Family of quadratic differential systems with a Darboux invariant and an invariant conic

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Abstract

In this paper we provide affine invariant necessary and sufficient conditions for a non-degenerate quadratic differential system to have an invariant conic and a Darboux invariant of the form $f(x,y)^{\lambda}e^{st}$ with $\lambda, s \in \mathbb{R}$ and $s \neq 0$. The family of all such systems has a total of seven topologically distinct phase portraits. For each one of these seven phase portraits we provide necessary and sufficient conditions in terms of affine invariant polynomials for a non-degenerate quadratic system in this family to possess this phase portrait.

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Résume

Dans ce travail nous donnons des conditions nécessaires et suffisantes pour un système différentiel quadratique non-dégénéré d'avoir une conique invariante et un invariant de Darboux de la forme $f(x,y)e^{\lambda}$ tel que $\lambda, s \in \mathbb{R}$ et $s \neq 0$. La famille de tous ces systèmes possède le nombre total sept de portraits de phases topologiquement distincts. Pour chacun de ces sept portraits de phases nous donnons des conditions nécessaires et suffisantes en termes de polynômes invariants, pour qu'un système différentiel quadratique non-dégénéré de cette famille possède ce portrait de phases.

1 Introduction and statement of the main results

We consider the family of real quadratic differential systems

$$\begin{aligned} \dot{x} &= p_0 + p_1(\tilde{a}, x, y) + p_2(\tilde{a}, x, y) \equiv P(\tilde{a}, x, y), \\ \dot{y} &= q_0 + q_1(\tilde{a}, x, y) + q_2(\tilde{a}, x, y) \equiv Q(\tilde{a}, x, y) \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} &p_0 = a, \quad p_1(\tilde{a}, x, y) = cx + dy, \quad p_2(\tilde{a}, x, y) = gx^2 + 2hxy + ky^2, \\ &q_0 = b, \quad q_1(\tilde{a}, x, y) = ex + fy, \quad q_2(\tilde{a}, x, y) = lx^2 + 2mxy + ny^2. \end{aligned}$$

and with $\max(\deg(p), \deg(q)) = 2$.

Here we denote by $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$ the 12-tuple of the coefficients of systems (1.1). We denote the class of all quadratic differential systems with **QS**.

Definition 1.1. *Let Ω be an open and dense subset of $\mathbb{R}^2$. An* ***invariant*** *of a system* (1.1) *in Ω is a nonconstant C^1 function I in the variables x, y and t such that $I(x(t), y(t), t)$ is constant on all solution curves $(x(t), y(t))$ of system* (1.1) *contained in Ω, i.e.*

$$\frac{\partial I}{\partial x} P + \frac{\partial I}{\partial y} Q + \frac{\partial I}{\partial t} = 0,$$

for all $(x, y) \in \Omega$.

For $f \in \mathbb{C}[x, y]$ we say that the curve $f(x, y) = 0$ is *an invariant algebraic curve* of system (1.1) if there exists $K \in \mathbb{C}[x, y]$ such that $P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf$.

We shall consider here a particular case of a Darboux invariant: we say that $I(x, y, t) = f(x, y)^\lambda e^{st}$ with $\lambda, s \in \mathbb{R}$ and $s \neq 0$ is a *Darboux invariant* of a system (1.1) if $f(x, y) = 0$ is an invariant algebraic curve and $I(x, y, t)$ is an invariant of the system (see the above definition).

In the paper [8] the family of systems (1.1) possessing an invariant conic and a Darboux invariant is considered using the above definitions and the authors classified topologically this family. The work in [8] is part of the program to classify topologically the set of all quadratic differential systems. There are numerous works in which specific families of quadratic differential systems were classified topologically. For a survey of what has been done so far on this program we refer the reader to [11] and [2]. Many of the papers mentioned in [11] or in [2] are about systems which possess specific types of invariant algebraic curves. Systematic studies of quadratic differential systems possessing invariant straight lines were done in the following articles: [12],[14], [16], [18], [19], [15], [17], [16], [20]. In [9] the authors classify the family of systems possessing at least one invariant hyperbola in terms of the configurations of hyperbolas and invariant lines the system may possess. The problem of topologically classifying the family of quadratic systems possessing an invariant parabola is still open. The particular class we are concerned with here is formed by systems which have an invariant conic and a Darboux invariant. As the authors proved in [8], in this case necessarily the conic must be a parabola and hence their work classified topologically a part of the family of systems in **QS** which possess an invariant parabola. The problem of classifying topologically the family of all systems in **QS** possessing an invariant ellipse is still open.

Our goal in this article is: (i) to provide affine invariant necessary and sufficient conditions for a non-degenerate quadratic differential system to have an invariant conic and a Darboux invariant;
(ii) for each one of the seven phase portraits of this family we provide necessary and sufficient conditions in terms of affine invariant polynomials for a non-degenerate quadratic system to possess this phase portrait.

