# Heat kernels on metric measure spaces

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**Examples:** Manifolds, $\mathbb{R}^d$, domains in $\mathbb{R}^d$, Sierpinski carpets, 'cable systems' for graphs, now often called quantum graphs.

Let $(\mathcal{X}, d, \mu)$ be a measure metric space. We assume that $(\mathcal{X}, d)$ is complete, $d$ is a length metric, $\mu$ is Radon, $\mathcal{X}$ has infinite radius, and the balls

$$B(x,r) = \{y : d(x,y) < r\}, \quad x \in \mathcal{X},\ r > 0 \text{ are precompact.}$$

Set $V(x,r) = \mu(B(x,r))$.

**Definition.** $\mathcal{X}$ satisfies $V_\alpha$ if

$$c_1 r^\alpha \le V(x,r) \le c_2 r^\alpha, \quad x \in \mathcal{X},\ r > 0.$$

$\mathcal{X}$ satisfies volume doubling (D) if

$$V(x,2r) \le c_D V(x,r), \quad x \in \mathcal{X},\ r > 0.$$

$V_\alpha$ implies D.

**Example.** The pre-SC in $\mathbb{R}^d$ satisfies $V(x,r) \asymp r^d$ for $r \in (0,1)$ and $V(x,r) \asymp r^\alpha$ for $r \in (1,\infty)$. It is easy to check that D holds, but clearly $V_\theta$ fails for any $\theta$.

**Dirichlet forms.** Let $(\mathcal{E}, \mathcal{F})$ be a regular strongly local Dirichlet form on $L^2(\mathcal{X}, \mu)$.

This means:

(1) $\mathcal{E}(f,g)$ is a symmetric bilinear form defined on a subspace $\mathcal{F} \subset L^2(\mathcal{X})$.
(2) $\mathcal{E}$ is 'Markov': $\mathcal{E}(f_+ \wedge 1, f_+ \wedge 1) \le \mathcal{E}(f,f)$.
(3) $\mathcal{E}$ is closed: if $||f||_H = \mathcal{E}(f,f) + ||f||_2^2$ then $\mathcal{F}$ is a complete Hilbert space in $||.||_H$.
(4) $\mathcal{E}$ is regular: $\mathcal{F} \cap C_0(\mathcal{X})$ is dense in $\mathcal{F}$ w.r.t. $||.||_H$ and dense in $C_0(\mathcal{X})$ w.r.t. $||.||_\infty$.
(5) $\mathcal{E}$ is strongly local: if $f_1, f_2$ have compact support and $f_1$ is constant on an open set $U_1$ containing $\mathrm{supp}(f_2)$, then $\mathcal{E}(f_1, f_2) = 0$.

For $f, g \in \mathcal{F}$ there exists a measure $d\Gamma(f,g)$ (called the 'energy measure') which gives $\mathcal{E}$ by integration:

$$\mathcal{E}(f,g) = \int_{\mathcal{X}} d\Gamma(f,g).$$

For a manifold $d\Gamma(f,g) = (\nabla f.\nabla g)d\mu$, but in general $\Gamma$ need not be absolutely continuous with respect to $\mu$. $d\Gamma$ satisifes Liebnitz type rules – e.g.

$$d\Gamma(fg,h) = fd\Gamma(g,h) + gd\Gamma(f,h).$$

We call a MMS with such a Dirichlet form a MMD space.

**Examples.**
(1) Manifolds.
(2) Divergence form operators on domains in $\mathbb{R}^d$: $D$ a domain in $\mathbb{R}^d$,

$$\mathcal{E}(f,f) = \int \nabla f.a.\nabla f$$

where $a = a(x)$ is bounded and uniformly elliptic.

(3) Sierpinski carpets.

(4) 'Cable system' for a graph. If $G = (V, E)$ is a graph, let $\mathcal{X}$ be the metric space obtained by replacing each edge $e \in E$ by a unit line segment $I_e$, joined at vertices. Then

$$\mathcal{E}(f, f) = \sum_e \int_{I_e} (f')^2 dx.$$

(These are now often called 'quantum graphs').

