Notes for Mini-course at CRM, Montréal July 2011

Felix Otto, MPI Leipzig

Parts 1, 2, 3, and 4

Main result of Mini-course

Time-discrete version of *two-phase* Wasserstein gradient flow: $\theta_k(z) \in [0,1]$ $(z \in \mathbb{R})$ minimizes

$$\frac{1}{2} d^2(\theta_{k-1}, \theta_k) + \frac{r}{2} d^2(1-\theta_{k-1}, 1-\theta_k) + h \int_{\mathbb{R}} z\theta_k \, dz.$$

Then the interpolation $\theta_h(t,x)$ in time converges for $h \downarrow \infty$ to the *entropy solution* of the scalar conservation law

$$\partial_t \theta + \partial_z f(\theta) \;=\; 0$$

with "flux function" given by $f(\theta) \;=\; -\frac{\theta(1-\theta)}{r\theta+(1-\theta)}$

Equivalently: The sequence of functions $V_h(t,z) = \int_{-\infty}^{z} \theta_h(t,z')\, dz'$ converges the *viscosity solution* of the Hamilton Jacobi equation

$$\partial_t V + f(\partial_z V) \;=\; 0.$$

Topics of this lecture

Where does this problem come from?
Density-driven two phase flow in porous medium.

Details of the proof:
An exercise in optimal transportation and viscosity solutions.

Why is the result natural?
Gradient flows in non-convex energy landscapes and C. Dafermos' maximal entropy dissipation rate criterion.

Main references

F. Otto: Evolution of microstructure in unstable porous media flow: a relaxational approach, *Comm. Pure Appl. Math.* **52 (7)** (1999), 873-915.

F. Otto: Evolution of microstructure: an example. In *Ergodic theory, analysis, and efficient simulation of dynamical systems*, Springer, Berlin, 2001, 501-522.

N. Gigli & F. Otto: Entropic Burgers' equation via a minimizing movements scheme based on the Wasserstein metric,

Density-driven two-phase flow in a porous medium

Phase distribution $\chi \in \{0,1\}$:
$\chi = 1 \leftrightarrow$ heavy phase, $\chi = 0 \leftrightarrow$ light phase

Common incompressible velocity u:
$\partial_t \chi + \nabla \cdot (\chi u) = 0$, $\nabla \cdot u = 0$

Density of $\left\{ \begin{array}{rcl} \text{heavy phase} & = & 1 \\ \text{light phase} & = & 0 \end{array} \right\}$: gravity force $= -\chi \binom{0}{1}$

mobility of $\left\{ \begin{array}{rcl} \text{heavy phase} & = & 1 \\ \text{light phase} & = & m \end{array} \right\}$: mobility $= \chi + m(1-\chi)$

Darcy's law involves pressure p:
$u \;=\; (\chi + m(1-\chi))(-\nabla p - \chi \binom{0}{1})$

An ill-posed free boundary problem

Formally defines an evolution of χ: Given $\chi(t)$, the elliptic system

$$\nabla \cdot u \;=\; 0 \quad \text{and} \quad \nabla \times \left(\frac{1}{\chi + m(1-\chi)} u + \chi \binom{0}{1} \right) \;=\; 0$$

uniquely determines $u(t)$. Update χ via $\partial_t \chi + \nabla \cdot (\chi u) = 0$.

Linear instability of flat interface with heavy phase above: growth rate of sinusoidal perturbation $\sim \frac{1}{wavelength}$ [Saffman-Taylor].

$\exists$ special solutions in channel geometry (complex analysis): connect flat interface at $t = -\infty$ to traveling wave (finger) at $t = +\infty$.

Analysis of growth of mixing zone

Consider an infinite channel $(y, z) \in (0, 2\pi)^{d-1} \times \mathbb{R}$ with lateral periodic boundary conditions. Suppose that the initial configuration χ_0 is unstable in the sense that there exist $-\infty < M \leq N < \infty$ such that

$$\chi_0 = 0 \quad \text{for } z \leq M \quad \text{and} \quad \chi_0 = 1 \quad \text{for } z \geq N.$$

Consider the *horizontal average* $\theta(t, z)$ of the solution $\chi(t, x)$:

$$\theta(t, z) := \frac{1}{(2\pi)^{d-1}} \int_{(0,2\pi)^{d-1}} \chi(t, y, z)\, dy.$$

Consider the *accumulated volume* $V(t, z)$: $V(t, z) := \int_{-\infty}^{z} \theta(t, z')\, dz'$.

Rescale $V_\epsilon(t, z)$ for time asymptotics: $V_\epsilon(\epsilon t, \epsilon z) = \epsilon V(t, z)$.

A first connection with Hamilton Jacobi equation

Theorem 1. *[Otto '01] For any sequence* $\epsilon \downarrow 0$*, there exists a subsequence such that* $V_\epsilon(t,z)$ *converges locally uniformly to a Lipschitz continuous* $\bar{V}(t,z)$ *that is a subsolution of the Hamilton Jacobi equation*

$$\partial_t \bar{V} + f(\partial_z \bar{V}) \;\leq\; 0 \quad \textit{almost everywhere}$$

with initial data

$$\bar{V}(t=0,z) \;=\; \max\{0,z\}.$$

Here the flux function $f(\theta)$ *is given by*

$$f(\theta) \;:=\; -\frac{\theta(1-\theta)}{r\theta + (1-\theta)} \quad \textit{for } \theta \in [0,1]$$

where $r := \frac{1}{m}$ *is the friction constant of the light phase.*

A standard result on Hamilton Jacobi equations with convex flux

Lemma 1. *Let* $V^*(t,z)$ *denote the pointwise supremum of all Lipschitz subsolutions* $V(t,z)$ *of the Hamilton-Jacobi equation* $\partial_t V + f(\partial_z V) \le 0$ *with* $\partial_z V \in [0,1]$ *and initial data* $\max\{0,z\}$*.*

It is given by

$$V^*(t,z) \;=\; f^*(\frac{z}{t}),$$

where $f^*(\xi)$ *is the Legendre transform of* $f(\theta)$ *(extended by* $+\infty$ *for* $\theta \notin [0,1]$*).*

$V^*(t,x)$ *itself is a Lipschitz solution of* $\partial_t V^* + f(\partial_z V^*) = 0$ *with initial data* $\max\{0,z\}$*.*

An asymptotic bound on the mixing zone

The "wave speed" $\frac{df}{d\theta}$ satisfies $\frac{df}{d\theta}(\theta = 0) = -1$, $\frac{df}{d\theta}(\theta = 1) = m$:

Corollary 1. *Suppose that the initial configuration* $\chi_0(x)$ *is unstable in the sense that there exist* $-\infty < M \leq N < \infty$ *such that*

$$\chi_0 \;=\; 0 \quad \textit{for}\, z \leq M \quad \textit{and} \quad \chi_0 \;=\; 1 \quad \textit{for}\, z \geq N.$$

Then we have for any weak solution $\chi(t, x)$*:*

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_{(0,2\pi)^{d-1}\times(-\infty,-t)} \chi(t,x)\,dx \;=\; 0 \quad \textit{and}$$

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_{(0,2\pi)^{d-1}\times(mt,\infty)} (1-\chi)(t,x)\,dx \;=\; 0.$$

... and its interpretation

Interpretation: The mixing zone is confined to $\{-t < z < mt\}$.

Optimality: By the method of convex integration, L. Székelyhidi ("Relaxation of the incompressible porous medium equation") constructs rough weak solutions that exactly saturate this bound.

Main ingredient for Theorem 1

Proposition 1 (Otto '01). *Let* $\chi(t,x)$ *be a weak solution. Define* $V(t,z)$ *as above, that is,*

$$V(t,z) \;:=\; -\frac{1}{(2\pi)^{d-1}}\int_{(0,2\pi)^{d-1}\times(-\infty,z)} \chi(t,x)\,dx.$$

For given length L *define the convolution kernel* $\zeta_L(z)$ *via*

$$\zeta_L(z) \;:=\; \frac{1}{2L}\exp(-\frac{|z|}{L}).$$

Let $(V*\zeta_L)(t,z)$ *denote the convolution of* $V(t,z)$ *with* $\zeta_L(z)$ *in* z*. We have*

$$\partial_t(V*\zeta_L)+\frac{1}{1-\frac{\sqrt{m}}{L}}f(\partial_z(V*\zeta_L)) \;\le\; 0.$$

HEURISTICS: Gradient flow structure

Formally, the evolution is the gradient flow with respect to:

- The functional E given by

$$E(\chi) \;=\; \int z\chi\,dx + const \;=\; \int z(\chi - H(z))\,dx,$$

  for any configuration $\chi(x) \in \{0, 1\}$ with volume constraint $\int \chi\,dx = const$, say $\int (\chi - H)\,dx = 0$.