We denote by $\boldsymbol{QS_{DI}^{par}}$ the class of quadratic systems possessing an invariant parabola and a Darboux invariant of the form indicated above.

We define in Section 2 the following 12 polynomials in x, y and the coefficient of systems (1.1) with coefficients in $\mathbb{R}$:

$$\eta, \quad \mu_2, \quad \mu_3, \quad \widetilde{M}, \quad \widetilde{K}, \quad \widetilde{R}, \quad K_1, \quad K_3, \quad \mathbf{D}, \quad V_1, \quad V_2, \quad V_3,$$

which are invariant with respect to the affine group action.

The phase portraits of the family of quadratic differential systems possessing an invariant conic and a Darboux invariant were given in [8].

Our main results are stated in the following theorem.

Main Theorem. ***(A)*** *A non-degenerate quadratic differential system in the class* **QS** *(i.e. $\sum_{i=0}^4 \mu_i^2 \neq 0$) possesses an invariant parabola $f(x, y) = 0$ and a Darboux invariant of the form $f(x, y)^\lambda e^{st}$ with $\lambda, s \in \mathbb{R}$ and $s \neq 0$ if and only if $\eta = \widetilde{K} = \widetilde{R} = 0$ and one of the following three sets of conditions holds:*

$$\begin{aligned} &(i)\ \widetilde{M}\mu_2 \neq 0,\ V_1 = 0; \\ &(ii)\ \widetilde{M} = \mu_2 = 0,\ K_3 V_2 \neq 0; \\ &(iii)\ \widetilde{M} = \mu_2 = 0,\ K_3 \neq 0,\ \ V_2 = V_3 = 0. \end{aligned} \tag{1.2}$$

Moreover this system has an one-parameter family of invariant parabolas if an only if $\widetilde{M} = \mu_2 = V_2 = V_3 = 0$, *whereas in all other cases the invariant parabola is unique.*

(*B*) *Assume that a non-degenerate system in* **QS** *possesses an invariant parabola and a Darboux invariant, i.e. the conditions provided by the statement* **(*A*)** *are satisfied. Then the phase portrait of this system corresponds to one of the given in Figure 1 if and only if the corresponding additional conditions are satisfied as follows:*

Port.1 $\Leftrightarrow$ $\widetilde{M} \neq 0$, $\mathbf{D} < 0$;
Port.2 $\Leftrightarrow$ $\widetilde{M} \neq 0$, $\mathbf{D} > 0$;
Port.3 $\Leftrightarrow$ $\widetilde{M} \neq 0$, $\mathbf{D} = 0$;
Port.4 $\Leftrightarrow$ $\widetilde{M} = 0$, $\mu_3 K_1 < 0$;
Port.5 $\Leftrightarrow$ $\widetilde{M} = 0$, $\mu_3 K_1 > 0$, $K_3 < 0$;
Port.6 $\Leftrightarrow$ $\widetilde{M} = 0$, $\mu_3 K_1 > 0$, $K_3 = 0$;
Port.7 $\Leftrightarrow$ $\widetilde{M} = 0$, $\mu_3 K_1 = 0$.

All the invariant polynomials which classify this subfamily of quadratic systems are defined in Subsection 2.1.