## Semigroup and heat kernel.

On a MMD space we have a Laplacian type operator $\mathcal{L}$ which satisfies

$$\mathcal{E}(f, g) = \int (-\mathcal{L} f) g d\mu \quad \text{for } f \in \mathcal{D}(\mathcal{L}),\, g \in \mathcal{F}.$$

Associated with $\mathcal{L}$ is a semigroup, formally given by $P_t = \exp(t\mathcal{L})$. This semigroup defines a Markov process $(X_t, P^x)$ on $\mathcal{X}$, and

$$E^x f(X_t) = E(f(X_t)|X_0 = x) = P_t f(x), \quad x \in \mathcal{X},\, t \geq 0.$$

If $P_t$ has a density $p_t(x,y)$ with respect to $\mu$ then this is the *heat kernel*

$$P^x(X_t \in A) = P_t 1_A(x) = \int_A p_t(x,y)\mu(dy).$$

The heat kernel is symmetric: $p_t(x,y) = p_t(y,x)$, satisfies the Chapman-Kolgogorov equation

$$\int p_s(x,y)p_t(y,z)\mu(dy) = p_{s+t}(x,z),$$

and the heat equation

$$\frac{\partial}{\partial t}p_t(x,y) = \mathcal{L}p_t(x,y).$$

## Regularity and bounds on heat kernel

This problem for divergence form operators in $\mathbb{R}^d$ was solved by de Giorgi, Moser and Nash in the late 1950s. The methods were extended to manifolds by Bombieri and Giusti. Followed by work of Aronsen, Li and Yau, Fabes and Stroock, Grigoryan, Saloff-Coste, Sturm, and others.

**Theorem 1.** *(Grigoryan, Saloff-Coste, Sturm.) Let $(\mathcal{X}, \mathcal{E})$ be a MMD space. The following are equivalent:*
*(1) $p_t(x, y)$ satisfies GB= Gaussian bounds,*
*(2) A parabolic Harnack inequality (PHI) holds on $\mathcal{X}$,*
*(3) $\mathcal{X}$ satisfies D plus PI (Poincaré inequality).*

**Definition.** $(p_t)$ satisfies Gaussian bounds (GB) if

$$p_t(x, y) \overset{(c)}{\asymp} V(x, ct^{1/2})^{-1} \exp(-cd(x, y)^2/t).$$

Here $\overset{(c)}{\asymp}$ means that upper bound hold with constants $c_1, c_2$ and lower bounds with $c_3, c_4$.

**Poincaré inequality** $(\mathcal{X}, \mathcal{E})$ satisfies PI if for all balls $B = B(x,r)$, $f : B \to \mathbb{R}$,

$$\min_a \int_B (f-a)^2 d\mu = \int_B (f - \overline{f})^2 d\mu \le C_P \, r^2 \int_B d\Gamma(f,f).$$

**Harnack inequalities**

We need to define harmonic functions (solutions of $\mathcal{L}h = 0$) and caloric functions (solutions of $\partial_t u(x,t) = \mathcal{L}u$) in the MMD context.

If $D \subset \mathcal{X}$ is open, we say $h : D \to \mathbb{R}$ is *harmonic* in $D$ if

$$\mathcal{E}(f,h) = 0 \text{ for all } f \in C_0(X) \cap \mathcal{F}, \ \operatorname{supp} f \subset D.$$

There are various definitions of caloric, all of which essentially say that

$$\frac{\partial}{\partial t} u(x,t) = \mathcal{L}u(x,t), \ (x,t) \in Q \subset \mathcal{X} \times \mathbb{R}.$$

Lack of regularity in general means they may not be exactly equivalent, but in all cases the heat kernel $p_t(x,y)$ is caloric.

Definition. $(\mathcal{X}, \mathcal{E})$ satisifes the elliptic Harnack inequality (EHI) if whenever $B = B(x,R)$ and $h : \overline{B} \to \mathbb{R}_+$ is harmonic in $B$ then if $B' = B(x,R/2)$

$$\sup_{B'} h \le C_E \inf_{B'} h.$$

Easy iteration arguments show that if EHI holds then harmonic functions are Hölder continuous, with index $\delta = \delta(C_E)$.

