ANOSOV FLOWS IN DIMENSION 3
PRELIMINARY VERSION

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1. Introduction

These are lecture notes made for a mini-course given at the *School on contemporary dynamical systems*, held in Montréal, Québec, in July 2017. The goal of these lectures is to present some of the results that links the topology of a 3-manifold M with the dynamics of Anosov flows supported on M. More importantly, we aim to present the basic tools that have been successfully used to study this relationship, which are a mix of foliation theory, metric geometry, and dynamics.

In particular, we want to present what I want to call the "theory of lozenges", which is done in Section 3, and then some of its applications. Our particular emphasis will be to deduce properties on orbits which are freely homotopic to each other. The notion of a lozenge was introduced by Fenley in [Fen94], and, despite having proven itself to be very useful time and time again, it does not appear to be so widely known, even among people interested in Anosov, or related, flows. As far as I am aware, the only published articles using these lozenges admit as co-author at least one of only three people: Barbot, Fenley, or myself. I hope with these lectures to popularize a bit more this notion. Among the many applications of lozenges, we will present only two types: one relating to rigidity results for Anosov flow, and one relating to counting periodic orbits.

Although lozenges as such are an object that arise purely in Anosov flows, the ideas and tools around it, which is in some sense just the study of foliations via their leaf space, have been used in many contexts: for instance in the study of the geometry of foliations and their link to 3-manifold topology [Thu, Cal07], quasigeodesic flows [Fra15b, Fra15a], and recently partially hyperbolic diffeomorphisms [BFFP17].

Anosov flows are the archetypal examples of a uniformly hyperbolic system and have been intensely studied since their definition in the 60' by D. Anosov [Ano63, Ano67], and going back at least to Poincaré and Hadamard for the prototypical example: the geodesic flow of a hyperbolic manifold (see for instance, [Had98]). Although the dynamical and ergodic theoretical properties of Anosov flows are generally very well understood irrespective of the dimension, the links to the topology of the subjacent manifold have started to be unearthed only in dimension 3, and are completely unknown in higher dimension. However, what we know so far already points to a deep and beautiful relationship between the properties of an Anosov flow and the topology of the manifold it lives on.

There are many topics that should really be included in a review on the study of Anosov flows in 3-manifolds, but that we unfortunately won't have time to cover. Among those I will mention only one, that is useful in a more general context: the study of group actions on *non-Hausdorff* 1-manifolds. We will see in Section 3 how these manifolds come about in our context, but won't have time to cover their study in general.

Contents

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1.1. Definitions.

Definition 1.1. *Let M be a closed manifold and $\phi^t\colon M \to M$ a C^1 flow on M. The flow ϕ^t is called Anosov if there exists a splitting of the tangent bundle $TM = \mathbb{R}\cdot X \oplus E^{ss} \oplus E^{uu}$ preserved by $D\phi^t$ and two constants $a, b > 0$ such that:*

1. *X is the generating vector field of ϕ^t;*
2. *For any $v \in E^{ss}$ and $t > 0$,*
$$\|D\phi^t(v)\| \leq be^{-at}\|v\| ;$$
3. *For any $v \in E^{uu}$ and $t > 0$,*
$$\|D\phi^{-t}(v)\| \leq be^{-at}\|v\| .$$

In the above, $\|\cdot\|$ is any Riemannian (or Finsler) metric on M.

During these lectures, we will mainly be interested in Anosov flow up to *orbit equivalence*

Definition 1.2. *Let $\phi^t\colon M \to M$ and $\psi^t\colon N \to N$ be two flows. They are* orbit equivalent *if there exists an homeomorphism $h\colon M \to N$ such that the flow $h^{-1}\circ\psi^t\circ h$ is a time-change of ϕ^t, i.e., if there exists a real-valued function $\tau\colon \mathbb{R}\times M \to \mathbb{R}$ such that, for all $x \in M$ and $t \in \mathbb{R}$*
$$\phi^{\tau(t,x)}(x) = h^{-1}\circ\psi^t\circ h(x),$$
and such that $\tau(t,x) \geq 0$ if $t \geq 0$.

So an orbit equivalence between two flows send the orbit of one onto the orbit of the other while preserving the orientation, but not a priori the parametrization. This is not to be confused with the much more restrictive concept of topological *conjugacy*, in which the homeomorphism has to also preserve the parametrization of orbits.

The classical criterion to prove that a given flow is Anosov is the cone criterion. To provide a bit of diversity to the usual statement of that criterion, we follow Barbot [Bar05], and state it, for 3-manifolds, in terms of Lorentzian metrics, i.e., pseudo-Riemannian metric of signature $(-,+,+)$.

Lemma 1.3. *Let M be a closed 3-manifold and $\phi^t\colon M \to M$ a flow. The flow ϕ^t is Anosov if and only if there exists two Lorentzian metrics Q^+ and Q^-, and three constants $a, b, T > 0$ such that*

1. *For all $x \in M$, the cones $C^+(x) = \{(x,v) \in T_xM \mid Q^+_x(v) < 0\}$ and $C^-(x) = \{(x,v) \in T_xM \mid Q^-_x(v) < 0\}$ are disjoint;*
2. *The values of Q^+ and Q^- on X are equal, constant and positive, where X is the vector field generating ϕ^t;*
3. *For all $t \geq 0$, and all $x \in M$,*
$$D_x\phi^t\left(\overline{C^+(x)} \smallsetminus \{0\}\right) \subset C^+\left(\phi^t(x)\right)$$
$$D_x\phi^{-t}\left(\overline{C^-(x)} \smallsetminus \{0\}\right) \subset C^-\left(\phi^{-t}(x)\right);$$
4. *For all $x \in M$, and $t > T$, we have*
$$Q^+\left(D_x\phi^t(v)\right) < ae^{bt}Q^+(v) \text{ for all } v \in C^+(x)$$
$$Q^-\left(D_x\phi^{-t}(v)\right) < ae^{-bt}Q^-(v) \text{ for all } v \in C^-(x).$$

The most fundamental objects in our study of Anosov flows are the invariant foliations:

Proposition 1.4 (Anosov [Ano63])**.** *If ϕ^t is an Anosov flow, then the distributions E^{ss}, E^{uu}, $E^{ss}\oplus\mathbb{R}\cdot X$, and $E^{uu}\oplus\mathbb{R}\cdot X$ are uniquely integrable. The associated foliations are denoted respectively by $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$, $\mathcal{F}^s$, and $\mathcal{F}^u$ and called the strong stable, strong unstable, (weak) stable, and (weak) unstable foliations, respectively.*

Notice that the strong foliations are *not* preserved under orbit equivalence, whereas the weak stable foliations are. Hence, all of the results we will present focus on the weak foliations, and only relies sometimes on the existence of strong foliations.

Following Mosher [Mos92a, Mos92b], we also define *topological Anosov flows.*

Definition 1.5. *Let M be a compact 3-manifold and $\phi^t\colon M \to M$ a flow on M. We say that ϕ^t is a* topological Anosov flow *if the following conditions are verified:*

(1) For every $x \in M$, the flow lines $t \mapsto \phi^t(x)$ are C^1 and not constant, and X, the tangent vector field, is C^0.
(2) There exists two transverse foliations $\mathcal{F}^s, \mathcal{F}^u$, which are two-dimensional, with leaves saturated by the flow, and such that the leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ intersect exactly along the flow lines of ϕ^t.
(3) If x, y are on the same leaf of $\mathcal{F}^s$, then, there exists t_0 such that, we have

$$\lim_{t\to+\infty} d(\phi^{t+t_0}(x), \phi^t(y)) = 0 \quad \textit{and} \quad \lim_{t\to-\infty} d(\phi^t(x), \phi^t(y)) = +\infty,$$

where d is an induced metric on the leaf.
(4) If x, y are on the same leaf of $\mathcal{F}^u$, then, there exists t_0 such that, we have

$$\lim_{t\to+\infty} d(\phi^{-t-t_0}(x), \phi^{-t}(y)) = 0 \quad \textit{and} \quad \lim_{t\to-\infty} d(\phi^{-t}(x), \phi^{-t}(y)) = +\infty,$$

where d is an induced metric on the leaf.

Remark 1.6. Mosher actually defined *pseudo-Anosov flows.* In the definition, the foliations are instead singular foliations with the restriction that there are only a finite number of singular orbits, all periodic, and that the singular foliations are p-prongs, $p \geq 3$, in a neighborhood of the singular orbits.

A number of results that we will see in these lectures can be extended to pseudo-Anosov flows, and there are many reason why one would want to study pseudo-Anosov flows (one reason being that they are much more common then Anosov flow). However, we will stick to Anosov flows here.

Clearly, Anosov flows are topological Anosov flows. However it is a long-standing (yet unnamed) conjecture that any topological Anosov flow is orbit equivalent to an Anosov flow. As far as I am aware, the only case where this conjecture is known to be true is on Seifert manifolds and torus bundles [Bru93].

2. Examples

2.1. The geodesic flow.

Example 1 (The geodesic flow)**.** Let Σ be a closed manifold, equipped with a (C^2) Riemannian metric g. Let $S\Sigma$ be the unit tangent bundle of Σ. Let $\phi^t\colon S\Sigma \to S\Sigma$ be the geodesic flow of g, i.e., for any $t \in \mathbb{R}$ and $v \in S\Sigma$,

$$\phi^t(v) = (\exp(tv), \frac{d}{dt}\exp(tv)),$$

where $\exp\colon T\Sigma \to \Sigma$ is the exponential map of g.

The fundamental remark of Anosov was that, if g is negatively curved, then its geodesic flow ϕ^t is an Anosov flow.

Despite geodesic flows being clearly dependant on the choice of metric, we will always say *the* geodesic flow of a surface. This is because any two geodesic flows on the same surface are orbit equivalent ([Ghy84, Gro00]). One can easily see that in the case of negatively curved Riemannian metrics on surfaces using structural stability. Notice first that two distinct geodesic flows a priori live on two different manifolds: the unit tangent bundle of Σ associated with two different metrics. But we can easily remedy to this: just define the geodesic flows on the *homogenized bundle* (or *bundle of directions*) $H\Sigma = T\Sigma \smallsetminus \{0\}/\mathbb{R}^+$. Now the space of negatively curved Riemannian metrics on a surface is connected (travel in the conformal class to the hyperbolic metric and then travel in the Teichmüller space), hence structural stability gives the result.

When the Riemannian metric has constant negative curvature, then $\Sigma = \mathbb{H}^2/\Gamma$ and $S\Sigma = PSL(2, \mathbb{R})/\Gamma$. Then, the geodesic flow can be identified with the action of the diagonal matrices

$$g^t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

Therefore, geodesic flows on surfaces will be referred to as an algebraic Anosov flow.

2.2. **Suspension of Anosov diffeomorphisms.** We start by recalling what a general suspension is. If $f\colon \Sigma \to \Sigma$ is a diffeomorphism, then the *mapping torus* of f is the manifold

$$\Sigma_f = (\Sigma \times [0,1])/(x,1) \sim (f(x),0).$$

The suspension of f is then the flow $\phi_f^t\colon \Sigma_f \to \Sigma_f$ given (locally) by

$$\phi_f^t(x,s) = (x, s+t)$$

Example 2. Let $f\colon \Sigma \to \Sigma$ be an Anosov diffeomorphism. Let ϕ_f^t be its suspension, then ϕ_f^t is an Anosov flow.

Theorem 2.1 (Franks [Fra70], Newhouse [New70])**.** *Any Anosov diffeomorphism of codimension one is topologically conjugated to a linear Anosov diffeomorphism on the torus* $\mathbb{T}^n$.

Remark 2.2. One can show that if M is a manifold that support a suspension of an Anosov diffeomorphism, then, up to orbit equivalence, M supports at most two Anosov flow (the suspension of a linear automorphism and the suspension of its inverse).

P. Tomter [Tom68] proved that geodesic flows in constant curvature and suspensions of Anosov automorphisms are the only algebraic flows in dimension 3. These two family of examples are also essentially the only two known types of examples of Anosov flows in higher dimension (see Remark 5.5 for more about what is known and unknown in higher dimensions).

In dimension 3, we know of a huge variety of different types of Anosov flows. There are now two techniques to build Anosov flows in dimension 3: either by doing surgery on existing Anosov flows, or by gluing together "building blocks". We will now start describing these tools.

2.3. **Fried surgery.** We start by recalling/defining some notions from low-dimensional topology: Dehn twists, Dehn surgeries, and Dehn fillings.

Definition 2.3. *A* (p,q)-Dehn twist *on the torus* $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ *is any homeomorphism isotopic to the map* $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ *given by*

$$f(s,t) = (s+pt, qt).$$

A $(p,1)$-Dehn twist *on an annulus* $S^1 \times [0,1]$ *is any homeomorphism isotopic to the map* $f\colon S^1 \times [0,1] \to S^1 \times [0,1]$ *given by*

$$f(s,t) = (s+pt, t).$$

Remark 2.4. Given a torus T, to define a Dehn twist on T, one needs to choose a meridian and a parallel on T, i.e., a pair of curves that generates the fundamental group of T. Notice that two different choices will give two different (p,q)-Dehn twists.

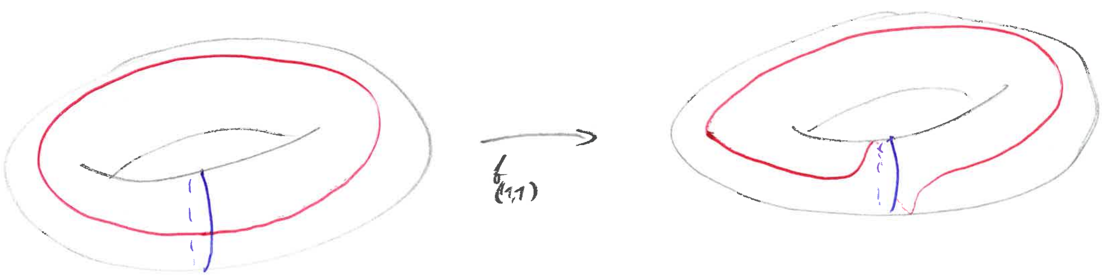


FIGURE 1. A $(1,1)$-Dehn twist

Notice, since it will be important later, that one can always do a $(p,1)$-Dehn twist f on a torus in such a way that f is the identity everywhere except on a small annulus around a meridian.

A *Dehn surgery* on a 3-manifold M is the action of removing a solid torus $\mathbb{D}^2 \times S^1$ from M and gluing it back in via a Dehn-twist. In this case, the meridian is always uniquely defined (up to homotopy) as a curve in $\mathbb{T}^2 = \partial\mathbb{D}^2 \times S^1$ that bounds a disc in $\mathbb{D}^2 \times S^1$. Note however that the choice of parallel is not unique: any (homotopy class) of a closed, homotopically non-trivial curve in $\mathbb{D}^2 \times S^1$ that intersect the meridian only once will do.

If N is a manifold with boundary a union of tori $T_1, \ldots, T_n$. Suppose that meridians and parallels are chosen on each torus T_i. A *Dehn filling* of N consists of gluing solid tori to the boundary of N using Dehn twists.

One acclaimed result of Thurston is that, if N admits a complete hyperbolic metric, then all but a finite number of Dehn fillings of N gives a hyperbolic manifold.

Example 3 (Fried's surgery [Fri83])**.** Let ϕ^t be an Anosov flow on a manifold M. Let α be a periodic orbit. Assume that $\mathcal{F}^s(\alpha)$ and $\mathcal{F}^u(\alpha)$ are both annuli (for a generic periodic orbit, one or both could be a Möbius band). Let M^* be the manifold obtained by blowing-up M along the normal bundle of α. That is, if V is a tubular neighborhood of α and (r,θ,t) are polar coordinates on V such that $\alpha = (0,0,t)$, then M^* is obtained by replacing each point $x \in \alpha$ by the circle $\{(r,\theta,t) \mid r = 0,\, \theta \in \mathbb{R}/\mathbb{Z}\}$.

The manifold M^* has one torus boundary T_α and contains $M \smallsetminus \alpha$ in its interior. The flow ϕ^t extends to a flow ϕ^t_* on M^* (see Figure 2). The flow ϕ^t_* has four periodic orbits on T_α, two attractive and two repelling.

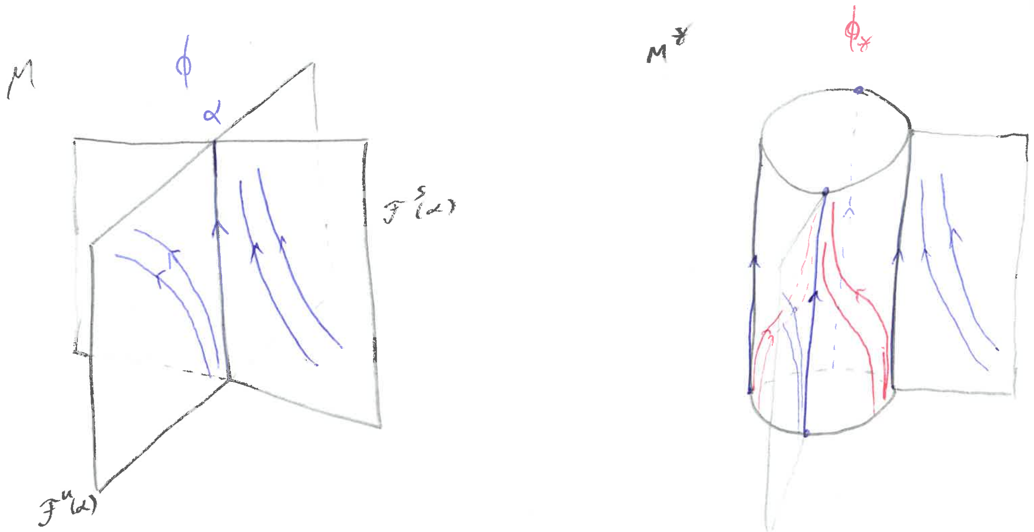


FIGURE 2. Blowing-up the orbit α

On T_α, choose a foliation $\mathcal{C}$ by circles, transverse to ϕ^t_*, and such that each circle leaf intersect the periodic orbits in only four points. Then define the manifold $M_\mathcal{C}$ by identifying every circle leaf of $\mathcal{C}$ to a point.

The flow ϕ^t_* naturally descends to a flow $\phi_\mathcal{C}{*}^t$ on $M_\mathcal{C}$, and one easily checks that it is a *topological* Anosov flow (because the flow and foliations of M and $M_\mathcal{C}$ are the same outside of the orbit α, and the new orbit $\alpha_\mathcal{C}$ is clearly hyperbolic).