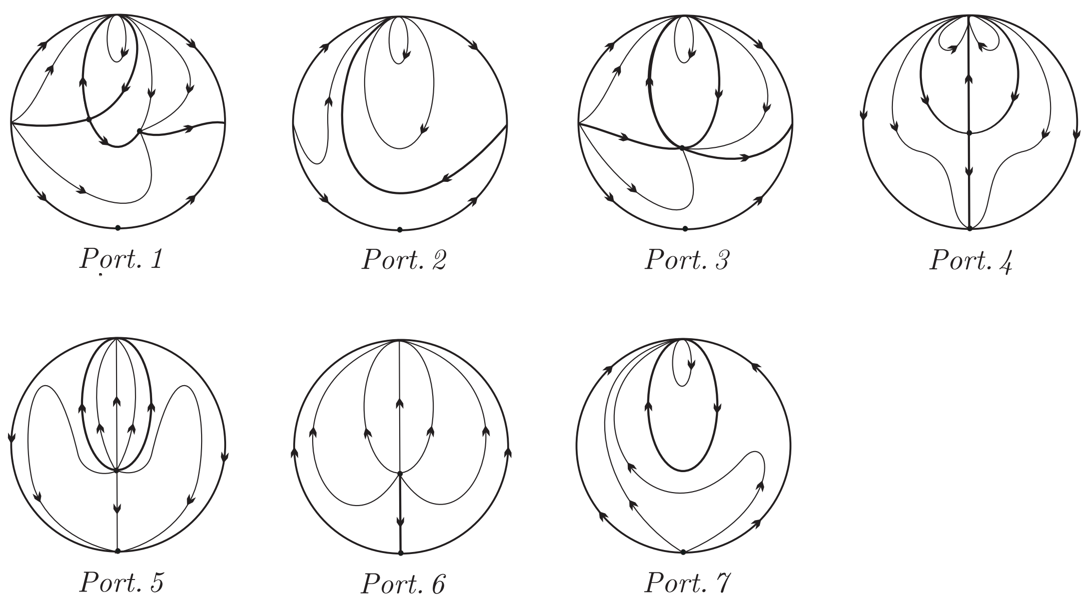

Figure 1: **The phase portraits**

2 Preliminaries

In the paper [8] the authors have investigated the class of quadratic systems possessing an invariant conic and in addition a Darboux invariant. They proved the next proposition:

Proposition 2.1. *Assume that a quadratic system possesses an invariant conic and a Darboux invariant. Then this conic must be an invariant parabola. Moreover via an affine transformation this system could be brought to the normal form:*

$$\dot{x} = px + qy + r, \quad \dot{y} = c(y - x^2) + 2x(px + qy + r), \tag{2.1}$$

possessing the invariant parabola $y - x^2 = 0$ *and the Darboux invariant of the form* $I(x, y, t) = (y - x^2)e^{-ct}$.

According to [6] we have the next lemma:

Lemma 2.1. *If a quadratic system* (1.1) *possesses an invariant parabola then it could be brought via an affine transformation to the following canonical form:*

$$\begin{aligned} \dot{x} &= c(y - x^2) + (a + bx + gy) + exy, \\ \dot{y} &= d(y - x^2) + 2x(a + bx + gy) + 2ey^2 \end{aligned} \tag{2.2}$$

having the invariant parabola $y - x^2 = 0$.

2.1 The main invariant polynomials associated to the class QS_{DI}^{par}

Consider real quadratic systems of the form (1.1). It is known that on the set **QS** acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [13]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (1.1) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS}\longrightarrow \mathbb{R}^{12}$ which associates to each system (1.1) the 12–tuple $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL–comitants, the T–comitants and the CT–comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [13] (see also [2]).

Here we construct the mentioned invariant polynomials in the following way.

First we need the GL-comitants of degree one with respect to the coefficients of systems (1.1):

$$C_i(x,y) = yp_i(x,y) - xq_i(x,y),\ i = 0,1,2; \quad D_i(x,y) \ = \frac{\partial}{\partial x}p_i(x,y) + \frac{\partial}{\partial y}q_i(x,y),\ i = 1,2 \tag{2.3}$$

and the so-called *transvectant of order k* (see [7], [10]) of two polynomials $f,\ g \in \mathbb{R}[\tilde{a}, x, y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

Using this differential operator we construct now the following GL-comitants which are of degree two with respect to the coefficients of systems (1.1):

$$\begin{aligned}
&T_1 = (C_0,C_1)^{(1)}, \quad T_2 = (C_0,C_2)^{(1)}, \quad T_3 = (C_0,D_2)^{(1)}, \quad T_4 = (C_1,C_1)^{(2)}, \quad T_5 = (C_1,C_2)^{(1)},\\
&T_6 = (C_1,C_2)^{(2)}, \quad T_7 = (C_1,D_2)^{(1)}, \quad T_8 = (C_2,C_2)^{(2)}, \quad T_9 = (C_2,D_2)^{(1)}.
\end{aligned}$$

Consider the differential operator $\mathcal{L} = x\cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ constructed in [5], acting on $\mathbb{R}[\tilde{a}, x, y]$, where

$$\begin{aligned}
\mathbf{L}_1 = &\quad 2a\frac{\partial}{\partial c} + c\frac{\partial}{\partial g} + \frac{1}{2}d\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial e} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}f\frac{\partial}{\partial m},\\
\mathbf{L}_2 = &\quad 2a\frac{\partial}{\partial d} + d\frac{\partial}{\partial k} + \frac{1}{2}c\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial f} + f\frac{\partial}{\partial n} + \frac{1}{2}e\frac{\partial}{\partial m}.
\end{aligned}$$

Using this operator and the affine invariant $\mu_0 = \mathrm{Resultant}[p_2(\tilde{a},x,1), q_2(\tilde{a},x,1), x]$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i = 1,..,4,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

