REGULARITY RESULTS FOR MINIMAL SETS

(with attempts at the boundary)

Montreal, June 2011

Guy David, Université de Paris Sud 11 (Orsay)

Many pictures from Ken Brackke's home page
http://www.susqu.edu/brakke/

1. THE PLATEAU PROBLEM (some attempts)

Plateau was interested in soap films. The simplest statement of Plateau's problem: describe the soap films $E \subset \mathbb{R}^3$ bounded by a set Γ (for instance a smooth curve). Existence and regularity?

1. THE PLATEAU PROBLEM (some attempts)

Plateau was interested in soap films. The simplest statement of Plateau's problem: describe the soap films $E \subset \mathbb{R}^3$ bounded by a set Γ (for instance a smooth curve). Existence and regularity?

Apparently, a soap film is composed of two layers of some molecules with a water-attracting head and a water-repelling tail, which align themselves head to head; the width of the film is roughly equal to the length of two molecules.

1. THE PLATEAU PROBLEM (some attempts)

Plateau was interested in soap films. The simplest statement of Plateau's problem: describe the soap films $E \subset \mathbb{R}^3$ bounded by a set Γ (for instance a smooth curve). Existence and regularity?

Apparently, a soap film is composed of two layers of some molecules with a water-attracting head and a water-repelling tail, which align themselves head to head; the width of the film is roughly equal to the length of two molecules.

So we want to minimize the area of a surface E spanned by Γ. But different notions of "surface" and "spanned" exist. We shall mention some.

A soap bubble is slightly different: due to different pressures on both sides, it only "almost-minimizes" the area. In the smooth case, it has constant (instead of vanishing) mean curvature.

We may also study higher dimensions or different boundaries.

Hausdorff measure

Our sets E will not always be smooth. So we measure their size with the d-dimensional Hausdorff measure $\mathcal{H}^d(E)$ given by

$$\mathcal{H}^d(E) = \lim_{\delta \to 0} \mathcal{H}^d_\delta(E), \tag{1}$$

where

$$\mathcal{H}^d_\delta(E) = c_d \inf \Big\{ \sum_{j \in \mathbb{N}} \operatorname{diam}(D_j)^d \Big\}, \tag{2}$$

and the infimum is taken over all coverings of E by a countable collection $\{D_j\}$ of sets, with $\operatorname{diam}(D_j) \le \delta$ for all j.

We may choose the normalizing constant c_d so that $\mathcal{H}^d$ coincides with Lebesgue measure on subsets of $\mathbb{R}^d$.

1.a. Douglas' solution of Plateau's problem (1931)

Take a smoth curve Γ, and look for $E = f(D)$, where D is the unit disk. Try to minimize the area of the image $\int_D J_f(x)dx$.

Difficulty: too many possible parameterizations to hope for compactness.

1.a. Douglas' solution of Plateau's problem (1931)

Take a smoth curve Γ, and look for $E = f(D)$, where D is the unit disk. Try to minimize the area of the image $\int_D J_f(x)dx$.

Difficulty: too many possible parameterizations to hope for compactness.

Bright idea: try to take f to be harmonic in D, and choose f on ∂D so that $\int_D J_f(x)dx$ is minimal. Luckily, this amounts to solving a variational problem for $f_{|\partial D}$ which is much simpler: minimize

$$A(f) = \int\int \frac{\sum_{j=1}^{n} |f_j(\theta) - f_j(\varphi)|^2}{\sin^2\left(\frac{\theta-\varphi}{2}\right)} d\theta d\varphi. \tag{3}$$

But the method does not see whether $f(D)$ crosses itself, so the solutions are not physical.

1.b. Currents and the Plateau problem

We start with the most celebrated and successful model, provided by currents. Work by Federer, Fleming, De Giorgi, and others.

A d-dimensional **current** is a continuous linear form on the space of smooth d-forms. Almost the same as a form-valued distribution.

Main example: if S is smooth, oriented surface of dimension d, the current S' of integration on S is defined by $\langle S', \omega \rangle = \int_S \omega$. But we want a much larger class with compactness properties.

1.b. Currents and the Plateau problem

We start with the most celebrated and successful model, provided by currents. Work by Federer, Fleming, De Giorgi, and others.

A d-dimensional **current** is a continuous linear form on the space of smooth d-forms. Almost the same as a form-valued distribution.

Main example: if S is smooth, oriented surface of dimension d, the current S' of integration on S is defined by $\langle S', \omega \rangle = \int_S \omega$. But we want a much larger class with compactness properties.

Other useful example, the rectifiable current T defined on a d-dimensional rectifiable set E such that $\mathcal{H}^d(E) < +\infty$, with a measurable orientation τ, and an integrable integer-valued multiplicity m:

$$\langle T, \omega \rangle = \int_E m(x)\, \omega(x) \cdot \tau(x)\, d\mathcal{H}^d(x). \tag{4}$$

Recall (4) :
$$\langle T, \omega \rangle = \int_E m(x)\, \omega(x) \cdot \tau(x)\, d\mathcal{H}^d(x).$$

Again $\mathcal{H}^d = d$-dimensional Hausdorff measure $\simeq$ surface measure.

Rectifiable implies that E has a tangent (approximate) d-plane at $\mathcal{H}^d(E)$-a.e. point. We choose in a measurable way a unit d-vector $\tau(x)$ that spans this tangent plane, and use it to define $\omega(x) \cdot \tau(x)$.

This would give the same definition on a smooth oriented manifold.

Recall (4) : $$\langle T, \omega \rangle = \int_E m(x)\, \omega(x) \cdot \tau(x)\, d\mathcal{H}^d(x).$$

Again $\mathcal{H}^d = d$-dimensional Hausdorff measure $\simeq$ surface measure.

Rectifiable implies that E has a tangent (approximate) d-plane at $\mathcal{H}^d(E)$-a.e. point. We choose in a measurable way a unit d-vector $\tau(x)$ that spans this tangent plane, and use it to define $\omega(x) \cdot \tau(x)$.

This would give the same definition on a smooth oriented manifold.

The boundary ∂T of a d-dimensional current T is defined by

(5) $$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle \ \text{ for every } (d-1)\text{-form } \omega.$$

[d is the exterior derivative.] When S is a smooth oriented surface with boundary Γ, Green says that $\partial S' = \Gamma'$. Notice: $\partial\partial = 0$ because $dd = 0$.

Boundary condition for currents:

Take a $(d-1)$-dimensional current Γ, with $\partial\Gamma = 0$, and solve $\partial T = \Gamma$.

Two main problems: minimize MASS or SIZE.

The mass $\mathrm{Mass}(T)$ is the operator norm of T, where we put a L^∞-norm on forms. When T is a rectifiable current given by

Recall (4) :
$$\langle T, \omega \rangle = \int_E m(x)\, \omega(x) \cdot \tau(x)\, d\mathcal{H}^d(x).$$

(6)
$$\mathrm{Mass}(T) = \int_E |m(x)|\, d\mathcal{H}^d(x).$$

The size of T is the $\mathcal{H}^d$-measure of its closed support

(7)
$$\mathrm{Size}(T) = \mathcal{H}^d\big(\{x \in E\,;\, m(x) \neq 0\}\big).$$

Plateau problem for currents, conclusion

Great existence results, in all dimensions, when we minimize Mass(T) under the condition $\partial T = \Gamma$.

