Burgers' equation as a gradient flow on two-phase Wasserstein space

The flow of two phases through a porous medium is known to be unstable to "fingering" if the heavier phase is on top of the lighter one — in fact, the evolution is ill-posed as a free boundary problem. Nevertheless, it can be formally written as a gradient flow, time-discretized (leading to a "minimizing movement" scheme) and relaxed. The time-discrete gradient flow scheme involves the potential energy w. r. t. gravity E and the Wasserstein distance *in both phases*. For stratified initial data, the solution of the relaxed time-discrete gradient flow scheme converges to the unique *entropy* solution of Burgers' equation. In particular, the rarefaction waves of Burgers' equation captures the mixing of the two phases.

This relaxation process by time-discretization has been justified on the level of the original model: L. Szekelihydi has shown by "convex integration" that there are solutions of the original model that weakly converge to *any weak* solution of Burgers' equation.

The fact that the relaxed time-discrete gradient flow converges to the unique *entropy* solution of Burgers' equation can be related to an observation of C. Dafermos on entropy solutions: Indeed, E is *not* semi concave on the *two-phase* Wasserstein space. Even in a finite-dimensional situation, this would lead to multiple solutions of the initial value problem for the gradient flow of E. Intuitively, the *time-discrete* gradient flow scheme selects among the time-continuous solution the one which instantaneously decreases energy E fastest – or equivalently, which instantaneously has the largest energy dissipation rate $|\nabla E|^2$. This is just a reformulation of Dafermos' observation that the entropy solution is the weak solution of Burgers' equation that decreases entropy fastest among all weak solutions.

This is (in parts) joint work with Nicola Gigli.