The construction of $M_\mathcal{C}$ is the same thing as doing some $(n,1)$-Dehn filling of M^*, where the meridian of T_α is chosen to be a curve $\{r = 0,\, t = \text{const.}\}$ and the parallels are the curves $\{r = 0,\, \theta = \text{const.}\}$.

Remark 2.5. As far as I am aware, it is not yet proven that every flow obtained by Fried surgery is orbit equivalent to a true (differentiable) Anosov flow. It has been fairly generally assumed that this was the case, despite any clear evidence supporting it so far.

Another lingering open question regarding Fried's surgery is to know whether every Anosov flow is orbit equivalent to one obtained by (multiple) Fried surgery on a suspension of an Anosov diffeomorphism.

2.4. **Handel–Thurston, Goodman, and Foulon–Hasselblatt surgeries.** Fried surgery, described above, has the great advantage of letting us keep a good control on the stable and unstable foliations, but at the cost of losing regularity. The construction we will describe here gives differentiable Anosov flows, but we lose the control on the foliations.

2.4.1. *The Handel–Thurston construction* [HT80]. Let Σ be a surface of genus $g \geq 2$ equipped with a hyperbolic metric. Consider α a separating geodesic, and call Σ_1, Σ_2 the two surfaces that α bounds.

The unit tangent bundle $S\Sigma$ then splits into the two unit bundles $S\Sigma_1$ and $S\Sigma_2$, which are 3-manifolds with boundary a torus: the unit tangent bundle $S\alpha$ over α (see Figure 3).

Let ϕ^t be the geodesic flow of Σ. The torus $S\alpha$ is *quasi-transverse* to ϕ^t, that is, $S\alpha$ decomposes into two (open) annulus V, W on which ϕ^t is transverse, and two periodic orbits α_1, α_2 of ϕ^t (which are the geodesic α described in either directions).

Let ϕ^t_1, ϕ^t_2 be the restriction of the geodesic flow ϕ^t to, respectively, $S\Sigma_1$ and $S\Sigma_2$. Then each ϕ^t_i is transverse to the annuli V_i and W_i of $\partial S\Sigma_i$. Moreover, if the flow ϕ^t_1 is, say, exiting on V_1, then ϕ_2 is entering on V_2.

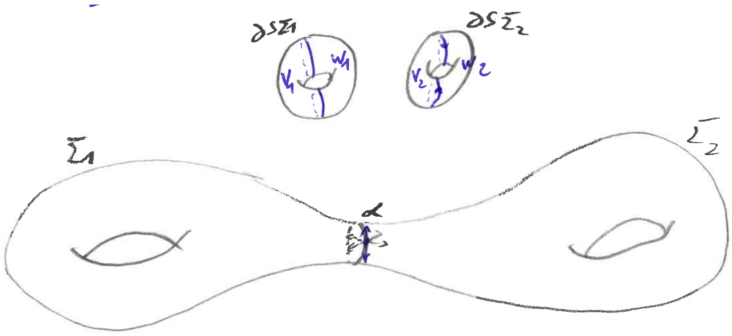


FIGURE 3. Handel–Thurston setting

We can use any homeomorphism $f\colon \partial S\Sigma_1 \to \partial S\Sigma_2$ to build a new manifold M_f by gluing $S\Sigma_1$ and $S\Sigma_2$ via f. Moreover, if f sends V_1 to V_2 and W_1 to W_2, then one can actually glue the flows ϕ_1^t and ϕ_2^t, to get a flow ϕ_f^t on M_f.

Now in general, the flow ϕ_f^t has no reason to be Anosov. Handel and Thurston [HT80] proved that for a good choice of a Dehn twist f, the flow ϕ_f^t is Anosov (see Figure 4).

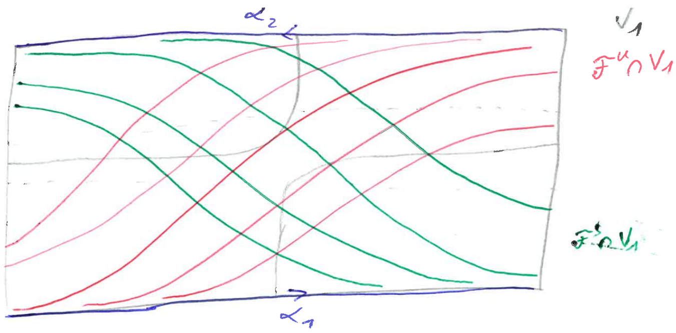


FIGURE 4. A Dehn twist in the correct direction will push the cones more onto the unstable direction, hence making the surgered flow hyperbolic

Remark 2.6. Topologically, the Handel–Thurston construction gives a *graph-manifold*, i.e., a 3-manifold M such that its torus decomposition consists of Seifert-fibered pieces.

Notice also that the geodesic α does not have to be separating for the construction to work. It does have to be simple though.

2.4.2. *Goodman's surgery* [Goo83]. Goodman's surgery is very similar to Handel and Thurston's: It is also a Dehn twist on an annulus transverse to an Anosov flow, but it has the added advantage of leading to all sort of manifolds, not just graph-manifolds.

Let α be a periodic orbit of an Anosov flow. Now we can take a small enough tubular neighborhood of α in such a way that its boundary contains an annulus that is transverse to the flow (see Figure 5). Then, one can do a $(p, 1)$-Dehn twist on that annulus, and for either $p \geq 0$ or $p \leq 0$, the flow obtained will be Anosov.

Using Goodman's surgery, one can obtain Anosov flow for instance on some hyperbolic manifold. Indeed, according to Thurston theorem, if $M \smallsetminus \{\alpha\}$ is atoroidal, then all but a finite number of Dehn filling on it will yield a hyperbolic manifold. If $M = S\Sigma$ is the unit tangent bundle of a negatively curved surface, and α is a periodic orbit of the geodesic flow that projects to a *filling* geodesic (a closed geodesic on a surface is *filling* if any other closed geodesic has to intersect it), then a folkloric result of 3-manifold topology implies that $M \smallsetminus \{\alpha\}$ is atoroidal (see for instance [FH13, Appendix B]).

2.4.3. *Foulon–Hasselblatt surgery* [FH13]. The Foulon–Hasselblatt surgery follows the same principle as Handel–Thurston's or Goodman's: finding an annulus transverse to the flow and doing a Dehn twist on it. The added feature is that, for a well chosen annuli, one can obtain a *contact* Anosov flow.

A flow $\phi^t\colon M \to M$ is called *contact* if it preserves a contact form, i.e., a 1-form $\eta\colon M \to T^*M$ such that $\eta \wedge d\eta$ is a volume form. The typical example of such a flow being a geodesic flow.

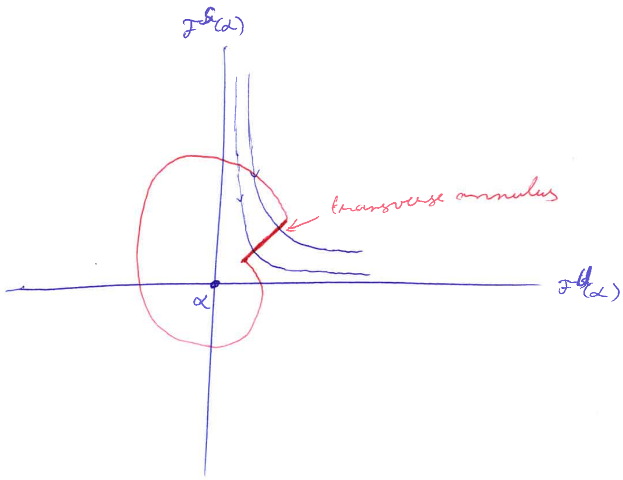


FIGURE 5. A torus around the periodic orbit α containing a transverse annulus

Remark 2.7. Notice that the property of being contact is a differential property, hence it is not necessarily preserved by orbit equivalency. In fact, even a time change can lose the contact property (if the time change is not C^1).

Foulon and Hasselblatt proved the following

Theorem 2.8 (Foulon and Hasselblatt [FH13])**.** *Let ϕ^t be an Anosov geodesic flow on a unit tangent bundle of a surface $M = S\Sigma$. Suppose that γ is a simple closed curve in M, obtained by rotating the vector direction of a geodesic by $\pi/2$. Then, for any small tubular neighborhood U of γ, half of the Dehn surgeries on U yields a manifold N that supports a contact Anosov flow ψ^t.*

Moreover, the orbits of ϕ^t that never enter the surgery locus U are still orbits of the new flow ψ^t and the contact form of ϕ^t and ψ^t are the same on $M \smallsetminus U = N \smallsetminus U$.

In particular, Foulon–Hasselblatt result shows that the Handel–Thurston surgery can in fact be done in such a way that the resulting flows are contact.

2.5. **Building blocks; from Franks–Williams to Béguin–Bonatti–Yu.** In [BBY], Béguin, Bonatti, and Yu designed a new way of building Anosov flows, starting from "building blocks" that they call *hyperbolic plugs*. This method finds its roots in Franks and Williams work [FW80], as well as examples by Bonatti and Langevin [BL94] (see also [Bar98]). A different construction, also using building blocks instead of surgery, was designed by Barbot and Fenley in [BF13].

The idea of a building block is to consider a (non-complete) flow ϕ^t on a manifold V with boundary a union of tori such that ϕ^t satisfies to the property an Anosov flow would have to satisfy, and then prove that by gluing such building blocks in a proper way, one obtains an actual Anosov flow if the manifold is closed. One of the difference between Béguin–Bonatti–Yu approach versus Barbot–Fenley is that Béguin–Bonatti–Yu requires less from the flow itself, but they then have to require that the flow be transverse to the boundary tori.

In order to save time, we will here undersell the work of Béguin, Bonatti, and Yu (and not talk about Barbot and Fenley's work), and explain just one way of using their result to build new Anosov flows. However, the part we do present already allows one to build many new examples.

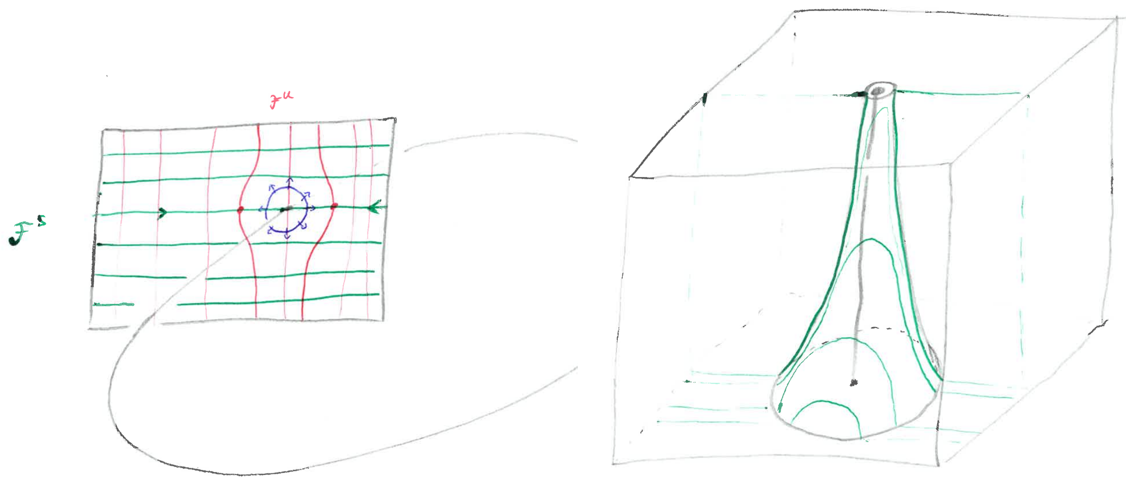
2.5.1. *DA flows.* We first need to recall the *DA* (Derived from Anosov) construction of Smale [Sma67] (that was later fleshed out by Williams [Wil70]).

Let ϕ^t be an Anosov flow on a 3-manifold M and α a periodic orbit. Then one can modify the flow in such that a way that α becomes a repelling periodic orbit, by "pushing away" from α along its stable foliation. Let us call $\bar{\phi}^t$ this modified flow. One can show (see for instance [GHS97])

Proposition 2.9. *For a good choice of modification, the flow $\bar{\phi}^t$ satisfy:*

(1) The periodic orbit α is a repeller;
(2) There is a tubular neighborhood V of α such that the set $\Lambda = M \smallsetminus \cup_{t\geq 0}\bar{\phi}^t(V)$ is an invariant hyperbolic attractor for $\bar{\phi}^t$. Moreover, ∂V is a torus transverse to $\bar{\phi}^t$;
(3) The weak stable manifolds of $\bar{\phi}^t|_\Lambda$ are contained in the weak stable manifolds of ϕ^t.

The flow $\bar{\phi}^t$ is called a (repelling) *DA flow*. We also say that $\bar{\phi}^t$ is derived from ϕ^t by a repelling DA bifurcation. One can obviously do a similar construction and have the orbit α become an attractor. The flow thus obtained is said to be derived from ϕ^t by an *attracting* DA bifurcation.



(A) A repelling DA bifurcation on α (B) The transverse torus for a DA on a suspension

Figure 6. The DA bifurcation

2.5.2. *The Franks–Williams example.* The Franks–Williams [FW80] consists in the following.

Consider two Anosov flows $\phi_1^t\colon M_1 \to M_1$ and $\phi_2^t\colon M_2 \to M_2$, given by the suspension of, respectively, an Anosov automorphism A and its inverse. Produce $\bar{\phi}_1^t$, a repelling DA on an orbit α of ϕ_1^t and, $\bar{\phi}_2^t$, an attracting DA on the corresponding orbit of ϕ_2^t. Consider V_1 and V_2 the tubular neighborhood given by Proposition 2.9. Franks and Williams showed that for a particular gluing of the manifolds $M_1 \smallsetminus V_1$ to $M_2 \smallsetminus V_2$ equipped with the flows $\bar{\phi}_1^t$ and $\bar{\phi}_2^t$, the flow obtained is Anosov.

In this construction each flow $\bar{\phi}_i^t$ restricted to $M_i \smallsetminus V_i$ is a "building block", and one can glue these two blocks together to obtain an Anosov flow.

For the resulting flow to be Anosov, one needs (and it is actually also enough) that the gluing $f\colon \partial V_1 \to \partial V_2$ sends the stable foliation of $\bar{\phi}_1^t$ on ∂V_1 to a foliation *transverse* to the unstable foliation of $\bar{\phi}_2^t$ on ∂V_2.

Note that the resulting Franks–Williams flow is not transitive as orbits that enter $M_1 \smallsetminus V_1$ can never go back to $M_2 \smallsetminus V_2$.

2.5.3. *The Béguin–Bonatti–Yu construction.* What Béguin, Bonatti, and Yu proved is that one can do the Franks–Williams type constructions starting with, essentially, any number of periodic orbits and doing both attracting and repelling DA bifurcations, i.e., doing any number of repelling or attracting DA bifurcations on an Anosov flow will give a "building block" (a hyperbolic plug in the language of [BBY]) and any kind of "reasonable" gluing will yield an Anosov flow.

The actual result is, as should be expected, not quite as general, so we will be more precise and restrictive.

Start with ϕ^t an Anosov flow on a 3-manifold M. Let α_1 and α_2 be two periodic orbits. Let $\bar{\phi}^t$ be the flow obtained by doing an attracting DA bifurcation on α_1 and a repelling DA bifurcation on α_2. Let V_1, V_2 be the associated tubular neighborhoods. Then, up to some modifications by isotopies, any map $f\colon \partial V_1 \to \partial V_2$ can be used to glue $M \smallsetminus (V_1 \cup V_2)$ to itself and obtain an Anosov flow. The only condition is that f sends the stable foliation on ∂V_1 to a foliation transverse to the unstable foliation on ∂V_2 (see [BBY, Section 7]).

Notice that, contrarily to the Franks–Williams example, one can obtain transitive Anosov flows using the above method. Indeed, if the starting flow is transitive, the flow obtained after surgery is again transitive ([BBY]).

Béguin–Bonatti–Yu method gives another generalization of Franks–Williams. Start now with two Anosov flows $\phi_1^t\colon M_1 \to M_1$ and $\phi_2^t\colon M_2 \to M_2$. Choose $\alpha_1, \dots, \alpha_k$ periodic orbits of ϕ_1^t and $\beta_1, \dots, \beta_k$ periodic orbits of ϕ_2^t. Let $\bar{\phi}_1^t$ be obtained by doing attracting DA bifurcations on all the α_i. Similarly, let $\bar{\phi}_2^t$ be obtained by doing repelling DA bifurcations on all the β_i. Call V_i the good tubular neighborhoods of α_i and W_i those for β_i. Then one can glue $M_1 \smallsetminus \cup_{i=1}^k V_i$ to $M_2 \smallsetminus \cup_{i=1}^k W_i$ to obtained a (non transitive) Anosov flow.

Remark 2.10. For the moment, the only construction that has been extended to higher dimension is Franks and Williams' (although not exactly in the way that Franks and Williams claim it can be done in [FW80], see [BBGRH17] and Remark 5.5). This raises hopes of developing the whole Béguin–Bonatti–Yu technology, but it would be a good bit more technical to do.

It is also noteworthy that, for the moment, the only know examples of Anosov flows with different stable-unstable dimensions are suspensions of Anosov diffeomorphisms (since the examples suggested in [FW80] turned out not to be Anosov).

3. The topology of Anosov flows

In Section 2, we saw that one can build a very wide variety of Anosov flows in a large range of 3-manifolds. We will now start studying what these manifolds, supporting Anosov flows, must have in common, and develop some of the tools allowing the study and the (attempts at) classification of Anosov flows, up to orbit equivalency.

Among the people that have worked on these questions, E. Ghys, J. Plante, W. Thurston and A. Verjovsky [Ghy84, Ghy88, Pla72, PT72, Ver74] have been among the precursors in the 70' and 80'. Starting in the 90', T. Barbot and S. Fenley have immensely improved our understanding (see for instance [Bar92, Bar95a, Bar95b, Bar96, Bar98, Bar01, BF13, BF15, Fen94, Fen95a, Fen95b, Fen97, Fen98, Fen99]).

3.1. **The leaf spaces.**

Let ϕ^t be an Anosov flow on a 3-manifold M. A. Verjovsky [Ver74] was the first to study Anosov flows via the topology of the associated foliations and their *leaf spaces* that we will define shortly. But before that we start by stating some basic results about the stable and unstable foliations.