Very good regularity too (smoothness up to dimension 7, etc.).

Plateau problem for currents, conclusion

Great existence results, in all dimensions, when we minimize $\mathrm{Mass}(T)$ under the condition $\partial T = \Gamma$.

Very good regularity too (smoothness up to dimension 7, etc.).

But the solutions are not so physical, and for soap films we should rather minimize $\mathrm{Size}(T)$.

And then, no general existence result, even when $n = 3$, $d = 2$, and Γ is a smooth curve. The difficulty is that we have no bounds on the masses in a minimizing sequence, so the good compactness theorems do not apply.

Plateau problem for currents, conclusion

Great existence results, in all dimensions, when we minimize Mass(T) under the condition $\partial T = \Gamma$.

Very good regularity too (smoothness up to dimension 7, etc.).

But the solutions are not so physical, and for soap films we should rather minimize Size(T).

And then, no general existence result, even when $n = 3$, $d = 2$, and Γ is a smooth curve. The difficulty is that we have no bounds on the masses in a minimizing sequence, so the good compactness theorems do not apply.

Also (even if we minimize Size(T)) why should our set be orientable?

For some other examples, the problem $\partial T = \Gamma$ does not fit.

Interesting partial results by R. Hardt and T. De Pauw. General idea: minimize Size+εMass, control the minimizers uniformly, and let $\varepsilon \to 0$.

1.c. Directly with sets and homology

Return to $d = 2$, $n = 3$, and E is a "surface" (a set) spanned by a curve Γ. We want to minimize $\mathcal{H}^2(E)$, but what does "spanned" mean?

For **Reifenberg** (1960), E is a compact set that contains Γ, and the boundary condition is in terms of Čech homology on some commutative group G. We require the inclusion $i : \Gamma \to E$ to induce a trivial homomorphism from $H_1(\Gamma, G)$ to $H_1(E, G)$. Then we minimize $\mathcal{H}^2(E)$.

1.c. Directly with sets and homology

Return to $d = 2$, $n = 3$, and E is a "surface" (a set) spanned by a curve Γ. We want to minimize $\mathcal{H}^2(E)$, but what does "spanned" mean?

For **Reifenberg** (1960), E is a compact set that contains Γ, and the boundary condition is in terms of Čech homology on some commutative group G. We require the inclusion $i : \Gamma \to E$ to induce a trivial homomorphism from $H_1(\Gamma, G)$ to $H_1(E, G)$. Then we minimize $\mathcal{H}^2(E)$.

Reifenberg proved the existence of minimizers when $G = \mathbb{Z}_2$ or $\mathbb{R}/\mathbb{Z}$. Beautiful proof with minimizing sequences and haircuts.

De Pauw obtained the 2-dimensional case when $G = \mathbb{Z}$ (with currents).

The equivalence with the size minimizing problem is not clear, but the infimum is the same [De Pauw].

Quite nice. The list of minimizers may depend on G, so we may need to choose it correctly.

1.d. Sliding Almgren minimizers

We propose a third definition, where we minimize $\mathcal{H}^2(E)$ among all compact sets E obtained by deformation of an initial candidate E_0 with a sliding boundary condition. [Think about a rubber shower curtain.]

We give ourselves a finite collection of smooth boundary pieces Γ_j, $j \in J$. For instance, just one curve or surface Γ.

1.d. Sliding Almgren minimizers

We propose a third definition, where we minimize $\mathcal{H}^2(E)$ among all compact sets E obtained by deformation of an initial candidate E_0 with a sliding boundary condition. [Think about a rubber shower curtain.]

We give ourselves a finite collection of smooth boundary pieces Γ_j, $j \in J$. For instance, just one curve or surface Γ.

A sliding deformation of the closed set E_0 is a set $F = \varphi_1(E_0)$, where $\varphi_t : E_0 \to \mathbb{R}^3$, $0 \leq t \leq 1$, is a one-parameter family of functions such that:

$$(x,t) \to \varphi_t(x) \text{ is continuous: } E_0 \times [0,1] \to \mathbb{R}^3, \tag{8}$$

$$\varphi_0(x) = x \text{ for } x \in E_0, \tag{9}$$

$$\varphi_t(x) \in \Gamma_j \text{ when } x \in \Gamma_j, \tag{10}$$

$$\varphi_1 \text{ is Lipschitz.} \tag{11}$$

Then we give ourselves an initial closed set E_0, with $\mathcal{H}^d(E_0) < +\infty$, and try to minimize $\mathcal{H}^d(F)$, among all the sliding deformations of E_0.

Remarks.

- Deformations do not need to be injective: pinching is allowed.
- We need choose E_0 so that $\inf(\mathcal{H}^d(F)) > 0$ (involves some topology).
- We required (101) for safety, but no bound on the Lipschitz constant is attached. We may also drop (11).

Then we give ourselves an initial closed set E_0, with $\mathcal{H}^d(E_0) < +\infty$, and try to minimize $\mathcal{H}^d(F)$, among all the sliding deformations of E_0.

Remarks.

- Deformations do not need to be injective: pinching is allowed.
- We need choose E_0 so that $\inf(\mathcal{H}^d(F)) > 0$ (involves some topology).
- We required (101) for safety, but no bound on the Lipschitz constant is attached. We may also drop (11).

Advantages.

- No need to orient E or choose a group.
- More flexible a priori (different E_0 can give different solutions).

Bad news: the usual difficulties with parameterizations.
No general existence result yet. Even when $d = 2$, $n = 3$, and the boundary is a smooth curve Γ. Less bad : should we forbid unrealistic deformations that first extend the film too far? Some real films can be deformed into a point through a long homotopy.

1.e. Other possible definitions of Plateau's problem.

For instance, let $\Omega \subset \mathbb{R}^n$ be closed, choose $V_1, V_2 \subset \mathbb{R}^n \setminus \Omega$ open and disjoint, and minimize $\mathcal{H}^{n-1}(E)$ under the constraint that

$$E \subset \Omega \text{ (still) } \underline{\text{separates}} \; V_1 \text{ from } V_2 \text{ in } \mathbb{R}^n.$$

Easier to get existence results, because of compactness in BV; some regularity away from $V_1 \cup V_2$.

1.e. Other possible definitions of Plateau's problem.

For instance, let $\Omega \subset \mathbb{R}^n$ be closed, choose $V_1, V_2 \subset \mathbb{R}^n \setminus \Omega$ open and disjoint, and minimize $\mathcal{H}^{n-1}(E)$ under the constraint that

$$E \subset \Omega \text{ (still) } \underline{\text{separates}}\ V_1 \text{ from } V_2 \text{ in } \mathbb{R}^n.$$

Easier to get existence results, because of compactness in BV; some regularity away from $V_1 \cup V_2$.

In higher codimensions, replace separation with homology conditions: let the $\gamma_j \subset \mathbb{R}^n \setminus \Omega$ be spheres or surfaces of dimension $n-d-1$, require that the γ_j define nonzero elements in the homology of $\mathbb{R}^n \setminus E$, and minimize $\mathcal{H}^d(E)$. Existence results of Xiangyu Liang.