And finally we construct the remaining invariant polynomials which will be needed:

$$\begin{aligned}
\mathbf{D}(\tilde{a}) =&\Big[3\big((\mu_3,\mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48;\\
\widetilde{M}(\tilde{a},x,y) =&(C_2,C_2)^{(2)} \equiv 2\mathrm{Hess}\,\big(C_2(\tilde{a},x,y)\big);\\
\eta(\tilde{a}) =&(\widetilde{M},\widetilde{M})^{(2)}/384 \equiv \mathrm{Discrim}\,\big(C_2(\tilde{a},x,y)\big);\\
\widetilde{K}(\tilde{a},x,y) =&(T_8 + 4T_9 + 4D_2^2)/18 \equiv \mathrm{Jacob}\,\big(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\big);\\
K_1(\tilde{a},x,y) =&p_1(\tilde{a},x,y)q_2(\tilde{a},x,y) - p_2(\tilde{a},x,y)q_1(\tilde{a},x,y);\\
K_3(\tilde{a},x,y) =&C_2^2(4T_3 + 3T_4) + C_2(3C_0\widetilde{K} - 2C_1T_7) + 2K_1(3K_1 - C_1D_2);\\
\tilde{L}(\tilde{a},x,y) =&4\widetilde{K} - \widetilde{M} + 4(T_8 - 8T_9 - 2D_2^2)/9;\\
\widetilde{R}(\tilde{a},x,y) =& \tilde{L} + 8\widetilde{K}.\\
\kappa(\tilde{a}) =&(\widetilde{M},\widetilde{K})^{(2)}/4.
\end{aligned}$$

These invariant polynomials were constructed earlier and could be found, for example, in [3] and [4] (we keep the notations from these papers). However for our gal we also use three new invariant polynomials V_1, V_2 and V_3, constructed here as follows:

$$\begin{aligned}
V_1 =&\big(\big((\widehat{D},\widehat{D})^{(2)}, D_2\big)^{(1)}, D_2\big)^{(1)} - 4\big((\widehat{D},\widehat{F})^{(2)}, D_2\big)^{(1)},\\
V_2 =&T_5, \quad V_3 = 4T_2 + 3C_1D_1,
\end{aligned}$$

where

$$\begin{aligned}\widehat{D}(\tilde{a},x,y) = \ & \big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6)-(C_1,T_5)^{(1)}-9D_1^2C_2\\ & +6D_1(C_1D_2-T_5)\big]/36,\\ \widehat{F}(\tilde{a},x,y) = \ & \big[6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)+48C_0(D_2,T_9)^{(1)}-9D_2^2T_4\\ & +288D_1\widehat{E}-24(C_2,\widehat{D})^{(2)}+120(D_2,\widehat{D})^{(1)}-36C_1(D_2,T_7)^{(1)}\\ & +8D_1(D_2,T_5)^{(1)}\big]/144,\end{aligned}$$

2.2 Preliminary results involving the use of polynomial invariants

According to [13] we have the next result.

Lemma 2.2 ([13])**.** *The number of infinite singularities (real and complex) of a quadratic system in* **QS** *is determined by the following conditions:*

(i) 3 *real if* $\eta > 0$*;*

(ii) 1 *real and* 2 *imaginary if* $\eta < 0$*;*

(iii) 2 *real if* $\eta = 0$ *and* $\widetilde{M} \neq 0$*;*

(iv) 1 *real if* $\eta = \widetilde{M} = 0$ *and* $C_2 \neq 0$*;*

(v) ∞ *if* $\eta = \widetilde{M} = C_2 = 0$*.*

Moreover, the quadratic systems (1.1)*, for each one of these cases, can be brought via a linear transformation to the corresponding case of the following canonical systems* $(\mathbf{S}_I)-(\mathbf{S}_V)$*:*

$$\begin{cases}\dot{x} &= a+cx+dy+gx^2+(h-1)xy,\\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2;\end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases}\dot{x} &= a+cx+dy+gx^2+(h+1)xy,\\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2;\end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases}\dot{x} &= a+cx+dy+gx^2+hxy,\\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2;\end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases}\dot{x} &= a+cx+dy+gx^2+hxy,\\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2;\end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases}\dot{x} &= a+cx+dy+x^2,\\ \dot{y} &= b+ex+fy+xy.\end{cases} \qquad (\mathbf{S}_V)$$

Using Lemma 2.1 and Proposition 2.1we prove the next result.

Lemma 2.3. *Assume that a quadratic system possesses an invariant parabola* $f(x,y)=0$*. Then this system has a Darboux invariant of the form* $I(x,y,t)=f(x,y)e^{-ct}$ *if and only if* $\widetilde{K}=0=\widetilde{R}$*.*

Proof: Consider a quadratic system possessing an invariant parabola. By Lemma 2.1 this systems could be brought via an affine transformation to form (2.2). It remains to prove that these systems have the form (2.1) if and only if the conditions $\widetilde{K}=0=\widetilde{R}$ are satisfied. Indeed, first of all we observe that for systems (2.1) these conditions hold, i.e. the necessity is evident.