Let $\widetilde{M}$ be the universal cover of M. The foliations associated with ϕ^t on M, as well as the flow itself, lifts to foliations on $\widetilde{M}$. Let $\tilde{\phi}^t$ designate the lifted flow and $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^{ss}, \widetilde{\mathcal{F}}^u, \widetilde{\mathcal{F}}^{uu}$ the lifts of the (weak)-stable, strong stable, (weak)-unstable, and strong unstable foliations.

In order to study the leaf and orbit spaces, we need some basic results about the leaves of the stable and unstable foliations.

One result that we will use over and over again is the following:

Lemma 3.1. *Let ϕ^t be an Anosov flow on M. Any closed loop that is transverse to either $\mathcal{F}^s$ or $\mathcal{F}^u$ is not null-homotopic. Hence there does not exists any closed transversal of $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$.*

Proof sketch. This follows from a result of Haefliger [Hae62], which states that, if there exists a closed transversal to a codimension one foliation that is null-homotopic, then there exists a closed loop in one of the leaf that has trivial holonomy on one side and non-trivial on the other. Now, for an Anosov flow, the holonomy of a, say, unstable leaf is exponentially contracting. Hence, there cannot exists any null-homotopic transversal to either $\mathcal{F}^s$ or $\mathcal{F}^u$. □

Proposition 3.2 (Verjovsky [Ver74])**.** *Let ϕ^t be an Anosov flow on a 3-manifold M. Then:*

1. *The periodic orbits of ϕ^t are not null-homotopic.*
2. *The leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ are homeomorphic to either $\mathbb{R}^2$, an annulus or a Möbius band. The leaves of $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$ are homeomorphic to $\mathbb{R}^2$.*
3. *The universal cover $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$.*

Remark 3.3. The above results hold for any *codimension-one* Anosov flow, i.e., an Anosov flow such that either the strong stable or strong unstable distribution has dimension one.

Sketch of proof. The first item follows from Lemma 3.1. Indeed, any periodic orbit of ϕ^t can be homotoped to a loop that is transverse to say $\mathcal{F}^s$, it suffice to deform the periodic orbit along the strong unstable foliation, closing up using a piece of unstable (see Figure 7). Therefore, they cannot be null-homotopic.

The second item then comes easily: A weak leaf is the saturated of a strong leaf by the flow. Now, a strong leaf is homeomorphic to $\mathbb{R}$ (since any part of a strong leaf can be shrunk to a small interval by flowing forwards or backwards for long enough). This gives the three possibilities for stable and unstable leaves in M, depending on whether a leaf contains a periodic orbit or not, and whether or not it is orientable. Now, since periodic orbits are not null-homotopic, their lifts are homeomorphic to the real line, hence a lifted weak leaf is homeomorphic to $\mathbb{R}^2$.

The last item follows from a result of Palmeira [Pal78] stating that $\mathbb{R}^n$ is the only simply connected manifold of dimension n that admits a foliation by leaves diffeomorphic to $\mathbb{R}^{n-1}$. □

Definition 3.4.

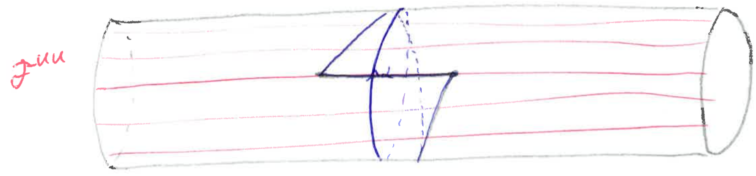


FIGURE 7. Pushing a periodic orbit along the strong unstable foliation makes it transverse to the weak stable foliation

- *The* orbit space *of ϕ^t is the quotient space of $\widetilde{M}$ by the relation "being on the same orbit of $\tilde{\phi}^t$", equipped with the quotient topology. We denote it by $\mathcal{O}$.*
- *The* stable *(resp.* unstable*) leaf space of ϕ^t is the quotient of $\widetilde{M}$ by the relation "being on the same leaf of $\widetilde{\mathcal{F}}^s$ (resp. $\widetilde{\mathcal{F}}^u$)", equipped with the quotient topology. We denote them by $\mathcal{L}^s$ and $\mathcal{L}^u$ respectively.*

The foliations $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$ of $\widetilde{M}$ descends to (1-dimensional) foliations of the orbit space $\mathcal{O}$, that we will still denote by $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$.

The fundamental group $\pi_1(M)$ of M acts on the orbit space, as well as the leaf spaces, since its action on $\widetilde{M}$ preserves the foliations. One can easily verify the following

Proposition 3.5. *Let ϕ^t be an Anosov flow on M.*
(1) The stabilizer in $\pi_1(M)$ of a point in $\mathcal{O}$, $\mathcal{L}^s$, or $\mathcal{L}^u$ is either trivial or cyclic.
(2) If $g \in \pi_1(M)$, $g \neq \mathrm{Id}$, fixes an orbit $\alpha \in \mathcal{O}$, then α is an hyperbolic fixed point.
(3) If $g \in \pi_1(M)$, $g \neq \mathrm{Id}$, fixes a leaf l in $\mathcal{L}^s$ or $\mathcal{L}^u$, then l is either attracting or repelling.

Sketch of proof. Let us prove just part of the first item, since everything else follows from similar considerations. Suppose that $g \in \pi_1(M)$ fixes an element $l \in \mathcal{L}^s$, and assume that $g \neq \mathrm{Id}$. Then l is a leaf in $\widetilde{M}$ such that its projection $\hat{l}$ in M has non-trivial fundamental group. Then, by Proposition 3.2, $\hat{l}$ is homeomorphic to either a cylinder or a Möbius band, so in particular, it has cyclic fundamental group. □

Using the fact that all closed transversal to $\mathcal{F}^s$ or $\mathcal{F}^u$ cannot be null-homotopic (Lemma 3.1), one deduce the following proposition

Proposition 3.6. *Any leaf of $\widetilde{\mathcal{F}}^{ss}$ (resp. $\widetilde{\mathcal{F}}^{uu}$) intersects any leaf of $\widetilde{\mathcal{F}}^u$ (resp. $\widetilde{\mathcal{F}}^s$) in* at most *one point. Moreover, leaves of $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$ intersect at most along one orbit.*

Remark 3.7. Note that we are talking about the *lifts* of the foliations to the universal cover. This result is obviously wrong on the manifold itself (strong leaves can even be dense on M).

Proof. Suppose a strong stable leaf l^{ss} intersects a weak unstable leaf F^u twice, then one can construct a loop transverse to $\widetilde{\mathcal{F}}^u$ by first closing the segment of l^{ss} between the two intersections via a path on F^u, and then locally deform that path on F^u to make it transverse (see Figure 8).

From the first point, we immediately deduce the second point since on a say stable leaf any two orbits are intersected by a common strong stable leaf. □

Notice that this result does *not* say that every leaf of $\widetilde{\mathcal{F}}^{ss}$ intersects every leaf of $\widetilde{\mathcal{F}}^u$. Indeed, we will see later that if *one* leaf of $\widetilde{\mathcal{F}}^{ss}$ intersects every leaf of $\widetilde{\mathcal{F}}^u$, then the flow is a suspension of a Anosov diffeomorphism! (see Theorem 3.20 and Remark 3.26).

Anosov [Ano63] proved that periodic orbits are dense in the set of non-wandering points. Now, since the set of non-wandering points contains the ω- and α-limit points of any point, we deduce that the set of stable (or unstable) leaves that contains a periodic orbit is dense in M. Writing this in terms of leafs spaces gives

Proposition 3.8. *The set of points in $\mathcal{L}^s$ or $\mathcal{L}^u$ with non-trivial $\pi_1(M)$-stabilizer is dense.*

We can now start discussing the topology of the orbit spaces and leaf spaces. First, given that $\widetilde{\mathcal{F}}^s$ are foliations by planes of $\mathbb{R}^3$, we deduce (see [Pal78])

Proposition 3.9. *The leaf spaces $\mathcal{L}^s$ and $\mathcal{L}^u$ are connected, simply connected,* not necessarily Hausdorff *1-manifold.*

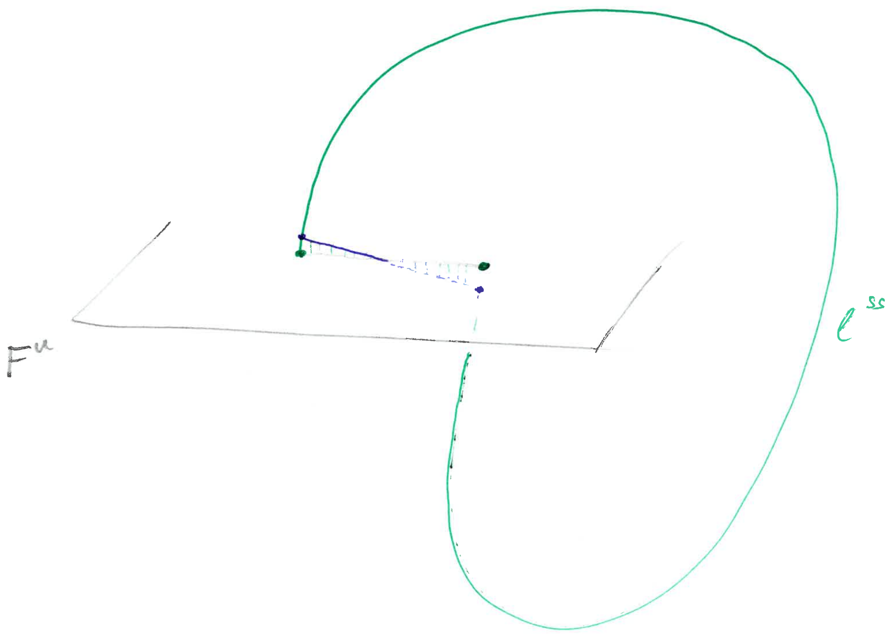


FIGURE 8. A strong stable leaf intersecting twice an unstable leaf leads to a closed transversal

The topology of the orbit space on the other hand is much nicer:

Theorem 3.10 (Barbot [Bar95a], Fenley [Fen94]). *The orbit space of an Anosov flow on a 3-manifold is homeomorphic to* $\mathbb{R}^2$.

Remark 3.11. This result is again true for codimension one Anosov flow, i.e., $\mathcal{O}$ is then homeomorphic to $\mathbb{R}^{n-1}$ [Bar95a]

We will give here the proof of Barbot [Bar95a].

Sketch of proof. First, we remark that $\mathcal{O}$ is a 2-manifold (not necessarily Hausdorff): It suffice to show that any orbit $\widetilde{\alpha}$ of $\tilde{\phi}^t$ admits a 2-dimensional transversal to $\tilde{\phi}^t$ such that any orbit of $\tilde{\phi}^t$ intersect it in at most one point. Taking a flow-box gives a small transversal, and if an orbit were to hit that transversal twice, then we could once again build a closed transversal to either $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$, which is impossible as we have already seen.

The main difficulty is to prove that $\mathcal{O}$ is Hausdorff. Let $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ be two orbits of $\tilde{\phi}^t$. We need to show that there exists two open sets V_1, V_2, saturated by orbits of $\tilde{\phi}^t$, that contain respectively $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$, and with empty intersection.

For each $i = 1, 2$, let $\widetilde{F}^s_i$ be the (weak) stable leaf that contain $\widetilde{\alpha}_i$, and define U_i as the saturated of $\widetilde{F}^s_i$ by the strong unstable foliation $\widetilde{\mathcal{F}}^{uu}$. The sets U_i are open and saturated by the orbits of $\tilde{\phi}^t$. If they are disjoint, we proved what we wanted.

So assume that $U_1 \cap U_2 \neq \emptyset$. Then, there exists $\widetilde{F}^{uu}$ that intersect both $\widetilde{F}^s_i$, and the intersection must be in a unique point. Call $x_i = \widetilde{F}^{uu} \cap \widetilde{F}^s_i$.

If $x_1 \neq x_2$, then there exists two neighborhoods $u_1, u_2 \subset \widetilde{F}^{uu}$ of x_1 and x_2 that are disjoint (since $\widetilde{F}^{uu}$ is a real line). Then take V_i to be the saturated of u_i by $\widetilde{\mathcal{F}}^s$ (see Figure 9). The sets V_i are open, saturated by the orbits of $\tilde{\phi}^t$ and disjoint (because a strong unstable leaves intersect weak stable in at most one point), which proves the result in this case.

So we are left with dealing with the case $x_1 = x_2$. But this means that $\widetilde{F}^s_1 = \widetilde{F}^s_2$. Then it is easy to find neighborhoods s_1, s_2 of $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ inside $\widetilde{F}^s_1$ that are disjoint and saturated by the orbits of $\tilde{\phi}^t$ (just take any two disjoint segments of a strong stable leaf centered at a point of $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$, and saturate it by the flow). Then, if we consider V_i to be the saturated of s_i by $\widetilde{\mathcal{F}}^u$, we get disjoint open sets that are saturated by the flow. Therefore, the space $\mathcal{O}$ is Hausdorff.

Since $\mathcal{O}$ is a simply connected, open, Hausdorff manifold of dimension 2, it is homeomorphic to $\mathbb{R}^2$.

In the case of codimension one Anosov flows, the same proof holds (if $\mathcal{F}^s$ is the codimension one foliation, otherwise, one need to switch $\mathcal{F}^s$ and $\mathcal{F}^u$), but to finish, we use Palmeira's theorem again: The

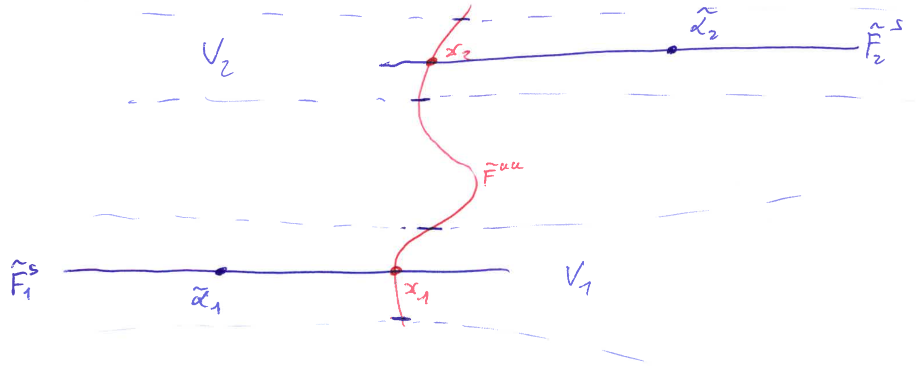


FIGURE 9. Construction of disjoint neighborhoods

foliation $\widetilde{\mathcal{F}}^s$ (or $\widetilde{\mathcal{F}}^u$) is a codimension one foliation by planes of the simply connected manifold $\mathcal{O}$, hence $\mathcal{O}$ is homeomorphic to $\mathbb{R}^n$. □

Remark 3.12. This proof is also nice in that it almost applies to dynamically coherent partially hyperbolic diffeomorphisms: it can be used to show that either the central-leaf space is homeomorphic to $\mathbb{R}^2$ or there exists two non-separated central leaf *on the same central-stable or central-unstable leaf.*

The orbit space and the action of the fundamental group on it is enough to recover the Anosov flow. Indeed, using the Ghys–Gromov trick [Ghy84, Gro00], Barbot proved the

Theorem 3.13 (Barbot [Bar95a])**.** *Let ϕ^t and ψ^t be Anosov flows on respectively M and N. Then ϕ^t is topologically conjugate to ψ^t or ψ^{-t} if and only if there exists an isomorphism $f_* \colon \pi_1(M) \to \pi_1(N)$ and a homeomorphism $\bar{f} \colon \mathcal{O}_\phi \to \mathcal{O}_\psi$ that is f_*-equivariant, i.e., such that, for $g \in \pi_1(M)$ and $\alpha \in \mathcal{O}_\phi$, we have*

$$\bar{f}(g \cdot \alpha) = f_*(g) \cdot \bar{f}(\alpha).$$

In fact, one can go one-dimension lower, that is

Theorem 3.14 (Brunella [Bru92])**.** *Two Anosov flows on a 3-manifold are orbit equivalent if and only if their stable or unstable leaf spaces are conjugated.*

3.1.1. $\mathbb{R}$-*covered flows.* We saw for the moment that the orbit space is topologically simple, whereas the leaf space might not be. However, for some Anosov flows, the leaf spaces are indeed Hausdorff, hence homeomorphic to $\mathbb{R}$.

Definition 3.15. *An Anosov flow is said to be* $\mathbb{R}$-covered *if its stable or unstable leaf space is homeomorphic to $\mathbb{R}$.*

Example 4. Both suspensions and geodesic flows are $\mathbb{R}$-covered Anosov flows.

Let ψ^t be a suspension of an Anosov automorphism $A \in SL(2, \mathbb{Z})$. Then it is easy to see that the foliations $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$ are the product of the stable and unstable foliations of A on $\mathbb{R}^2 = \widetilde{\mathbb{T}}^2$ with $\mathbb{R}$. So, in particular, suspensions ar e$\mathbb{R}$-covered.

Now consider $\phi^t \colon S\Sigma \to S\Sigma$ a geodesic flow on a hyperbolic surface. Classical results in geometry shows that the strong stable and strong unstable foliations lifts to stable and unstable horocycles in $\widetilde{S\Sigma} = S\mathbb{H}^2$. Now, a, say, weak stable leaf in $S\mathbb{H}^2$ corresponds to all the points of $\mathbb{H}^2$ equipped with the vector such that all the geodesics determined by that vector ends at the same endpoint on the circle at infinity $\partial_\infty \mathbb{H}^2$. In other words, each weak stable leaf of the lifted flow on $S\mathbb{H}^2$ corresponds to a point on $\partial_\infty \mathbb{H}^2 \simeq S^1$. Lifting to $\widetilde{S\Sigma} \simeq \mathbb{R} \times \mathbb{H}^2$ corresponds to unrolling the circle $\partial_\infty \mathbb{H}^2$. In particular, we see that $\mathcal{L}^s \simeq \widetilde{\partial_\infty \mathbb{H}^2} \simeq \mathbb{R}$.

Since both of the "classical" examples of Anosov flows are $\mathbb{R}$-covered, we could wonder whether all Anosov flows are. But this is far from being true. Theorem 3.23 will in particular imply that all the examples obtained by a Béguin–Bonatti–Yu construction cannot be $\mathbb{R}$-covered.