1.e. Other possible definitions of Plateau's problem.

For instance, let $\Omega \subset \mathbb{R}^n$ be closed, choose $V_1, V_2 \subset \mathbb{R}^n \setminus \Omega$ open and disjoint, and minimize $\mathcal{H}^{n-1}(E)$ under the constraint that

$$E \subset \Omega \text{ (still) } \underline{\text{separates}}\ V_1 \text{ from } V_2 \text{ in } \mathbb{R}^n.$$

Easier to get existence results, because of compactness in BV; some regularity away from $V_1 \cup V_2$.

In higher codimensions, replace separation with homology conditions: let the $\gamma_j \subset \mathbb{R}^n \setminus \Omega$ be spheres or surfaces of dimension $n-d-1$, require that the γ_j define nonzero elements in the homology of $\mathbb{R}^n \setminus E$, and minimize $\mathcal{H}^d(E)$. Existence results of Xiangyu Liang.

Weak form: ask that E is a limit of deformations of E_0, and only that $\mathcal{H}^d(E) \leq \mathcal{H}^d(F)$ for all deformations of E in a compact subset of the boundary. Or work on a manifold without boundary [V. Feuvrier].

Almgren sliding almost mimimal sets.

Anyway, we claim that the Plateau problem has not been solved entierely, even for curves in $\mathbb{R}^3$.

Probably, some insight on the regularity properties of potential minimizers near the boundary will be useful. So (for the sliding Plateau problem above) we want to study Almgren sliding almost mimimal sets:

Almgren sliding almost mimimal sets.

Anyway, we claim that the Plateau problem has not been solved entierely, even for curves in $\mathbb{R}^3$.

Probably, some insight on the regularity properties of potential minimizers near the boundary will be useful. So (for the sliding Plateau problem above) we want to study Almgren sliding almost mimimal sets:

We choose boundaries Γ_j as above, and a small gauge function $h(r)$ such that $\lim_{r\to 0} h(r) = 0$; typically, $h(r) = Cr^\alpha$.

Almgren sliding almost mimimal sets.

Anyway, we claim that the Plateau problem has not been solved entierely, even for curves in $\mathbb{R}^3$.

Probably, some insight on the regularity properties of potential minimizers near the boundary will be useful. So (for the sliding Plateau problem above) we want to study Almgren sliding almost mimimal sets:

We choose boundaries Γ_j as above, and a small gauge function $h(r)$ such that $\lim_{r\to 0} h(r) = 0$; typically, $h(r) = Cr^\alpha$.

A sliding deformation of E in the ball $B \subset \mathbb{R}^n$ is a set $F = \varphi_1(E)$, where $\{\varphi_t\}_{0\le t\le 1}$ satisfies (8)-(11), and $\varphi_t(x) = x$ for $x \in E \setminus B$ and $\varphi_t(E \cap B) \subset B$ for $0 \le t \le 1$.

Almgren sliding almost mimimal sets.

Anyway, we claim that the Plateau problem has not been solved entierely, even for curves in $\mathbb{R}^3$.

Probably, some insight on the regularity properties of potential minimizers near the boundary will be useful. So (for the sliding Plateau problem above) we want to study Almgren sliding almost mimimal sets:

We choose boundaries Γ_j as above, and a small gauge function $h(r)$ such that $\lim_{r\to 0} h(r) = 0$; typically, $h(r) = Cr^{\alpha}$.

A <u>sliding deformation</u> of E in the ball $B \subset \mathbb{R}^n$ is a set $F = \varphi_1(E)$, where $\{\varphi_t\}_{0\leq t\leq 1}$ satisfies (8)-(11), and $\varphi_t(x) = x$ for $x \in E \setminus B$ and $\varphi_t(E \cap B) \subset B$ for $0 \leq t \leq 1$.

An <u>Almgren sliding almost mimimal set</u> is a closed set E such that

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(F \cap B) + h(0)r^d \tag{12}$$

whenever F is a sliding deformation of E in a ball B of radius r.

2. INTERIOR REGULARITY : J. TAYLOR'S THEOREM

We now focus on regularity properties of potential solutions away from the boundary. J. Taylor's theorem will be the best example.

First define local minimal and almost-minimal sets in a domain $U \subset \mathbb{R}^n$. We essentially use Almgren's definition (good for soap films and bubbles).

We consider sets $E \subset U$, of dimension d, and even such that $\mathcal{H}^d(E \cap B) < +\infty$ for every compact ball $B \subset U$.

2. INTERIOR REGULARITY : J. TAYLOR'S THEOREM

We now focus on regularity properties of potential solutions away from the boundary. J. Taylor's theorem will be the best example.

First define local minimal and almost-minimal sets in a domain $U \subset \mathbb{R}^n$. We essentially use Almgren's definition (good for soap films and bubbles).

We consider sets $E \subset U$, of dimension d, and even such that $\mathcal{H}^d(E \cap B) < +\infty$ for every compact ball $B \subset U$.

Let B be a compact ball in U. A competitor for E in B is a set $F = \varphi(E)$, where $\varphi : U \to U$ is Lipschitz (but no bounds required), and

$$\varphi(y) = y \text{ for } y \in U \setminus B, \text{ and } \varphi(B) \subset B. \tag{1}$$

Same thing as before, but $B \subset U$, we have no boundary piece, and there is an obvious deformation given by $\varphi_t(x) = (1-t)x + t\varphi(x)$.

An almost minimal set of dimension d in U, with gauge function h, is a closed set E, with $\mathcal{H}^d(E \cap B) < +\infty$ for closed balls $B \subset U$, such that

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(F \cap B) + r^d h(r) \tag{2}$$

whenever B is a closed ball, $B \subset U$, r is its radius, and F is a competitor for E in B.

An almost minimal set of dimension d in U, with gauge function h, is a closed set E, with $\mathcal{H}^d(E \cap B) < +\infty$ for closed balls $B \subset U$, such that

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(F \cap B) + r^d h(r) \tag{2}$$

whenever B is a closed ball, $B \subset U$, r is its radius, and F is a competitor for E in B.

Comments.

1. Minor variant of Almgren's definition. Other variants exist.
2. If E is a sliding almost minimal set with the Γ_j, $j \in J$, then it is almost minimal in $U = \mathbb{R}^n \setminus \bigcup_{j \in J} \Gamma_j$.
3. Note that φ_1 is still **not** required to be injective. Pinching is allowed.
4. The support of a size-minimizing current T is almost minimal away from the support of ∂T, I think.

2.a. Simple examples (of Minimal cones)

That is, cones that are also minimal sets, i.e., such that

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(\varphi(E) \cap B)$$

for $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz s.t. $\varphi(x) = x$ for $x \in \mathbb{R}^n \setminus B$ and $\varphi(B) \subset B$.

• $d = 1$: A line, a Y (three half lines in a plane, with 120° angles).

• $d = 1$: But not two lines (even perpendicular), or 4 half-lines in $\mathbb{R}^3$.

• $d = 2$: A plane, a $\mathbb{Y}$ (product of Y by a perpendicular line = three half planes that make 120° angles).