We consider now systems (2.2) and calculate

$$\widetilde{K} = -2(2be-de+2cg)x^2-8cexy+4e^2y^2.$$

Therefore the condition $\widetilde{K}=0$ yields $e=cg=0$. On the other hand for $e=0$ we have $\widetilde{R}=8c(c-2g)x^2$ and hence the conditions $\widetilde{K}=0=\widetilde{R}$ are equivalent to $c=e=0$. Evidently we arrived at the 4-parameter family of systems of the form (2.1) and this completes the proof of the lemma. ∎

Remark 2.1. *For systems (4) we calculate* η *and* C_2 *and we obtain*

$$\eta = 0, \quad C_2 = x^2\big[(c-2p)x-2qy\big].$$

So we observe that the condition $C_2=0$ *gives* $c=2p$ *and* $q=0$ *and this leads to linear systems. Therefore we conclude that the conditions* $\eta=0$ *and* $C_2\neq 0$ *are necessary for a quadratic system* (1.1) *to possess an invariant parabola and a Darboux invariant of the mentioned type. Moreover we deduce that the invariant parabola* $y-x^2=0$ *of systems* (2.1) *has the double point* $N_1[0:1:0]$ *as an intersection point with the line* $Z=0$ *at infinity.*

Assume that the conic

$$\Phi(x,y) \equiv p + qx + ry + sx^2 + 2vxy + uy^2 = 0 \tag{2.4}$$

is an invariant curve for systems (1.1), i.e. the following identity holds:

$$\frac{\partial \Phi}{\partial x} P(x,y) + \frac{\partial \Phi}{\partial y} Q(x,y) = \Phi(x,y)(Ux + Vy + W),$$

where the cofactor $K = Ux + Vy + W \in \mathbb{C}[x,y]$. This identity yields a system of ten algebraic equations for determining the 9 unknown parameters p,q,r,s,u,v,U,V,W:

$$\begin{aligned}
Eq_1 &\equiv s(2g-U) + 2lv = 0,\\
Eq_2 &\equiv 2v(g+2m-U) + s(4h-V) + 2lu = 0,\\
Eq_3 &\equiv 2v(2h+n-V) + u(4m-U) + 2ks = 0,\\
Eq_4 &\equiv u(2n-V) + 2kv = 0,\\
Eq_5 &\equiv q(g-U) + s(2c-W) + 2ev + lr = 0,\\
Eq_6 &\equiv r(2m-U) + q(2h-V) + 2v(c+f-W) + 2(ds+eu) = 0,\\
Eq_7 &\equiv r(n-V) + u(2f-W) + 2dv + kq = 0,\\
Eq_8 &\equiv q(c-W) + 2(as+bv) + er - pU = 0,\\
Eq_9 &\equiv r(f-W) + 2(bu+av) + dq - pV = 0,\\
Eq_{10} &\equiv aq + br - pW = 0.
\end{aligned} \tag{2.5}$$

3 The proof of the Main Theorem

By Remark 2.1 the conditions $\eta = 0$ and $C_2 \neq 0$ are necessary for a quadratic system (1.1) to have an invariant parabola and a Darboux invariant. On the other hand according to Lemma 2.2 in the case $\eta = 0$ and $C_2 \neq 0$ any quadratic system (1.1) could be brought via a linear transformation either to systems $(\mathbf{S}_{III})$ (if $\widetilde{M} \neq 0$) or to systems $(\mathbf{S}_{IV})$ (if $\widetilde{M} = 0$). We consider each one of these two cases.

3.1 Proof of the statement $(\boldsymbol{A})$

3.1.1 The case $\widetilde{M} \neq 0$

According to Lemma 3 the conditions $\widetilde{K} = 0 = \widetilde{R}$ are necessary for a quadratic system (1.1) to have an invariant parabola and a Darboux invariant. For systems $(\mathbf{S}_{III})$ we calculate

$$\widetilde{K} = 2g(g-1)x^2 + 4ghxy + 2h^2y^2, \quad \widetilde{R} = 8g(2g-1)x^2 + 16h(2g-1)xy + 216h^2y^2$$

and evidently the conditions $\widetilde{K} = 0 = \widetilde{R}$ yield $h = g = 0$. So due to an additional translation we may assume $e = f = 0$ and we get the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b - xy \tag{3.1}$$

for which we have $C_2 = x^2y$.