The parabolic Harnack inequality (PHI) is more complicated to state. Let $T = R^2$, $B = B(x,R)$, $Q = B \times (0,T)$, and

$$Q_- = B' \times [\tfrac{1}{4}, \tfrac{1}{2}T], \quad Q_+ = B' \times [\tfrac{3}{4}, T).$$

Then $(\mathcal{X}, \mathcal{E})$ satisfies PHI if whenever $u = u(x,t)$ is non-negative and caloric in $Q$ then

$$\sup_{Q_-} u \le C_H \inf_{Q_+} u.$$

**Remarks.** 1. Note that PHI implies EHI since if $h$ is harmonic then $u(x,t) = h(x)$ is caloric.

2. The PHI involves time and gives control on the amount of time it takes heat (or the diffusion $X$) to cross $B(x, R)$.

3. Recall Theorem 1: GB $\Leftrightarrow$ PHI $\Leftrightarrow$ D + PI.

This gives necessary and sufficient conditions for PHI. It also proves that GB and PHI are *stable*: if $\mathcal{E}'$ is another Dirichlet form and $\mathcal{E}' \asymp \mathcal{E}$ then PHI holds for $(\mathcal{X}, \mathcal{E}')$ iff it holds for $(\mathcal{X}, \mathcal{E})$.

The PI is clearly stable, while the stability of PHI or GB is far from evident.

4. Necessary and sufficient conditions for EHI are not known. It is also not known whether or not EHI is stable.

**Extensions to (fractal) MMD spaces.**

We can replace the GB on the heat kernel by more general bounds. Since I want to discuss spaces, such as the pre-Sierpinksi carpet or quantum graphs, where the local and global structures are different, introduce the function:

$$\Psi(r) = \begin{cases} r^{\beta_L} & \text{if } 0 \le r \le 1, \\ r^{\beta} & \text{if } r \ge 1. \end{cases}$$

(So for the pre-SC $\beta_L = d$ and for a quantum graph $\beta_L = 1$.)

We say $(p_t)$ satsifies HK($\Psi$) if

$$p_t(x,y) \overset{(c)}{\asymp} V(x, ct^{1/\beta_L})^{-1} \exp(-c(d(x,y)^{\beta_L}/t)^{1/(\beta_L-1)}), \text{ in } I_1$$
$$p_t(x,y) \overset{(c)}{\asymp} V(x, ct^{1/\beta})^{-1} \exp(-c(d(x,y)^{\beta}/t)^{1/(\beta-1)}), \text{ in } I_2.$$

Here

$$I_1 = \{(t,x,y) : t \le 1 \vee d(x,y)\}, \quad I_2 = \{(t,x,y) : t \ge 1 \vee d(x,y)\}.$$

**Examples.** (1) The pre-SC satisfies HK($\Psi$) with $\beta_L = d$, $\beta = d_w$.
(2) The true (infinite) SC satisfies HK($\Psi$) with $\beta_L = \beta = d_w$ – call this HK($\beta$).
(3) GB are just HK(2).
(4) Many regular true fractals satisfy HK($\beta$), always with $\beta > 2$.
(5) The quantum/cable graph associated with the Sierpinski gasket graph satisfies HK($\Psi$) with $\beta_L = 1$, $\beta = \log 5/\log 2$.

Suppose a MMD space $(\mathcal{X}, \mathcal{E})$ satisfies $V_\alpha$ and HK$(\beta)$. What values can $(\alpha, \beta)$ take?

**Theorem 2.** *We have $\alpha \geq 1$, $2 \leq \beta \leq 1 + \alpha$, and all these values are possible.*

I stated the bounds on $\alpha$, $\beta$ (without proof) in my St. Flour notes.

(i) Consistency conditions for HK$(\beta)$ imply that $\beta > 1$.

(ii) $\alpha \geq 1$ is obvious from the existence of points $x_n$ with $d(x_n, x_0) = n$.

(iii) The bound $\beta \leq 1 + \alpha$ comes from the fact that when $\beta > \alpha$ then harmonic functions are Hölder continuous of order $\beta - \alpha$.

(iv) Hino proved $\beta \geq 2$ by showing in general that

$$\liminf_{t \downarrow 0} t \log p_t(x, y) \geq -C > -\infty.$$

If HK$(\beta)$ holds then the LHS is $ct \log t - c' d(x, y)^c t^{(\beta-2)/(\beta-1)}$, so would diverge to $-\infty$ if $\beta < 2$.