• $d = 2$: Less trivial to check: in $\mathbb{R}^3$, the (closed positive) cone $\mathbb{T}$ of dimension 2 over the union of the edges of a regular tetrahedron centered at the origin. Six faces that meet by sets of three along four half lines (the spine). [Morgan-Lawlor with calibrations.]

This is the full list in $\mathbb{R}^3$, known by Lamarle, Heppes, Taylor.

Reduction (= cleaning)

For $E \subset \mathbb{R}^n$ closed, with locally finite $\mathcal{H}^d$ measure, denote by E^* the closed support of E. That is,

$$E^* = \big\{x \in E\,;\, \mathcal{H}^d(E \cap B(x,r)) > 0 \text{ for all } r > 0\big\}. \tag{3}$$

Say that E is <u>reduced</u> (or coral) when $E = E^*$.

Reduction (= cleaning)

For $E \subset \mathbb{R}^n$ closed, with locally finite $\mathcal{H}^d$ measure, denote by E^* the closed support of E. That is,

$$E^* = \{x \in E\,;\, \mathcal{H}^d(E \cap B(x,r)) > 0 \text{ for all } r > 0\}. \tag{3}$$

Say that E is reduced (or coral) when $E = E^*$.

If E is almost minimal, then E^* is almost minimal, with the same gauge h, because $\mathcal{H}^d(E \setminus E^*) = 0$. So it is safe to focus on reduced sets.

This simplifies things; otherwise we would get ugly statements because if E is almost minimal, then $E \cup Z$ is also almost minimal for any closed Z such that $\mathcal{H}^d(Z) = 0$.

We should keep in mind that $E^* \setminus E$ can play a role in some topological problems. But from now on, all our sets will be reduced.

2.2. J. Taylor's theorem

Theorem (JT, 1976). Let E be a reduced local almost minimal set of dimension 2 in some open set $U \subset \mathbb{R}^3$, with gauge function $h(r) = Cr^\alpha$ ($\alpha > 0$). Then for each $x \in E$, there is a ball $B(x, r)$ inside which E is the image of a minimal cone by a C^1-diffeomorphism of $\mathbb{R}^3$.

2.2. J. Taylor's theorem

Theorem (JT, 1976). Let E be a reduced local almost minimal set of dimension 2 in some open set $U \subset \mathbb{R}^3$, with gauge function $h(r) = Cr^\alpha$ $(\alpha > 0)$. Then for each $x \in E$, there is a ball $B(x, r)$ inside which E is the image of a minimal cone by a C^1-diffeomorphism of $\mathbb{R}^3$.

Comments

• The cone is a plane, a $\mathbb{Y}$, or a $\mathbb{T}$, as in the list above. This also means that near $B(x, r)$, E is composed of C^1 faces, which meet along C^1 arcs with 120° angles and the same combinatorics as in a minimal cone.

• The singularies above occur in real soap films.

• Alas, the result does not give a concrete way to estimate r.

• Some (Dini) condition on h is needed, but $h(r) = Cr^\alpha$ is not optimal.

Theorem (JT, 1976). Let E be a reduced local almost minimal set of dimension 2 in some open set $U \subset \mathbb{R}^3$, with gauge function $h(r) = Cr^\alpha$ ($\alpha > 0$). Then for each $x \in E$, there is a ball $B(x, r)$ inside which E is the image of a minimal cone by a C^1-diffeomorphism of $\mathbb{R}^3$.

• You can get more regularity than this, especially if h is very small or vanishing and in the regions where E is C^1-diffeomorphic to a plane.

• There is a generalization to 2-dimensional almost minimal sets in $\mathbb{R}^n$, but we only get the Hölder equivalence with a minimal cone, and the list of 2-dimensional minimal cones in $\mathbb{R}^n$ is not known. Just a first description of the structure.

• For higher dimensions and codimensions, much less is known (local Ahlfors-regularity, uniform rectifiability, theorems about limits, local structure of 3-cones in $\mathbb{R}^4$).

2.3. Bits of proofs

Exercise 1. If E is minimal of dimension 1 in $\Omega \supset B$, $[a,b]$ is a diameter of B, and and if $E \cap \partial B = \{a,b\}$ then $E \cap B = [a,b]$.
If E is only almost-minimal, $E \cap B(x,r)$ stays within $Ch(r)^{1/2} r$ of $[a,b]$.

2.3. Bits of proofs

Exercise 1. If E is minimal of dimension 1 in $\Omega \supset B$, $[a,b]$ is a diameter of B, and and if $E \cap \partial B = \{a,b\}$ then $E \cap B = [a,b]$.
If E is only almost-minimal, $E \cap B(x,r)$ stays within $Ch(r)^{1/2}r$ of $[a,b]$.

Exercise 2. Y is a minimal set of dimension 1.
If E is a minimal set of dimension 1 in $\Omega \supset B$ and $E \cap \partial B = \{a,b,c\}$, what is $E \cap B$? What if E is only almost-minimal?

2.3. Bits of proofs

Exercise 1. If E is minimal of dimension 1 in $\Omega \supset B$, $[a, b]$ is a diameter of B, and and if $E \cap \partial B = \{a, b\}$ then $E \cap B = [a, b]$.
If E is only almost-minimal, $E \cap B(x, r)$ stays within $Ch(r)^{1/2} r$ of $[a, b]$.

Exercise 2. Y is a minimal set of dimension 1.
If E is a minimal set of dimension 1 in $\Omega \supset B$ and $E \cap \partial B = \{a, b, c\}$, what is $E \cap B$? What if E is only almost-minimal?

With arguments like this, one gets a first description of minimal cones E of dimension 2 in $\mathbb{R}^n$, as a finite union of arcs of great circles, that meet only by sets of 3 and with $120°$ angles.

In $\mathbb{R}^3$, we get a collection of nets on the sphere, eliminate some, and we are reduced to P, $\mathbb{Y}$, and $\mathbb{T}$.

In $\mathbb{R}^4$ or more, we don't know the list. Xiangyu Liang showed that $P_1 \cup P_2$ is minimal when P_1 and $P_2 \subset \mathbb{R}^4$ are almost orthogonal. We suspect that $Y \times Y$ is minimal. Anything other one?

