Thermodynamic formalism of one-dimensional maps, I

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Integral means and geometric pressure

1. Integral means spectrum;
2. Quadratic JULIA sets;
3. Geometric pressure function.

Integral means spectrum

$\phi : \mathbb{D} \to \overline{\mathbb{C}}$: Univalent,

$$\phi(z) = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots .$$

$$\beta_\phi(t) := \limsup_{r \to 1^-} \frac{\log\left(\int_0^{2\pi} \left|\phi'(re^{i\theta})\right|^t \, d\theta\right)}{|\log(1-r)|}.$$

Integral means spectrum.

$$B(t) := \sup_\phi \beta_\phi(t).$$

Universal spectrum.

Conjeture: For $|t| < 2, B(t) = \frac{t^2}{4}$.

$B(1) =$ LITTLEWOOD's constant;

$\Rightarrow$ HÖLDER domains and BRENAN's conjectures.

LITTLEWOOD's constant

$\phi : \mathbb{D} \to \overline{\mathbb{C}}$: Univalent, $\phi(z) = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots$.

$$\beta_\phi(1) = \limsup_{r \to 1^-} \frac{\log \mathrm{Length}\left(\phi\left(\{z \in \mathbb{D} : |z| = r\}\right)\right)}{|\log(1-r)|}.$$

Length = EUCLIDEAN length in $\mathbb{C}$.

Theorem (LITTLEWOOD, 1925; CARLESON–JONES, 1992)
For every $\phi(z) = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots$,

$$|b_n| \lesssim n^{B(1)}.$$

Moreover, $B(1)$ is the least constant with this property.

$B(1) < 0.46$, HEDENMALM–SHIMORIN, 2005.

$B(1) > 0.2308$, BELIAEV–SMIRNOV, 2010;

Littlewood's constant

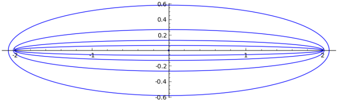


Figure: Equipotentials of $\phi(z) = \frac{1}{z} + z$, for $r = 1 - \frac{1}{2^2}$, $1 - \frac{1}{2^3}$, $1 - \frac{1}{2^4}$, and $1 - \frac{1}{2^5}$.

Littlewood's constant

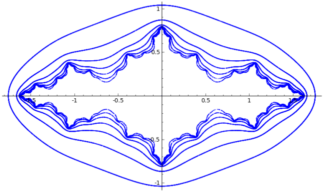


Figure: Extremal functions must have a fractal nature

Quadratic Julia sets

For $c \in \mathbb{C}$:

$$f_c : \mathbb{C} \to \mathbb{C}, \quad z \mapsto f_c(z) := z^2 + c$$

$$K_c := \left\{ z_0 \in \mathbb{C} : (f_c^n(z_0))_{n \geq 1} \text{ is bounded} \right\}$$

= complement of the attracting basin of infinity;
Filled Julia set of f_c.

$$J_c := \partial K_c$$

Julia set of f_c.

Quadratic Julia sets

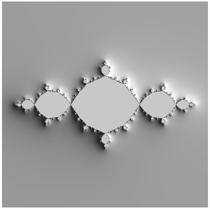


Figure: Quadratic Julia set; from Tomoki Kawahira's gallery.

Quadratic JULIA sets

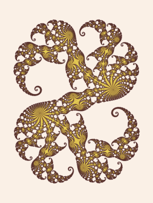


Figure: Another quadratic JULIA set, from Arnaud CHÉRITAT's gallery.

The spectrum as a pressure function

$c \in \mathbb{C}$: Such that J_c is connected;

$\phi_c : \mathbb{D} \to \overline{\mathbb{C}}$: Conformal representation of $\overline{\mathbb{C}} \setminus K_c$,

$$\phi_c(z) = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots.$$

The universal spectrum can be computed with JULIA sets of arbitrary degree (BINDER, JONES, MAKAROV, SMIRNOV, ...)

$$P_c(t) := \left(\beta_{\phi_c}(t) - t + 1\right) \log 2;$$

Geometric pressure function of f_c.

$$= \lim_{n\to\infty} \frac{1}{n} \log \sum_{z \in f_c^{-n}(z_0)} |Df_c^n(z)|^{-t};$$

= spectral radius of the transfer operator.

Multifractal analysis

ρ_c: Harmonic measure of J_c

= Maximal entropy measure of f_c.

$D_c(\alpha) := \mathrm{HD}(\{z \in J_c : \rho_c(B(z,r)) \sim r^\alpha\})$.

Local dimension spectrum.

Theorem (PESIN, WEISS, 1997)

f_c uniformly hyperbolic $\Rightarrow$ D_c and P_c are analytic and

$$D_c(\alpha) = \inf_{t\in\mathbb{R}} \left\{ t + \alpha \frac{P_c(t)}{\log 2} \right\}.$$

~ LEGENDRE transform;
Morally: P_c is analytic $\Leftrightarrow$ D_c is analytic.