Next we determine the conditions in terms of the parameters (a,b,c,d) for the existence of an invariant parabola of the above systems. Considering Remark 2.1 we deduce that the homogeneous quadratic part of the conic (2.4) must be of the form sx^2 with $s \neq 0$. So without loss of generality we may assume $u = v = 0, s = 1$ and we arrive at the conic

$$\Phi(x,y) \equiv p + qx + ry + x^2 = 0 \tag{3.2}$$

with $r \neq 0$ (in order to have a parabola). Then considering the equations (2.5) we obtain

$$\begin{gathered}
U = 0 = V,\ W = 2c,\ r = 2d,\ Eq_8 = 2a - cq,\\
Eq_9 = -d(4c - q), \quad Eq_{10} = 2bd - 2cp + aq,\\
Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_7 = 0.
\end{gathered}$$

Since $r \neq 0$ we must have $d \neq 0$ and therefore the equation $Eq_9 = 0$ gives $q = 4c$. Then we obtain:

$$Eq_8 = 2(a - 2c^2),\ Eq_{10} = 2(2ac + bd - cp).$$

So the condition $Eq_8 = 0$ yields $a = 2c^2$. We observe that $c \neq 0$, otherwise due to $d \neq 0$ the equations $Eq_8 = Eq_{10} = 0$ imply $a = b = 0$ and this leads to degenerate systems. So $c \neq 0$ and we may assume $c = 1 = d$ due to the rescaling

$(x, y, t) \mapsto (cx, c^2y/d, t/c)$ (in this case we get $a = 2c^2 = 2$). Then the equation $Eq_{10} = 2(4+b-p) = 0$ gives $p = 4-b$ and we arrive at the systems

$$\dot{x} = 2 + x + y, \quad \dot{y} = b - xy \tag{3.3}$$

which possess the invariant parabola $\Phi(x, y) = b+2y+(x+2)^2 = 0$ and the Darboux invariant $I(x, y, t) = \Phi(x, y)e^{-2t}$.

Thus we have proved that non-degenerate systems (3.3) possess an invariant parabola if and only if the conditions $cd \neq 0$ and $a = 2c^2$ are satisfied. In order to determine the corresponding affine invariant conditions, for these systems we calculate

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = -cdxy, \quad V_1 = (a - 2c^2)d^2/8.$$

So it is clear that the conditions $cd \neq 0$ and $a = 2c^2$ are equivalent to $\mu_2 \neq 0$ and $V_1 = 0$. This completes the proof of the statement **(*A*)** of Main Theorem in the case $\widetilde{M} \neq 0$.

3.1.2 The case $\widetilde{M} = 0$

According to Lemma 2.2 in this case we consider systems $(\mathbf{S}_{IV})$ for which we calculate

$$\widetilde{K} = 2(g^2 + h)x^2 + 4ghxy + 2h^2y^2.$$

Evidently the condition $\widetilde{K} = 0$ gives $h = g = 0$ and this implies $\widetilde{R} = 0$. So applying a translation (in order to annihilate the parameter e) we arrive at the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b + fy - x^2, \tag{3.4}$$

for which we have $C_2 = x^3$. So at infinity we have a unique singular point namely $N_1[0 : 1 : 0]$ and therefore the above systems could only have an invariant parabola in the form (3.2). Thus considering the equations (2.5) for systems (3.4) we obtain

$$U = 0 = V, \ W = 2c - r, \ Eq_6 = 2d, \ Eq_8 = 2a - cq + qr,$$
$$Eq_9 = dq - 2cr + fr + r^2, \quad Eq_{10} = -2cp + aq + br + pr,$$
$$Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0.$$

We observe that the equation $Eq_6 = 0$ gives $d = 0$ and then we have $Eq_9 = r(f - 2c + r) = 0$. Since we must have $r \neq 0$ this implies $r = 2c - f \neq 0$ and we obtain

$$Eq_8 = 2a + (c - f)q, \quad Eq_{10} = 2bc - bf - fp + aq.$$

First of all we remark that for a non-degenerate system (3.4) the condition $f \neq 0$ must holds. Indeed, supposing $f = 0$ we have $c \neq 0$ (due to $2c - f \neq 0$) and then we calculate

$$\text{Res}_{\,q}(Eq_8, Eq_{10}) = 2(bc^2 - a^2) = 0 \ \Rightarrow \ b = a^2/c^2$$

and this leads to the degenerate systems

$$\dot{x} = a + cx, \quad \dot{y} = (a - cx)(a + cx)/c^2.$$

So $f \neq 0$ and then for systems (3.4) with $d = 0$ we may consider $b = 0$ due to the change $y \to y - b/f$. Then the equation $Eq_{10} = aq - fp = 0$ yields $p = aq/f$. It remains to examine the equation $Eq_8 = 0$ with respect to the parameter q and we have to consider two possibilities: $c - f \neq 0$ and $c - f = 0$.