Monotonicity of density

Theorem. Let E be a minimal set of dimension d in $U \subset \mathbb{R}^n$. Set

$$\theta(x,r) = r^{-d}\mathcal{H}^d(E \cap B(x,r)) \tag{1}$$

for $x \in E$ and $r > 0$ such that $B(x,r) \subset U$. Then for each x,

$$\theta(x,\cdot) \ \text{ is nondecreasing.} \tag{2}$$

Monotonicity of density

Theorem. Let E be a minimal set of dimension d in $U \subset \mathbb{R}^n$. Set

$$\theta(x,r) = r^{-d}\mathcal{H}^d(E \cap B(x,r)) \tag{1}$$

for $x \in E$ and $r > 0$ such that $B(x,r) \subset U$. Then for each x,

$$\theta(x,\cdot) \ \text{ is nondecreasing.} \tag{2}$$

Idea of proof. Observe that $r \to \mathcal{H}^d(E \cap B(x,r))$ is nondecreasing, so it is the integral of its derivative (seen as a Stiljes measure), which is no less than its almost-everywhere derivative. Thus it is enough to check that for almost every r,

$$r^{-d}\,\frac{\partial}{\partial r}\big(\mathcal{H}^d(E \cap B(0,r))\big) \geq d\,r^{-d-1}\mathcal{H}^d(E \cap B(0,r)) \tag{3}$$

... enough to check that for almost every r,

$$r^{-d}\,\frac{\partial}{\partial r}\big(\mathcal{H}^d(E\cap B(0,r))\big) \geq d\,r^{-d-1}\mathcal{H}^d(E\cap B(0,r)). \tag{3}$$

But

$$\frac{\partial}{\partial r}\big(\mathcal{H}^d(E\cap B(0,r))\big) \geq \mathcal{H}^{d-1}(E\cap \partial B(0,r)) \tag{4}$$

[Think about C^1 surfaces], so it is enough to show that

$$\mathcal{H}^d(E\cap B(0,r)) \leq \frac{r}{d}\,\mathcal{H}^{d-1}(E\cap \partial B(0,r)). \tag{5}$$

... enough to show that

$$\mathcal{H}^d(E \cap B(0,r)) \leq \frac{r}{d}\, \mathcal{H}^{d-1}(E \cap \partial B(0,r)). \tag{5}$$

Notice that

$$\frac{r}{d}\, \mathcal{H}^{d-1}(E \cap \partial B(0,r)) = \mathcal{H}^d(\Gamma \cap B(0,r)), \tag{6}$$

where Γ denotes the cone over $E \cap \partial B(0,r))$.

Now $[\Gamma \cap B(0,r)] \cup E \setminus B(0,r)]$ is not directly a competitor for E, but we can approximate it by Lipschitz deformations of E in $B(0,r)$ [Expand a lot near $\partial B(0,r)$ and contract most of $B(0,r)$ to the origin.] The comparison yields (5) and the monotonicity of θ.

... enough to show that

$$\mathcal{H}^d(E \cap B(0,r)) \leq \frac{r}{d}\, \mathcal{H}^{d-1}(E \cap \partial B(0,r)). \tag{5}$$

Notice that

$$\frac{r}{d}\, \mathcal{H}^{d-1}(E \cap \partial B(0,r)) = \mathcal{H}^d(\Gamma \cap B(0,r)), \tag{6}$$

where Γ denotes the cone over $E \cap \partial B(0,r))$.

Now $[\Gamma \cap B(0,r)] \cup E \setminus B(0,r)]$ is not directly a competitor for E, but we can approximate it by Lipschitz deformations of E in $B(0,r)$ [Expand a lot near $\partial B(0,r)$ and contract most of $B(0,r)$ to the origin.] The comparison yields (5) and the monotonicity of θ.

Last comments: 1. Almost-monotonicity for almost-minimal sets.

2. When $r \to \theta(x,r)$ is constant on (r_1, r_2), E coincides with a minimal cone on $B(x,r_2) \setminus \overline{B}(x,r_1)$. Complicated, but not too surprising proof.

Limiting theorems

Theorem [D]. Let $\{E_k\}$ be a sequence of reduced almost minimal sets in U, with the same gauge function h. Suppose that $\{E_k\}$ converges to the closed set E (locally in Hausdorff distance). Then

$$\mathcal{H}^d(E \cap V) \leq \liminf_{k\to+\infty} \mathcal{H}^d(E_k \cap V) \text{ for every open set } V \subset U, \tag{7}$$

and

$$E \text{ is a reduced almost minimal set, with gauge function } h. \tag{8}$$

Limiting theorems

Theorem [D]. Let $\{E_k\}$ be a sequence of reduced almost minimal sets in U, with the same gauge function h. Suppose that $\{E_k\}$ converges to the closed set E (locally in Hausdorff distance). Then

$$\mathcal{H}^d(E \cap V) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap V) \text{ for every open set } V \subset U, \tag{7}$$

and

$$E \text{ is a reduced almost minimal set, with gauge function } h. \tag{8}$$

• See the definition of convergence below.

• The main part is (7); for the rest, we use a competitor for the limit to construct competitors for the E_k.

• (7) follows from other regularity properties (local Ahlfors-regularity, uniform rectifiability) and a result of Dal Maso, Morel, and Solimini. Proofs of Almgren, D., and Semmes.

• *Limsup* inequality for minimal sets and V closed (similar proof).

Definition of local convergence of $\{E_k\}$ to E in the open set U:

$$\lim_{k\to+\infty} d_{x,r}(E,E_k) = 0 \quad \text{for every ball } B(x,r) \subset\subset U, \tag{9}$$

where we set

$$\begin{aligned} d_{x,r}(E,E_k) &= r^{-1}\sup\big\{\mathrm{dist}(y,E)\,;\, y\in E_k\cap B(x,r)\big\} \\ &\qquad + r^{-1}\sup\big\{\mathrm{dist}(y,E_k)\,;\, y\in E\cap B(x,r)\big\}. \end{aligned} \tag{10}$$

[The right way to write local Hausdorff distances between sets; also the only notion of local convergence if we want to be able to extract convergent subsequences.]

Blow-up limits are cones

A blow-up limit of E at x is any limit

$$E_\infty = \lim_{k \to +\infty} \frac{E - x}{\varepsilon_k}, \quad \text{where} \quad \lim_{k \to +\infty} \varepsilon_k = 0. \tag{11}$$

Blow-up limits are cones

A blow-up limit of E at x is any limit

$$E_\infty = \lim_{k \to +\infty} \frac{E - x}{\varepsilon_k}, \quad \text{where} \lim_{k \to +\infty} \varepsilon_k = 0. \tag{11}$$

By limiting theorems, the monotonicity (or almost-monotonicity) of density, and Ahlfors-regularity,

$$\begin{aligned}
\theta_{E_\infty}(0, r) &= r^{-d}\mathcal{H}^d(E_\infty \cap B(0, r)) = r^{-d} \lim_{k \to +\infty} \mathcal{H}^d\Big(\frac{E - x}{\varepsilon_k} \cap B(0, r)\Big) \\
&= \lim_{k \to +\infty} r^{-d}\varepsilon_k^{-d}\mathcal{H}^d(E \cap B(x, \varepsilon_k r)) = \lim_{k \to +\infty} \theta_E(x, \varepsilon_k r) \\
&:= \theta_E(x) > 0.
\end{aligned}$$

Blow-up limits are cones

A blow-up limit of E at x is any limit

$$E_\infty = \lim_{k\to+\infty} \frac{E-x}{\varepsilon_k}, \quad \text{where} \quad \lim_{k\to+\infty} \varepsilon_k = 0. \tag{11}$$

By limiting theorems, the monotonicity (or almost-monotonicity) of density, and Ahlfors-regularity,

$$\begin{aligned}\theta_{E_\infty}(0,r) &= r^{-d}\mathcal{H}^d(E_\infty \cap B(0,r)) = r^{-d} \lim_{k\to+\infty} \mathcal{H}^d\Big(\frac{E-x}{\varepsilon_k} \cap B(0,r)\Big)\\ &= \lim_{k\to+\infty} r^{-d}\varepsilon_k^{-d}\mathcal{H}^d(E\cap B(x,\varepsilon_k r)) = \lim_{k\to+\infty} \theta_E(x,\varepsilon_k r)\\ &:= \theta_E(x) > 0.\end{aligned}$$

Then E_∞ is a minimal set with constant density; it is a minimal cone.