- The metric tensor g given by

$$\begin{aligned} g_\chi(\delta\chi, \delta\chi) \;&=\; \inf_u \{ \int (\chi + r(1-\chi))|u|^2\,dx \,| \\ &\qquad \partial\chi + \nabla\cdot(\chi u) = 0,\ \nabla\cdot u = 0\,\}, \end{aligned}$$

  where $\delta\chi$ denotes an infinitesimal perturbation of χ, i. e. a normal velocity of the boundary $\partial\{\chi = 1\}$.

Check gradient flow structure

In the abstract framework, $\partial_t\chi$ is determined as the solution of

$$\text{minimize} \quad \frac{1}{2}g_\chi(\delta\chi,\delta\chi) + \mathsf{diffE}_{|\chi}.\delta\chi \quad \text{in } \delta\chi.$$

Using $\delta\chi + \nabla\cdot(\chi u) = 0$, this turns into

$$\text{minimize} \quad \frac{1}{2}\int(\chi + r(1-\chi))|u|^2\,dx - \int z\nabla\cdot(\chi u)\,dx$$
$$\text{among all } u \text{ with } \nabla\cdot u = 0,$$

and is solved by the u for which there exists a pressure p such that $u = (\chi + m(1-\chi))(-\nabla p - \chi\binom{0}{1})$. The pressure acts as the Lagrange multiplier of the incompressibility constraint $\nabla\cdot u = 0$.

Physical interpretation of gradient flow structure

Function $E(\chi)$ & metric tensor $g_\chi(\delta\chi, \delta\chi)$ have physical meaning:

- $E(\chi)$ is the potential energy w. r. t. gravity of the configuration χ. The energy is driving the evolution.

- $g_\chi(\delta\chi, \delta\chi)$ is the rate of energy dissipation that is necessary to generate the infinitesimal perturbation $\delta\chi$ of the configuration χ. It comes from the friction in the porous medium when the two phases flow as described by an incompressible velocity field u. The expression $\chi + r(1-\chi)$ accounts for the different friction in the phases. The dissipation is limiting the evolution.

How to guess gradient flow structure

Typically, easier to guess energy functional $E(\chi)$. Then metric tensor $g_\chi(\delta\chi,\delta\chi)$ can be guessed from relation $g_{\chi(t)}(\frac{d\chi}{dt}(t),\frac{d\chi}{dt}(t)) = -\frac{d}{dt}E(\chi(t))$ along gradient flow trajectory. In our case:

$$
\begin{aligned}
g_\chi(\partial_t\chi,\partial_t\chi) &= -\frac{d}{dt}E(\chi) \\
&= -\int z\,\partial_t\chi\,dx \quad \text{by } E=\int z\chi\,dx + const \\
&= -\int \binom{0}{1}\cdot(\chi u)\,dx \quad \text{by } \partial_t\chi + \nabla\cdot(\chi u) = 0 \\
&= \int \left(\frac{1}{\chi+m(1-\chi)}u + \nabla p\right)\cdot u\,dx \quad \text{by } u = (\chi+m(1-\chi))\left(-\nabla p - \chi\binom{0}{1}\right) \\
&\overset{r=m^{-1}}{=} \int (\chi + r(1-\chi))|u|^2\,dx \quad \text{by } \nabla\cdot u = 0.
\end{aligned}
$$

Metric tensor comes from "ambient" weighted two-phase Wasserstein space

Rewrite metric tensor: $g_\chi(\delta\chi,\delta\chi)$

$$= \inf_u \{ \int (\chi + r(1-\chi))|u|^2\,dx \,\Big|\, \delta\chi + \nabla\cdot(\chi u) = 0,\ \nabla\cdot u = 0 \}$$

$$= \inf_j \{ \int \frac{1}{\chi}|j|^2\,dx \,\Big|\, \delta\chi + \nabla\cdot j = 0 \}$$

$$+ r \inf_q \{ \int \frac{1}{1-\chi}|q|^2\,dx \,\Big|\, -\delta\chi + \nabla\cdot q = 0 \} \quad \text{by } (j,q) = (\chi u, (1-\chi)u).$$

Note that for both terms: $\inf_j \{ \int \frac{1}{\theta}|j|^2\,dx \,|\, \delta\theta + \nabla\cdot j = 0 \}$

$$= \int \theta|\nabla\phi|^2\,dx \quad \text{where } \delta\theta - \nabla\cdot(\theta\nabla\phi) = 0$$

$$= g_\theta^{Wass}(\delta\theta,\delta\theta).$$

Hence $g_\chi(\delta\chi,\delta\chi) = g_\chi^{Wass}(\delta\chi,\delta\chi) + r\, g_{1-\chi}^{Wass}(-\delta\chi,-\delta\chi)$.

Time discretization

A gradient flow has a natural time discretization: Fix time step size $h > 0$. Given solution χ_{k-1} at time step $k-1$ get solution χ_k at time step k by minimizing

$$\frac{1}{2}\text{dist}^2(\chi_{k-1}, \chi_k) + hE(\chi),$$

where the *induced distance* $\text{dist}(\chi_0, \chi_1)$ between two configurations is the minimal length of connecting curves; its square $\text{dist}^2(\chi_0, \chi_1)$ is the minimal energy of connecting curves:

$$\begin{aligned} \text{dist}^2(\chi_0, \chi_1) &= \inf_{\chi(t)} \{ \int_0^1 g_\chi(\partial_t \chi, \partial_t \chi) \, dt \, | \\ & \quad \chi(t=0) = \chi_0, \quad \chi(t=1) = \chi_1 \}. \end{aligned}$$

Induced distance in our case

In our case

$$\mathrm{dist}^2(\chi_0,\chi_1) \;=\; \inf_{\chi(t,x),u(t,x)} \Big\{ \int_0^1 \int (\chi + r(1-\chi))|u|^2\, dx\, dt \;\Big|$$
$$\partial_t\chi + \nabla\cdot(\chi u) = 0, \quad \nabla\cdot u = 0, \quad \chi\in\{0,1\},$$
$$\chi(t=0) = \chi_0, \quad \chi(t=1) = \chi_1 \Big\},$$

— and there is no explicit representation.

A proxy for the induced distance

Strategy: Take induced distance from ambient space (see above). This makes no difference for consistency of discretization in finite-d smooth case.
Taking induced distance from ambient space means relaxing constraints on the connecting curves.

Step 0. Rewrite (as above) metric tensor in terms of *fluxes* $(j(x), q(x))$:

$$\begin{aligned} g_\chi(\delta\chi, \delta\chi) &= \inf_u \{ \int (\chi + r(1-\chi))|u|^2 \, dx \,\Big|\, \\ &\qquad \delta\chi + \nabla \cdot (\chi u) = 0, \, \nabla \cdot u = 0 \} \\ &= \inf_{j,q} \{ \int \frac{1}{\chi}|j|^2 + \frac{r}{1-\chi}|q|^2 \, dx \,\Big|\, \\ &\qquad \delta\chi + \nabla \cdot j = 0, \quad -\delta\chi + \nabla \cdot q = 0 \}. \end{aligned}$$

Step 1. Relax *non-convex* constraint $\chi \in \{0,1\} \rightsquigarrow \theta \in [0,1]$:

$$\mathrm{dist}^2(\chi_0,\chi_1) \rightsquigarrow \inf_{j,q}\Big\{\int_0^1\int \frac{1}{\theta}|j|^2 + \frac{r}{1-\theta}|q|^2\,dx\,dt\,\Big|\, \partial_t\theta + \nabla\cdot j = 0, \quad -\partial_t\theta + \nabla\cdot q = 0, \quad \theta\in[0,1], \quad \theta(t=0)=\chi_0, \quad \theta(t=1)=\chi_1\Big\}.$$

(Cf. relaxation of Euler's equation, vortex sheets [Brenier])

Step 2. Relax *incompressibility* constraint $(\theta, 1-\theta) \rightsquigarrow (\theta, \rho)$:

$$
\begin{aligned}
&\mathrm{dist}^2(\chi_0, \chi_1) \\
&\quad \rightsquigarrow \quad \inf_{j,q}\Big\{\int_0^1 \int \frac{1}{\theta}|j|^2 + \frac{r}{\rho}|q|^2 \, dx \, dt \,\Big| \\
&\qquad\qquad \partial_t \theta + \nabla \cdot j = 0, \quad \partial_t \rho + \nabla \cdot q = 0, \quad \theta, \rho \in [0,1], \\
&\qquad\qquad \theta(t=0) = \chi_0, \ \rho(t=0) = 1 - \chi_0 \\
&\qquad\qquad \theta(t=1) = \chi_1, \ \rho(t=1) = 1 - \chi_1 \Big\} \\
&\quad = \quad \inf_{j}\Big\{\int_0^1 \int \frac{1}{\theta}|j|^2 \, dx \, dt \,\Big|\, \partial_t \theta + \nabla \cdot j = 0, \quad \theta \in [0,1], \\
&\qquad\qquad \theta(t=0) = \chi_0, \quad \theta(t=1) = \chi_1 \Big\} \\
&\quad + \quad r \inf_{q}\Big\{\int_0^1 \int \frac{1}{\rho}|q|^2 \, dx \, dt \,\Big|\, \partial_t \rho + \nabla \cdot q = 0, \quad \rho \in [0,1], \\
&\qquad\qquad \rho(t=0) = 1 - \chi_0, \quad \rho(t=1) = 1 - \chi_1 \Big\} \\
&\quad = \quad d^2(\chi_0, \chi_1) + r d^2(1 - \chi_0, 1 - \chi_1) \quad \text{by Brenier-Benamou}
\end{aligned}
$$

yields Wasserstein metric d on product space with weight r
— this is the ambient space.

