polynomial by Ozsvath, Szabo, and Rasmussen, categorification of HOMFLYPT polynomial by Khovanov and Rozansky. More recently, Witten-Reshetikhin-Turaev invariants of links and tangles associated to arbitrary simple Lie algebras were categorified by Webster.

There are now many constructions of link homologies, utilizing Fukaya-Floer categories of Lagrangians on quiver varieties, derived categories of coherent sheaves on quiver varieties and on convolution varieties of affine Grassmannians, concrete constructions via diagrammatically defined rings, as well as a variety of approaches via gauge theory and string theory.

These constructions boast many deep interconnections, involving mirror symmetry and the Langlands program, and exhibit the remarkable feature of fusing together many disctinct areas of mathematics and mathematical physics. Some of these connections originate appealingly from ideas in physics, topological gauge theory and string theory.

Among the most recent works, we mention Kronheimer-Mrowka's take on link homology via gauge theory as well as Witten's interpretation and generalization of link homologies via certain 6-dimensional quantum field theory.

The aim of this school is to bring together leading researchers in mathematics and mathematical physics interested in knot homologies and related fields, in order to provide the opportunity to educate a new generation of scientists in this growing field.

The basic idea of categorification is to turn integers into vector spaces (abelian groups), vector spaces into abelian or triangulated categories, operators into functors between these categories. The number becomes the dimension of the vector space, while the vector space becomes the Grothendieck group of the category (tensored with a field). In low-dimensional topology, categorification led to many remarkable developments. In particular, a number of polynomial knot invariants were lifted to homological invariants. These new invariants provide a refinement of the familiar polynomial invariants. More importantly, they often lift to functors. Each such theory is a doubly graded knot homology whose graded Euler characteristic with respect to one of the gradings gives the corresponding knot invariant,

$$P(q) = \sum_{i,j} (-1)^i q^j \dim H_{i,j} \qquad (1)$$

For example, the Jones polynmial can be obtained in this way as the graded Euler characteristic of the Khovanov homology Similarly, the so-called knot Floer homology provides a categorification of the Alexander polynomial $\Delta(q)$.

"Categorification" and "geometrization" is one of the origins of the modern connection between gauge theory and geometric representation theory. There are several ways in which this can be understood as a natural part of the framework of topological quantum field theories and string theory.

From the point of view of mathematical physics (specifically, topological quantum theory), categorification has to do with understanding higher-dimensional origin of topological field theories (TFTs). Roughly speaking, a d-dimensional TFT associates to a d-dimensional manifold M a number, the partition function. A $d+1$-dimensional TFT will associate to M a vector space, of states, and to a cobordism a map of vector spaces, the transition amplitude. A $d+2$-dimensional TFT will associate to M a category, of boundary conditions, functors to cobordisms, etc.

Concretely, polynomial invariants of knots and links can be realized as expectation values of Wilson lines in three-dimensional Chern-Simons gauge theory. Their homological lift can be understood in terms of defect operators in (topological) gauge theory in 4 (or higher) dimensions.

Categorification is also closely related to recent developments in string theory, more specifically, the homological algebra of mirror symmetry. Following the original proposal of M.Kontsevich, it has been understood by now that a mathematically accurate description of D-branes (which properly accounts for open strings) in topological string theory is in terms of objects in triangulated categories. This description naturally appears in gauge theory on manifolds with boundaries and corners, so that all the insights and techniques developed in string theory enter the connection between gauge theory and the geometric representation theory.

The embedding in string theory provides central insights into the structure of knot homologies that cannot be obtained from gauge theory realization alone. A key idea in this context is that the vector spaces appearing in (1) are Hilbert spaces of BPS states in a supersymmetric physical theory associated with the geometric setup in question. Among other things, this suggests a (partial) unification of knot homologies in a "super"-homology theory (Dunfield-Gukov-Rasmussen).

In their physical realization, homological knot invariants provide an important bridge between the construction of topological field theories of cohomological and Chern-Simons type. The construction is uniform (in the dependence on the gauge group and the representation), and makes the higher-dimensional symmetries manifest. The embedding in string theory is suggestive of further unification, and additional symmetries.

A central goal of the school is to provide a pedagogical review of the current state of the various constructions of knot homologies, and to encourage interactions between the communities in order to facilitate development of the unified picture.

3 Timeliness of the proposed event

In recent years, homological knot invariants have been gaining momentum in both mathematics and physics. We feel it may be just the right time to organize a summer school on this subject, with a healthy mix of physicists and mathematicians (of course, encouraging interactions and flow of ideas in both directions).