Almost constant density

Another consequence of monotonicity and the theorem on limits is:

Let $\delta > 0$ be small. Suppose E is almost minimal in $B(x,r)$, and that h, $h(2r)$, and $\theta(x,2r) - \theta(x)$ are small enough.

Then inside $B(x,r)$, E is as close as we want to a minimal cone Z centered at x with density $\theta(x)$:

$$d_{x,r}(E,Z) \leq \delta \tag{12}$$

and, for $B(y,t) \subset B(x,r)$,

$$\left|\mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(Z \cap B(y,t))\right| \leq C\delta r^d. \tag{13}$$

The proof uses a standard compactness argument.
Recall that E is a cone when E is minimal and $\theta(x,\cdot)$ is constant.

Reifenberg parameterizations

This extension of Reifenberg's topological disk theorem is useful to get a parameterization. [Joint work with T. de Pauw and T. Toro.]

For every small $\tau > 0$, we can find $\varepsilon_0 > 0$ such that: Let $E \subset \mathbb{R}^3$ be a closed set, with $0 \in E$. Suppose that for each $x \in E \cap B(0, 2)$ and $r \in (0, 2]$, we can find a minimal cone $Z = Z(x, r)$ such that

$$d_{x,r}(E, Z) \leq \varepsilon_0. \tag{14}$$

Then

Reifenberg parameterizations

This extension of Reifenberg's topological disk theorem is useful to get a parameterization. [Joint work with T. de Pauw and T. Toro.]

For every small $\tau > 0$, we can find $\varepsilon_0 > 0$ such that: Let $E \subset \mathbb{R}^3$ be a closed set, with $0 \in E$. Suppose that for each $x \in E \cap B(0,2)$ and $r \in (0,2]$, we can find a minimal cone $Z = Z(x,r)$ such that

$$d_{x,r}(E,Z) \leq \varepsilon_0. \tag{14}$$

Then there is a minimal cone Z_0 and a bi-Hölder homeomorphism $f : B(0,1) \to f(B(0,1))$ such that

$$E \cap B(0,1-\tau) \subset f(Z_0 \cap B(0,1)) \subset E \cap B(0,1+\tau).$$

$$|f(x) - x| \leq \tau \quad \text{for } x \in B(0,1),$$

$$(1-\tau)|x-y|^{1+\tau} \leq |f(x)-f(y)| \leq (1+\tau)|x-y|^{1-\tau} \quad \text{for } x,y \in B(0,1),$$

Comments.

When all Z are planes, we get the well known theorem of Reifenberg.

Main case when $Z(0,2)$ is centered at 0. Then we can take $Z_0 = Z(0,2)$.

Same sort of proof here: identify the "spine" of E, show that it is Reifenberg-flat, construct f first on the spine of Z_0, and then extend to Z_0 and to $\mathbb{R}^3$. Do all this scale by scale (from large ones to small ones); each time push $f_k(Z_0)$ in the direction of E, using the minimal cones Z and a partition of $\mathbf{1}$.

Some coherence between the various $Z(x,r)$ is forced by (14).

When $d_{x,r}(E,Z)$ tends to 0 with some definite faster speed, we get a better (for instance C^1) parameterization.

Not surprising but useful: global metric and topological information is derived from approximate information at all scales and locations. [Compare with John-Nirenberg and Cheeger-Colding]

Return to the proof of Jean Taylor's theorem

We want to say how to use the tools above to prove (part of) J. Taylor's theorem.

Recall, we are given an almost minimal set $E \subset U$ and a point $x_0 \in E$, and we want to find $r_0 > 0$ such that E is bi-Hölder equivalent to a minimal cone (centered at x) in $B(x_0, r_0)$.

Return to the proof of Jean Taylor's theorem

We want to say how to use the tools above to prove (part of) J. Taylor's theorem.

Recall, we are given an almost minimal set $E \subset U$ and a point $x_0 \in E$, and we want to find $r_0 > 0$ such that E is bi-Hölder equivalent to a minimal cone (centered at x) in $B(x_0, r_0)$.

We may assume that $x_0 = 0$. We want to apply the extension of Reifenberg's theorem, so it is enough to find minimal cones $Z(x, r)$, when $x \in B(0, 2r_0)$ and $0 < r \leq 2r_0$, such that $d_{x,r}(E, Z) \leq \varepsilon_0$.

Return to the proof of Jean Taylor's theorem

We want to say how to use the tools above to prove (part of) J. Taylor's theorem.

Recall, we are given an almost minimal set $E \subset U$ and a point $x_0 \in E$, and we want to find $r_0 > 0$ such that E is bi-Hölder equivalent to a minimal cone (centered at x) in $B(x_0, r_0)$.

We may assume that $x_0 = 0$. We want to apply the extension of Reifenberg's theorem, so it is enough to find minimal cones $Z(x, r)$, when $x \in B(0, 2r_0)$ and $0 < r \leq 2r_0$, such that $d_{x,r}(E, Z) \leq \varepsilon_0$.

Notice that we get approximating cones because all blow-up limits of E are cones, but this is not enough: we need some uniformity.

We whall assume that E is minimal (to simplify). Otherwise, add error terms everywhere.

a. Regularity of E near a P-point

We set $\theta(y,t) = t^{-2}\mathcal{H}^2(E \cap B(y,t))$ for $y \in E$ and $t > 0$.

We start with the simplest special case when 0 is a P-point, i.e., when $d(0) = \lim_{t\to 0} \theta(0,t) = \pi$.

Pick $\varepsilon > 0$ very small, and choose r_0 so that $\theta(0, 16r_0) \leq d(0) + \varepsilon$.

a. Regularity of E near a P-point

We set $\theta(y,t) = t^{-2}\mathcal{H}^2(E \cap B(y,t))$ for $y \in E$ and $t > 0$.

We start with the simplest special case when 0 is a P-point, i.e., when $d(0) = \lim_{t\to 0} \theta(0,t) = \pi$.

Pick $\varepsilon > 0$ very small, and choose r_0 so that $\theta(0, 16r_0) \le d(0) + \varepsilon$.

By almost-constant density, E is very close to a plane in $B(0, 8r_0)$ (as in (12) and (13)). In particular,

$$\theta(x, 4r_0) \le d(0) + \tau \quad \text{for } x \in B(0, 2r_0), \tag{15}$$

where $\tau > 0$ is as small as we want.

a. Regularity of E near a P-point

We set $\theta(y,t) = t^{-2}\mathcal{H}^2(E \cap B(y,t))$ for $y \in E$ and $t > 0$.

We start with the simplest special case when 0 is a P-point, i.e., when $d(0) = \lim_{t\to 0} \theta(0,t) = \pi$.

Pick $\varepsilon > 0$ very small, and choose r_0 so that $\theta(0, 16r_0) \leq d(0) + \varepsilon$.