Relaxation of the time discretization

There is an unambiguous relaxation procedure for variational problems that display a microstructure: Take lower semi-continuous envelope w. r. t. weak topology.
When used on time discretization one obtains:
Given the solution $\theta_{k-1} \in [0,1]$ at time step $k-1$ get solution $\theta_k \in [0,1]$ at time step k by minimizing

$$\frac{1}{2}d^2(\theta_{k-1},\theta_k) + \frac{r}{2}d^2(1-\theta_{k-1},1-\theta_k) + hE(\theta_k).$$

A relaxed evolution equation

The above is natural time discretization of gradient flow of Functional E given by

$$E(\theta) \;=\; \int z(\theta - H(z))\,dx,$$

on all configurations $\theta \in [0,1]$ of prescribed total volume $\int(\theta - H(z))\,dx = 0$.

Metric tensor g given by

$$g_\theta(\delta\theta,\delta\theta) \;=\; g_\theta^{Wass}(\delta\theta,\delta\theta) + r g_{1-\theta}^{Wass}(-\delta\theta,-\delta\theta),$$

where $g_\theta^{Wass}(\delta\theta,\delta\theta)$ is standard metric tensor behind Wasserstein metric, i. e.

$$g_\theta^{Wass}(\delta\theta,\delta\theta) \;=\; \inf_j\{\int \frac{1}{\theta}|j|^2\,dx \,\Big|\, \delta\theta + \nabla\cdot j = 0\}.$$

One space dimension and accumulated volume

Assume weak limits only depend on vertical variable z: $\theta = \theta(z)$.
Then metric tensor can be rewritten as

$$g_\theta(\delta\theta, \delta\theta) \;=\; \int (\frac{1}{\theta} + \frac{r}{1-\theta})|j|^2\, dz \quad \text{where } \delta\theta + \frac{dj}{dz} = 0.$$

Convenient new variables: Accumulated volume

$$V(z) \;=\; \int_{-\infty}^{z} \theta(z')\, dz' \quad \longleftrightarrow \quad \frac{dV}{dz} \;=\; \theta.$$

functional $E:$ $E(V) = \int (V - \max\{0, z\})\, dx$,
metric tensor $g:$ $g_V(\delta V, \delta V) \;=\; \int (\frac{1}{\frac{dV}{dz}} + \frac{r}{1-\frac{dV}{dz}})(\delta V)^2\, dz.$

Identification of gradient flow equation

Abstract form of gradient flow evolution: $\partial_t V$ is determined by

$$g_V(\partial_t V, \delta V) + \mathrm{diff} E_{|V}.\delta V \;=\; 0 \quad \text{for all } \delta V.$$

This yields

$$(\frac{1}{\partial_z V} + \frac{r}{1 - \partial_z V})\partial_t V \;+\; 1 \;=\; 0,$$

that is, the Hamilton-Jacobi equation

$$\partial_t V + f(\partial_x V) \;=\; 0,$$

with the (convex) flux function

$$f(\theta) \;:=\; -\frac{1}{\frac{1}{\theta} + \frac{r}{1-\theta}} \;=\; -\frac{\theta(1-\theta)}{r\theta + (1-\theta)}.$$

Non-uniqueness

The initial value problem for the Hamilton Jacobi equation has many Lipschitz continuous solutions that satisfy the equation almost everywhere.

Example: Both $V^*(t, z)$ in Lemma 1 and $V(t, z) = \max\{z, 0\}$ are Lipschitz solutions of the Hamilton Jacobi equation with initial data $V_0(z) = \max\{z, 0\}$.

However, since f is convex, the *maximal* Lipschitz continuous *sub*solution can be characterized as the unique "viscosity solution".

The notion of viscosity solution only requires continuity of $V(t, z)$ and relies on the (formal) maximum principle of the equation.

Notion of viscosity solution

Definition 1. *A continuous function* $V(t, z)$ *is called viscosity solution of the Hamilton-Jacobi equation*

$$\partial_t V + f(\partial_x V) \;=\; 0,$$

if for any continuously differentiable function $\zeta(t, z)$ *that satisfies for some* (t_1, z_1)

$$(V - \zeta)(t, x) \;\substack{\geq \\ \leq}\; (V - \zeta)(t_1, z_1) \quad \textit{for all } (t, z)$$

we have

$$(\partial_t \zeta + f(\partial_z \zeta))(t_1, z_1) \;\substack{\geq \\ \leq}\; 0,$$

where the upper sign corresponds to the super-solution property whereas the lower sign to the subsolution property.

Get uniqueness of initial value problem.

Short notation

A function $\theta(z)$ will be called admissible if

- $\theta \in [0, 1]$

- Bounded mixing zone: There exist $-\infty < M \leq N < \infty$ with

$$\theta_0(z_0) \ = \ 0 \quad \text{for } z_0 \leq M \quad \text{and} \quad \theta_0(z_0) \ = \ 1 \quad \text{for } z_0 \geq N$$

- Volume constraint: $\int (\theta_0 - H)\, dz_0 = 0$

The main result of Mini-course

Theorem 2 (Otto '99, Otto&Gigli '11). *Let* θ_0 *be admissible. Then for any* $h > 0$*, there exists a unique sequence of admissible* $\{\theta_k\}_{k\in\mathbb{N}}$ *such that for any* $k \in \mathbb{N}$*,* θ_k *minimizes the functional*

$$\frac{1}{2}d^2(\theta_{k-1},\theta_k) + \frac{r}{2}d^2(1-\theta_{k-1}, 1-\theta_k) + h\int z(\theta_k - H)\,dz$$

in the class of all admissible θ_k*.*

Consider $V_k(z) := \int_{-\infty}^{z}\theta_k(z')\,dz'$*,*
and $V_h(t,z) := V_k(z) \quad$ *for* $t \in [kh,(k+1)h)$*.*

Then $V_h(t,z)$ *converges locally uniformly to the unique viscosity solution of* $\partial_t V + f(\partial_z V) = 0$ *with initial data* V_0*.*

Key ingredient for Theorem 2

Proposition 2. *[Otto '99, Otto&Gigli '11]* $h > 0$ *and* θ_0 *admiss.*

Then $\exists!$ *minimizer* θ_1 *in admissible class of*

$$\frac{1}{2}d^2(\theta_0,\theta_1) + \frac{r}{2}d^2(1-\theta_0, 1-\theta_1) + h\int z_1(\theta_1 - H)\,dz_1.$$

Consider $V_0(z_0) := \int_{-\infty}^{z_0}\theta_0\,dz_0'$, $V_1(z_1) := \int_{-\infty}^{z_1}\theta_1\,dz_1'$,

and $V(t,z) := \left\{\begin{array}{lll} V_0(z) & \textit{for} & t\in[0,h) \\ V_1(z) & \textit{for} & t = h \end{array}\right\}.$

Suppose $\zeta \in C^1$ *and* z_1*:* $(V-\zeta)(t,x) \genfrac{}{}{0pt}{}{\geq}{\leq} (V-\zeta)(h,z_1)$.

Then $(\partial_t\zeta + f(\partial_z\zeta))(h,z_1) \genfrac{}{}{0pt}{}{\geq}{\leq} o(1)$, *where* $o(1)$ *vanishes for* $h\downarrow 0$ *and only depends on the modulus of continuity of the partial derivatives of* ζ*.*

PROOF OF PROPOSITION 2

Step 1. Definition of the Wasserstein metric. Squared Wasserstein metric $d^2(\theta_0, \theta_1)$ between two densities $\theta_0\, dz_0$ and $\theta_1\, dz_1$ on $\mathbb{R}$:

$$\frac{1}{2} d^2(\theta_0, \theta_1) \;=\; \inf_{\lambda} \int \frac{1}{2}(z_0 - z_1)^2\, \lambda(dz_0 dz_1),$$

where infimum is taken over all "transference plans" λ.

A non-negative (Borel) measure $\lambda(dz_0 dz_1)$ on the product space $\mathbb{R} \times \mathbb{R}$ is called transference plan if

$$\int \zeta(z_0)\, \lambda(dz_0 dz_1) \;=\; \int \zeta(z_0)\, \theta_0(z_0)\, dz_0 \quad \text{and}$$

$$\int \zeta(z_1)\, \lambda(dz_0 dz_1) \;=\; \int \zeta(z_1)\, \theta_1(z_0)\, dz_1$$

holds for all smooth and compactly supported functions ζ.