1) Assume first $c - f \neq 0$. Then $Eq_8 = 0$ implies $q = 2a/(c - f)$ and we obtain that the systems

$$\dot{x} = a + cx, \quad \dot{y} = fy - x^2$$

with $f(2c - f)(c - f) \neq 0$ possess the invariant parabola

$$\Phi(x, y) = -\frac{2a^2}{f(c - f)} - \frac{2a}{c - f}x + (2c - f)y + x^2 = 0.$$

We observe that for the above systems we may assume $f = 1$ and $a \in \{0, 1\}$ due to the rescaling $(x, y, t) \mapsto (ax/f, a^2y/f^3, t/f)$ in the case $a \neq 0$ and $(x, y, t) \mapsto (x, y/f, t/f)$ in the case $a = 0$. So we arrive at the family of systems

$$\dot{x} = a + cx, \quad \dot{y} = y - x^2, \quad c \in \mathbb{R}/\{1/2, 1\}, \quad a \in \{0, 1\} \tag{3.5}$$

possessing the invariant parabola

$$\Phi(x, y) = -\frac{2a^2}{c - 1} - \frac{2a}{c - 1}x + (2c - 1)y + x^2 = 0. \tag{3.6}$$

and the Darboux invariant $I(x,y,t)=\Phi(x,y)e^{-t}$.

2) Suppose now $c-f=0$, i.e. $c=f\neq 0$. Then $Eq_8=2a=0$ gives $a=0$ and assuming $f=1$ (due to a rescaling) we arrive at the system

$$\dot{x}=x,\quad \dot{y}=y-x^2 \tag{3.7}$$

possessing the one-parameter family of invariant parabolas $\Phi(x,y)=y+qx+x^2=0$ and the Darboux invariant $I(x,y,t)=\Phi(x,y)e^{-t}$.

So in the case $\widetilde{M}=0$ we proved that systems (3.4) possess at least one invariant parabola if and only if $d=0$ and $f(2c-f)\neq 0$. Moreover these systems possess exactly one invariant parabola if $c-f\neq 0$ and they possess an one-parameter family of invariant parabolas if $c-f=a=0$. In order to determine the corresponding affine invariant conditions for systems (3.4) we calculate:

$$\mu_0=\mu_1=0,\quad \mu_2=d^2x^2,\quad K_3|_{d=0}=6(2c-f)fx^6,$$
$$V_2|_{d=0}=-3(c-f)x^3,\quad V_3|_{\{d=0,c=f\}}=-12ax^2.$$

So we deduce that the conditions $d=0$ and $f(2c-f)\neq 0$ are equivalent to $\mu_2=0$ and $K_3\neq 0$. Moreover if $\mu_2=0$ then the condition $c-f\neq 0$ (respectively $c-f=a=0$) is equivalent to $V_2\neq 0$ (respectively $V_2=0=V_3$). This completes the proof of the statement ***(A)*** of the Main Theorem.

3.2 Proof of the statement $(\boldsymbol{B})$

For determining the phase portrait of a non-degenerate quadratic system which possesses an invariant parabola and a Darboux invariant we again examine two cases: $\widetilde{M}\neq 0$ and $\widetilde{M}=0$.

3.2.1 The case $\widetilde{M}\neq 0$

We consider the normal form (3.1) for this case, possessing the invariant parabola $\Phi(x,y)=b+2y+(x+2)^2=0$. These systems have two finite singular points $M_{1,2}\big(-1\pm\sqrt{1-b},\,-1\mp\sqrt{1-b}\big)$ which are distinct real (respectively complex) if $1-b>0$ (respectively $1-b<0$) and they coincide if $1-b=0$.

On the other hand for these systems we have $\mathbf{D}=192(b-1)$ and hence the above mentioned cases concerning the finite singularities of systems (3.1) are described by the invariant polynomial $\mathbf{D}$. It is easy to detect that one of the real singular points is a node and another is a saddle and in the case when they coalesced we have a saddle-node. For systems (3.1) we have $C_2=x^2y$ and hence at infinity we have the singular points $N_1[0:1:0]$ in the y-axis direction and and $N_2[1:0:0]$ in the xy-axis direction. Regarding the behavior of the trajectories in the neighborhood of infinity, for systems (3.1), following [13] we calculate

$$\eta=0,\quad \widetilde{M}=-8x^2,\quad \mu_0=\mu_1=0,\quad \mu_2=-xy,\quad \kappa=\tilde{L}=0.$$

So, according to [13] we deduce that at the infinity the behavior of the trajectories corresponds to *Config. 11* (see Figure 2). We point out that the singular point $N_1[0:1:0]$ is an elliptic saddle.