(v) I conjectured that if $\beta = 2$ then only $d \in \mathbb{N}$ was possible. Bourdon and Pajot, Laakso gave examples which showed otherwise. In the graph case I gave constructions for all $\alpha$, $\beta$ using Laakso's method.

The condition $\beta \geq 2$ means that heat (or the diffusion $X_t$) can move at most distance $O(t^{1/2})$ in time $t$. So Euclidean space gives the fastest possible order in terms of heat diffusion. Obstacles etc. as in fractals can slow things down, but not speed them up.

For regular fractals with scaling factors $L$, $M$, $\rho$ one has

$$\alpha = d_f = \frac{\log M}{\log L}, \qquad \beta = d_w = \frac{\log M\rho}{\log L}.$$

Since $L$ and $M$ are immediate from the construction, one can calculate $\alpha$ easily.

The constant $\rho$ is somehow deeper, and seems to require some analysis on the set or its approximations. Loosely one can say that $\alpha$ is a 'geometric' constant, while $\beta$ is an analytic constant. One may guess that in some sense $\beta$ is inaccessible by any purely geometric argument.

**EHI and PHI.** The pre-SC gives an example where EHI holds but PHI fails. Let $R \gg 1$, $Q = B(x_0, R) \times [0, T]$ where $T = R^2$, and let $d(x_1, x_0) = R/2 - 1 \approx R/2$. Set

$$u(x,t) = p_t(x_0, x).$$

Then

$$\sup_{Q_-} u = p_{T/2}(x_0, x_0) \simeq cT^{-\alpha/\beta} \approx cR^{-2\alpha/\beta},$$

while

$$\inf_{Q_+} u \le p_T(x_0, x_1) \simeq cT^{-\alpha/\beta} \exp\Big( - c(R^\beta/T)^{1/(\beta-1)}\Big)$$
$$\le \exp(-cR^{(\beta-2)/(\beta-1)}) \ll \sup_{Q_-} u.$$

The usual PHI fails because heat needs time $O(R^\beta)$ rather than $O(R^2)$ to flow from $x_0$ to $x_1$.

The fix is fairly clear – define a modified PHI to take account of the space time scaling $t = \Psi(r)$.

Definition. $(\mathcal{X}, \mathcal{E})$ satisfies PHI($\Psi$) if when $R > 0$, $T = \Psi(R)$, $B = B(x,R)$, $Q = B \times (0,T)$, and $Q_- = B' \times [\frac{1}{4}, \frac{1}{2}T]$, $\quad Q_+ = B' \times [\frac{3}{4}, T)$, and $u = u(x,t)$ is non-negative and caloric in $Q$ then

$$\sup_{Q_-} u \le C_H \inf_{Q_+} u.$$

In the same way one defines the rescaled Poincaré inequality PI($\Psi$):

$$\min_a \int_{B(x,r)} (f-a)^2 d\mu = \int_{B(x,r)} (f-\overline{f})^2 d\mu \le C_P \, \Psi(r) \int_{B(x,r)} d\Gamma(f,f).$$

If $\Psi(r) = r^\beta$ we write PHI($\beta$) etc. The old PI and PHI are just PHI(2) and PI(2).

Natural first guess: one has HK($\Psi$) $\Leftrightarrow$ PHI($\Psi$) $\Leftrightarrow$ $D +$ PI($\Psi$).

The first double implication is correct.

Introduce another condition $T(\Psi)$. Let $\tau_{x,r}$ be the first exit time of the process $X$ from $B(x,r)$. Then

$$E^x \tau_{x,R} \asymp \Psi(R). \qquad T(\Psi)$$

**Theorem 3.** *(Hebisch & Saloff-Coste, Grigoryan & Telcs). The following are equivalent:*
*(a)* $(\mathcal{X}, \mathcal{E})$ *satisfies* $\mathrm{HK}(\Psi)$.
*(b)* $(\mathcal{X}, \mathcal{E})$ *satisfies* $\mathrm{PHI}(\Psi)$.
*(c)* $(\mathcal{X}, \mathcal{E})$ *satisfies D, EHI and* $T(\Psi)$.
*(d)* $(\mathcal{X}, \mathcal{E})$ *satisfies D, EHI and* $\mathrm{RES}(\Psi)$*:*
*These all imply* $\mathrm{PI}(\Psi)$.