Note: θ_0 and θ_1 admissible $\Longrightarrow \exists$ admissible transference plan of finite cost.

Indeed: Boundedness of mixing zone: $\exists\ M < 0 < N$
$\theta_0 = \theta_1 = 0$ on $(-\infty, M)$ and $\theta_0 = \theta_1 = 1$ on (N, ∞).
Volume constraint: $\int_M^N \theta_0 \, dz_0 = N = \int_M^N \theta_1 \, dz_1$.
Then

$$\begin{aligned} &\int \zeta(z_0, z_1)\, \lambda(dz_0 dz_1) \\ &\quad = \frac{1}{N} \int_M^N \int_M^N \zeta(z_0, z_1)\, \theta_0(z_0)\, \theta_1(z_1)\, dz_0\, dz_1 + \int_N^\infty \zeta(z_0, z_0)\, dz_0 \end{aligned}$$

defines transference plan, which has finite cost.

Direct method of calculus of variations (with soft compactness and lower semi-continuity arguments under weak convergence): $\exists$ *optimal* transference plan.

Step 2. Heuristics and dual formulation of Wasserstein.

Saddle point formulation: incorporates admissibility constraints on transference plan:

$$
\begin{aligned}
&\frac{1}{2}d^2(\theta_0,\theta_1)\\
&\quad = \inf_{\lambda}\sup_{\phi_0,\phi_1}\Big(\int (z_0-z_1)^2\,\lambda(dz_0dz_1)\\
&\qquad + \int (\frac{1}{2}z_0^2-\phi_0(z_0))\,\theta_0(z_0)\,dz_0 - \int (\frac{1}{2}z_0^2-\phi_0(z_0))\,\lambda(dz_0dz_1)\\
&\qquad + \int (\frac{1}{2}z_1^2-\phi_1(z_1))\,\theta_1(z_1)\,dz_1 - \int (\frac{1}{2}z_1^2-\phi_1(z_1))\,\lambda(dz_0dz_1)\Big)\\
&\quad = \inf_{\lambda}\sup_{\phi_0,\phi_1}\Big(\int (\phi_0(z_0)+\phi_1(z_1)-z_0z_1)\,\lambda(dz_0dz_1)\\
&\qquad + \int (\frac{1}{2}z_0^2-\phi_0(z_0))\,\theta_0(z_0)\,dz_0 + \int (\frac{1}{2}z_1^2-\phi_1(z_1))\,\theta_1(z_1)\,dz_1\Big),
\end{aligned}
$$

where infimum runs over all non-negative measures $\lambda(dz_0dz_1)$ and supremum runs over all (continuous) functions $\phi_0(z_0)$, $\phi_1(z_1)$.

Dual variational formulation: 1) switch supremum and infimum by the minimax theorem, 2) carry out the supremum over λ:

$$\frac{1}{2}d^2(\theta_0, \theta_1)$$
$$= \sup_{\phi_0, \phi_1} \left(\int (\frac{1}{2}z_0^2 - \phi_0(z_0))\, \theta_0(z_0)\, dz_0 + \int (\frac{1}{2}z_1^2 - \phi_1(z_1))\, \theta_1(z_1)\, dz_1 \right),$$

where supremum now runs over all pairs of functions $\phi_0(z_0)$ and $\phi_1(z_1)$ with

$$z_0 z_1 \leq \phi_0(z_0) + \phi_1(z_1) \quad \text{for all } z_0, z_1.$$

Solution of saddle point problem: $(\lambda, (\phi_0, \phi_1))$.
Have by optimality

$$\begin{aligned} \phi_0(z_0) + \phi_1(z_1) \;&=\; z_0 z_1 \qquad \text{for } \lambda\text{-a. e. } (z_0, z_1), \\ \phi_0 \text{ and } \phi_1 \qquad &\text{are Legendre transforms of each other} \end{aligned}$$

Thus for sub-gradients $\Phi_0 = \partial\phi_0$ and $\Phi_1 = \partial\phi_1$:

$$\begin{aligned} z_1 \;\in\; \Phi_0(z_0) \quad \text{and} \quad z_0 \;\in\; \Phi_1(z_1) \qquad &\text{for } \lambda\text{-a. e. } (z_0, z_1), \\ \Phi_0 \text{ and } \Phi_1 \qquad &\text{are inverses of each other.} \end{aligned}$$

Have

$$\begin{aligned}
\frac{1}{2}d^2(\theta_0,\theta_1) &= \int \frac{1}{2}(z_0 - \Phi_0(z_0))^2\, \theta_0(z_0)\, dz_0 \\
&= \int \frac{1}{2}(\Phi_1(z_1) - z_1)^2\, \theta_1(z_1)\, dz_1, \\
\int \zeta(\Phi_1(z_1))\, \theta_1(z_1)\, dz_1 &= \int \zeta(z_0)\, \theta_0(z_0)\, dz_0, \\
\int \zeta(\Phi_0(z_0))\, \theta_0(z_0)\, dz_0 &= \int \zeta(z_1)\, \theta_1(z_0)\, dz_1,
\end{aligned}$$

for all ζ. On level of accumulated volume almost everywhere

$$V_1(z_1) = V_0(\Phi_1(z_1)) \quad \text{and} \quad V_0(z_0) = V_1(\Phi_0(z_0)).$$

We now use dual formulation for $\frac{1}{2}d^2(\theta_0, \theta_1)$ and for $\frac{1}{2}d^2(1-\theta_0, 1-\theta_1)$ to reformulate our variational problem as *another* saddle point problem:

$$\inf_{\theta_1} \sup_{(\phi_0,\phi_1),(\psi_0,\psi_1)} \Big(\int (\frac{1}{2}z_0^2 - \phi_0(z_0))\,\theta_0(z_0)\,dz_0 + \int (\frac{1}{2}z_1^2 - \phi_1(z_1))\,\theta_1(z_1)\,dz_1$$
$$+ \; r \int (\frac{1}{2}z_0^2 - \psi_0(z_0))\,(1-\theta_0)(z_0)\,dz_0 + r \int (\frac{1}{2}z_1^2 - \psi_1(z_1))\,(1-\theta_1)(z_1)\,dz_1$$
$$+ \; h \int z_1(\theta_1(z_1) - H(z_1))\,dz_1 \Big).$$

Solution of saddle point problem: $(\theta_1, ((\phi_0, \phi_1), (\psi_0, \psi_1)))$.
Optimality in $((\phi_0, \phi_1), (\psi_0, \psi_1))$: sub-gradients

$$\Phi_0 \;:=\; \partial\phi_0,\; \Phi_1 \;:=\; \partial\phi_1,\; \Psi_0 \;:=\; \partial\psi_0,\; \Psi_1 \;:=\; \partial\psi_1$$

satisfy

$$\begin{aligned} V_1(z_1) &= V_0(\Phi_1(z_1)), \quad & V_0(z_0) &= V_1(\Phi_0(z_0)), \\ W_1(z_1) &= W_0(\Phi_1(z_1)), \quad & W_0(z_0) &= W_1(\Phi_0(z_0)), \end{aligned}$$

where W accumulated volume w. r. t. $1-\theta$:

$$\begin{aligned} W_0(z_0) &:= -\int_{z_0}^{\infty}(1-\theta_0)\,dz_0' \overset{\text{volume constraint}}{=} z_0 - V_0(z_0), \\ W_1(z_1) &:= -\int_{z_1}^{\infty}(1-\theta_1)\,dz_1' \overset{\text{volume constraint}}{=} z_1 - V_1(z_1). \end{aligned}$$

Optimality in θ_1. Respect volume constraint but ignore constraint $\theta_1 \in [0,1]$:

$$(\frac{1}{2}z_1^2 - \phi_1(z_1)) - r(\frac{1}{2}z_1^2 - \psi_1(z_1)) + hz_1 \;=\; const \quad \text{for all } z_1.$$

Derivative:

$$(z_1 - \Phi_1(z_1)) - r(z_1 - \Psi_1(z_1)) + h \;=\; 0 \quad \text{for all } z_1.$$

Rewrite ($m = \frac{1}{r}$)

$$(1-m)z_1 + m\Phi_1(z_1) - mh \;=\; \Psi_1(z_1) \quad \text{for all } z_1.$$

Difficulty: variational inequality instead of equality, derivative?.

Instead, will take opposite direction in proof:

Step 3 through Step 6: *construct* two maximally monotone maps Φ_1, Ψ_1 that satisfy

$$V_0(\Phi_1(z_0)) + W_0(\Psi_1(z_1)) = z_1 \quad \forall\, z_1 \quad \text{“incompressibility”},$$

$$\Psi_1(z_1) = (1-m)z_1 + m\Phi_1(z_1) - mh \quad \forall\, z_1 \quad \text{“relation”}.$$

Step 7 and Step 8: $V_1(z_1) = V_0(\Phi_1(z_1))$ is discrete viscosity solution.