However, a lot of non-algebraic examples are still $\mathbb{R}$-covered. Fenley [Fen94] showed that some Fried surgeries on $\mathbb{R}$-covered flows yield $\mathbb{R}$-covered flows. Also, Barbot proved the following

Proposition 3.16 (Barbot [Bar01])**.** *If ϕ^t is a contact Anosov flow, then it is $\mathbb{R}$-covered.*

As a corollary, all the examples obtain by Foulon–Hasselblatt surgery are $\mathbb{R}$-covered. In fact, I believe that we have a much stronger relationship between $\mathbb{R}$-covered and contact Anosov flows:

Conjecture 3.17. *If ϕ^t is $\mathbb{R}$-covered, then it is orbit equivalent to a contact Anosov flow.*

Remark 3.18. Barbot [Bar96] proved that every $\mathbb{R}$-covered Anosov flow on a graph-manifold is orbit equivalent to one obtained by *generalized* Handel–Thurston surgery (see [Bar96] for the definition of generalized Handel–Thurston). Now, by Foulon–Hasselblatt [FH13], any flow obtained by (non generalized) Handel–Thurston surgery is orbit equivalent to a contact flow. If one can prove that every generalized Handel–Thurston surgery is actually just a classical Handel–Thurston surgery, then the conjecture above would be proven for graph-manifolds.

We will state without proofs two important results on $\mathbb{R}$-covered Anosov flows

Proposition 3.19 (Barbot [Bar95a], Fenley [Fen94])**.** *Let ϕ^t be an Anosov flow on a 3-manifold. We have $\mathcal{L}^s \simeq \mathbb{R}$ if and only if $\mathcal{L}^u \simeq \mathbb{R}$.*

So, in particular, it makes sense not to differentiate between the stable and unstable leaf space in the definition of $\mathbb{R}$-covered.

One of the most important result about $\mathbb{R}$-covered Anosov flows is the following:

Theorem 3.20 (Barbot [Bar95a])**.** *Let ϕ^t be a $\mathbb{R}$-covered Anosov flow on a closed 3-manifold M.*

- *either no leaf of $\widetilde{\mathcal{F}}^s$ intersect every leaf of $\widetilde{\mathcal{F}}^u$ (and vice-versa),*
- *or ϕ^t is orbit equivalent to a suspension of an Anosov diffeomorphism*

Definition 3.21. *If ϕ^t is a $\mathbb{R}$-covered Anosov flow and is not the suspension of an Anosov diffeomorphism, then ϕ^t is said to be* skewed.

The previous result implies that the structure of the orbit space and the stable and unstable foliations are particularly nice for skewed $\mathbb{R}$-covered Anosov flows: Consider a leaf $\lambda^s \in \mathcal{L}^s$. Then the set

$$I^u(\lambda^s) \ := \ \{\lambda^u \in \mathcal{L}^u \mid \lambda^u \cap \lambda^s \neq \emptyset\}$$

is an open, non-empty, connected and bounded set in $\mathcal{L}^u \simeq \mathbb{R}$. Hence it admits an upper and lower bound. Let $\eta^s(\lambda^s) \in \mathcal{L}^u$ be the upper bound and $\eta^{-u}(\lambda^s) \in \mathcal{L}^u$ be the lower bound. Similarly, for any $\lambda^u \in \mathcal{L}^u$, define $\eta^u(\lambda^u)$ and $\eta^{-s}(\lambda^u)$ as, respectively, the upper and lower bounds in $\mathcal{L}^s$ of the set of stable leaves that intersects λ^u. We have the following result (see Figure 10):

Proposition 3.22 (Fenley [Fen94], Barbot [Bar95a, Bar01])**.** *Let ϕ^t be a skewed $\mathbb{R}$-covered Anosov flow in a 3-manifold M, where $\mathcal{F}^s$ is transversely orientable. Then, the functions $\eta^s \colon \mathcal{L}^s \to \mathcal{L}^u$ and $\eta^u \colon \mathcal{L}^u \to \mathcal{L}^s$ are Hölder-homeomorphisms and $\pi_1(M)$-equivariant. We have $(\eta^u)^{-1} = \eta^{-u}$, and $(\eta^s)^{-1} = \eta^{-s}$. Furthermore, $\eta^u \circ \eta^s$ and $\eta^s \circ \eta^u$ are strictly increasing homeomorphisms and we can define $\eta \colon \mathcal{O} \to \mathcal{O}$ by*

$$\eta(o) := \eta^u\left(\widetilde{\mathcal{F}}^u(o)\right) \cap \eta^s\left(\widetilde{\mathcal{F}}^s(o)\right).$$

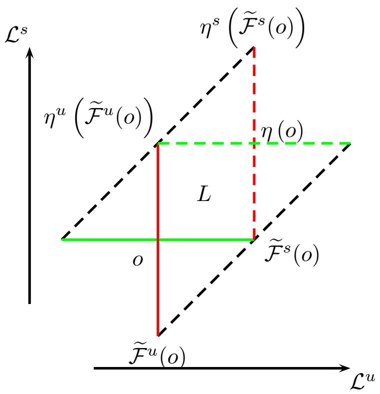


FIGURE 10. The orbit space in the $\mathbb{R}$-covered case

If $\mathcal{F}^s$ is not transversely orientable the homeomorphisms η^s, η^u are twisted $\pi_1(M)$-equivariant, i.e., if $g \in \pi_1(M)$ reverses the orientation of $\widetilde{M}$, then

$$\eta^s(g \cdot l) = g \cdot \eta^{-s}(l) \text{ and } \eta^u(g \cdot l) = g \cdot \eta^{-u}(l)$$

We state without proof some more properties of $\mathbb{R}$-covered Anosov flows.

Theorem 3.23 (Barbot [Bar95a], Fenley [Fen94]). *Let ϕ^t be an $\mathbb{R}$-covered Anosov flow on a 3-manifold M that is not a suspension of an Anosov diffeomorphism. Then*

(1) The flow ϕ^t is topologically transitive;
(2) The manifold M is orientable;
(3) If one of the weak foliation is transversely orientable, then ϕ^t is orbit equivalent to its inverse ϕ^{-t}, moreover, the orbit equivalence can be taken homotopic to identity;
(4) The flow ϕ^t does not admit any transverse embedded surface.

Remark 3.24. In particular, no examples build using Béguin–Bonatti–Yu are $\mathbb{R}$-covered, since they all have embedded transverse tori.

When Anosov flows are not $\mathbb{R}$-covered flows, the following definition becomes relevant

Definition 3.25. *A leaf of $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$ is called a non-separated leaf if it is non-separated in its respective leaf space ($\mathcal{L}^s$ or $\mathcal{L}^u$) from a distinct leaf. A leaf of $\mathcal{F}^s$ or $\mathcal{F}^u$ is called a* branching *leaf if it is the projection of a non-separated leaf in $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$.*

Remark 3.26. If there exists non-separated, say, stable leaves, then no unstable leaf can intersect every stable ones. Indeed, if F_1, F_2 are non-separated stable leaves and L is an unstable leaf that intersect both, then there exists F close to both F_1 and F_2 and $L^{uu} \subset L$ a strong unstable leaf in L such that L^{uu} intersects F twice, which is impossible (by Proposition 3.6). Hence, Theorem 3.20 implies that if an Anosov flow is such that one of its lifted stable leaf intersect every unstable, then the flow is a suspension of an Anosov diffeomorphism.

3.2. **Lozenges.** Studying the orbit space has over the years proven to be extremely useful. One of the most important object, introduced by Fenley in [Fen94] is the notion of a *lozenge*.

Before introducing lozenges, we define a *half leaf* of a stable or unstable leaf L as a connected component of the complement of an orbit in L.

Definition 3.27. *Let $\widetilde{\alpha}, \widetilde{\beta}$ be two orbits in $\mathcal{O}$ and let $A \subset \widetilde{\mathcal{F}}^s(\widetilde{\alpha})$, $B \subset \widetilde{\mathcal{F}}^u(\widetilde{\alpha})$, $C \subset \widetilde{\mathcal{F}}^s(\widetilde{\beta})$ and $D \subset \widetilde{\mathcal{F}}^u(\widetilde{\beta})$ be four half leaves satisfying:*

- *For any $\lambda^s \in \mathcal{L}^s$, $\lambda^s \cap B \neq \emptyset$ if and only if $\lambda^s \cap D \neq \emptyset$,*
- *For any $\lambda^u \in \mathcal{L}^u$, $\lambda^u \cap A \neq \emptyset$ if and only if $\lambda^u \cap C \neq \emptyset$,*
- *The half-leaf A does not intersect D and B does not intersect C.*

A lozenge L with corners $\widetilde{\alpha}$ and $\widetilde{\beta}$ *is the open subset of $\mathcal{O}$ given by (see Figure 11):*

$$L := \{p \in \mathcal{O} \mid \widetilde{\mathcal{F}}^s(p) \cap B \neq \emptyset,\ \widetilde{\mathcal{F}}^u(p) \cap A \neq \emptyset\}.$$

The half-leaves A, B, C and D are called the sides.

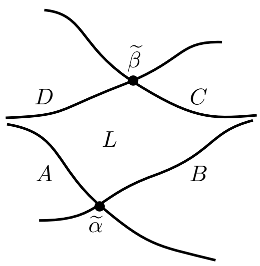


FIGURE 11. A lozenge with corners $\widetilde{\alpha}$, $\widetilde{\beta}$ and sides A, B, C, D

Remark 3.28. From Proposition 3.22, we already see that lozenges are everywhere in skewed $\mathbb{R}$-covered Anosov flows. Indeed, for any orbit $o \in \mathcal{O}$, o and $\eta(o)$ are the two corners of a lozenge (see Figure 10). But lozenges turns out to be even more ubiquitous than that. We will soon see that the *only* type of Anosov flows that does not admit at least one lozenge is the suspensions.

In the above definition, the sides A and D, as well as B and C, are in a special configuration: they do not intersect but "almost" do, that is, every close-by leaf on one side of one will intersect the other. This behavior deserves a definition

Definition 3.29. *Two leaves (or half-leaves) $F^s \in \mathcal{L}^s$ and $L^u \in \mathcal{L}^u$ are said to make a* perfect fit *if $F^s \cap L^u = \emptyset$ and there exists $x \in F$ and $y \in L$ such that for two half-open segments $\tau^u \subset \widetilde{\mathcal{F}}^u(x)$, ending at x, and $\tau^s \subset \widetilde{\mathcal{F}}^s(y)$, ending at y, we have*

- *For any $\lambda^s \in \mathcal{L}^s$, $\lambda^s \cap \tau^u \neq \emptyset$ if and only if $\lambda^s \cap L^u \neq \emptyset$,*
- *For any $\lambda^u \in \mathcal{L}^u$, $\lambda^u \cap \tau^s \neq \emptyset$ if and only if $\lambda^u \cap F^s \neq \emptyset$.*

In the rest of the text, we will say that an orbit $\widetilde{\alpha}$ in $\mathcal{O}$ is *periodic* if it is the lift of a periodic orbit of ϕ^t. Equivalently, $\widetilde{\alpha}$ is periodic if there exists $g \in \pi_1(M)$, $g \neq \mathrm{Id}$ such that $g \cdot \widetilde{\alpha} = \widetilde{\alpha}$.

Lozenges such that their corners are periodic will be particularly important. One can already easily notice the following

Lemma 3.30. *Let L be a lozenge with corners $\widetilde{\alpha}$ and $\widetilde{\beta}$. If $g \in \pi_1(M)$ fixes $\widetilde{\alpha}$, then g or g^2 fixes $\widetilde{\beta}$.*

Proof. Since g fixes $\widetilde{\alpha}$, it also fixes $\widetilde{\mathcal{F}}^s(\widetilde{\alpha})$. Let A be the half-leaf of $\widetilde{\mathcal{F}}^s(\widetilde{\alpha})$ that is a side of L. Then, either g preserves A or it sends A to the opposite half-leaf of $\widetilde{\mathcal{F}}^s(\widetilde{\alpha})$, so, in any case, g^2 preserves A. Moreover, as a deck transformation preserves globally the foliations, g^2 also preserves the set

$$I^u = \{l^u \in \mathcal{L}^u \mid l^u \cap A \neq \emptyset\}.$$

The same reasoning with the side B, half-leaf in $\widetilde{\mathcal{F}}^u(\widetilde{\alpha})$ shows that g^2 preserves the set

$$I^s = \{l^s \in \mathcal{L}^s \mid l^s \cap B \neq \emptyset\}.$$

Hence, g^2 preserves $L = I^u \cap I^s$, and therefore ∂L. Therefore, g^2 must also fix the two last sides C and D, and the opposite corner $\widetilde{\beta}$. □

Remark 3.31. Note that if both foliations are transversely oriented, which can always be done by taking an at most 4-fold cover, then we do not need to take g^2 in the above lemma.

We can rephrase the above lemma in terms of *free homotopy*

Lemma 3.32. *Let α, β be two indivisible periodic orbits of ϕ^t (i.e., periodic orbits that are not a flow multiple of another periodic orbit), such that there exists lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ that are the two corners of a lozenge L. Then, there exists $i \in \{1, 2\}$ and $j \in \{-2, -1\}$ such that α^i is freely homotopic to β^j.*

Remark 3.33. Once again, if the flow is transversely oriented, then we can eliminate three of the possibilities and we have α freely homotopic to β^{-1}.

Proof. Let $g \in \pi_1(M)$ be the representative of the curve α that fixes $\widetilde{\alpha}$, then by Lemma 3.30, g or g^2 fixes $\widetilde{\beta}$. So there must exists $n \in \mathbb{Z}$ such that α or α^2 is freely homotopic to β^n. Now the same argument with β shows that β or β^2 is freely homotopic to α^m. So, we deduce that there exists $i, j \in \{-2, -1, 1, 2\}$ such that α^i is freely homotopic to β^j.

Now, to see that j has to be negative when i is positive, we look at the action of g on the leaves $\widetilde{\mathcal{F}}^s(\widetilde{\alpha})$ and $\widetilde{\mathcal{F}}^s(\widetilde{\beta})$: they must be of opposite type (i.e., if $\widetilde{\alpha}$ is, say, an attracting fixed point, then $\widetilde{\beta}$ is a repelling one). □

Lozenges do not necessarily stands by themselves, that is, a given orbit can be the corner of several (at most 4) lozenges. If two lozenges share a corner, we call them *consecutive*.

There are basically two configurations for consecutive lozenges: either they share a side, or they do not. The first case is characterized by the fact that there exists a leaf intersecting the interior of both lozenges, while it cannot happen in the second case because a stable leaf in $\widetilde{M}$ cannot intersects an unstable leaf in more than one orbit (by Proposition 3.6). (See Figure 12).

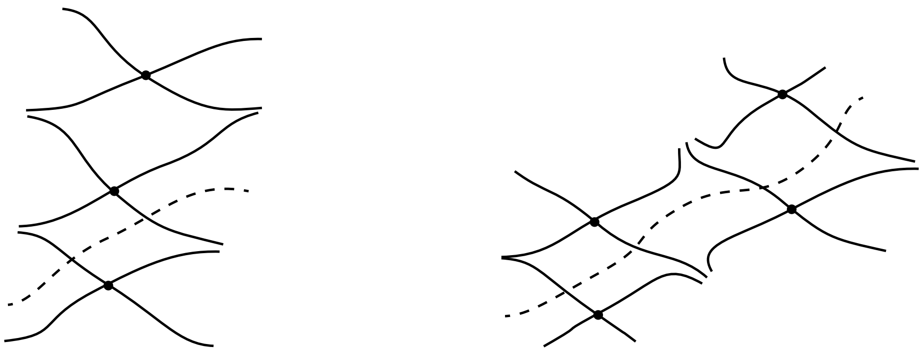
We define the two following types of union of lozenges:

Definition 3.34. *A* chain of lozenges *is a (finite or infinite) union of lozenges, such that, for any two lozenges λ_1, λ_2 in the chain, there exists a sequence of lozenges $L_1, \dots, L_n$ such that $L_1 = \lambda_1$, $L_n = \lambda_2$, and for all i, L_i and L_{i+1} are consecutive.*

A string of lozenges *is a chain of lozenges, such that no two lozenges share a side.*

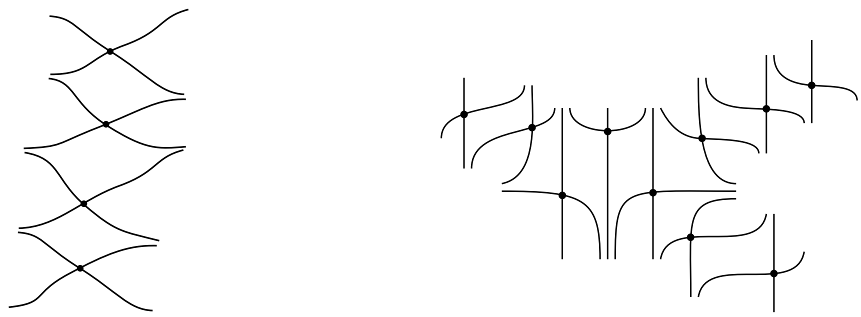
So according to the Lemmas 3.30 and 3.32, in a chain of lozenge such that one corner is periodic, then all the corners are periodic and they all come from orbits that are freely homotopic to each other or to their inverse. But the link between lozenges and free homotopy does not stop here. Indeed, Fenley proved

Proposition 3.35 (Fenley [Fen95b])**.** *If α and β are two freely homotopic orbit of an Anosov flow, then there exists two lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ that are two corners in a chain of lozenges.*



(A) Lozenges sharing only a corner (B) Lozenges sharing sides

Figure 12. The two types of consecutive lozenges



(A) A string of lozenges (B) A chain of lozenges

Figure 13. Chain and string of lozenges

Remark 3.36. As we will see in the proof, the proposition is in fact a bit stronger: It shows that, if α and β are two orbits such that α is freely homotopic to β or β^{-1}, then some coherent lifts can be joined by a chain of lozenges.

Proof sketch. Let $\widetilde{\alpha}$ be a lift of α and $\widetilde{\beta}$ a coherent lift of β, i.e., $\widetilde{\beta}$ is the unique lift obtained from $\widetilde{\alpha}$ by a lift of the free homotopy between α and β. Let $g \in \pi_(M)$ be a non-trivial and indivisible element that fixes both $\widetilde{\alpha}$ and $\widetilde{\beta}$.