It is easy to see that one cannot have $\mathrm{HK}(\Psi) \Leftrightarrow D + \mathrm{PI}(\Psi)$. If $\Psi' \geq \Psi$ then $\mathrm{PI}(\Psi)$ implies $\mathrm{PI}(\Psi')$, while if $\Psi'(r)/\Psi(r) \to \infty$ then $\mathrm{HK}(\Psi)$ and $\mathrm{HK}(\Psi')$ cannot both hold.

The inequality $\mathrm{PI}(\Psi)$ tells us, roughly, that heat homogenizes in a ball size $R$ in time at most $\Psi(R)$, but does not preclude the possibility it might homogenize more quicky. To 'capture' $\mathrm{HK}(\Psi)$ one needs to control the rate of heat diffusion. (This is not necessary in the classical case if $\Psi(r) = r^2$ because this is the fastest possible.)

Note that Theorem 3 does not give stability of $\mathrm{PHI}(\Psi)$. The conditions D and $\mathrm{RES}(\Psi)$ are clearly stable, but this is not apparent for any of the other ones.

**Sketch proof of Theorem 3:**

$\mathrm{HK}(\Psi) \Leftrightarrow \mathrm{PHI}(\Psi) \;\; \Leftrightarrow \mathrm{D} + \mathrm{EHI} + T(\Psi) \Leftrightarrow \mathrm{D} + \mathrm{EHI} + \mathrm{RES}(\Psi).$

(1) The implication $\mathrm{HK}(\Psi) \Leftrightarrow \mathrm{PHI}(\Psi)$ is proved very much as in the classical $\Psi(r) = r^2$ case.

(2) Given $\mathrm{HK}(\Psi) + \mathrm{PHI}(\Psi)$ one gets EHI immediately. $T(\Psi)$ is easy since integration of the heat kernel bounds gives

$$P^x\big(X_t \notin B(x, \lambda t^{1/\beta})\big) \le \exp(-c\lambda^{\beta/(\beta-1)}).$$

(3) A general result using potential theory gives that if $B' = B(x, r/2)$, $B = B(x, r)$ then there exists a probability measure $\pi$ on $\partial B'$ such that if $\tau_B$ is the first exit by $X$ from $B$ and $T'$ the first hit on $B'$ then

$$\int \pi(dz) E^z \tau_B = R_{\text{eff}}(B', B^c) \int_{B-B'} \mu(dy) P^y(T' \le \tau_B).$$

Using the regularity from EHI one can then show that

$$E^z \tau_B \asymp R_{\text{eff}}(B', B^c) V(x, r), \quad z \in \partial B'.$$

So $\mathrm{D} + \mathrm{EHI} + T(\Psi) \Leftrightarrow \mathrm{D} + \mathrm{EHI} + \mathrm{RES}(\Psi)$.

What remains is the hard (and most useful) implication:

$$D + \text{EHI} + T(\Psi) \Rightarrow \text{HK}(\Psi).$$

Like most heat kernel bounds, this proceeds in several steps.

Step 1. Obtain the upper bound

$$\sup_x p_t(x,x) \le \frac{c}{V(x,t^{1/\beta})}. \qquad (DUE)$$

Rough idea: write $g_B(x,y)$ for the Green function in a domain $B \subset \mathcal{X}$. Then $T(\Psi)$ gives control of

$$E^x \tau_{x,r} = \int_{B(x,r)} g_{B(x,r)}(x,y)\mu(dy).$$

EHI enables one to pass from integrals to pointwise bounds on Green's functions in balls.

These then lead to the Faber-Krahn inequality which is equivalent to DUE.

Step 2. Obtain the full upper bound:

$$p_t(x,y) \le V(x, c_1 t^{1/\beta})^{-1} \exp(-c_2 (d(x,y)^\beta / t)^{1/(\beta-1)}).$$

Historically going from DUE toUE has proved quite tricky. However, in this situation the estimate $T(\Psi)$ makes it rather straightforward.