Step 9 through Step 12: anti-derivatives ϕ_1 and ψ_1 satisfy

$$\left(\frac{1}{2}z_1^2 - \phi_1(z_1)\right) - r\left(\frac{1}{2}z_1^2 - \psi_1(z_1)\right) + h = const \,\forall\, z_1 \text{ “relation”}.$$

$\theta_1 = \frac{dV_1}{dz_1}$ is unique solution of variational problem.

Suggested quick reading: **Step 7** & **Step 8**.

Step 3. From θ_0 to (V_0, W_0). Accumulated volume:

$$V_0(z_0) \;:=\; \int_{-\infty}^{z_0} \theta_0 \, dz_0' \quad \text{and} \quad W_0(z_0) \;:=\; -\int_{z_0}^{\infty} (1-\theta_0)\, dz_0'.$$

From $\theta_0 \in [0,1]$:

$$\frac{dV_0}{dz_0},\; \frac{dW_0}{dz_0} \;\in\; [0,1] \;\text{ almost everywhere.}$$

From volume constraint:

$$V_0(z_0) + W_0(z_0) \;=\; z_0.$$

From bounded mixing zone:

$$(V_0, W_0)(z_0) \;=\; \left\{ \begin{array}{lll} (0, z_0) & \text{for} & z_0 \le M \\ (z_0, 0) & \text{for} & z_0 \ge N \end{array} \right\}.$$

Step 4. From (V_0, W_0) to (Φ_0, Ψ_0).
Heuristics. Want:

$$\begin{aligned} V_0(\Phi_1(z_1)) + W_0(\Psi_1(z_1)) &= z_1 && \text{incompressibility,} \\ \Phi_1(z_1) &= (1-r)z_1 + r\Psi_1(z_1) + h && \text{relation} \iff \\ \Psi_1(z_1) &= (1-m)z_1 + m\Phi_1(z_1) - mh && \text{relation.} \end{aligned}$$

Express in terms of inverses $\Phi_0 = \Phi_1^{-1}$ and $\Psi_0 = \Psi_1^{-1}$.
Set $z_1 = \Phi_0(z_0)$ in above:

$$\begin{aligned} V_0(z_0) + W_0(\Psi_1(\Phi_0(z_0))) &= \Phi_0(z_0) && \text{incompressibility,} \\ \Psi_1(\Phi_0(z_0)) &= (1-m)\Phi_0(z_0) + mz_0 - mh && \text{relation.} \end{aligned}$$

Combine to implicit equation for $\Phi_0(z_0)$:

$$\Phi_0(z_0) = V_0(z_0) + W_0((1-m)\Phi_0(z_0) + mz_0 - mh).$$

Likewise from $z_1 = \Psi_0(y_0)$:

$$\Psi_0(y_0) = W_0(y_0) + V_0((1-r)\Psi_0(y_0) + ry_0 + h).$$

Because of $\frac{dV_0}{dz_0}, \frac{dW_0}{dz_0} \in [0,1]$:
$\forall\ z_0, y_0 \quad \exists\,!\ \Phi(z_0), \Psi_0(y_0)$ such that

$$\begin{aligned} \Phi_0(z_0) &= V_0(z_0) + W_0((1-m)\Phi_0(z_0) + mz_0 - mh), \\ \Psi_0(y_0) &= W_0(y_0) + V_0((1-r)\Psi_0(y_0) + ry_0 + h). \end{aligned}$$

Moreover,

Φ_0, Ψ_0 are continuous and non-decreasing.

From bounded mixing zone:

$$\begin{aligned} \Phi_0(z_0) &= z_0 - h \quad \text{for } z_0 \le M \quad \text{and} \quad \Phi_0(z_0) = z_0 \quad \text{for } z_0 \ge N + mh, \\ \Psi_0(y_0) &= y_0 \quad \text{for } y_0 \le M - h \quad \text{and} \quad \Psi_0(y_0) = y_0 + mh \quad \text{for } y_0 \ge N. \end{aligned}$$

Get desired relation by construction:

$$\begin{aligned} \Phi_0(z_0) &= \Psi_0((1-m)\Phi_0(z_0) + mz_0 - hm), \\ \Psi_0(y_0) &= \Phi_0((1-r)\Psi_0(y_0) + ry_0 + h). \end{aligned}$$

Step 5. From (Φ_0, Ψ_0) **to** (Φ_1, Ψ_1).

Set $\Phi_1 := \Phi_0^{-1}$, $\Psi_1 := \Psi_0^{-1}$ = maximally monotone graphs.

Φ_0, Ψ_0 are continuous with range $\mathbb{R} \Longrightarrow$

Φ_1, Ψ_1 strictly monotone, defined on all $\mathbb{R}$.

Bounded mixing zone $\Longrightarrow$

$$(\Phi_1, \Psi_1)(z_1) = \left\{ \begin{array}{ll} (z_1 + h, z_1) & \text{for } \; z_1 \leq M - h \\ (z_1, z_1 - mh) & \text{for } \; z_1 \leq M - h \end{array} \right\}.$$

By construction $\Longrightarrow$

$$\begin{aligned} (1-m)z_1 + m\Phi_1(z_1) - mh &= \Psi_1(z_1) \quad \text{relation,} \\ V_0(\Phi_1(z_1)) + W_0(\Psi_1(z_1)) &= \{z_1\} \quad \text{incompressibility.} \end{aligned}$$

Step 6. From (Φ_1, Ψ_1) to (V_1, V_0).

Incompressibility, i. e. $V_0(\Phi_1(z_1)) + W_0(\Psi_1(z_1)) = \{z_1\} \quad \Longrightarrow$

$$\{V_1(z_1)\} = V_0(\Phi_1(z_1)), \quad \{W_1(z_1)\} = W_0(\Psi_1(z_1))$$

defines pair of functions V_1, W_1 with

$$V_1(z_1) + W_1(z_1) = z_1.$$

Φ_1, Ψ_1 monot. $\quad \& \quad \frac{dV_0}{dz_0}, \frac{dW_0}{dz_0} \geq 0 \quad \& \quad V_1 + W_1 = z_1 \qquad \Longrightarrow$

$$\frac{dV_1}{dz_1}, \frac{dW_1}{dz_1} \in [0, 1].$$

Bounded mixing zone $\Longrightarrow$

$$(V_1, W_1)(z_1) = \left\{ \begin{array}{lll} (0, z_1) & \text{for} & z_0 \leq M - h \\ (z_1, 0) & \text{for} & z_0 \geq N + mh \end{array} \right\}.$$

Step 7. Relation between V_1 and V_0.

Claim: $\forall\, z_1 \quad \exists\, z_0$ such that

$$\begin{aligned} V_1(z_1) &= V_0(z_0), \\ V_1(z_1) &= V_0\Big((1-m)z_1 + mz_0 - mh\Big) + m(z_1 - z_0 + h), \\ z_0 &\in [z_1, z_1 + h]. \end{aligned}$$

Choose $z_0 \in \Phi_1(z_1)$.
Definition of $V_1 \implies V_1(z_1) = V_0(z_0)$.

Relation $\Psi_1(z_1) = (1-m)z_1 + m\Phi_1(z_1) - hm \implies$

$$\exists\ y_0 \in \Psi_1(z_1): \quad y_0 = (1-m)z_1 + mz_0 - mh.$$

Definition of $W_1 \implies W_1(z_1) = W_0(y_0)$.

Substitute y_0 and use $V_i(z) + W_i(z) = z$.