Let $F_0 = \widetilde{\mathcal{F}}^s(\widetilde{\alpha})$. Then define $\mathcal{H}_0$ the half-space in $\widetilde{M}$ bounded by F_0 and containing $\widetilde{\mathcal{F}}^s(\widetilde{\beta})$. Let

$$I_0 = \{x \in \widetilde{M} \mid \widetilde{\mathcal{F}}^s(x) \subset \mathcal{H}_0,\, \widetilde{\mathcal{F}}^s(x) \cap \widetilde{\mathcal{F}}^u(\widetilde{\alpha}) \neq \emptyset\}.$$

The set I_0 is open, saturated by leaves of $\widetilde{\mathcal{F}}^s$ and invariant by g. Moreover, since $\widetilde{\mathcal{F}}^s(\widetilde{\beta}) \cap I_0 = \emptyset$, the set I_0 is not the whole half-space $\mathcal{H}_0$. Hence, its boundary ∂I_0 is a non-empty union of leaves of $\widetilde{\mathcal{F}}^s$ and it is invariant under g. Therefore, either $\widetilde{\mathcal{F}}^s(\widetilde{\beta}) \subset \partial I_0$, or there exists a unique leaf F_1 in ∂I_0 that separates F_0 from $\widetilde{\mathcal{F}}^s(\widetilde{\beta})$. Suppose we are in the second case, then, since $\widetilde{\mathcal{F}}^s(\widetilde{\beta})$ is fixed by g, the leaf F_1 also has to be fixed by g.

In either case, we obtained a leaf F_1 such that $g \cdot F_1 = F_1$. Thus, F_1 contains a periodic orbit $\widetilde{\alpha}_1$, and we can redo the above argument with $\widetilde{\alpha}_1$ and $\widetilde{\beta}$. So, by recurrence, we get a sequence of stable leaves F_i, $i = 1, 2, \ldots$, containing a periodic orbit $\widetilde{\alpha}_i$ fixed by g.

Claim 3.37. *The process above ends in finite steps, i.e., there exists n such that $F_n = \widetilde{\mathcal{F}}^s(\widetilde{\beta})$.*

Proof. Suppose that it is not the case. Then for each i, call $\mathcal{C}_i$ the half-space bounded by F_i and containing F_{i-1}. Let $\mathcal{C} = \mathcal{C}_i$. The set $\mathcal{C}$ is open, saturated by leaves of $\widetilde{\mathcal{F}}^s$, g-invariant (since each $\mathcal{C}_i$ is) and does not contain $\widetilde{\mathcal{F}}^s(\widetilde{\beta})$. So, as above, we deduce that there exists a stable leaf $F_\infty \subset \partial\mathcal{C}$ that is also fixed by g (either $F_\infty = \widetilde{\mathcal{F}}^s(\widetilde{\beta})$ or it is the unique leaf that separates $\widetilde{\mathcal{F}}^s(\widetilde{\beta})$ from $\mathcal{C}$).

Let $\widetilde{\alpha}_\infty$ be the periodic orbit on F_∞. Since F_i accumulates on F_∞, we have that $\widetilde{\mathcal{F}}^u(\widetilde{\alpha}_\infty) \cap F_n \neq \emptyset$ for n big enough. This is a contradiction because a given leaf can have only one periodic orbit on it. □

Now we are going to prove that for each i, the orbits $\widetilde{\gamma}_i$ and $\widetilde{\gamma}_{i+1}$ are connected by a chain of lozenges (that all intersect a given stable leaf hence all share a side). We do the proof for $\widetilde{\alpha} = \widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$. Recall that, by definition of the F_i, there exists a stable leaf F^s that intersect both $\widetilde{\mathcal{F}}^u(\widetilde{\gamma}_0)$ and $\widetilde{\mathcal{F}}^u(\widetilde{\gamma}_1)$.

Redoing the above proof but switching the roles of stable and unstable foliations, we deduce that there exists a finite number of unstable leaves L_i, $i = 0, \dots, k$, that are fixed by g and such that $L_0 = \widetilde{\mathcal{F}}^u(\widetilde{\gamma}_0)$ and $L_k = \widetilde{\mathcal{F}}^u(\widetilde{\gamma}_1)$. We call $\widetilde{\delta}_i$ the periodic orbits on L_i.

Now, the Proposition follows from the following claim

Claim 3.38. *The orbits $\widetilde{\gamma}_0$ and $\widetilde{\delta}_1$ are the two corners of a lozenge.*

Proof. First recall that, by definition of the F_i, there exists a stable leaf F^s that intersect both $\widetilde{\mathcal{F}}^u(\widetilde{\gamma}_0)$ and $\widetilde{\mathcal{F}}^u(\widetilde{\gamma}_1)$. Call A^u the side of $\widetilde{\mathcal{F}}^u(\widetilde{\gamma}_0)$ determined by $\widetilde{\gamma}_0$ that intersects F^s.

Since $\widetilde{\mathcal{F}}^u(\delta_1)$ separates $\widetilde{\mathcal{F}}^u(\widetilde{\gamma}_0)$ from $\widetilde{\mathcal{F}}^u(\widetilde{\gamma}_1)$, we also have $F^s \cap \widetilde{\mathcal{F}}^u(\delta_1) \neq \emptyset$. Call C^u the side of $\widetilde{\mathcal{F}}^u(\widetilde{\delta}_1)$ determined by $\widetilde{\delta}_1$ that intersects F^s.

Using the action of g on A^u and C^u, we deduce that every stable leaf that intersects A^u must also intersect C^u (because such a leaf will be in between $g^n \cdot F^s$ and $g^{n+1} \cdot F^s$ for some n).

Now, by definition of $\widetilde{\delta}_1$, there exists a leaf L^u that intersects $\widetilde{\mathcal{F}}^s(\widetilde{\delta}_1)$ and $\widetilde{\mathcal{F}}^s(\widetilde{\gamma}_0)$. Call B^s and D^s the sides of respectively $\widetilde{\mathcal{F}}^s(\widetilde{\delta}_1)$ and $\widetilde{\mathcal{F}}^s(\widetilde{\gamma}_0)$ that intersect L^u. Again, using the action of g, we deduce that any unstable leaf intersecting B^s must intersect D^s.

Hence, A^u, B^s, C^u, and D^u are the four sides of a lozenge with corners $\widetilde{\gamma}_0$ and $\widetilde{\delta}_1$. □

To finish the Proposition, remark that either $\widetilde{\delta}_1 = \widetilde{\gamma}_1$ or we can reapply the claim to the orbits $\widetilde{\delta}_1$ and $\widetilde{\gamma}_1$. □

A consequence of Proposition 3.35 is that if one is interested in studying anything having to do with freely homotopic orbits, then understanding the lozenges in the orbit space will be essential as well as sufficient.

We've already seen that lozenges are everywhere in $\mathbb{R}$-covered flows. Now, we will try to determine which other flows admits lozenges. First, notice the following fact

Lemma 3.39. *If L_1 and L_2 share a side, then the two leaves making perfect fits with that side are non-separated leaves.*

Proof. The two leaves abutting to the shared side are not separated since any leaf in the neighborhood of one of them is in the neighborhood of the other. See Figure 14. □

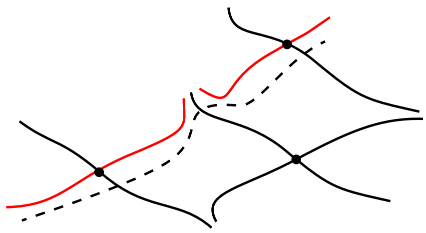


FIGURE 14. Two lozenges sharing a side. The red leaves are not separated.

So having lozenges sharing sides imply the existence of non-separated leaves. As with free homotopy, Fenley proved that it is actually an equivalence, and that moreover the lozenges have to be periodic.

Theorem 3.40 (Fenley [Fen98])**.** *Suppose F_1 and F_2 are two non-separated leaves of $\widetilde{\mathcal{F}}^s$ (or $\widetilde{\mathcal{F}}^u$). Then there exists an even chain of lozenges connecting F_1 and F_2. Moreover, F_1 and F_2 are periodic leaves.*

Before giving an idea of the proof of this theorem, let us state one of its "rigidity-type" corollary.

Corollary 3.41. *Let ϕ^t be an Anosov flow on a 3-manifold M. Either there exists at least two (not-necessarily indivisible) periodic orbits α and β of ϕ^t such that α is freely homotopic to β^{-1}, or ϕ^t is a suspension of an Anosov diffeomorphism.*

Proof. If ϕ^t is not a suspension, then by Theorem 3.20 it is either skewed R-covered, or not $\mathbb{R}$-covered. In both case, we saw that there exists periodic lozenges (by Proposition 3.22 in the skewed $\mathbb{R}$-covered case, and by Theorem 3.40 otherwise). Lemma 3.32 then gives the result. □

Sketch of proof of Theorem 3.40. There are actually two results that we gathered under one Theorem, and their proofs are separate. The first one is that two non-separated leaves must be periodic, i.e., fixed by some non-trivial element of the fundamental group. The second result is the existence of a chain of lozenges between the two leaves.

First part. We start by explaining the proof of the first part, i.e., periodicity.

Define
$$I^u = \{x \in \widetilde{M} \mid \widetilde{\mathcal{F}}^u(x) \cap F_1 \neq \emptyset\}.$$
Then the set I^u is open, saturated by $\widetilde{\mathcal{F}}^u$, and does not intersect F_2 (because otherwise there would be a leaf L^u of $\widetilde{\mathcal{F}}^u$ intersecting both F_1 and F_2, however, since F_1 and F_2 are non-separated, it implies that there is a leaf L^s of $\widetilde{\mathcal{F}}^s$ that intersect L^u in two orbits, a contradiction). Therefore, there exists a unique unstable leaf $L^u \in \partial I^u$ that separates F_1 from F_2.

By definition, L^u makes a perfect fit with F_1. Indeed, since L^u separates F_1 from F_2, all stable leaves close to F_1 on one side must intersect L^u, and since L^u is obtained as a limit of unstable leaves intersecting F_1, we get the second condition of a perfect fit. Our goal will be to show that L^u is periodic, hence F_1 will have to be too (by uniqueness of perfect fits).

Suppose L^u is not periodic. Let F^s be a stable leaf such that the unstable saturation of F^s, that is the set $\{x \in \widetilde{M} \mid \widetilde{\mathcal{F}}^u(x) \cap F^s \neq \emptyset\}$, intersects both F_1 and F_2. This is possible since F_1 and F_2 are non-separated (see Figure 15).

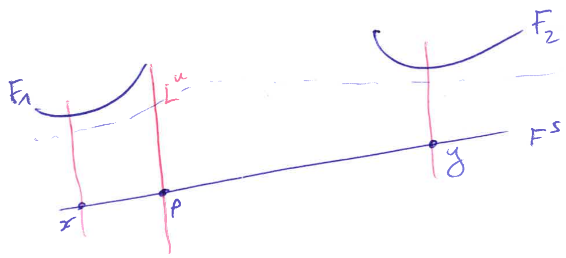


FIGURE 15. Two non-separated leaves

Let $p = L^u \cap F^s$. Since L^u is not periodic, we can find a sequence $p_n := g_n \cdot \widetilde{\phi}^{-t_n}(p)$, with $g_n \in \pi_1(M)$, and t_n positive and unbounded, that converges to some point $p_\infty \in \widetilde{M}$. Pick $x \in F^s = \widetilde{\mathcal{F}}^s(p)$ such that $\widetilde{\mathcal{F}}^u(x) \cap F_1 \neq \emptyset$. Similarly, Pick $y \in F^s$ such that $\widetilde{\mathcal{F}}^u(y) \cap F_2 \neq \emptyset$. Such x, y exist because of our choice of F^s. Call $x_n = g_n \cdot \widetilde{\phi}^{-t_n}(x)$ and $y_n = g_n \cdot \widetilde{\phi}^{-t_n}(y)$. Then the distances $d(x_n, p_n)$ and $d(y_n, p_n)$ both go to infinity.

Fix a flow-box around p_∞. Then, for N big enough we have that for all $n \geq N$, p_n is in the flow box and x_n, y_n are outside. Pick two integer $n, m > N$. Up to switching n and m, we can assume that $\widetilde{\mathcal{F}}^u(p_m)$ intersects $\widetilde{\mathcal{F}}^s(p_n)$ on the side making a perfect fit with $g_n \cdot F_1$, i.e., the side containing x_n. It implies that $\widetilde{\mathcal{F}}^u(p_m)$ must intersect $g_n \cdot F_1$ (because all the stable leaves between p and x intersects F_1). But then stable leaves close enough to $g_m \cdot F_1$ cannot be close to $g_m \cdot F_2$, because they cannot intersect the stable leaf $g_n \cdot F_1$, a contradiction (see Figure 16). Therefore, we proved that non-separated leaves must be periodic.

Second part. We will not give the whole proof of the second part, i.e., the existence of an even chain of lozenges, as it is fairly long, and necessitate to deal with a few cases, but we will just give the general plan.

Note that it does not follow directly from (the proof) of Proposition 3.35. It would if we knew that F_1 and F_2 are fixed by the same element of the fundamental group. But for the moment, all we know is that each leaves are fixed but by possibly two different elements. This gives us the strategy for the proof: one wants to show that, if an element g fixes F_1, then g or g^2 fixes F_2, and then we can just use the arguments of Proposition 3.35 to conclude.

Let $g \in \pi_1(M)$ be the element generating the stabilizer of F_1. Above, we proved that there exists L^u making a perfect fit with F_1 and that is invariant under g. Similarly, one shows that there exists L^u_2 making a perfect fit with F_2 and that separates F_1 and F_2.

Suppose that g does not fix F_2. Then, up to taking g^{-1} instead, we can assume that $g \cdot L^u_2$ is in between L^u_1 and L^u_2. Moreover, one can show that they all intersect a common stable leaf L^s. Iterating, one shows that $g^i \cdot L^u_2$ needs to accumulate on a unstable leaf L^u_3 that has to be fixed by g, and also intersects L^s. So there exists a periodic orbit on L^u_3, call it γ that is freely homotopic to the periodic orbit on L^u_1 or

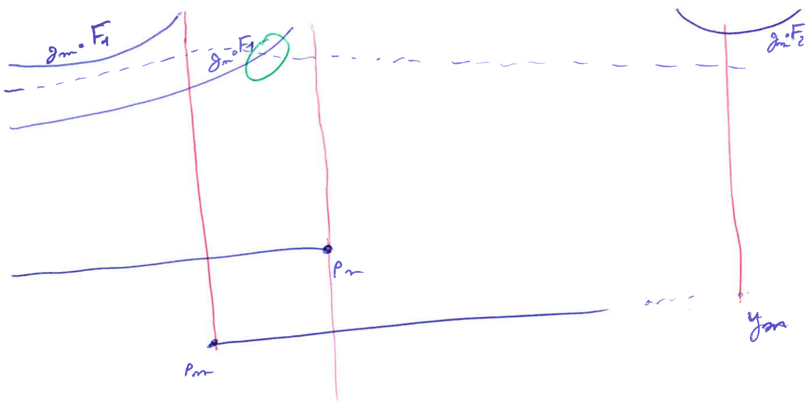


FIGURE 16. An impossible configuration

its inverse. Hence there exists a chain of lozenges that connect them and all these lozenges must share sides, since L^s intersects both L^u_1 and L^u_3.

Now, one uses that to prove that $\widetilde{\mathcal{F}}^s(\gamma)$ has to intersect the half-leaf of L^u_2 determined by its periodic orbit and making a perfect fit with F_2. Calling α the periodic orbit of L^u_2, one can also show that $\widetilde{\mathcal{F}}^s(\alpha)$ intersects a half-leaf of $L^u_3 = \widetilde{\mathcal{F}}^u(\gamma)$, forcing stable leaves to cross. □

So far, we saw that the orbit space of $\mathbb{R}$-covered flows is particularly simple, while having branching leaves makes it more complicated, but with still some fairly rigid fundamental pieces made up by lozenges. Another important fact that limit the complexity of the orbit and leaf spaces of an Anosov flow is the following

Theorem 3.42 (Fenley [Fen98])**.** *For any Anosov flow in a* 3*-manifold, there are only a finite number of branching leaves.*

Remark 3.43. Proposition 3.19 shows that if $\mathcal{F}^s$ does not have branching leaves, then $\mathcal{F}^u$ doesn't either (and vice-versa). As far as I am aware, the following is not yet known: does there exists an Anosov flow with a different number of branching leaves in its stable and unstable foliations?

Proof. Suppose it is not the case, i.e., there exists an infinite family of distinct, say stable, leaves $L_i \in \mathcal{F}^s$ that are branching. Consider lifts $\widetilde{L}_i$. By Theorem 3.40, these leaves makes perfect fits with unstable leaves $\widetilde{F}_i$ that are periodic. Then, up to the action by the fundamental group, the periodic orbits $\widetilde{\gamma}_i$ in $\widetilde{F}_i$ must accumulate somewhere. That is, up to taking different lifts $\widetilde{L}_i$, we can choose points $p_i \in \gamma_i \subset \widetilde{F}_i$ that are all in a given flow box for i big enough. Then we obtain a contradiction as in the proof of Theorem 3.40. □

It follows from Theorem 3.42 and Lemma 3.39 that only a finite number of lozenges, up to deck transformations, can share a side.

An orbit can in general be the corner of anything from 0 to 4 lozenges, but translating Lemma 3.39 to corners gives:

Lemma 3.44. *Suppose that $\widetilde{\alpha}$ is the corner of* 3 *or* 4 *lozenges, then the opposite corners are on non-separated leaves.*

So up to deck transformations, there are only a finite number of orbits that can be the corner of more than 2 lozenges.