$T(\Psi)$ states that the time taken to leave a ball $B(x,r)$ is of order $\Psi(r)$. It follows that there exists $p_0$, $\delta > 0$ such that

$$P^x(\tau_{x,r} \le \delta\Psi(r)) \le \tfrac{1}{2}.$$

Given a ball $B(x,R)$ divide the 'journey' of $X$ from $x$ to $B(x,R)^c$ into (at least) $n$ steps of length $r = R/n$. Call a part of the journey (i.e. across a ball $B(y,r)$) 'quick' if it takes time less than $\delta\Psi(r)$, and 'slow' otherwise. Then

$$P(\text{ more than } 2n/3 \text{ quick }) \le e^{-cn}.$$

If the number of quick journeys out of the first $n$ is less than $2n/3$ then the time to exit the ball is at least

$$t(n) = \tfrac{1}{3}\delta\Psi(r)n = \tfrac{1}{3}\delta\Psi(R/n)n.$$

So if we want to control rapid escape from a big ball, we have if $T = t(n)$ that

$$P^x(\tau_{x,R} < T) \le \exp(-cn).$$

Given $R$ and $T$, in the case $\Psi(r) = r^\beta$ we obtain

$$t(n) = T \asymp n(R/n)^\beta, \qquad \text{so } n^{\beta-1} = c\Big(\frac{R^\beta}{t}\Big).$$

Note that

$$r = R/n \asymp (T/R)^{1/(\beta-1)}.$$

So if $T > R$ then $r \ge 1$ and the small balls take time $R^\beta$ to cross, but if $T < R$ then $r \le 1$ and the small balls take $r^{\beta_L}$ to cross.

We thus obtain

$$\log P^x(\tau_{x,R} < T) \le \begin{cases} -c(R^\beta/T)^{1/(\beta-1)} & \text{if } R < T, \\ -c(R^{\beta_L}/T)^{1/(\beta_L-1)} & \text{if } R \ge T. \end{cases}$$

One can then combine this bound with DUE to obtain the full upper bound.

Step 3. Lower bounds.

The upper bound leads easily to:

$$p_t(x,x) \geq cV(x,t^{1/\beta})^{-1}.$$

The key is to extend this inequality to a ball around $x$:

$$p_t(x,y) \geq cV(x,t^{1/\beta})^{-1}, \quad y \in B(x,\delta t^{1/\beta}).$$

EHI gives Hölder continuity of harmonic functions. With other estimaes, this extends also to give control of oscillations of $p_t(x,y)$, and hence control of

$$|p_t(x,x) - p_t(x,y)|.$$

A standard chaining argument then gives the full lower bound.

**Theorem 4.** *(MB, Bass, Kumagai.) The following are equivalent:*
*(a)* $(\mathcal{X}, \mathcal{E})$ *satisfies* $\mathrm{HK}(\Psi)$*.*
*(b)* $(\mathcal{X}, \mathcal{E})$ *satisfies* $D$, $\mathrm{PI}(\Psi)$ *and* $CS(\Psi)$*.*

The condition $CS(\Psi)$ states that there exist 'low energy' cut-off functions.

**Definition.** A function $\psi : \mathcal{X} \to \mathbb{R}$ is a *cutoff function* for $B(x, R/2) \subset B(x, R)$ if $\varphi(y) \geq 1$ on $B(x, r/2)$ and is zero on $B(x, r)^c$ .

$(\mathcal{X}, \mathcal{E})$ satisfies $CS(\Psi)$ if there exists $\theta \in (0, 1)$ such that for all $x, R$ there exists a Hölder continuous cutoff function $\varphi$ for $B(x, R/2) \subset B(x, R)$ such that if $s \in (0, R)$ and $f : B = B(y, s) \to \mathbb{R}$ then

$$\int_{B(y,s/2)} f^2 d\Gamma(\varphi, \varphi) \leq c(s/R)^{2\theta} \Big( \int_B d\Gamma(f, f) + \Psi(s)^{-1} \int_B f^2 d\mu \Big).$$

Note that if $\Psi_1 \geq \Psi_2$ then $CS(\Psi_2)$ implies $CS(\Psi_1)$. (Increasing $\Psi$ weakens PI but strengthens CS).