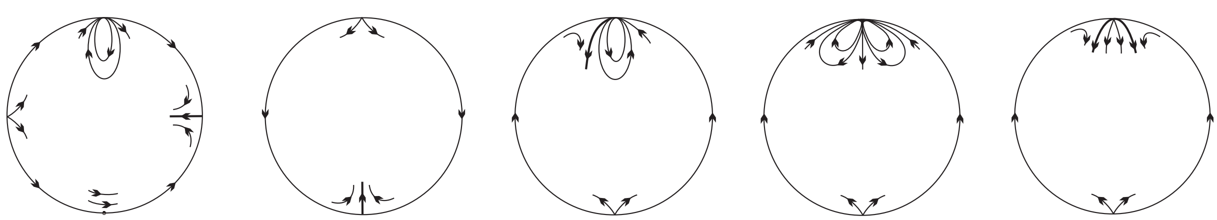

Figure 2: **Configurations of infinite singularities**

We observe that both finite singularities $M_{1,2}$ of systems (3.1) are located on the invariant parabola $\Phi(x,y)=b+2y+(x+2)^2=0$. Moreover the singular point $N_1[0:1:0]$ (which is an elliptic saddle) is a common point of the invariant parabola and of the line $Z=0$ at infinity. Therefore considering the position of the invariant parabola we get the phase portrait *Port.1* (LlibOliv-3) if $\mathbf{D}<0$; *Port.2* (LlibOliv-5) if $\mathbf{D}>0$ and *Port.3* (LlibOliv-4) if $\mathbf{D}=0$.

3.2.2 The case $\widetilde{M}=0$

Considering the results obtained above in Subsection 3.1.2 we will examine two canonical forms: systems (3.5) (in the case $\mu_2=0, K_3\neq 0$ and $V_2\neq 0$) and system (3.7) (in the case $\mu_2=0, K_3\neq 0$ and $V_2=0=V_3$).

3.2.2.1 Systems (3.5). For these systems following [13] we calculate

$$\begin{gathered}\mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 = -c^2x^3,\ \mu_4 = ax^3(ax+cy),\\ \widetilde{K} = 0,\ K_1 = -cx^3,\ K_3 = 6(2c-1)x^6.\end{gathered} \tag{3.8}$$

and we consider two possibilities: $\mu_3 \neq 0$ and $\mu_3 = 0$.

3.2.2.1.1 The possibility $\mu_3 \neq 0$. In this case $c \neq 0$ and systems (3.5) have one finite singular point $M_1(-a/c, a^2/c^2)$, which is located on the invariant parabola (3.6). Moreover we detect that M_1 is a saddle if $c < 0$ and it is a node if $c > 0$. At the same time we have $\mu_3K_1 = c^3x^6$, i.e. $\operatorname{sign}(c) = \operatorname{sign}(\mu_3K_1)$ and hence the invariant polynomial μ_3K_1 governs the type of the finite singularity M_1.

On the other hand according to [13] we conclude that at the infinity the behavior of the trajectories corresponds to *Config. 33* if $\mu_3K_1 < 0$ (see Figure 2). In the case $\mu_3K_1 > 0$ we have *Config. 38* if $K_3 < 0$ (i.e. $2c-1 < 0$) and *Config. 31* if $K_3 > 0$ (i.e. $2c-1 > 0$). We remark that for systems (3.5) the condition $c \neq 1/2$ has to be satisfied.

So considering the position of the invariant parabola and the type of the finite singular point we arrive at the phase portrait given by *Port.4* (LlibOliv-7) if $\mu_3K_1 < 0$; by *Port.5* (LlibOliv-8) if $\mu_3K_1 > 0$ and $K_3 < 0$; by *Port.6* (LlibOliv-6) if $\mu_3K_1 > 0$ and $K_3 > 0$.

3.2.2.1.2 The possibility $\mu_3 = 0$. Then $c = 0$ and $a \neq 0$ (otherwise we get a degenerate system) and considering systems (3.5) with $a = 1$ we obtain the system

$$\dot{x} = 1, \quad \dot{y} = y - x^2$$

possessing the invariant parabola $\Phi(x,y) = 2 + 2x + x^2 - y = 0$. At infinity we have a singular point of multiplicity 7 and following [13] we calculate

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = x^4,\ \widetilde{K} = 0,\ K_3 = -6x^6.$$

Thus we have $\mu_4 > 0$, $K_3 < 0$ and by [13] at the infinity the behavior of the trajectories corresponds to *Config. 32*. So considering the position of the invariant parabola we get the phase portrait given by *Port.7* (LlibOliv-9).

3.2.2.2 System (3.7). We observe that this system belongs to the family (3.5) for $a = 0$ if we allow the parameter c to take the value 1. So considering (3.8) and following [13] we calculate

$$\mu_0 = \mu_1 = \mu_2 = 0,\ \widetilde{K} = 0,\ \mu_3K_1 = x^6 > 0,\ K_3 = 6x^6 > 0$$

and according to [13] we have at infinity *Config. 31*. Taking into account that the system (3.7) has the family of invariant parabolas $\Phi(x,y) = y + qx + x^2 = 0$ we arrive again at the phase portrait given by *Port.6* (LlibOliv-6).

This completes the proof of the Main Theorem. ∎

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