Another fact can also limit the number of lozenges abutting to a particular orbit:

Lemma 3.45. *Suppose that $\widetilde{\alpha}$ is an orbit inside a lozenge L. Then $\widetilde{\alpha}$ is the corner of at most two lozenges.*

Proof. If $\widetilde{\alpha}$ is an orbit inside a lozenge L, then at least two of the quadrants that the stable and unstable leaves of $\widetilde{\alpha}$ define cannot be part of a lozenge as can be seen in Figure 17: The quadrant containing the red leaves cannot be part of a lozenge, since otherwise two stable leaves (and two unstable leaves) would intersect. The other two quadrants can however define lozenges, as can be seen with the blue leaves in Figure 17. □

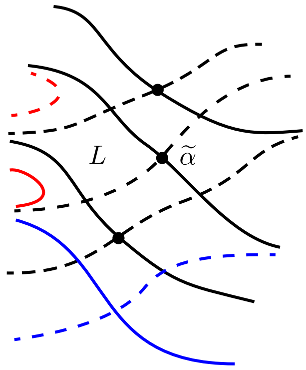


FIGURE 17. An orbit in a lozenge cannot be the corner of more than two lozenges

Recall that one of the properties of Theorem 3.23 is that skewed $\mathbb{R}$-covered Anosov flows (with transversely orientable foliations) are orbit equivalent to their inverse, and that the orbit equivalency can be taken to be homotopic to the identity. It turns out that one can completely describe self orbit equivalences of any Anosov flow that are homotopic to identity:

Theorem 3.46 (Barthelmé and Gogolev [BG17]). *Let M be a 3-manifold and $\phi^t\colon M \to M$ be an Anosov flow. Let $h\colon M \to M$ be a self orbit equivalence of ϕ^t which is homotopic to identity. Then we have:*

(1) If ϕ^t is not $\mathbb{R}$-covered, then h preserves every orbit of ϕ^t.
(2) If ϕ^t is a suspension of an Anosov diffeomorphism or the geodesic flow of a negatively curved surface, then h preserves every orbits of ϕ^t.
(3) If ϕ^t is $\mathbb{R}$-covered but not one of the above two cases, then there exist $\eta\colon M \to M$ (independent of h) a self orbit equivalence of ϕ^t which is homotopic to identity, $\eta \neq \mathrm{Id}$, and $k \in \mathbb{Z}$ (depending on h) such that $h \circ \eta^{2k}$ preserves every orbits of ϕ^t. Moreover, there are two possible sub-cases
  (a) Either, for all i, $\eta^i \neq \mathrm{Id}$, in which case k above is unique,
  (b) Or, there exists q such that $\eta^q = \mathrm{Id}$ in which case ϕ^t is the q-cover of a geodesic flow on a negatively curved surface. In this case, the number k is unique modulo q.

Remark 3.47. The above result is only proven for transitive Anosov flows in [BG17], because the proof is slightly simpler in that case. However, one can use the same techniques to prove that result without the transitivity assumption.

The map η in case 3) above is the map given by Proposition 3.22

As a corollary, we obtain another characterization of skewed $\mathbb{R}$-covered flows:

Corollary 3.48. *Let ϕ^t be an Anosov flow on a 3-manifold M. Suppose that one of the weak foliations is transversely orientable, then ϕ^t is orbit equivalent to its inverse by a homeomorphism homotopic to the identity if and only if ϕ^t is skewed $\mathbb{R}$-covered.*

3.3. **Modified JSJ decomposition.** We will now try to understand how lozenges can relate to tori in toroidal manifolds. We will also describe works of Barbot [Bar95b] and Barbot and Fenley [BF13, BF15] on how tori interacts with the flow.

Definition 3.49. *Let ϕ^t be an Anosov flow on M. A* Birkhoff annulus *is an (a priori only immersed) annulus $A \subset M$ such that its boundary ∂A consists of two (periodic) orbits of ϕ^t, and its interior $\mathring{A}$ is transverse to ϕ^t.*

The weak stable and weak unstable foliations gives 1-dimensional foliations on any Birkhoff annulus. A Birkhoff annulus is called *elementary* if the induced foliations do not have Reeb components, and the only closed leaves are the two orbits of the flow.

Example 5. Consider $\phi^t\colon S\Sigma \to S\Sigma$ a geodesic flow on a surface. Let $c(t) \subset \Sigma$ be a closed geodesic. Then $(c(t), \dot{c}(t)) \in S\Sigma$ is a periodic orbit of ϕ^t. Define the annulus

$$A := \{(c(t), \dot{c}(t) + \theta) \mid 0 \leq \theta \leq \pi\},$$

where $v + \theta$ is the vector obtained by rotating v by an angle θ. Then A is a Birkhoff annulus.

Proposition 3.50 (Barbot [Bar95b]). *Let A be a Birkhoff annulus and $\widetilde{A}$ a lift of it to $\widetilde{M}$. Then, $\pi_{\mathcal{O}}(\widetilde{A})$ is a finite chain of lozenges (only containing the corners of the first and last), where $\pi_{\mathcal{O}}\colon \widetilde{M} \to \mathcal{O}$ is the* *projection. Moreover, if there are more than one lozenge in the chain, then each lozenge, except for the first and last, has two neighbors and share a side with both of its neighbors.*

Sketch of the proof of 3.50. The proof consists of splitting A along subannuli A_i, such that each boundary of A_i is a closed leaf of either $\mathcal{F}^s \cap A$ or $\mathcal{F}^u \cap A$, then showing that each lift $\widetilde{A}_i \subset \widetilde{A}$ projects to a lozenge. □

We also have a sort of inverse of Proposition 3.50

Proposition 3.51 (Barbot [Bar95b]). *Let L be a periodic lozenge. Then there exists either an elementary Birkhoff annuli or a Klein bottle A such that $\pi_{\mathcal{O}}(\widetilde{A}) = L$ for some lift $\widetilde{A}$ of A.*

Remark 3.52. The case when A is a Klein bottle would happen if the two corners of L projects to the same periodic orbit in M. This can actually happen, for instance in the geodesic flow of a non-orientable surface.

Proof sketch. Choose a curve $\bar{c}$ in L joining the two corners, that does not have self-intersection and that is also disjoint from its iterates under g, the indivisible element of $\pi_1(M)$ that fixes the lozenge L. Let $\widetilde{c}$ be any lift of $\bar{c}$ to $\widetilde{M}$. Then $\widetilde{c}$, $g \cdot \widetilde{c}$, and the two segments along the corner orbits between the extremities of $\widetilde{c}$ and $g \cdot \widetilde{c}$ bound a rectangle that is transverse to the flow except along the periodic orbits.

Now, this rectangle projects to an annulus, that is transverse to the flow in its interior and tangent on its boundary. Unfortunately, this annulus is a priori only immersed. One needs to make some modifications to make it embedded. □

The main results of Barbot in [Bar95b] shows that any embedded tori in a 3-manifold that supports an Anosov flow can be put in "optimal" position with respect to the flow. That is, an embedded torus can be homotoped, along the flow lines of ϕ^t, to be either transverse to ϕ^t or *quasi-transverse* to it, i.e., such that it is a finite union of Birkhoff annuli. A quasi-transverse torus will be called a *Birkhoff torus.*

Instead of stating Barbot's result right away, we will give a slightly more general version that Barbot and Fenley obtained some years later [BF13, BF15].

In order to do that, we need to recall some facts about 3-manifold topology.

A 3-manifold M is called *irreducible* if every embedded sphere bounds a ball. A fundamental result of Jaco–Shalen and Johannson states that 3-manifolds are decomposed into simple pieces. A 3-manifold M is *Seifert fibered* if it has a foliation by circles [Hem76, Eps66]. A 3-manifold M is *atoroidal* if every π_1-injective map from the torus $f : T^2 \to M$ is homotopic into the boundary of M. A Seifert fibered manifold usually has many π_1-injective tori that are not homotopic to the boundary, so Seifert fibered and hyperbolic manifolds are, in some sense, opposites. The Jaco-Shalen-Johannson decomposition theorem also called the JSJ, or torus, decomposition states the following:

Theorem 3.53 (Jaco–Shalen, Johannson). *Let M be a compact, irreducible, orientable 3-manifold. Then there is a finite collection $\{T_j\}$ of π_1-injective, embedded tori which cut M into pieces $\{P_i\}$ such that the closure of each component P_i of $M - \cup T_j$ is either Seifert fibered or atoroidal. In addition except for a very small and completely specified class of simple manifolds, the decomposition (in other words the T_j or the P_i up to isotopy) is unique if the collection $\{T_j\}$ is minimal.*

Any manifold M supporting an Anosov flow is irreducible, since its universal cover is $\mathbb{R}^3$. However it may not be orientable, so, in this section, we will have to lift the flow to an orientable double cover.

Our goal here is to obtain a JSJ-decomposition that is also well-adapted to the flow, i.e., such that each of the tori are as transverse as possible to the flow. One can use the properties of lozenges and there relationship with Birkhoff annuli to show the following

Theorem 3.54 (Barbot and Fenley [BF13]). *Let ϕ^t be an Anosov flow on an orientable manifold M. Suppose that ϕ^t is not a suspension of an Anosov diffeomorphism. Let T be an embedded π_1-injective torus. Then one of the following happens*

(1) T is isotopic to an embedded Birkhoff torus.
(2) T is homotopic to a weakly embedded Birkhoff torus, and contained in a periodic Seifert piece of the JSJ decomposition. Here, weakly embedded *means that the torus is embedded outside of the tangent orbits.*
(3) T is isotopic to the boundary of a tubular neighborhood of an embedded quasi-transverse Klein bottle, and contained in a free Seifert piece of the JSJ decomposition.

The definitions of free and periodic Seifert pieces are given below in Definition 3.58

Note that this Theorem says that, if the flow is not a suspension, then an embedded torus T is always homotope to a *quasi-transverse* torus T'. This holds even when T is transverse.

Using the relationship between Birkhoff tori and chain of lozenges, the above theorem implies in particular that, if the flow is not a suspension, then any $\mathbb{Z}^2$-subgroup of $\pi_1(M)$ preserves a chain of lozenges.

We can use Barbot and Fenley work to obtain what we call the *modified JSJ decomposition*.

Theorem 3.55 ([BF13], Sections 5 and 6)**.** *Let ϕ^t be an Anosov flow in M orientable, which is not orbit equivalent to a suspension Anosov flow. Let $\{T'_j\}$ be a collection of disjoint, embedded tori given by the JSJ decomposition theorem. Then each torus T'_j is homotopic to a weakly embedded quasi-transverse torus T_j. In case T_j is not unique up to flow homotopy then T'_j is also isotopic to a transverse torus, which will then be denoted by T_j.*

Moreover, the collection $\{T_j\}$ is also weakly embedded*, that is, embedded outside the union of the orbits tangent to the tori T_j that are quasi-transverse to the flow.*

With these choices the tori T_j are unique up to flow homotopy and unique up to flow isotopy outside the tangent orbits. The closure of the complementary components P_i of $\cup T_j$ are called the pieces of the modified JSJ decomposition.

Furthermore, if P_i is not a manifold, then there are arbitrarily small neighborhoods of P_i that are representatives of the corresponding piece P'_j of the torus decomposition of M.

The fact that P_i may not always be a submanifold is due to the possible collapsing of tangent orbits in the union of the tori T_j. For example it could be that two distinct "boundary" components T_j and T_k of P_i have a common tangent orbit γ (and this is quite common as can be seen in [BF13]). Then, along γ, the piece P_i is not a manifold with boundary, since two "sheets" of the boundary of P_i intersect at γ.

In addition, notice that to ensure the flow uniqueness of the T_j, we need to choose the transverse tori in the case that there are two essentially distinct quasi-transverse tori homotopic to a given T'_j.

Let us now describe how the flow intersects a piece P_i: An orbit intersecting ∂P_i intersects it either in the tangential or transverse part of ∂P_i. If it is tangent then it is *entirely* contained in T_j for some j and so entirely contained in ∂P_i. Otherwise it either enters or exits P_i. Hence the fact that P_i may not be a manifold only affects the orbits that are entirely contained in ∂P_i.

The advantage of this modified JSJ decomposition is that periodic orbits of the flow behaves very well with respect to it, in particular, we will see that orbits that are freely homotopic either are in the same piece or go through the same pieces. In order to explain this, we need a few definitions.

First, as we have noticed before in Lemma 3.32 and Proposition 3.35, on the orbit space it is nicer to group together orbits that are freely homotopic to each other *or* to the inverse of the other. So we introduce the following definition

Definition 3.56. *Two periodic orbits α and β are called* freely homotopic up to orientation*, or* FHUTO*, if α is freely homotopic to either β or β^{-1}.*

The FHUTO class *of a periodic orbit α consists of all the periodic orbits that are FHUTO with α. We denote by $\mathcal{FH}(\alpha)$ the FHUTO class of α.*

Definition 3.57 (intersecting a piece, crossing a piece)**.** *Let P be a piece of the torus decomposition of M and let P' be an associated piece of the modified JSJ decomposition. We say that a periodic orbit α intersects P if α is either a tangent orbit in P' or if it intersects $\partial P'$ transversely. In the second case we in addition say that α crosses the piece P. We may also refer to this as α intersects or crosses P', the associated piece of the modified JSJ decomposition.*

Notice that P is defined up to isotopy and P' is defined up to homotopy along flow lines and isotopy outside the tangent orbits. Therefore α intersects P or crosses P independently of the particular modified JSJ representative P' and depends only on the isotopy class of P.

Definition 3.58 (periodic piece, free piece)**.** *With respect to an Anosov flow, a Seifert fibered piece S of the torus decomposition of M can have one of two possible behaviors:*

- *Either there exists a Seifert fibration of S and up to powers there exists a periodic orbit in M which is freely homotopic to a regular fiber of S in this Seifert fibration; in which case the piece is called* periodic*;*
- *Or no periodic orbit is freely homotopic to a regular fiber (even up to powers) of any Seifert fibration of S; and the piece S is then called* free.

Note that, S is periodic if and only if there is a Seifert fibration of S such that if $h \in \pi_1(S)$ represents a regular fiber of S, then h does not act freely in at least one of the leaf spaces of stable/unstable leaves in $\widetilde{M}$.

Example 6. The geodesic flow on a negatively curved surface is a free Seifert piece since no periodic orbit is freely homotopic to a circle of directions. Similarly, all the examples obtained by Handel–Thurston surgery only have free Seifert pieces [Bar96]. More generally, any $\mathbb{R}$-covered Anosov flow admits only free Seifert pieces in its JSJ decomposition [BF13]. However, there are a lot of Anosov flows having periodic Seifert pieces [BF15].

We can finally state how FHUTO periodic orbits interact with a modified JSJ decomposition.

Lemma 3.59 (Barthelmé and Fenley [BF17])**.** *Let ϕ^t be an Anosov flow on an orientable 3-manifold M. Let $M = \cup_j N_j$ be a modified JSJ decomposition. Let α_0 be a periodic orbit and $\mathcal{FH}(\alpha_0)$ its FHUTO class. Suppose that some orbit $\beta \in \mathcal{FH}(\alpha_0)$ crosses a piece $N = N_k$. Then all the orbits $\alpha \in \mathcal{FH}(\alpha_0)$ also cross N_k.*

In addition, if there exists a connected component β_1 of $\beta \cap N$ between two boundary torus T_1 and T_2 (where we also allow $T_1 = T_2$), then, for any $\alpha \in \mathcal{FH}(\alpha_0)$, there exists a connected component α_1 of $\alpha \cap N$ between T_1 and T_2 that is in the same free homotopy class as β_1 modulo boundary.

Furthermore, the free homotopy between two segments of orbits can always be realized inside the pieces of the decomposition that the orbits crosses.

Lemma 3.60 (Barthelmé and Fenley [BF17])**.** *Suppose that α and β are contained in a piece N of the torus decomposition and that they are in the same FHUTO class. Let H be a free homotopy between them. Then we can choose a free homotopy from α to β entirely contained in N, unless, possibly, the image of H intersects a periodic Seifert piece.*

3.4. **From chains to strings of lozenges.** In general, strings of lozenges are easier to deal with than more general chains of lozenges. We will see here that a chain of lozenge can actually be decomposed into a finite number of strings plus a (uniformly) finite part. All the results in these subsection are taken from [BF17].

We fix some terminology first. Let α be a periodic orbit of an Anosov flow ϕ^t. Let $\mathcal{FH}(\alpha)$ be the FHUTO class of α. We define a *coherent lift* of $\mathcal{FH}(\alpha)$ in the following way: Let g be an element of the fundamental group that represents α (so any other element of the conjugacy class of g would also represent α). A coherent lift of $\mathcal{FH}(\alpha)$ is the set of all the lifts of orbits in $\mathcal{FH}(\alpha)$ that are invariant under g. Notice that there may be distinct orbits (possibly infinitely many) in a coherent lift of $\mathcal{FH}(\alpha)$ that project to the same orbit in $\mathcal{FH}(\alpha)$.

Definition 3.61. *We say that $\{\alpha_i\}_{i\in I}$ is a* string of orbits *in $\mathcal{FH}(\alpha)$, if it satisfies to the following conditions:*

- *All the α_i are distinct and contained in $\mathcal{FH}(\alpha)$;*
- *For a coherent lift of $\mathcal{FH}(\alpha)$, the orbits $\{\alpha_i\}_{i\in I}$ are the projections of the corners of a string of lozenges $\{\widetilde{\alpha}_i\}$ (see Definition 3.34 above);*
- *Each $\widetilde{\alpha}_i$ is the corner of at most two lozenges in $\widetilde{M}$.*
- *Here I is an interval in $\mathbb{Z}$, which could be finite, isomorphic to $\mathbb{N}$ or $\mathbb{Z}$ itself.*

There are several slightly different types of string of orbits:

- A string of orbits $\{\alpha_i\}$ is *infinite* if it is indexed by $i \in \mathbb{N}$. We call it *bi-infinite* if it is indexed by $\mathbb{Z}$.
- A string of orbits $\{\alpha_i\}$ is *finite and periodic* if it is finite but the collection $\{\alpha_i\}$ is the projection of corners of an *infinite* string of lozenges. In other words the collection $\{\widetilde{\alpha}_i\}_{i\in\mathbb{Z}}$ is infinite, but there is an element $h \in \pi_1(M)$ and a integer $k > 0$ such that $h \cdot \widetilde{\alpha}_i = \widetilde{\alpha}_{i+k}$. Note that all the orbits in a periodic string are non-trivially freely homotopic to themselves (up to powers).
- A string of orbits $\{\alpha_i\}$ is *finite and non-periodic* otherwise. In other words the string $\{\alpha_i\}$ is *finite*, and it is not the projection of an infinite string $\{\widetilde{\alpha}_i\}, i \in \mathbb{N}$.

Example 7. Suppose that ϕ^t is $\mathbb{R}$-covered and that $\mathcal{F}^s$ is transversely orientable. Let α be a periodic orbit. Choose $\widetilde{\alpha}$ a lift of α and, set $\alpha_i = \pi\left(\eta^i(\widetilde{\alpha})\right)$ (where η is the map on the orbit space defined in Proposition 3.22). Then $\{\alpha_i\}$ is either a finite periodic string of orbits or a bi-infinite string of orbits. In addition the FHUTO class of α is exactly the collection $\{\alpha_i\}$.