Classification of phase transitions

1. Basic properties of the geometric pressure function;
2. Negative spectrum;
3. Phase transitions are of freezing type;
4. Positive spectrum tricothomy;
5. Phase transitions at infinity.

Geometric pressure function

Variational Principle

$$P_c(t) = \sup_{\mu \text{ invariant probability on } J_c} \left(h_\mu - t \int \log|Df_c| \, d\mu \right).$$

h_μ = measure-theoretic entropy.

Definition

- Equilibrium state for the potential $-t\log|Df_c|$: A measure μ realizing the supremum.
- Phase transition: A parameter at which P_c is not analytic.

Comparison with statistical mechanics.

Geometric pressure function

$$P_c(t) = \sup_{\mu \text{ invariant probability on } J_c} \left(h_\mu - t \int \log|Df_c| \, d\mu \right).$$

- P_c is convex, LIPSCHITZ, and non-increasing;
- $P_c(0) = \log 2$ topological entropy of f_c;
- $P_c(t) \geq \max\{-t\chi_{\inf}(c), -t\chi_{\sup}(c)\}$, where

$$\chi_{\sup}(c) := \lim_{t\to+\infty} \frac{P_c(t)}{-t};$$

= Supremum of LYAPUNOV exponents.

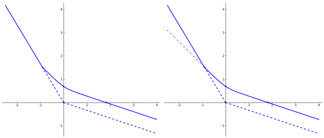
$$\chi_{\inf}(c) := \lim_{t\to-\infty} \frac{P_c(t)}{-t}.$$

= Infimum of LYAPUNOV exponents.

Theorem (Generalized BOWEN formula, PRZYTYCKI, 1998)

$$\inf\{t \in \mathbb{R} : P_c(t) = 0\} = \mathrm{HD}_{\mathrm{hyp}}(J_c).$$

Negative spectrum



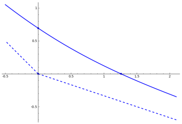
Mechanism: Gap in the LYAPUNOV spectrum.

$\Leftrightarrow$ there is a finite set Σ such that $f(\Sigma) = \Sigma$, $f^{-1}(\Sigma) \setminus \Sigma \subset \mathrm{Crit}(f)$.

These phase transitions are removable.

MAKAROV–SMIRNOV, 2000.

Phase transitions are of freezing type



Theorem (PRZYZYCKI–RL, 2011)

$P_c(t_0) > \max\{-t_0\chi_{\inf}(c), -t_0\chi_{\sup}(c)\}$

$\Rightarrow$ *P_c is analytic at $t = t_0$.*

Positive spectrum tricothomy

$$\chi_{\mathrm{crit}}(c) := \liminf_{m \to +\infty} \frac{1}{m} \log |Df_c^m(c)|.$$

1. $\chi_{\mathrm{crit}}(c) < 0 \quad \Leftrightarrow \quad f_c$ is uniformly hyperbolic;
   Levin–Przytycki–Shen, 2014.
2. $\chi_{\mathrm{crit}}(c) = 0 \quad \Leftrightarrow \quad$ Phase transition at the first zero of P_c;
   $\Leftrightarrow \quad \chi_{\mathrm{inf}}(c) = 0$
   Przytycki–RL–Smirnov (2003),
   "High-temperature phase transition"

Mechanism: Lack of expansion.

3. $\chi_{\mathrm{crit}}(c) > 0 \quad \Leftrightarrow \quad f_c$ is Collet–Eckmann
   Non-uniformly hyperbolic in a strong sense;
   Any phase transition in this case must be at "low-temperature":
   After the first zero of the geometric pressure function.

Positive spectrum tricothomy

Theorem (Coronel–RL, 2013)

There is $c \in \mathbb{R}$ such that $\chi_{\mathrm{crit}}(c) > 0$ and such that f_c has a phase transition at some $t_* > \mathrm{HD}_{\mathrm{hyp}}(J_c)$.
Moreover, c can be chosen so that the critical point of f_c is non-recurrent.

The phase transition can be of first order, or of "infinite order".

Mechanism: Irregularity of the critical orbit.

Phase transitions at infinity

Theorem (Coronel–RL)

There is a quadratic-like map f such that:

- *For every $t > 0$ there is a unique equilibrium state ρ_t for $-t \log |Df|$;*
- $\lim_{t \to +\infty} \rho_t$ *does not exists.*

Theorem (Sensitive dependence of equilibria)

There is a quadratic-like map f such that, for every sequence $(t(\ell))_{\ell \geq 1}$ going to infinity, there is $\widetilde{f}$ arbitrarily close to f such that

- *For every $t > 0$ there is a unique equilibrium state $\widetilde{\rho}_t$ of $\widetilde{f}$ for $-t \log |D\widetilde{f}|$;*
- $\lim_{\ell \to +\infty} \widetilde{\rho}_{t(\ell)}$ *does not exists.*