Part (3) in **claim**: $\forall\, z_1 \quad \exists\, z_0$ such that

$$V_1(z_1) = V_0(z_0), \tag{1}$$

$$V_1(z_1) = V_0\Big((1-m)z_1 + mz_0 - mh\Big) + m(z_1 - z_0 + h), \tag{2}$$

$$z_0 \in [z_1, z_1 + h]. \tag{3}$$

Have identity

$$m(z_1 - z_0 + h) \overset{(2)}{=} V_1(z_1) - V_0\Big((1-m)z_1 + mz_0 - mh\Big)$$
$$\overset{(1)}{=} V_0(z_0) - V_0\Big((1-m)z_1 + mz_0 - mh\Big).$$

Together with $\frac{dV_0}{dz_0} \left\{ \begin{array}{cc} \geq & 0 \\ \leq & 1 \end{array} \right\} \Longrightarrow$

$$m(z_1 - z_0 + h) \left\{ \begin{array}{ll} \geq & 0 \\ \leq & z_0 - \big((1-m)z_1 + mz_0 - mh\big) \end{array} \right\} \Longrightarrow \quad z_0 \left\{ \begin{array}{ll} \leq & z_1 + h \\ \geq & z_1 \end{array} \right\}.$$

Step 8. Discrete viscosity solution

Consider piecewise constant interpolation in time:

$$V(t,z) := \left\{ \begin{array}{lll} V_0(z) & \text{for} & t \in [0,h) \\ V_1(z) & \text{for} & t = h \end{array} \right\}.$$

Have: smooth function ζ "touches" V at (h, z_1) for some z_1:

$$(V-\zeta)(t,x) \ \begin{array}{c}\geq \\ \leq\end{array} \ (V-\zeta)(h,z_1) \quad \text{for all } (t,z).$$

Claim: inequality in Hamilton-Jacobi applied to ζ at (h, z_1):

$$(\partial_t \zeta + f(\partial_z \zeta))(h,z_1) \ \begin{array}{c}\geq \\ \leq\end{array} \ o(1).$$

Touching property at (h, z_1) & $\frac{dV_1}{dz} \in [0,1] \implies$

$$\partial_z \zeta(h, z_1) \in [0,1].$$

Have: $\forall\, z_1 \quad \exists\, z_0$ such that

$$\begin{aligned} V_1(z_1) &= V_0(z_0), \\ V_1(z_1) &= V_0\Big((1-m)z_1 + m z_0 - m h\Big) + m(z_1 - z_0 + h). \end{aligned}$$

Claim:

$$\partial_t \zeta(h, z_1) \quad \begin{matrix} \geq \\ \leq \end{matrix} \quad \partial_z \zeta(h, z_1)\, \tfrac{z_0 - z_1}{h} + o(1),$$

$$\partial_t \zeta(h, z_1) \quad \begin{matrix} \geq \\ \leq \end{matrix} \quad \Big(1 - \partial_z \zeta(h, z_1)\Big)\, m\, (1 - \tfrac{z_0 - z_1}{h}) + o(1).$$

$$
\begin{aligned}
&\partial_t \zeta(h, z_1) + o(1) \\
&\quad = \quad \frac{1}{h}(\zeta(h, z_1) - \zeta(0, z_1)) \\
&\quad = \quad \frac{1}{h}(\zeta(h, z_1) - \zeta(0, z_0)) + \frac{1}{h}(\zeta(0, z_0) - \zeta(0, z_1)) \\
&\quad \begin{matrix} \geq \\ \leq \end{matrix} \quad \frac{1}{h}(V_1(z_1) - V_0(z_0)) + \frac{1}{h}(\zeta(0, z_0) - \zeta(0, z_1)) \\
&\qquad\qquad \text{by touching property} \\
&\quad = \quad \frac{1}{h}(\zeta(0, z_0) - \zeta(0, z_1)) \\
&\qquad\qquad \text{by } V_1(z_1) = V_0(z_0) \\
&\quad = \quad \partial_z \zeta(h, z_1) \, \tfrac{z_0 - z_1}{h} + o(1).
\end{aligned}
$$

Have: $\forall\, z_1 \quad \exists\, z_0$ such that

$$
\begin{aligned}
\partial_z \zeta(h, z_1) \quad &\in \quad [0, 1], \\
\partial_t \zeta(h, z_1) \quad &\substack{\geq \\ \leq} \quad \partial_z \zeta(h, z_1)\, \frac{z_0 - z_1}{h} + o(1), \\
\partial_t \zeta(h, z_1) \quad &\substack{\geq \\ \leq} \quad \Big(1 - \partial_z \zeta(h, z_1)\Big)\, m\, (1 - \tfrac{z_0 - z_1}{h}) + o(1), \\
z_0 \quad &\in \quad [z_1, z_1 + h] \quad \Longrightarrow \quad \frac{z_0 - z_1}{h} \in [0, 1].
\end{aligned}
$$

Claim: $\partial_t \zeta(h, z_1) + f(\partial_z \zeta(h, z_1)) \;\substack{\geq \\ \leq}\; o(1).$

Follows from calculus inequality: $\forall\, \theta\,, \lambda \in [0, 1]$:

$\max\{\theta\, \lambda, (1 - \theta)\, m\, (1 - \lambda)\} \;\geq\; -f(\theta) \;\geq\; \min\{\theta\, \lambda, (1 - \theta)\, m\, (1 - \lambda)\},$

where $-f(\theta) = \frac{\theta(1-\theta)}{r\theta + (1-\theta)}$ by $\theta := \partial_z \zeta(h, z_1)$ and $\lambda := \frac{z_0 - z_1}{h}$.

Step 9. From (Φ_1, Ψ_1) to (ϕ_1, ψ_1) and to (ϕ_0, ψ_0)

Maximal monotone maps $\rightsquigarrow$ sub-gradients $\partial\phi_1 = \Phi_1$, $\partial\psi_1 = \Psi_1$.

Φ_1, Ψ_1 defined on all $\mathbb{R}$, strictly monotone $\Longrightarrow$:

ϕ_1, ψ_1 finite for every z_1, strictly convex.

Relation $(1-m)z_1 + m\Phi_1(z_1) - mh = \Psi_1(z_1) \quad \Longrightarrow$

$$(1-m)\tfrac{1}{2}z_1^2 + m\phi_1(z_1) - mhz_1 \;=\; \psi_1(z_1) + const,$$

$$\Longrightarrow \quad \tfrac{1}{2}z_1^2 - \phi_1(z_1) - r(\tfrac{1}{2}z_1^2 - \psi_1(z_1)) + hz_1 \;=\; const.$$

Bounded mixing zone $\Longrightarrow$ fixing additive constants

$\psi_1(z_1) = \frac{1}{2}z_1^2$ for $z_1 \le M-h$ and $\phi_1(z_1) = \frac{1}{2}z_1^2$ for $z_1 \ge N+mh$.

Step 10. From (ϕ_1, ψ_1) to (ϕ_0, ψ_0)

$(\phi_1, \psi_1) \rightsquigarrow (\phi_0, \psi_0)$ via Legendre transform

$$\phi_0(z_0) = \sup_{z_1}(z_1 z_0 - \phi_1(z_1)) \quad \text{and} \quad \psi_0(z_0) = \sup_{z_1}(z_1 z_0 - \psi_1(z_1)).$$

$\Phi_0 = \Phi_1^{-1}$, $\Psi_0 = \Psi_1^{-1}$ uni-valued $\Longrightarrow$ $\frac{d\phi_0}{dz} = \Phi_0$, $\frac{d\psi_0}{dz} = \Psi_0$.

(ϕ_1, ψ_1) strictly convex $\Longrightarrow$

$$z_0 z_1 \le \phi_0(z_0) + \phi_1(z_1) \quad \text{with equality iff } z_1 = \frac{d\phi_0}{dz}(z_0),$$

$$z_0 z_1 \le \psi_0(z_0) + \psi_1(z_1) \quad \text{with equality iff } z_1 = \frac{d\psi_0}{dz}(z_0).$$

Bounded mixing zone $\Longrightarrow$

$$\phi_0(z_0) = \tfrac{1}{2} z_0^2 \quad \text{for } z_0 \ge N + mh \quad \text{and} \quad \psi_0(z_0) = \tfrac{1}{2} z_0^2 \quad \text{for } z_0 \le M - h.$$

Step 11. From (V_1, W_1) to θ_1.

Set $\theta_1 := \frac{dV_1}{dz_1} \in [0,1]$.

Bounded mixing zone $\implies$

$\theta_1(z_1) = 0 \quad$ for $z_1 \le M - h \quad$ and $\quad \theta_1(z_1) = 1 \quad$ for $z_1 \ge N + mh$.

$$V_1 + W_1 = z_1$$

$$\implies \quad V_1(z_1) = \int_{-\infty}^{z_1} \theta_1 \, dz_1' \,, \quad W_1(z_1) = -\int_{z_1}^{\infty} (1 - \theta_1) \, dz_1'$$

$$\implies \quad \int (\theta_1 - H) \, dz_1 = 0 \text{ volume constraint.}$$

$V_1 \circ \Phi_0 = V_0$ and $W_1 \circ \Psi_0 = W_0 \quad \implies$

$$\theta_1 = \Phi_0 \# \theta_0 \quad \text{and} \quad (1 - \theta_1) = \Psi_0 \# (1 - \theta_0).$$

Step 12. θ_1 is unique solution of variational problem.

For any admissible $\tilde{\theta}_1$, i. e. $\tilde{\theta}_1(z_1) \in [0,1]$ measurable with bounded mixing zone and volume constraint:

$$\tilde{\theta}_1(z_1) \;=\; 0 \quad \text{for } z_1 \leq \tilde{M} \quad \text{and} \quad \tilde{\theta}_1(z_1) \;=\; 1 \quad \text{for } z_1 \geq \tilde{N},$$

$$\int (\theta_1 - H)\, dz_1 \quad = \quad 0.$$

we have

$$\begin{aligned} &\frac{1}{2}d^2(\theta_0, \theta_1) + \frac{r}{2}d^2(1-\theta_0, 1-\theta_1) + h\int z(\theta_1 - H)\, dz_1 \\ &\quad \leq \; \frac{1}{2}d^2(\theta_0, \tilde{\theta}_1) + \frac{r}{2}d^2(1-\theta_0, 1-\tilde{\theta}_1) + h\int z(\tilde{\theta}_1 - H)\, dz_1 \end{aligned}$$

with equality only if $\tilde{\theta}_1 = \theta_1$ almost everywhere.