If ϕ^t is a geodesic flow on an orientable surface, then, for any periodic orbit α its complete FHUTO class is α and α^{-1}. At the opposite, if ϕ^t is an $\mathbb{R}$-covered flow in a *hyperbolic* manifold, then for any periodic orbit α, its FHUTO class is (bi)-infinite. Indeed, if $\pi\left(\eta^i(\widetilde{\alpha})\right)$ is finite, it means that there exists i_0 and $h \in \pi_1(M)$ such that $h \cdot \eta^i(\widetilde{\alpha}) = \widetilde{\alpha}$. Now, if $g \in \pi_1(M)$ is and indivisible element fixing $\widetilde{\alpha}$, then one can show that the group generated by g and h is a $\mathbb{Z}^2$ ([Fen94]).

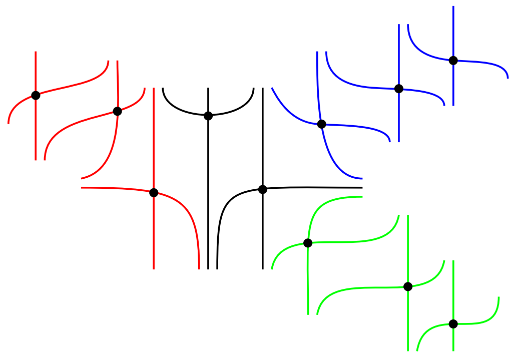


FIGURE 18. Three different strings of lozenges inside a chain of lozenges

So the above example shows that FHUTO class in a $\mathbb{R}$-covered Anosov flow are extremely simple. It turns out that, in general, it is not so much more complicated.

Proposition 3.62. *Let α be a closed orbit of an Anosov flow on a 3-manifold. The FHUTO class $\mathcal{FH}(\alpha)$ can be decomposed in the following way:*

- *A finite part $\mathcal{FH}_{finite}(\alpha)$,*
- *A finite number of disjoint strings of closed orbits (that could be infinite, finite and periodic or just finite).*

Moreover, there exists a uniform bound (i.e., depending only on the manifold and the flow) on the number of elements in $\mathcal{FH}_{finite}(\alpha)$. And there exists a uniform bound on the number of different strings that a FHUTO class can contain.

In fact, the statement about the uniform bounds can be made even stronger: Except for a finite number of FHUTO classes, each FHUTO class is either a finite, infinite, or bi-infinite string of orbits.

Notice however that this result does not say that there exists a uniform bound on the number of orbits inside a *finite* FHUTO class, but just a bound on the parts of a FHUTO class that are not strings of orbits.

Idea of proof. The idea of the proof of this proposition is fairly straightforward: given a FHUTO class $\mathcal{FH}(\alpha)$ and a coherent lift $\widetilde{\mathcal{FH}}(\alpha)$, this coherent lift is *not* a string of lozenges if and only if there exists two lozenges that share a side. By Lemma 3.39 this happens if and only if it contains some non-separated leaves. Now, according to Theorem 3.42, there are only a finite number of such non-separated leaves up to deck transformations. So, if we remove all the corners in $\widetilde{\mathcal{FH}}(\alpha)$ that are on non-separated leaves, or their neighbors, will be left with strings of orbits, and, up to deck transformation, there are only a finite number of different strings in a fixed coherent lift. □

It is noteworthy that strings of orbits that are finite and periodic are actually fairly special, in the sense that they are forced to stay in some topologically limited part of the manifold M:

Proposition 3.63. *Let $\{\alpha_i\}$ be a finite periodic string of orbits. Then $\{\alpha_i\}$ is a complete FHUTO class, i.e., $\{\alpha_i\} = \mathcal{FH}(\alpha_0)$. In addition they are either entirely contained in a Seifert piece of the modified JSJ decomposition, or are the orbits on one of the quasi-transverse decomposition tori.*

Proof. Let $\widetilde{\alpha}_i$ be a coherent lift of $\{\alpha_i\}$. Let $g \in \pi_1(M)$ be a generator of the stabilizer of all the $\widetilde{\alpha}_i$ and $h \in \pi_1(M)$ such that $h \cdot \widetilde{\alpha}_i = \widetilde{\alpha}_{i+k}$. First, applying h^n, $n \in \mathbb{Z}$, to α_0 shows that the indexation i needs to be bi-infinite, and since all the $\widetilde{\alpha}_i$ are, by definition, assumed to be the corners of at most two lozenges, the part $\mathcal{FH}_{\text{finite}}(\alpha_0)$ has to be empty and $\{\alpha_i\} = \mathcal{FH}(\alpha_0)$, which finishes the first part.

Now, since α_0 is freely homotopic to itself, there exists a π_1-injective immersed torus that contains α_0. Using Gabai's version of the Torus Theorem [Gab92], we see that this immersed torus is either embedded or the manifold is (a special case of) Seifert-fibered. If it is the second case, we are done, and if the torus is embedded, then it can be isotoped inside a Seifert piece or to one of the modified JSJ decomposition tori (see section 3.3 or [BF13]), which finishes the proof. □

3.4.1. *Strings of orbits and metric geometry.* One of the great advantages of string of orbits, that is used in an essential way for the counting results we will describein Section 5, is that the distance between orbits in a string is uniformly bounded both above and below:

Lemma 3.64. *There exists $A > 0$, depending only on the flow, such that, if $\{\alpha_i\}$ is a string of orbits and $\{\widetilde{\alpha}_i\}$ is a coherent lift, then, for all i,*

$$d(\widetilde{\alpha}_0, \widetilde{\alpha}_i) \geq A|i|.$$

Here d is the minimum distance between $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_i$. Notice that in this and in the following result we do not need to assume that M is orientable or any hypothesis on $\mathcal{F}^s, \mathcal{F}^u$.

Lemma 3.65. *There exists $B > 0$, depending only on the flow, such that, if $\{\alpha_i\}$ is a string of orbits, then, for all i, there exists an homotopy H_i between α_0 and α_i that moves points a distance at most $B|i|$.*

Proof of Lemma 3.64. First, notice that since we can cover M by a finite number of flow boxes, there exists a constant A, depending only on the flow and M, such that, if the minimum distance between two, say, stable leaves $\lambda_1, \lambda_2 \in \widetilde{\mathcal{F}}^s$ is less than A, then there exists an unstable leaf $l^u \in \widetilde{\mathcal{F}}^u$ intersecting both λ_1 and λ_2.Moreover, if the minimum distance between two orbits α and β of the lifted flow $\widetilde{\phi}^t$ is less than A, then the stable leaf through α intersects the unstable through β and vice-versa.

Now the lemma follows easily: Let $\widetilde{\alpha}_i$ be a coherent lift of the α_i. There exists a uniform constant $A > 0$ such that, since the stable leaf of $\widetilde{\alpha_1}$ does not intersect the unstable leaf of $\widetilde{\alpha}_0$, $d(\widetilde{\alpha}_1, \widetilde{\alpha}_0) \geq A$. Moreover, we can choose A such that the minimum distance between the stable leaves of $\widetilde{\alpha}_i$ and $\widetilde{\alpha}_{i+2}$ is at least $2A$, because no unstable leaf intersects both the stable leaf of $\widetilde{\alpha}_i$ and $\widetilde{\alpha}_{i+2}$.

So, using the facts that $\widetilde{M} \simeq \mathbb{R}^3$, that each leaves of the lifted flow is homeomorphic to $\mathbb{R}^2$ and that the stable leaf of $\widetilde{\alpha}_i$ separates $\widetilde{M}$ in two pieces, one containing $\widetilde{\alpha}_{i-1}$ and the other $\widetilde{\alpha}_{i+1}$, we immediately obtain

$$d(\widetilde{\alpha}_0, \widetilde{\alpha}_i) \geq A|i|. \qquad \square$$

Lemma 3.65 follows from the following non-trivial result

Proposition 3.66 (Fenley [Fen16])**.** *Let α, β be two freely homotopic orbits such that they admits lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ that are the corners of the same lozenge. Then, there exists $B > 0$ depending only on the flow and on the manifold such that there exists an homotopy H from α to β that moves each point by a distance at most B.*

In order to give an idea of the proof of the Proposition above, we need to introduce one more definition. A (stable) *product region* Q in $\mathcal{O}$ is an open set such that the restriction of the stable and unstable foliations is product (i.e., every stable in Q intersect every unstable and vice versa), and such that Q contains a complete stable half-leave. If Q contains a complete unstable half-leaf instead, then it is called an unstable product region. The requirement that Q is unbounded, i.e., contains a complete half-leaf is essential in the definition.

A product region can be very common for general foliations of the plane, but it turns out to be extremely specific for foliations coming from Anosov flows:

Proposition 3.67 (Fenley [Fen98])**.** *If an Anosov flow ϕ^t admits a product region, then ϕ^t is a suspension of an Anosov diffeomorphism.*

Sketch of proof. We only sketch the proof when we assume ϕ^t to be transitive, as it is extremely easy in that case. First notice that by the description of the structure of the orbit space in the skewed R-covered case given in Proposition 3.22, there are no product regions in skewed $\mathbb{R}$-covered Anosov flow. Hence, by Theorem 3.20, if ϕ^t is $\mathbb{R}$-covered and there exists a product region then ϕ^t is a suspension.

So suppose that ϕ^t is not $\mathbb{R}$-covered. Let Q be a stable product region. Then there exists a leaf $F^u \in \widetilde{\mathcal{F}}^u$ and a subspace $l^u \subset F^u$, such that Q contains the one-sided saturation of l^u by $\widetilde{\mathcal{F}}^s$. Here by one-sided, we mean all the half-leaves of $\widetilde{\mathcal{F}}^s$ on one side of l^u.

Since ϕ^t is not $\mathbb{R}$-covered, there exists F_1^s, F_2^s two non-separated stable leaves. Thanks to the transitivity of ϕ^t, up to the action by deck transformation, we can suppose that $F_1^s \cap l^u$. Then F_2^u also has to be in Q and we get a contradiction.

The non-transitive case is a bit harder but can still be dealt with thanks to the fact that the stable and unstable leaves of periodic orbits are dense in $\mathcal{O}$. $\square$

Using Proposition 3.67, one can prove Proposition 3.66

Sketch of proof of Proposition 3.66. We show more generally that there exists a uniform bound on the Hausdorff distance between two corners of a lozenge. Arguing by contradiction, suppose that there exists $\widetilde{\alpha}_i$, $\widetilde{\beta}_i$ a sequence of periodic corners of lozenges and $p_i \in \widetilde{\alpha}_i$ getting infinitely far from $\widetilde{\beta}_i$. Then, up to acting by deck transformations, one can suppose that p_i converges to a point p. Then, using the structure of lozenges, one can show that the quadrant determined by $\widetilde{\mathcal{F}}^s(p)$ and $\widetilde{\mathcal{F}}^u(p)$ that contains infinitely many p_i must be a product region, yielding a contradiction. $\square$

4. On rigidity and flexibility results

If the full classification of Anosov flows up to orbit equivalency is still far away in the future (if it ever happens), there are nonetheless a lot of results about the rigidity and/or flexibility of Anosov flows on a given manifold. We will gather some here, as well as some still open questions.

4.1. **Rigidity.** The first type of rigidity results that were obtained only requires a knowledge of the topology of the manifold, but, unsurprisingly, that topology has to be very special.

Theorem 4.1 (Plante [Pla81])**.** *Let ϕ^t be an Anosov flow on a 3-manifold M. If $\pi_1(M)$ is solvable, then ϕ^t is orbit equivalent to a suspension of an Anosov diffeomorphism.*

Theorem 4.2 (Ghys [Ghy84], Barbot [Bar96])**.** *Let ϕ^t be an Anosov flow on a 3-manifold M. If M is Seifert fibered, then up to taking a finite cover, ϕ^t is orbit equivalent to a geodesic flow.*

Remark 4.3. In [BF13], Barbot and Fenley prove that the above two results holds also for *pseudo-Anosov flow*, i.e., one does not have to assume that the flow is Anosov to get the rigidity.

These two results are the only ones where such a decisive answer can be given based on just the topology. All the other classification or rigidity results we have seen or will see requires more than just topology. But before that, we mention a few results that requires only topology, but gives *negative* results:

Calegari and Dunfield [CD03] proved that if a 3-manifold supports a (pseudo)-Anosov flow (more generally, admits an essential lamination), then $\pi_1(M)$ acts on a circle. In the same article they also prove that the fundamental group of the Weeks manifold (the Weeks manifold is the closed orientable hyperbolic 3-manifold with the smallest volume) cannot act on the circle, hereby proving

Theorem 4.4 (Calegari and Dunfield [CD03])**.** *The Weeks manifold does not admit any (pseudo)-Anosov flows.*

Around the same time, Roberts, Shareshian, and Stein [RSS03] exhibited a family of hyperbolic metrics that do not admit any Reebless foliations, hereby proving

Theorem 4.5 (Roberts, Shareshian, and Stein [RSS03])**.** *There exists infinitely many hyperbolic 3-manifolds that do not admit any Anosov flows.*

Remark 4.6. Fenley [Fen07] proved that a subfamily of the examples considered by Roberts, Shareshian, and Stein actually do not support essential laminations either. In particular, that subfamily does not support pseudo-Anosov flows.

Coming back to rigidity results, we will explain the proof of the following

Theorem 4.7 (Barthelmé and Fenley [BF17])**.** *Let ϕ^t be a $\mathbb{R}$-covered Anosov flow on a closed 3-manifold M. Suppose that every periodic orbit of ϕ^t is freely homotopic to* at most *a finite number of other periodic orbits. Then either ϕ^t is orbit equivalent to a suspension or ϕ^t is orbit equivalent to a finite cover of the geodesic flow of a negatively curved surface.*

We can prove that Theorem fairly easily using what we have seen so far. In fact, it is a consequence of the following

Theorem 4.8 (Barthelmé and Fenley [BF17])**.** *Let ϕ^t be a $\mathbb{R}$-covered Anosov flow on a closed, orientable 3-manifold M and suppose that ϕ^t is not orbit equivalent to a suspension. Suppose that $\mathcal{F}^s$ is transversely orientable. Let α be a periodic orbit of ϕ^t. Then α has only finitely many periodic orbits in its free homotopy class, if and only if α is either isotopic into one of the tori of the JSJ decomposition, or isotopic to a curve contained in a Seifert-fibered piece of the JSJ decomposition.*

Proof. Since ϕ^t is skewed $\mathbb{R}$-covered, there are no branching leaves and hence any chain of lozenges is in fact a string of lozenges. Moreover each lift $\widetilde{\alpha}$ of α generates an infinite string of lozenges $\mathcal{C}$ in $\widetilde{M}$. Since this is a string of lozenges then a closed orbit β is in $\mathcal{FH}(\alpha)$ if and only if there is a lift $\widetilde{\beta}$ that is a corner of $\mathcal{C}$. Hence $\mathcal{FH}(\alpha)$ is finite if and only if the string of orbits obtained by projecting the corners of $\mathcal{C}$ to M is finite, that is, $\mathcal{FH}(\alpha)$ is finite periodic. So in particular if $\mathcal{FH}(\alpha)$ is finite, then Proposition 3.63 implies the result.

Let us now deal with the other direction. Suppose that up to isotopy α is on one of the tori or entirely inside a Seifert piece of the JSJ decomposition.

If α is on one of the boundary tori then as an element of $\pi_1(M)$, α is in a $\mathbb{Z}^2$ subgroup of $\pi_1(M)$. If α is contained in a Seifert piece of the JSJ decomposition, then in $\pi_1(M)$, α^2 commutes with an element representing a regular fiber of the Seifert fibration in the piece. In either case α^2 is an element of a subgroup $G \sim \mathbb{Z}^2$ of $\pi_1(M)$. Let $g \in G$ associated with α^2, and $\widetilde{\alpha}$ a lift of α to $\widetilde{M}$ left invariant by g. Let $f \in G$ not leaving $\widetilde{\alpha}$ invariant. Then

$$g(f(\widetilde{\alpha})) = f(g(\widetilde{\alpha})) = f(\widetilde{\alpha}),$$

so $\widetilde{\alpha}$ and $f(\widetilde{\alpha})$ are distinct orbits of $\widetilde{\phi}^t$ that are invariant under g non trivial in $\pi_1(M)$. This implies that $\widetilde{\alpha}$ and $f(\widetilde{\alpha})$ are connected by a chain of lozenges $\mathcal{C}_0$. This chain is a part of a bi-infinite chain $\mathcal{C}$ that is invariant by g. The transformation f acts as a translation in the corners of $\mathcal{C}$, which shows that these corners project to only finitely many closed orbits of ϕ^t in M. Therefore the string of orbits associated to $\mathcal{C}$ is finite. On the other hand, using again that the flow is $\mathbb{R}$-covered, we have that any $\beta \in \mathcal{FH}(\alpha)$ has a coherent lift $\widetilde{\beta}$ to $\widetilde{M}$ such that $\widetilde{\beta}$ is a corner of this bi-infinite chain $\mathcal{C}$.

This ends the proof of Theorem 4.8. □

Now we prove Theorem 4.7.

Proof of Theorem 4.7. If a finite lift of ϕ^t is a suspension then ϕ^t itself is a suspension [Fen99]. So we assume from now on that M is orientable and both stable and unstable foliations are transversely orientable.

Suppose that every periodic orbit of ϕ^t is freely homotopic to at most a finite number of other periodic orbits and also that ϕ^t is not orbit equivalent to a suspension. We want to show that the flow is, up to finite covers, orbit equivalent to a geodesic flow. All we have to do is to prove that the manifold M is Seifert-fibered, as Theorem 4.2 then yields the orbit equivalence.