Our main motivation for organizing the event is that while there have been several specialized meetings on the subject in the recent years, the barrier for newcomers to enter the field is especially high because of the wide range of necessary background knowledge from a variety of areas. There is a sorely felt need for an introductory school aimed at intermediate to advanced graduate students. The aims and format of the Séminaire de Mathématiques Supérieures squarely fit those goals.

Moreover, most meetings on the subject in the recent past were mostly focused either on mathematics or on physics alone. With a unified approach, this school would the first of a kind, and now is probably the best time for it since the interaction between the two communities is taking off the ground, and recent developments (including the work of Kronheimer-Mrowka and a recent paper of Witten) promise it will only be growing. In fact, paying special attention to the interface with physics will also help to achieve the desired coherence of the school.

The 2012/13 academic year at the Centre de Recherches Mathématiques will be the Thematic Year on "Moduli Spaces, Extremality, and Global Invariants", which includes several topics and activities related to our school. We are loosely coordinating the event with the organizers of the thematic year. In particular, a two-week workshop entitled "moduli spaces and their invariants in mathematical physics" will be organized by Jacques Hurtubise, Lisa Jeffrey, and Johannes Walcher from June 3 to 14 (ending one week before the school), and a workshop entitled “low-dimensional topology after Floer” will be organized by Olivier Collin and Hans Boden from July 8 to 12 (directly after the school). These workshops are both broader in scope and at a higher level than the school itself. The concentration in the late spring/early Summer 2013 will amplify each of those events, to the benefit especially of the local and Canadian mathematical community.

4 Organization of Lecturers

4.1 General approach

Our approach to organizing the program is guided by the desire to maximize the benefit for the target audience, which are intermediate to advanced graduate students with a high academic profile, but from a variety of backgrounds. The subject matter of our school is likely rather more advanced than what is typically offered at comparable events. Our goal is precisely to provide the rare opportunity to present the material in a coherent fashion accessible to the beginning researcher. The success of our school will be measured several years down the road in future publications by the participants.

We propose to organize the program in a “top-down” fashion, with a few selected introductory lectures to cover the basics, some intermediate ones, and crowned with the most recent developments. We feel it would be most beneficial to have a mix of technical mathematical lectures interspersed with more intuitive overviews, although we would leave the detailed selection of the topics to the invited speakers.

We will ask lecturers to give between 2 and 5 hours of lectures, each sequence spreading over 2 days to one week.

4.2 Confirmed lecturers

The chronological order of the lecture will aim to follow the logical order. The following schedule might have to be adjusted depending on the availability of the speakers. The titles are tentative.

FIRST WEEK

As sketched above, the beginning of the School will be devoted to introductory lectures, and to providing the context for the recent developments. We also aim to emphasize strongly the connections between the mathematical and physical aspects of the subject. The organizers are planning to provide some of the introductory lectures themselves, **Mikhail Khovanov** covering the mathematical side, and **Sergei Gukov** and **Johannes Walcher** the physics side.

Dror Bar-Natan (University of Toronto), has been asked to provide a “Pedagogical, gentle introduction to categorification”. With a background in mathematical physics, Dror Bar-Natan is an expert expositor on knot theory and especially on the homological invariants (Khovanov homology). He is also an accomplished practitioner and has written several of the main computer packages used by the community. He has offered to share hands-on experience with the participants.

Marcos Mariño (Université de Genève) Invited speaker at ICM 2006, will give a "Pedagogical introduction to BPS invariants of Knots and Links". Marcos Mariño is a high-energy physicist with important contributions to the embedding of knot invariants into string theory via D-brane constructions and duality. He is the author of a standard monograph on the subject, and an excellent communicator between mathematics and physics.

Joel Kamnitzer (University of Toronto) Recipient of 2011 André-Aisenstadt Prize of CRM, will give a "Pedagogical introduction to derived categorification". He is an expert on geometric representation theory, with prize-winning contributions to homological knot invariants, algebraic combinatorics, as well as algebraic topology.

Anton Kapustin (California Institute of Technology) Invited speaker at ICM 2010, as agreed to lecture on "Line and surface operators in gauge theory". A high-energy physicist by training, Anton Kapustin is an expert on the categorification in the gauge and string theory context via (higher-codimension) defects. He recently initiated (with Edward Witten) the gauge theory approach to the geometric Langlands program.

SECOND WEEK

As the School progresses, the nature of the lectures will change from general overviews to more specialized topics, each of which will be presented by its creator or leading expert.