By almost-constant density, E is very close to a plane in $B(0, 8r_0)$ (as in (12) and (13)). In particular,

$$\theta(x, 4r_0) \leq d(0) + \tau \quad \text{for } x \in B(0, 2r_0), \tag{15}$$

where $\tau > 0$ is as small as we want. By monotonicity,

$$\theta(x, r) \leq d(0) + \tau \quad \text{for } x \in B(0, 2r_0) \text{ and } 0 < r \leq 4r_0. \tag{16}$$

a. Regularity of E near a P-point

We set $\theta(y,t) = t^{-2}\mathcal{H}^2(E \cap B(y,t))$ for $y \in E$ and $t > 0$.

We start with the simplest special case when 0 is a P-point, i.e., when $d(0) = \lim_{t\to 0} \theta(0,t) = \pi$.

Pick $\varepsilon > 0$ very small, and choose r_0 so that $\theta(0, 16r_0) \leq d(0) + \varepsilon$.

By almost-constant density, E is very close to a plane in $B(0, 8r_0)$ (as in (12) and (13)). In particular,

$$\theta(x, 4r_0) \leq d(0) + \tau \quad \text{for } x \in B(0, 2r_0), \tag{15}$$

where $\tau > 0$ is as small as we want. By monotonicity,

$$\theta(x, r) \leq d(0) + \tau \quad \text{for } x \in B(0, 2r_0) \text{ and } 0 < r \leq 4r_0. \tag{16}$$

Thus every $x \in E \cap B(0, 2r_0)$ is of type P.

And, for $x \in E \cap B(0, 2r_0)$, $\theta(x, \cdot)$ is almost-constant on $(0, 4r_0]$.

Thus every $x \in E \cap B(0, 2r_0)$ is of type P.

And, for $x \in E \cap B(0, 2r_0)$, $\theta(x, \cdot)$ is almost-constant on $(0, 4r_0]$.

If τ is chosen small enough, almost-constant density implies that for $x \in E \cap B(0, 2r_0)$ and $0 < r \leq 2r_0$, E is ε_0-close to a minimal cone (in fact, a plane) in $B(x, r)$.

The standard Reifenberg theorem applies, and $E \cap B(0, r_0)$ is bi-Hölder equivalent to a disk, as needed.

Thus every $x \in E \cap B(0, 2r_0)$ is of type P.

And, for $x \in E \cap B(0, 2r_0)$, $\theta(x, \cdot)$ is almost-constant on $(0, 4r_0]$.

If τ is chosen small enough, almost-constant density implies that for $x \in E \cap B(0, 2r_0)$ and $0 < r \leq 2r_0$, E is ε_0-close to a minimal cone (in fact, a plane) in $B(x, r)$.

The standard Reifenberg theorem applies, and $E \cap B(0, r_0)$ is bi-Hölder equivalent to a disk, as needed.

Here we got lucky because the density was already close to its absolute minimum. If the origin is a Y-point, we get that E is close to a $\mathbb{Y}$, and $\theta(x, r) \leq \pi + \tau$ for points that are far from the spine of the $\mathbf{Y}$. For the other points x, we cannot apply the almost constant-density principle as easily, so we need the next section.

b. The other cases, modulo the existence of Y-points

Existence Lemma. Suppose that there is a cone Y of type $\mathbb{Y}$, centered at 0, such that $d_{0,4}(E, Y) \leq \varepsilon$. Then (if ε small enough) $E \cap B(0, 1)$ contains at least a Y-point.

b. The other cases, modulo the existence of Y-points

Existence Lemma. Suppose that there is a cone Y of type $\mathbb{Y}$, centered at 0, such that $d_{0,4}(E, Y) \leq \varepsilon$. Then (if ε small enough) $E \cap B(0,1)$ contains at least a Y-point.

Once we have the existence lemma, we can prove the regularity of E near a Y-point or a T-point roughly as we did for the P-points.

The existence lemma is useful, because we find many balls where the density is close to $3\pi/2$ and E looks like a $\mathbb{Y}$, and we can apply the constant density argument as soon as we know that they are centered at a Y-point. [Think about balls centered near the spine of $\mathbb{Y}$.]

b. The other cases, modulo the existence of Y-points

Existence Lemma. Suppose that there is a cone Y of type $\mathbb{Y}$, centered at 0, such that $d_{0,4}(E, Y) \leq \varepsilon$. Then (if ε small enough) $E \cap B(0,1)$ contains at least a Y-point.

Once we have the existence lemma, we can prove the regularity of E near a Y-point or a T-point roughly as we did for the P-points.

The existence lemma is useful, because we find many balls where the density is close to $3\pi/2$ and E looks like a $\mathbb{Y}$, and we can apply the constant density argument as soon as we know that they are centered at a Y-point. [Think about balls centered near the spine of $\mathbb{Y}$.]

We do not have an existence lemma for T-points (as above), but for J. Taylor's theorem, we do not need one, just because the only T-point nearby is already given to us.

For higher-dimensional analogues, we could be in trouble here.

Existence Lemma. Suppose that there is a cone Y of type $\mathbb{Y}$, centered at 0, such that $d_{0,4}(E,Y) \leq \varepsilon$. Then (if ε small enough) $E \cap B(0,1)$ contains at least a Y-point.

Assume not and get a contradiction by topology. First we check that

$$\theta(x,1) \leq \frac{3\pi}{2} + \tau \quad \text{for } x \in B(0,2) \text{ and } 0 < r \leq 1, \tag{17}$$

with $\tau > 0$ as small as we want.

Existence Lemma. Suppose that there is a cone Y of type $\mathbb{Y}$, centered at 0, such that $d_{0,4}(E, Y) \leq \varepsilon$. Then (if ε small enough) $E \cap B(0,1)$ contains at least a Y-point.

Assume not and get a contradiction by topology. First we check that

$$\theta(x, 1) \leq \frac{3\pi}{2} + \tau \quad \text{for } x \in B(0,2) \text{ and } 0 < r \leq 1, \tag{17}$$

with $\tau > 0$ as small as we want.

The proof by compactness is as for the almost constant-density principle near (3.7): suppose not, find a sequence $\{E_k\}$ that converges to a $\mathbb{Y}$ in $B(0,4)$ but for which (16) fails. Then use the lower semicontinuity of Hausdorff measure and the proof of the limiting theorem to get an estimate like (3.11) and a contradiction.

$$(17) \qquad \theta(x,1) \le \frac{3\pi}{2} + \tau \text{ for } x \in E \cap B(0,2) \text{ and } 0 < r \le 1,$$

Choose τ so that $\frac{3\pi}{2} + \tau < d_+$, where d_+ denotes the density of a cone of type $\mathbb{T}$.

By (17) and monotonicity, $d(x) < d_+$ for $x \in E \cap B(0,2)$ so x is never a T-point.

By assumption, there is no Y-point, so x is a P-point, and E is bi-Hölder equivalent to a plane near x.

This is true for all $x \in E \cap B(0,2)$.

We may assume that the spine of Y is the vertical axis.

Call S the unit circle in the horizontal plane through 0, and denote by a_1, a_2, a_3 the points of $S \cap Y$.