Suppose that M is not Seifert-fibered. If M was hyperbolic then, as we've seen in Example 7, every free homotopy class is infinite, contrary to the hypothesis. It follows that M has at least one torus in its torus decomposition. As ϕ^t is $\mathbb{R}$-covered, it is transitive (Theorem 3.23), so there exists a periodic orbit α that is neither contained in one piece of the modified JSJ decomposition nor in one of the tori of the decomposition. To build such a periodic orbit, we can start from a dense orbit and pick a long orbit segment returning inside one of the interiors of the Birkhoff annuli in a Birkhoff torus of the torus decomposition. Using the Anosov closing lemma this orbit is shadowed by a periodic orbit with the same properties. By Theorem 4.8, α has to have an infinite free homotopy class, which gives us a contradiction. □

If we assume more knowledge about the free homotopy classes of an Anosov flow, one should be able to remove the $\mathbb{R}$-covered assumption, and prove

Conjecture 4.9. *Let ϕ^t be an Anosov flow on a 3-manifold M. Suppose that every FHUTO class of periodic orbits contain exactly k elements, with $k \geq 2$, then ϕ^t is a lift of a geodesic flow.*

Remark 4.10. Without further assumption, $k \geq 2$ is necessary. Indeed, suspensions are not the only Anosov flows such that each FHUTO class contains a unique element; the example of Bonatti and Langevin [BL94] also satisfy that. The difference is that in the Bonatti–Langevin example, there exists one (unique) orbit that is freely homotopic to itself (via a non-trivial homotopy), whereas this does not happen for suspensions.

The idea behind this conjecture is that having each FHUTO class with the same number of elements forces each chain of lozenges to have the same number of elements which should be impossible except in the case of $\mathbb{R}$-covered flows.

4.2. **Flexibility.** In this section, we will focus on a different type of question: Given a 3-manifold M that supports an Anosov flow. How many distinct Anosov flows can M support?

The first result around that question was obtained by Barbot [Bar98] when he gave examples of a family of (Graph)-manifolds supporting at least two distinct Anosov flows: one $\mathbb{R}$-covered and one not.

The best result in terms of flexibility, though, was recently obtained by Béguin, Bonatti and Yu [BBY]

Theorem 4.11 (Béguin, Bonatti and Yu [BBY])**.** *For any n, there exists a 3-manifold M_n supporting at least n distinct Anosov flows.*

The example they produce has two atoroidal and one Seifert piece in its JSJ decomposition. One can also build graph-manifolds with the same property. As previously noted, none of these examples are $\mathbb{R}$-covered.

As far as I am aware, there are many more questions than answers regarding flexibility of Anosov flows.

The first most natural question in light of Béguin, Bonatti and Yu's result is

Question 1. Does there exists a 3-manifold M supporting infinitely many non orbit equivalent Anosov flows?

My personal guess is no, but I am willing to state a conjecture only in a more restrictive context

Conjecture 4.12. *If M is a graph-manifold, then there exists a constant N such that M supports at most N non orbit equivalent Anosov flows.*

The works of Barbot [Bar96] and Barbot and Fenley [BF13, BF15] gives some supporting evidence for that conjecture, although they do not make it themselves. The idea is that in a Seifert piece, the flow is either free or periodic (see Definition 3.58). If the piece is free, then the flow inside the piece should be like a (finite lift of a) geodesic flow on a surface with boundary (see [Bar96, BF13]). If the piece is periodic, there is a bound (depending on the topology of the base surface) on the number of periodic orbits in it (see [BF17, Theorem 4.4]), which should imply a bound on the number of possible distinct flows in that piece. Both those results should in turn bound the number of closed leaves in the tori of the modified JSJ decomposition, which leads to a bound on the number of non equivalent ways of gluing them together.

There are however two even more basic questions that are still open

Question 2. Does there exist a *hyperbolic* 3-manifold that admits more then one Anosov flow?

Question 3. Does there exist a 3-manifold that admits more then one $\mathbb{R}$-covered Anosov flow?

5. Some counting results

In this section, we will present some new developments around an old question: counting the growth of periodic orbits (or something related to them) as their period grows.

In the late 60' and early 70', Bowen [Bow72] and Margulis [Mar04] obtained the first results on counting periodic orbits in general Anosov flows. They proved, among other things,

Theorem 5.1 (Bowen [Bow72], Margulis [Mar04])**.** *Let ϕ^t be a transitive Anosov flow in a manifold M. Let h_{top} be the topological entropy of ϕ^t. Then*

$$h_{\text{top}} = \lim_{t\to+\infty} \frac{1}{t} \ln N(t) > 0,$$

where $N(t)$ is the number of periodic orbits of ϕ^t with period less than t.

Remark 5.2. Much more precisely, Margulis proved that for weak-mixing flows,

$$N(t) \sim \frac{e^{h_{\text{top}}t}}{h_{\text{top}}t}, \text{ as } t \to +\infty.$$

Notice also that, being a purely dynamical result, the theorem above does not require any conditions on the manifold (dimensional or otherwise).

We now state another result from the early 70' by Plante and Thurston [PT72], also relating to counting. We first recall the definition of having exponential growth for a group. Let Γ be a finitely-generated group. Let S be a set of generators. The *word-length* of an element $\gamma \in \Gamma$ for S, is defined as the minimal number of elements of S needed to write γ. We denote the word-length of γ by $d_{\Gamma,S}(\gamma)$. The group Γ is said to have *exponential growth* if there exists $A, a > 0$ such that the function $\#\{\gamma \in \Gamma \mid d_{\Gamma,S}(\gamma) \leq n\}$ dominates $A\exp(an)$. Notice that this definition is independent of the choice of the generating set S. A different choice just changes the constants.

If Γ is the fundamental group of a manifold M, having exponential growth is equivalent to the number of homotopically distinct paths of length (for a fixed metric on M) less than r growing exponentially in r. For manifolds supporting a codimension one Anosov flow, Plante and Thurston proved

Theorem 5.3 (Plante and Thurston [PT72])**.** *Let ϕ^t be a codimension-one Anosov flow on a manifold M. Then $\pi_1(M)$ has exponential growth.*

There is obviously a similarity between the result on topological entropy and the exponential growth of the fundamental group, but they are in fact counting quite different things. Indeed, since there are no fixed base point, a periodic orbit of a flow does not represent a homotopy class of a curve, but a *free* homotopy class.

Hence, if one knows for instance that all periodic orbits are homotopically non-trivial and that all free homotopy class contain, say, at most a bounded number of different periodic orbits, then the Bowen-Margulis result implies Plante and Thurston's. An example where both these conditions are realized is the geodesic flow of a negatively curved manifold. However, this is far from generic: First, aside from the codimension one case, we do not know whether periodic orbits of Anosov flows are always homotopically non-trivial (see Remark 5.5 below). Second, as we have seen above (see for instance Theorem 4.7), many examples of Anosov flows in 3-manifold have at least some free homotopy classes containing infinitely many distinct periodic orbits.

So a closer topological analogue to Bowen and Margulis' result would be to measure the growth rate of the number of conjugacy classes in the fundamental group of a manifold supporting an Anosov flow, with respect to the period of the shortest orbit representative. Surprisingly enough, this problem, suggested by Plante and Thurston [PT72] in 1972 was settled only recently.

Theorem 5.4 (Barthelmé and Fenley [BF17])**.** *Let ϕ^t be an Anosov flow on M^3. Then the number of conjugacy classes in $\pi_1(M)$ grows exponentially fast with the length of a shortest representative closed orbit in the conjugacy class.*

Moreover, if the flow is transitive, then the exponential growth rate is given by the topological entropy of the flow. That is, if we write $\mathrm{Cl}(h)$ for the conjugacy class of an element $h \in \pi_1(M)$, $\alpha_{\mathrm{Cl}(h)}$ for a periodic orbit in the conjugacy class $\mathrm{Cl}(h)$ with smallest period (if such a periodic orbit exists), and

$$\mathrm{CCl}(t) := \left\{ \mathrm{Cl}(h) \mid h \in \pi_1(M),\ T\left(\alpha_{\mathrm{Cl}(h)}\right) < t \right\},$$

where $T\left(\alpha_{\mathrm{Cl}(h)}\right)$ is the period of $\alpha_{\mathrm{Cl}(h)}$, then we have

$$\limsup_{t\to+\infty} \frac{1}{t} \log \sharp \mathrm{CCl}(t) = h_{top},$$

where h_{top} is the topological entropy of the flow.

Furthermore, the Bowen-Margulis measure μ_{BM} of the transitive flow ϕ^t (i.e., measure of maximal entropy) can be obtained as

$$\mu_{BM} = \lim_{t\to+\infty} \frac{1}{\sharp \mathrm{CCl}(t)} \sum_{\mathrm{Cl}(h)\in \mathrm{CCl}(t)} \delta_{\alpha_{\mathrm{Cl}(h)}},$$

where $\delta_{\alpha_{\mathrm{Cl}(h)}}$ is the Lebesgue probability measure supported on $\alpha_{\mathrm{Cl}(h)}$.

Remark 5.5. Before talking about the proof of this Theorem, let us make some remarks about what is known for Anosov flows in higher dimension and codimension. In [PT72], Plante and Thurston conjecture that the exponential growth of the fundamental group should be true for any manifold supporting an Anosov flow, as is the case for the positivity of the topological entropy. But as we mentioned above, in higher codimension, it is not even known whether periodic orbits are non homotopically trivial (although there has been several attempts at proving this, see for instance [Pat15]). In fact, there is much more we do not know. For instance, it is still unknown whether $\mathbb{S}^n$ can support (or presumably cannot) an Anosov flow if $n \geq 6$. The proof that $\mathbb{S}^3$ and $\mathbb{S}^4$ cannot support one follows from codimension one consideration, while Gogolev, Ontaneda, and Rodriguez Hertz [GORH15] found an ad hoc topological reason why $\mathbb{S}^5$ cannot carry an Anosov flow.

One of the problems regarding Anosov flow in higher dimension, aside from the fact that foliation theoretical tools stop being effective, is the lack of examples. Indeed, in higher codimension there are three types of "algebraic" examples: Geodesic flows, suspension of Anosov diffeomorphisms and a certain specific example of Tomter [Tom68]. Not all these examples are algebraic per se: the geodesic flow of a negatively curved manifold that does not admit a hyperbolic structure is not algebraic, but they are still part of the "classical" examples.

Franks and Williams [FW80] claimed that they could build non transitive Anosov flows with any stable and unstable dimensions (strictly greater than 1), using their 3-dimensional example. Unfortunately, their construction does not give Anosov flows (see [BBGRH17]). Barthelmé, Bonatti, Gogolev, and Rodriguez Hertz [BBGRH17] proved that one can use Franks and Williams' construction in a different way to obtain non transitive Anosov flow. The big caveat is that, in the construction of [BBGRH17], the stable and unstable dimensions have to match. In particular, it is still unknown whether there exists Anosov flows with non matching stable and unstable dimension, except for suspension of Anosov diffeomorphisms.

From our discussion above, it is clear that the growth rate result in Theorem 5.4 will follow from Theorem 5.1 if one can show that the number of distinct orbits in the same free homotopy class has less than exponential growth rate. This is exactly what was done in [BF17] and it follows from a mix of metric geometry and from the results on chains and strings of lozenges that we presented above.

Theorem 5.6. *Let ϕ^t be an Anosov flow on a 3-manifold M, and $\mathcal{FH}(\alpha_0)$ be the FHUTO class of a closed orbit α_0 of ϕ^t. For any periodic orbit α, let $T(\alpha)$ be the period of α.*

(1) *If M is hyperbolic, then there exists a uniform constant $A_1 > 0$ and a constant C_1 depending on $\mathcal{FH}(\alpha_0)$ (or equivalently on α_0) such that, for t big enough,*

$$\sharp\{\alpha \in \mathcal{FH}(\alpha_0) \mid T(\alpha) < t\} \leq A_1 \log(t) + C_1.$$

(2) *If the JSJ decomposition of M is such that no decomposition torus bounds a Seifert-fibered piece on both sides (so in particular, if all the pieces are atoroidal), then there exists a constant C_1 depending on $\mathcal{FH}(\alpha_0)$ such that, for t big enough,*

$$\sharp\{\alpha \in \mathcal{FH}(\alpha_0) \mid T(\alpha) < t\} \leq C_1\sqrt{t}.$$

(3) *Otherwise, there exist constants $A_1 > 0$ and $B_1 \geq 0$, such that, for t big enough,*

$$\sharp\{\alpha \in \mathcal{FH}(\alpha_0) \mid T(\alpha) < t\} \leq A_1 t + B_1.$$

Furthermore, if M is a graph manifold, then A_1 and B_1 can be chosen independently of $\mathcal{FH}(\alpha_0)$.

So, in any case, the growth of the number of orbits inside a FHUTO class is at most linear in the period — but a priori with constants depending on the particular FHUTO class.

Moreover, independently of the topology of M, the growth of the number of orbits inside an infinite free homotopy class is at least logarithmic in the period. More precisely, there exists a uniform constant $A_2 > 0$ and a constant C_2 depending on $\mathcal{FH}(\alpha_0)$ such that, if $\mathcal{FH}(\alpha_0)$ is infinite, then for any t

$$\sharp\{\alpha \in \mathcal{FH}(\alpha_0) \mid T(\alpha) < t\} \geq \frac{1}{A_2}\log(t) - C_2.$$

Remark 5.7. We stated the results above for a FHUTO class, but they read exactly the same if we consider instead an actual free homotopy class, the constants are just going to be different. Indeed, a consequence of the structure of lozenges (see Section 3.2), is that the number of orbits in a free homotopy class differ from the number of orbits in a FHUTO class by a factor of at most 4 (in fact, one can easily see that this ratio tends to 2 as the number of orbits goes to infinity).

Another way to state the above Theorem is by replacing the periods $T(\alpha)$ by lengths $l(\alpha)$, for any fixed metrics on M. Again, the statement does not change in that case, only the constants do.

Notice that the above result gives very different conclusions depending on where in the JSJ decomposition the orbits are. This might be an artifact of the proof, which depends heavily on the topology of the manifold. However, it seems far more likely that the growth rate indeed depends on where in the manifold the orbits are.

Finally, remark that the above result is not quite enough to deduce Theorem 5.4, as one needs a control of the constants that is independent of the free homotopy class. This is done in [BF17, Theorem G], but we will not cover it here, as it gets more technical.

5.1. **Proof of Theorem 5.6 in the hyperbolic case.**

In these notes, we will present the proof of the upper bound in Theorem 5.6 when the manifold M is hyperbolic, and only succinctly discuss the other cases.

Before specializing to the case when M is a hyperbolic manifold, notice first that, thanks to Proposition 3.62, in order to count the number of elements in a FHUTO class, it is enough to restrict to strings of orbits. Indeed, a general FHUTO class is made up of a finite (uniform) number of strings of orbits and a finite (and again uniform) number of special orbits. Hence, Theorem 5.6 will follow from

Theorem 5.8. *Let $\{\alpha_i\}_{i\in\mathbb{N}}$ be an infinite string of orbits, with the indexation chosen so that α_0 is one of the orbits with the smallest period. Then the growth of the period is at least:*

(1) *Exponential in i if the manifold is hyperbolic;*
(2) *Quadratic in i if the $\{\alpha_i\}_{i\in\mathbb{N}}$ intersect an atoroidal piece;*
(3) *Linear in i if $\{\alpha_i\}_{i\in\mathbb{N}}$ goes through two consecutive Seifert-fibered pieces.*

Moreover, the growth of the period is at most exponential in i, independently of the topology of M.

More precisely, in the hyperbolic case, we have

Proposition 5.9. *Let $\{\alpha_i\}$ be a string of orbits of an Anosov flow on M. If M is hyperbolic, then there exists constants $A, B > 0$, independent of the homotopy class and D_{α_0} depending on α_0 such that*

$$l(\alpha_i) \geq Be^{-D_{\alpha_0}}e^{Ai}.$$

In order to prove this proposition, we recall the following classical lemma of hyperbolic geometry (see for instance [Kli95, Proposition 3.9.11])

Lemma 5.10. *Let $c(t)$, $t \in \mathbb{R}$ be a geodesic of $\mathbb{H}^n$. Let $c_1(t)$, $a \leq t \leq b$, be a curve. Let p, resp. q, be the orthogonal projection of $c_1(a)$, resp. $c_1(b)$, onto c. Suppose that $d(c_1(a), p) = d(c_1(b), q) \geq k$ and that $d(c_1(t), c) \geq k$, for all $a \leq t \leq b$. Then*

$$l(c_1) \ \geq \ d(p,q)\cosh k.$$

Proof of Proposition 5.9. We fix a hyperbolic metric on M. Up to reparametrization of the flow, we can assume that the flow is unit speed for that particular metric. Notice that reparametrizing the flow will not impact the *existence* of the constants claimed in Proposition 5.9, but only their values. Hence we can choose any reparametrization without loss of generality.

Let $\{\widetilde{\alpha}_i\}$ be a coherent lift of the $\{\alpha_i\}$ and g be a generator of the stabilizer in $\pi_1(M)$ of all $\widetilde{\alpha}_i$. Since g preserves all of the $\widetilde{\alpha}_i$, these curves have the same endpoints on the boundary at infinity $\partial_\infty \mathbb{H}^3$. Let c_g be the axis of g acting on $\mathbb{H}^3$, or equivalently the geodesic with the same two endpoints as the $\widetilde{\alpha}_i$. Since c_g and $\widetilde{\alpha}_0$ have the same endpoints on the boundary at infinity, they are a bounded Hausdorff distance from each other. We denote by D_{α_0} that distance, that is, $D_{\alpha_0} = d_{\mathrm{Haus}}(c_g, \widetilde{\alpha}_0)$.

By Lemma 3.64, there exists a constant $A > 0$, depending only on the flow, such that the minimal distance between $\widetilde{\alpha}_i$ and $\widetilde{\alpha}_0$ is at least Ai. Therefore, the distance between $\widetilde{\alpha}_i$ and c_γ is bounded below by $Ai - D_{\alpha_0}$. Let x be a point on c_g. Define x_i as the point on $\widetilde{\alpha}_i$ such that the orthogonal projection of x_i onto c_g is x, and (in case there is more than one such point), we take x_i to be a point closest to c_g (but any other choice works as well).

Now $l(\alpha_i)$ is equal to the length of the part of the curve $\widetilde{\alpha}_i$ between x_i and $g \cdot x_i$. Therefore, by Lemma 5.10, we get that

$$l(\alpha_i) \ \geq \ d(x, g\cdot x)\cosh(Ai - D_{\alpha_0}) \ \geq \ \frac{l(c_g)}{2} e^{Ai} e^{-D_{\alpha_0}}.$$

Replacing $l(c_g)$ by the length of the smallest geodesic in M, we obtain the existence of a universal constant $B > 0$ such that, for all i,

$$l(\alpha_i) \ \geq \ B e^{-D_{\alpha_0}} e^{Ai}.$$

□

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