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Infinite-dimensional dynamical systems and the Navier-Stokes equation

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Abstract

In this set of lectures I will describe how one can use ideas of dynamical systems theory to give a quite complete picture of the long time asymptotics of solutions of the two-dimensional Navier-Stokes equation. I will discuss the existence and properties of invariant manifolds for dynamical systems defined on Banach spaces and review the theory of Lyapunov functions, again concentrating on the aspects of the theory most relevant to infinite-dimensional dynamics. I will then explain how one can apply both of these techniques to the two-dimensional Navier-Stokes equation to prove that any solution with integrable initial vorticity will will be asymptotic to a single, explicitly computable solution known as an Oseen vortex. If time permits I will describe certain extensions of this theory to the three-dimensional Navier-Stokes equations.

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