CS(2) always holds on a manifold – just take $\varphi$ 'linear' between $B(x, R/2)$ and $B(x, R)$. Then

$$\begin{aligned}
\int_{B(y,s/2)} f^2 d\Gamma(\varphi,\varphi) &= \int_{B(y,s/2)} f^2 |\nabla\varphi|^2 d\mu \\
&\leq \int_{B(y,s/2)} f^2 d\mu ||\nabla\varphi||_\infty^2 \\
&\leq cR^{-2} \int_{B(y,s/2)} f^2 d\mu \\
&= c(s^2/R^2)s^{-2} \int_{B(y,s)} f^2 d\mu.
\end{aligned}$$

**Outline of proof:**

(1) HK($\Psi$) implies D, PI($\Psi$) and CS($\Psi$). The first two follow by known arguments. To prove CS($\Psi$) use properties of Green's functions to construct a suitable $\varphi$.

(2) In many arguments one needs to control expressions of the form

$$\int_A f^2 |\nabla \varphi|^2 d\mu,$$

where $A = B(x, r+h) - B(x, r)$ and $\varphi$ is a cutoff function between $B(x, r)$ and $B(x, r+h)$. The CS inequalities enable one to do this. The 'extra' term

$$\int_A d\Gamma(f, f)$$

often turns out to be harmless or controllable.

In particular, $CS(\Psi)$ provides a family of cutoff functions which enable one to run the first ('easy') part of Moser's argument. Thus one can use Moser's iteration argument to show that D, PI($\Psi$) and $CS(\Psi)$ imply EHI. Once one has EHI one can use Theorem 3, since PI($\Psi$) and CS($\Psi$) also imply RES($\Psi$).

## Rough isometries.

A map $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$ is a *rough isometry* if there exist constants $C_1$, $C_2$ such that

$$C_1^{-1}(d_1(x,y) - C_2) \le d_2(\varphi(x), \varphi(y)) \le C_1 d_1(x,y) + C_2,$$

$$\bigcup_{x \in X_1} B_{d_2}(\varphi(x), C_2) = X_2.$$

If there exists a rough isometry between two spaces they are said to be *roughly isometric*. (One can check that this is an equivalence relation.)

Application: stability of HK($\Psi$) under rough isometries. (One also needs some local regularity of both spaces.)

In particular if $M$ is a manifold which is roughly isometric to the pre-SC $\widetilde{F}$ then the heat kernel on $M$ also satisfies HK($\Psi$).

Unfortunately in examples the condition CS($\beta$) is hard to check. The only cases I know it holds are where it has been verified from HK($\Psi$).

It is quite possible that there exists a simpler condition $X(\Psi)$ such that $X(\Psi) + \mathrm{PI}(\Psi) + \mathrm{D} \Leftrightarrow \mathrm{HK}(\Psi)$.

Suppose that $V_\alpha$ holds and $\Psi(r) = r^\beta$, so HK($\beta$) takes the form

$$p_t(x,y) \overset{(c)}{\asymp} ct^{-\alpha/\beta}) \exp(-c(d(x,y)^\beta/t)^{1/(\beta-1)}).$$

In particular $p_t(x,x) \asymp t^{-\alpha/\beta}$. The behaviour of $X$ can be divided into two main cases.

(1) $\alpha < \beta$. In this case the process $X_t$ is recurrent, and in fact 'hits points', so that the range

$$R_t = \{X_s, 0 \le s \le t\}$$

has positive $\mu$ measure. Ultimately the process $X$ will hit every point in the space $\mathcal{X}$ infinitely often.

(2) $\alpha > \beta$. In this case the process $X_t$ is transient, and $R_t$ has zero $\mu$ measure. The probability that $X$ hits any specific point after time 0 is zero. The Green function

$$g(x,y) = \int_0^\infty p_t(x,y)dt \asymp \frac{1}{d(x,y)^{\alpha-\beta}}.$$

In the classical case of $\mathbb{R}^d$ the first case (1) $\alpha < \beta$ only arises when $d = 1$. In this setting we have a richer family of 'low dimensional' spaces. (The SC in $d = 2$ is recurrent, but is transient for $d \geq 3$. The resistence estimates on $R_n$ are good enough to show this.)