Claim 1: $\frac{1}{2}d^2(\theta_0, \theta_1) + \frac{r}{2}d^2(1-\theta_0, 1-\theta_1) + h\int z(\theta_1 - H)\,dz_1$

$$\begin{aligned}
\leq\ & \int (\tfrac{1}{2}z_0^2 - \phi_0)\,\theta_0\,dz_0 + \int (\tfrac{1}{2}z_1^2 - \phi_1)\,\theta_1\,dz_1 \\
&+ r\int (\tfrac{1}{2}z_0^2 - \psi_0)\,(1-\theta_0)\,dz_0 + r\int (\tfrac{1}{2}z_1^2 - \psi_1)\,(1-\theta_1)\,dz_1 \\
&+ h\int z_1\,(\theta_1 - H)\,dz_1.
\end{aligned}$$

Claim 2: $\frac{1}{2}d^2(\theta_0, \tilde{\theta}_1) + \frac{r}{2}d^2(1-\theta_0, 1-\tilde{\theta}_1) + h\int z(\tilde{\theta}_1 - H)\,dz_1$

$$\begin{aligned}
\geq\ & \int (\tfrac{1}{2}z_0^2 - \phi_0)\,\theta_0\,dz_0 + \int (\tfrac{1}{2}z_1^2 - \phi_1)\,\tilde{\theta}_1\,dz_1 \\
&+ r\int (\tfrac{1}{2}z_0^2 - \psi_0)\,(1-\theta_0)\,dz_0 + r\int (\tfrac{1}{2}z_1^2 - \psi_1)\,(1-\tilde{\theta}_1)\,dz_1 \\
&+ h\int z_1(\tilde{\theta}_1 - H)\,dz_1,
\end{aligned}$$

with equality only if $\tilde{\theta}_1 = \theta_1$ almost everywhere.

The θ_1 and $\tilde{\theta}_1$-dependent parts of r. h. s., i. e.

$$\int (\tfrac{1}{2}z_1^2 - \phi_1)\,\theta_1\,dz_1 + r\int (\tfrac{1}{2}z_1^2 - \psi_1)\,(1-\theta_1)\,dz_1 + h\int z_1\,\theta_1\,dz_1,$$
$$\int (\tfrac{1}{2}z_1^2 - \phi_1)\,\tilde{\theta}_1\,dz_1 + r\int (\tfrac{1}{2}z_1^2 - \psi_1)\,(1-\tilde{\theta}_1)\,dz_1 + h\int z_1\,\tilde{\theta}_1\,dz_1$$

coincide because of

$$(\tfrac{1}{2}z_1^2 - \phi_1) - r(\tfrac{1}{2}z_1^2 - \psi_1) + hz_1 = const \quad \text{relation,}$$
$$\int (\theta_1 - \tilde{\theta}_1)\,dz_1 = 0 \quad \text{volume constraint.}$$

First half of **claim 1**:

$$\frac{1}{2}d^2(\theta_0,\theta_1) \;\le\; \int (\tfrac{1}{2}z_0^2 - \phi_0)\,\theta_0\,dz_0 + \int (\tfrac{1}{2}z_1^2 - \phi_1)\,\theta_1\,dz_1$$

1) $\theta_1 = \Phi_0\#\theta_0 \implies$

$$\int \zeta(z_0,z_1)\,\lambda(dz_0dz_1) \;=\; \int \zeta(z_0,\Phi_0(z_0))\,\theta_0(z_0)\,dz_0$$

is transference plan for $d^2(\theta_0,\theta_1)$.

2) $\Phi_0 = \frac{d\phi_0}{dz_0}$ and $\phi_1 = \phi_0^* \implies$

$$\begin{aligned}\tfrac{1}{2}(z_0-z_1)^2 &= \tfrac{1}{2}z_0^2 - z_0z_1 + \tfrac{1}{2}z_1^2\\ &= (\tfrac{1}{2}z_0^2 - \phi_0(z_0)) + (\tfrac{1}{2}z_1^2 - \phi_1(z_1)) \quad \text{for } z_1 = \Phi_0(z_0).\end{aligned}$$

First half of **claim 2**:

$$\frac{1}{2}d^2(\theta_0, \tilde{\theta}_1) \;\geq\; \int (\tfrac{1}{2}z_0^2 - \phi_0)\,\theta_0\,dz_0 + \int (\tfrac{1}{2}z_1^2 - \phi_1)\,\tilde{\theta}_1\,dz_1$$

with equality only if $\tilde{\theta}_1 = \theta_1$.

$\Phi_0 = \frac{d\phi_0}{dz_0}$, $\phi_1 = \phi_0^*$ and ϕ_1 strictly convex $\Longrightarrow$

$$\begin{aligned} \tfrac{1}{2}(z_0 - z_1)^2 &= \tfrac{1}{2}z_0^2 - z_0 z_1 + \tfrac{1}{2}z_1^2 \\ &\geq (\tfrac{1}{2}z_0^2 - \phi_0(z_0)) + (\tfrac{1}{2}z_1^2 - \phi_1(z_1)) \end{aligned}$$

with equality only if $z_1 = \Phi_0(z_0)$.

For optimal transference plan $\tilde{\lambda}(dz_0 dz_1)$ in $d^2(\theta_0, \tilde{\theta}_1)$:

$$\int \tfrac{1}{2}(z_0 - z_1)^2\,\tilde{\lambda}(dz_0 dz_1) \;\geq\; \int (\tfrac{1}{2}z_0^2 - \phi_0)\,\theta_0\,dz_0 + \int (\tfrac{1}{2}z_1^2 - \phi_1)\,\tilde{\theta}_1\,dz_1$$

with equality only if $z_1 = \Phi_0(z_0)$ for $\tilde{\lambda}$-a. e. (z_0, z_1),
hence only if $\tilde{\theta}_1 = \Phi_0 \# \theta_0 = \theta_1$.

HEURISTIC DISCUSSION OF RESULT

Energy landscape is not semi-convex $\Longrightarrow$ initial value problem is non-unique

Time discretization provides selection criterion, picks solution that instantaneously decreases energy fastest

Connection with Dafermos' "entropy rate admissibility criterion" for conservation laws

A relation between energy and energy dissipation

Recall $E(\theta) := \int z(\theta - H)\,dz \le 0$,

formally $|\nabla E(\theta)|^2 \;=\; \int (-f(\theta))\,dz \;=\; \int \frac{\theta(1-\theta)}{r\theta+(1-\theta)}\,dz$.

Lemma 2 (Menon & O.). $\exists\ C_0 = C_0(m) \in (0,\infty)$:

$$|\nabla E(\theta)|^2 \;\le\; C_0(-E(\theta))^{1/2}. \tag{4}$$

Consider entropy solution of $\partial_t\theta^* + \partial_z f(\theta^*) = 0$ *with initial data*

$$H = \left\{ \begin{array}{lll} 1 & \textit{for} & z > 0 \\ 0 & \textit{for} & z < 0 \end{array} \right\}.$$

Note that $\theta^*(t,z) = \frac{df^*}{d\xi}(\frac{z}{t})$, *where* $f^*(\xi)$ *is Legendre transform of* $f(\theta)$ *(extended by* $+\infty$ *for* $\theta \notin [0,1]$*).*

There is equality in (4) for $\theta = \theta^*(t,\cdot)$ *and every* $t \ge 0$*.*

... and its consequence for the energy landscape

$\exists\ [0,\infty) \ni t \mapsto \theta^*(t)$ with $\frac{d\theta^*}{dt} = -\nabla E(\theta^*)$,

$$E(\theta^*(0)) = 0 \quad \text{and} \quad |\nabla E(\theta^*(t))|^2 = C_0(-E(\theta^*(t)))^{1/2}$$

$$\Longrightarrow \quad E(\theta^*(t)) = -\tfrac{C_0^2}{4}t^2 \ \text{ and } \ \int_0^t |\frac{d\theta^*}{dt'}|\,dt' = \tfrac{2^{1/2}}{3}C_0 t^{3/2}$$

$$\Longrightarrow \quad E(\theta^*(t)) = -\tfrac{3^{4/3}}{2^{8/3}}C_0^{2/3}\left(\int_0^t |\frac{d\theta^*}{dt'}|\,dt'\right)^{4/3}.$$

Hence, from the point of view of gradient flow:
∞-d energy landscape $\supset$ 1-d energy landscape given by

$$E(\Theta) = -\tfrac{3^{4/3}}{2^{8/3}}C_0^{2/3}\Theta^{4/3} \quad \Theta \in [0,\infty)$$

which is *not semi-convex* at $\Theta = 0$.