Since E is close to a plane near a_i, we can apply the proof of regularity for P-points, and get bi-Hölder equivalence of E to a plane in $B(a_i, 10^{-1})$.

We may assume that the spine of Y is the vertical axis.

Call S the unit circle in the horizontal plane through 0, and denote by a_1, a_2, a_3 the points of $S \cap Y$.

Since E is close to a plane near a_i, we can apply the proof of regularity for P-points, and get bi-Hölder equivalence of E to a plane in $B(a_i, 10^{-1})$.

Then we can modify S slightly, to get a simple arc γ that crosses E exactly three times (one near each a_i), transversally in bi-Hölder coordinates.

Since E is bi-Hölder equivalent to a plane near every point $x \in E \cap B(0,2)$, we can deform γ into a point (inside $B(0,2)$), and the number of intersections of γ with E only jumps by multiples of ± 2. [In the detailed argument, it is easier to discretize and get transversality.]

We may assume that the spine of Y is the vertical axis.

Call S the unit circle in the horizontal plane through 0, and denote by a_1, a_2, a_3 the points of $S \cap Y$.

Since E is close to a plane near a_i, we can apply the proof of regularity for P-points, and get bi-Hölder equivalence of E to a plane in $B(a_i, 10^{-1})$.

Then we can modify S slightly, to get a simple arc γ that crosses E exactly three times (one near each a_i), transversally in bi-Hölder coordinates.

Since E is bi-Hölder equivalent to a plane near every point $x \in E \cap B(0,2)$, we can deform γ into a point (inside $B(0,2)$), and the number of intersections of γ with E only jumps by multiples of ± 2. [In the detailed argument, it is easier to discretize and get transversality.]

But we started with 3 intersections, (odd) and end with 0 (even); this contradiction proves the lemma.

A modification of this degree argument works for 2-sets in $\mathbb{R}^n$.

3. BOUNDARY REGULARITY OF SLIDING ALMOST MINIMAL SETS

Let E be such a set. What is its regularity near the Γ_j? If very lucky, this could lead to existence results for Plateau.

3. BOUNDARY REGULARITY OF SLIDING ALMOST MINIMAL SETS

Let E be such a set. What is its regularity near the Γ_j? If very lucky, this could lead to existence results for Plateau.

Recall: <u>sliding almost minimal</u> means that

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(F \cap B) + h(r)r^d$$

when $F = \varphi_1(E)$ is a sliding deformation of E in B, and in particular $\varphi_t(E \cap \Gamma_j) \subset \Gamma_j$ for $j \in J$ and $t \in [0, 1]$.

Note that even in low dimensions, the behaviour of E near Γ_j seems harder to predict.

General battle plan (copied from the interior regularity program):

- Ahlfors-regularity of E near $x \in \Gamma_j$

- Rectifiability of E

- Uniform rectifiability of E?

- Lower semicontinuity of $\mathcal{H}^d$ along a sequence of (uniformly) sliding almost minimal sets: $\mathcal{H}^d(E_\infty) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k)$

- Almost-minimality of a limit

- Is $r^{-d} H^d(E \cap B(x,r))$ an almost nondecreasing function of r when $x \in \Gamma_j$?

- What can be said when x is merely close to some Γ_j?

- Is every blow-up limit of E at $x \in \Gamma_j$ a sliding minimal cone?

- What is the list of sliding minimal cones?

- Is there a Jean Taylor Theorem at $x \in \Gamma_j$?

Last comments :

Seems to be new questions!

Applies to other categories of minimizers.

$d = 2$ and $n = 3, 4$ would already be nice.

Example of simple question: is the cone over the vertices of a cube minimal, with a sliding boundary equal to the great diagonal?

Other topics:

Simpler variants of the Plateau problem:
- inside a manifold with complicated topology but no boundary,
- separation conditions,
- homology conditions on the complement.

REFERENCES

[Al] F. J. Almgren, Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, Memoirs of the Amer. Math. Soc. 165, volume 4 (1976), i-199.

[DMS] G. Dal Maso, J.-M. Morel, and S. Solimini, A variational method in image segmentation: Existence and approximation results, Acta Math. 168 (1992), no. 1-2, 89–151.

[D1] G. David, Limits of Almgren-quasiminimal sets, Proceedings of the conference on Harmonic Analysis, Mount Holyoke, A.M.S. Contemporary Mathematics series, Vol. 320 (2003), 119-145.

[D2] G. David, Singular sets of minimizers for the Mumford-Shah functional, Progress in Mathematics 233 (581p.), Birkhäuser 2005.

[D3] G. David, Quasiminimal sets for Hausdorff measures, in Recent Developments in Nonlinear PDEs, Proceeding of the second symposium on analysis and PDEs (June 7-10, 2004), Purdue University, D. Danielli editor, 81–99, Contemp. Math. 439, Amer. Math. Soc., Providence, RI, 2007.

[D4] G. David, Hölder regularity of two-dimensional almost-minimal sets in $\mathbb{R}^n$, Annales de la Faculté des Sciences de Toulouse, Vol 18, 1 (2009), 65–246.

[D5] G. David, C^{1+a}-regularity for two-dimensional almost-minimal sets in $\mathbb{R}^n$, J. Geom. Anal. 20 (2010), no. 4, 837954.

[DDT] G. David, T. De Pauw, and T. Toro, A generalization of Reifenberg's theorem in $\mathbb{R}^3$, to appear, Geometric And Functional Analysis.

[DS] G. David and S. Semmes, Uniform rectifiability and quasiminimizing sets of arbitrary codimension, Memoirs of the A.M.S. Number 687, volume 144, 2000.

[Fe] H. Federer, Geometric measure theory, Grundlehren der Mathematishen Wissenschaften 153, Springer Verlag 1969.

[He] A. Heppes, Isogonal sphärischen netze, Ann. Univ. Sci. Budapest Eötvös Sect. Math. 7 (1964), 41-48.

[La] E. Lamarle, Sur la stabilité des systèmes liquides en lames minces, Mém. Acad. R. Belg. 35 (1864), 3-104.

[LM] Gary Lawlor and Frank Morgan, Paired calibrations applied to soap films, immiscible fluids, and surfaces or networks minimizing other norms, Pacific J. Math. 166 (1994), no. 1, 55–83.

[Ma] P. Mattila, Geometry of sets and measures in Euclidean space, Cambridge Studies in Advanced Mathematics 44, Cambridge University Press l995.

[Mo1] F. Morgan, Size-minimizing rectifiable currents, Invent. Math. 96 (1989), no. 2, 333-348.

[Mo2] F. Morgan, Minimal surfaces, crystals, shortest networks, and undergraduate research, Math. Intelligencer 14 (1992), no. 3, 37–44.

[R1] E. R. Reifenberg, Solution of the Plateau Problem for m-dimensional surfaces of varying topological type, Acta Math. 104, 1960, 1–92.
[R2] E. R. Reifenberg, An epiperimetric inequality related to the analyticity of minimal surfaces, Ann. of Math. (2) 80, 1964, 1–14.
[St] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton university press 1970.
[Ta] J. Taylor, The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces, Ann. of Math. (2) 103 (1976), no. 3, 489–539.