In particular one has a nicer version of the stability result in the low dimensional situation. Introduce the condition $PR(\beta)$:

$$R_{\text{eff}}(x, y) = R_{\text{eff}}(\{x\}, \{y\}) \asymp \frac{d(x, y)^\beta}{V(x, d(x, y))}.$$

**Theorem 5.** *(MB, T. Coulhon, T. Kumagai). Let $\mathcal{X}$ satisfy $V_\alpha$, and let $\beta > \alpha$. Then the following are equivalent:*
*(1) $(\mathcal{X}, \mathcal{E})$ satisfies* HK($\Psi$).
*(2) $(\mathcal{X}, \mathcal{E})$ satisfies $PR(\beta)$.*

In the recurrent case the resistance $R_{\text{eff}}(x, y)$ is a metric, and gives a lot of information about the space. In the transient case one has $R_{\text{eff}}(x, y) = \infty$, and

$$\limsup_{d(x,y)\to\infty} R_{\text{eff}}(B(x, \varepsilon), B(y, \varepsilon)) < \infty.$$

This result also holds for graphs, and has proved useful in studying random walks on percolation clusters.

So in the 'low dimensional' setting Harnack inequalities are no longer difficult, and one just has to calculate resistance between points.

## Local structure.

In the MMD setting we have a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, \mu)$. Suppose that HK$(\beta)$ holds: then what can be said about regularity properties of the functions in $\mathcal{F}$ or the domain of the Laplacian $\mathcal{D}(\mathcal{L})$?

**Theorem 6.** *Let $\mathcal{X}$ be the Sierpinski carpet. Suppose that $f \in C^1(\mathbb{R}^2)$ and $g = f|_F \in \mathcal{F}$. Then $g$ is constant.*

However, $\mathcal{F}$ can be described explicitly as a Besov space. Set

$$N_\theta(f) = \sup_{0<r\leq 1} r^{-\alpha-\theta} \int_F \int_{B(x,r)} |f(x) - f(y)|^2 \mu(dx)\mu(dy),$$
$$W(\theta) = \{f \in L^2(F, \mu) : N_\theta(f) < \infty\}.$$

**Theorem 7.** *(Grigoryan, Hu, Lau). Suppose $V_\alpha$ and* HK$(\beta)$ *hold. Then $\mathcal{F} = W(\beta)$, and*

$$\mathcal{E}(f, f) \asymp N_\beta(f).$$

*Further,*

$$\beta = \sup\{\theta : \dim W(\theta) = \infty\}.$$

**Corollary.** *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be Dirichelt forms on a metric measure space $\mathcal{X}$. Suppose that $(\mathcal{X}, \mathcal{E}_i)$ satisfy* $\mathrm{HK}(\beta_i)$, *then $\beta_1 = \beta_2$.*

**Problem.** This result gives a characterization of $\beta = d_w$ which seems 'independent' of the Dirichlet form $\mathcal{E}$. Can one use it, for example, to bound $d_w$ for the SC?

## Energy measures.

These are the measures $d\Gamma(f,f)$, $f \in \mathcal{F}$. Suppose that $(\mathcal{X}, \mathcal{E})$ satisifes HK($\beta$). If one had $d\Gamma(f,f) \le C(f) d\mu$ then we could define a metric associated with $\mathcal{E}$ by setting

$$\mathcal{K} = \{f : d\Gamma(f,f) \le d\mu\},$$

and

$$d_{\mathcal{E}}(x,y) = \sup\{f(y) - f(x) : f \in \mathcal{K}\}.$$

One would then have Gaussian heat kernel bounds with respect to $d_{\mathcal{E}}$, and on-diagonal heat kernel decay of the form $t^{-\alpha/2}$.

**Theorem 8.** *(Hino) Let $\mathcal{X}$ be a Sierpinski carpet. If $f \in \mathcal{F}$ and $d\Gamma(f,f) \ll d\mu$, then $f$ is constant.*

Kusuoka (1989) proved a similar result for the Sierpinski gasket (SG).

So the energy measures and Hausdorff measure are mutually singular. Even for the SG not much is known about the dimension of the support of the energy measures.