Catharina Stroppel (Bonn), invited speaker at ICM 2010, "Highest weight categorification," Catharina Stroppel's interest range from geometric and combinatorial aspects of representation theory to connections with topology and category theory. Her main contributions to knot homology arise from quantum representation theory.

Mina Aganagic (UC Berkeley) "(Homological) Knot invariants and Chern-Simons theory". Mina Aganagic is one of the main developers of the physical theory of BPS states relevant for topological invariants. She has recently been approaching knot homologies from the point of view of the relation between Chern-Simons theory and topological strings.

Alexei Oblomkov (University of Massachusetts) "Introduction to knots and Hilbert schemes" A. Oblomkov is interested in representation theory, algebraic geometry, and related areas of mathematical physics.

Ben Webster (University of Oregon) "Categorification of Reshetikhin-Turaev invariants" Webster's research is on connections between representation theory, geometry and topology. His recent categorification of Reshetikhin-Turaev invariants for arbitrary Lie algebras is a landmark in the subject.

Peter Ozsvath(MIT) (invited speaker at ICM 2006) with Zoltan Szabo the creator of the subject of "Heegard-Floer homology"

Jacob Rasmussen (Oxford) "Super-knot homologies". J. Rasmussen is one of the leading experts on Heegard-Floer homology, and became interested very early on in the unified point of view provided by physics. The observations on super-polynomials for knots made with Dunfield and Gukov count among the main open problems in the area.

4.3 Further invited lecturers

Further invited lecturers whose participation will be confirmed closer to the actual event, include

Peter Kronheimer/Tomasz Mrowka (Harvard/MIT) "Link homology via Gauge theory"

Cumrun Vafa (Harvard University) "String theoretic unification of homological invariants"

Edward Witten (Institute for Advanced Study) "Gauge theory approach to Khovanov homology"

5 Participants

The audience we are aiming for consists of advanced Doctoral studens and young postdoctoral fellows, although local students who are less advanced will be encouraged to attend the more introductory lectures. We estimate to have around 60 students. In addition, other mathematicians at all level of their career may be interested in attending the school (without support).

We intend to advertise the school world-wide. Advertsiement of the School will be made via a publicity poster that will go around to 70 academic institutions in North America, and 50 in the rest of the world. The anouncement will also be distributed by e-mail using a wide list extracted from the general distribution list of the CRM, as well as via the CMS and AMS channels.

At about the same time, we estimate that the web page of the school will be added, and will constantly be maintained, at the url

`http://www.dms.umontreal.ca/ sms/index_e.php`

The interested students will apply using a web-based interface at

`http://www.crm.umontreal.ca/ sms/index_en.html`

or its French version. Students will be selected on the basis of academic excellence, with due consideration of the available budget in allocation of travel support.

6 Projected Budget (all numbers in CAD amounts)

- Lecturer reimbursement: The standard estimate to cover airfare, housing, and per diem meals is $2,500 per lecturer, depending on the distance travelled. Most of our lecturers reside in North America. On the other hand, we are considering inviting a few more than the standard 12 lecturers. Overall, we estimate our expenses for lecturer reimbursement to be around $32,000.

- Student support: The students will be housed in Université de Montréal dormitories. The cost of supporting a student, including housing and per diem meals, is $700 per student. With support for around 60 participants, this amounts to $42,000. We are counting on certain students receiving a higher level of support from their home institutions, which would allow us to extend some travel support to others.

- Auxiliary expenses: The cost of providing coffee and refreshments during breaks between the lectures (a very important occasion for lecturers and students to get together for informal discussions) is estimated around $4,500.
- Publicity: The cost of producing a high-quality poster and distributing it by mail to mathematics departments worldwide is estimated at $1,500.

The estimated total budget is $32,000+$42,000+$4,500+$1,500=$80,000.

6.1 SMS recurrent funding (expected to be renewed)

- Centre de Recherches Mathématiques: $20,000
- Fields Institute: $10,000 (approved)
- Pacific Institute for the Mathematical Sciences: $10,000
- Institut des Sciences Mathématiques: $7,500 (approved).
- Canadian Mathematical Society: $2,000 (approved).

6.2 Funding Applications

- Mathematical Science Research Institute: $20,000 (approved).
- CIRGET, Mathematical Physics labs: $5,000 each.

6.3 Additional funding application

- Simons Foundation: $20,000 (approved)

If this additional funding is available we intend to offer more travel support to the students as well as possibly offer support to around 70 students.