Selection by time discretization

For this non-convex 1-d energy landscape

$$E(\Theta) \;=\; -\tfrac{3^{4/3}}{2^{8/3}} C_0^{2/3} \Theta^{4/3} \quad \Theta \in [0, \infty),$$

the initial value problem

$$\frac{d\Theta}{dt}(t) \;=\; -\frac{dE}{d\Theta}(\Theta(t)), \quad \Theta(0) = 0$$

has infinitely many solutions.

The time discretization, $\Theta_0 = 0$ and

$$\text{minimize} \quad \tfrac{1}{2}(\Theta_{k-1} - \Theta_k)^2 + hE(\Theta_k) \quad \text{in } \Theta_k$$

picks the one that "instantaneously decreases energy fastest", i. e. for any time $t^* \geq 0$ and other solution $\tilde{\Theta}$ that agrees with Θ for $t \leq t^*$ we have $\frac{d}{dt}_{|t=t^*} \frac{d}{dt} E(\Theta(t)) \leq \frac{d}{dt}_{|t=t^*} \frac{d}{dt} E(\tilde{\Theta}(t))$.

Hessian not bounded from below

Recall metric tensor

$$|\delta\theta|^2 \;=\; \int \frac{1}{-f(\theta)} j^2 \, dz \quad \text{where} \quad \delta\theta + \frac{dj}{dz} = 0.$$

Hessian as growth rate of infinitesimal perturbations of gradient flow trajectory (O. & Westdickenberg):

$$\langle \delta\theta, \mathrm{Hess} E_{|\theta} \delta\theta \rangle \;=\; -\frac{1}{2} \int \frac{d}{dz}[\frac{df}{d\theta}(\theta)] \, (\frac{1}{-f(\theta)} j^2) \, dz$$

Not bounded below since $\frac{d}{dz}[\frac{df}{d\theta}(\theta)]$ can be arbitrarily positive.

... nonetheless, contraction along good trajectories

Oleinik's E condition for *entropy* solution of $\partial_t\theta + \partial_z f(\theta) = 0$
$\Longleftrightarrow$ semi-concavity for *viscosity* solutions of $\partial_t V + f(\partial_z V) = 0$:

$$\frac{\partial}{\partial z}\frac{df}{d\theta}(\theta(t,z)) \;=\; \frac{\partial}{\partial z}\frac{df}{d\theta}(\frac{\partial V}{\partial z}(t,z)) \;\le\; \frac{1}{2t}.$$

Hence along *entropy* solutions $\theta(t,z)$:

$$\mathrm{HessE}_{|\theta(t,\cdot)} \;\ge\; -\frac{1}{2t}\,\mathrm{id}.$$

Hence "contraction" property for entropy solutions $\theta_0(t,z)$, $\theta_1(t,z)$:

$$\mathrm{dist}^2(\theta_0(t,\cdot),\theta_1(t,\cdot)) \;\le\; \frac{t}{s}\mathrm{dist}^2(\theta_0(s,\cdot),\theta_1(s,\cdot)) \quad \text{for } t > s > 0.$$

Dafermos' "Entropy rate admissibility criterion"

scalar conservation law $\partial_t\theta + \partial_z f(\theta) = 0$, *convex* flux $f(\theta)$.

Lax criterion: A (piecewise smooth) distributional solution of $\partial_t\theta + \partial_z f(\theta) = 0$ is entropy solution iff all jumps are downwards (in positive z-direction).

Dafermos's criterion: Select strictly convex $\eta(\theta)$.
Consider "total entropy" $H(\theta) := \int \eta(\theta(z))\, dz$.
A (piecewise smooth) distributional solution of $\partial_t\theta + \partial_z f(\theta) = 0$ is entropy solution iff for any time $t^* \in [0, \infty)$ and any other (piecewise smooth) distributional solution $\tilde{\theta}$ that coincides with θ for $t \le t^*$ one has $\frac{d^+}{dt}_{|t=t^*} H(\theta) \le \frac{d^+}{dt}_{|t=t^*} H(\tilde{\theta})$.
"The entropy solution is *the* distributional solution that instantaneously decreases total entropy fastest."

Connection with selection by time discretization

"The limit $\theta(t,z)$ of time discretization is the gradient flow trajectory that instantaneously decreases energy fastest"

$\iff$ For any $t^* \in [0,\infty)$ and any other gradient flow trajectory $\tilde\theta(t)$ that coincides with θ for $t \le t^*$ we have $\frac{d^+}{dt}_{|t=t^*}\frac{d}{dt}E(\theta(t)) \le \frac{d^+}{dt}_{|t=t^*}\frac{d}{dt}E(\tilde\theta(t))$

$\overset{-|\nabla E(\theta)|^2=\int f(\theta)\,dz=H(\theta)}{\iff}$ For any $t^* \in [0,\infty)$ and any other distributional solution $\tilde\theta(t,z)$ of $\partial_t\tilde\theta + \partial_z f(\tilde\theta) = 0$ that coincides with θ for $t \le t^*$ we have $\frac{d^+}{dt}_{|t=t^*}H(\theta(t,\cdot)) \le \frac{d^+}{dt}_{|t=t^*}H(\tilde\theta(t,\cdot))$

$\iff$ θ is entropy solution of $\partial_t\theta + \partial_z f(\theta) = 0$

$\iff$ V is viscosity solution of $\partial_t V + f(\partial_z V) = 0$

For rarefaction wave: instantaneous = finite

Lemma 3.
Let $\eta(\theta)$, $\theta \in [0,1]$, *strictly convex with* $\eta(0) = \eta(1) = 0$.
Denote by $V^*(t,z)$ *the viscosity solution of* $\partial_t V + f(\partial_z V) = 0$ *with initial data* $\max\{0, z\}$;
given by $V^*(t,z) = t f^*(\frac{z}{t})$, *where* $f^*(\xi)$ *Legendre transform of* $f(\theta)$ *(extended by* $+\infty$ *for* $\theta \notin [0,1]$*).*

Let $\tilde{V}(t,z)$ *be Lipschitz function which satisfies* $\partial_t \tilde{V} + f(\partial_z \tilde{V}) = 0$ *almost everywhere and has initial data* $\max\{0, z\}$.

Then we have for all $t \geq 0$

$$\int \eta(\partial_z V^*(t,z))\, dz \;\leq\; \int \eta(\partial_z \tilde{V}(t,z))\, dz$$

with equality only if $\tilde{V} = V^*$ *on* $[0,t] \times \mathbb{R}$.

Open problem: Multi-dimensional case

$\theta(x) \in [0,1]$, $x \in (0,2\pi)^{d-1} \times \mathbb{R}$

Scheme as before:

$\frac{1}{2}d^2(\theta_{k-1},\theta_k) + \frac{r}{2}d^2(1-\theta_{k-1}, 1-\theta_k) + h\int z\,(\theta_k - H(z))\,dx.$

Convergence to "Buckley Leverett system"

$$
\begin{aligned}
\partial_t\theta + \nabla\cdot\Big(g(\theta)u + f(\theta)\tbinom{0}{1}\Big) &= 0 \quad \text{in entropy sense} \\
&\quad \text{with } g(\theta) := \tfrac{\theta}{\theta+m(1-\theta)}, \\
\nabla\cdot u &= 0, \\
u &= (\theta + m(1-\theta))\Big(-\nabla p - \theta\tbinom{0}{1}\Big).
\end{aligned}
$$

Existence for PDE unclear (lack of compactness in θ).
Consistency of scheme ok (strong convergence of $\theta_h \Longrightarrow$ limit satisfies PDE).

Open problem: More than two phases

Three phases: $\left\{ \begin{array}{lll} \theta(z) \geq 0, & \text{density } 1 & \text{mobility } 1 \\ \rho(z) \geq 0, & \text{density -1} & \text{mobility } 1 \\ 1-\theta-\rho, & \text{density } 0 & \text{mobility } 1 \end{array} \right\}$.

Scheme as before:

$$\frac{1}{2}d^2(\theta_{k-1},\theta_k)+\frac{1}{2}d^2(\rho_{k-1},\rho_k)+\frac{1}{2}d^2(1-\theta_{k-1}-\rho_{k-1},1-\theta_k-\rho_k)$$
$$+h\int z\,(\theta_k-H)\,dz+h\int z\,(\rho_k-(1-H))\,dz.$$

Convergence to 2×2 system of conservation laws

$$\partial_t\binom{\theta}{\rho}+\partial_z\binom{-\theta(1-\theta+\rho)}{\rho(1-\rho+\theta)} \;=\; 0 \quad \text{in entropy sense.}$$

Theory for PDE subtle (hyperbolic with umbilic point at $\binom{0.5}{0.5}$.
Consistency of scheme open.