ASYMPTOTIC DENSITIES IN NUMBER THEORY. PART II: PLANE ASYMPTOTIC DENSITY OF ARITHMETIC FUNCTIONS

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Abstract. The chance and the number do not cease intervening, almost daili, in all the scientific disciplines. In this paper, we make an extension of the asymptotic density by introducing a new network of probability in form $\boldsymbol{\nu} := \{\boldsymbol{\nu}_{n,\ m},\ n,\ m \geq 1\}$. While passing in extreme cases on n, m when (n, m) tends towards $+\infty$, therefore by diffusing measurement considered, we obtain what we will call plane asymptotic density of arithmetic function f.

In this work, we will state the principal results established in support of this philosophy, and will present somme application of them.

1. Introduction

The approach that we will consider in this paper is completely different from the other approaches studied until now ([1], [2], [3], [4], [5], [6] and [7]).

One of the great difficulties which is posed in the calculation of plane asymptotic density is that there does not exist only one method for the multidirectional summation, like in the case of one-way summation.

Let $f : \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{R}^+$ be a positive arithmetic function and consider

$$S_{j,\ k} = \sum_{j,\ k \geq 1} f(j, k).$$

Date: December 26, 2017.

1991 *Mathematics Subject Classification.* (2010)Primary 11B05, 11K65 ; Secondary 11R45, 60A10, 60E62.

Key words and phrases. Arithmetic functions, plane asymptotic density, Criteria, Applications.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Given a network of points which we can represent by a matrix T of numbers :

$$T : \begin{pmatrix} f(1,1) & f(1,2) & f(1,3) & f(1,4) & f(1,5) & . & . & . \\ f(2,1) & f(2,2) & f(2,3) & f(2,4) & f(2,5) & . & . & . \\ f(3,1) & f(3,2) & f(3,3) & f(3,4) & f(3,5) & . & . & . \\ f(4,1) & f(4,2) & f(4,3) & f(4,4) & f(4,5) & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix},$$

the terms of $S_{j,k}$ are the elements of this matrix T.

Before introducing the notion of plane asymptotic density, here is some methods used to calculate $S_{j,\ k}$.

Let be

$$\sum_{s=2}^{n}(\sum_{j+k=s} f(j,k)) \tag{1.1}$$

$$\sum_{j+k\leq n} f(j,k). \tag{1.2}$$

In this first case, by making aim n towards $+\infty$, we are returned to add by "triangle method",

$$\sum_{j+k=s} f(j,k) = f(1,s-1) + f(2,s-2) + ... + f(s-1,1).$$

Let be

$$\sum_{j=n,\ k<n} f(j,k) + \sum_{j<n,\ k=n} f(j,k) + f(n,n) \tag{1.3}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{m} f(j,k). \tag{1.4}$$

In the case of the expression (1.3), by making aim n towards $+\infty$, we are returned to add by "square method"

$$\sum_{j=n,\ k<n} f(j,k) = f(n,1) + f(n,2) + ... + f(n,n-1).$$

In the case of the expression (1.4), by making aim n and m towards the infinity, we are returned to add by "rectangle method".

Let be

$$\sum_{j=1}^{n}(\sum_{k=1}^{m} f(j,k)). \tag{1.5}$$

In this case, by making aim m and n towards the infinity, we are returned to add by "line method".

Let be

$$\sum_{k=1}^{m}(\sum_{j=1}^{n} f(j,k)) \tag{1.6}$$

In this case, by making aim n and m towards the infinity, we are returned to add by "column method".

In the last two cases, we transformed a "double" sum in a "repeated" sum.

Thus, we have the following theorem :

Theorem 1.1. *Let $f : \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{R}^+$ be a positive arithmetic function and $\ell \in \mathbb{R}^+$. Put*

$$S_{j,\ k} = \sum_{j,\ k\geq 1} f(j,k); \quad V_{j,\ k} = \sum_{j\geq 1}\sum_{k\geq 1} f(j,k); \quad W_{j,\ k} = \sum_{k\geq 1}\sum_{j\geq 1} f(j,k).$$

Then, $S_{j,\ k}$, $V_{j,\ k}$, $W_{j,\ k}$ converge at the same time and have a common limit ℓ, or diverge.

Proof. see ([8]). □

2. Plane Asymptotic Density of Theoretic Arithmetic Function

We make an extension of the asymptotic density ([1], [2], [3], [4], [5] and [6]) by introducing a new network of probability in the form

$$\boldsymbol{\nu} := \{\boldsymbol{\nu}_{n,\ m}, \quad n,\ m \geq 1\}.$$

with, more generally, for any positive arithmetic function $f : \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{R}^+$, put

$$\boldsymbol{\nu}_{n,\ m}(f) := \frac{1}{n\ m}\sum_{j,\ k=1}^{m,\ n} f(j,\ k).$$

Taking the limit on n, m when (n,m) tends to $+\infty$, we diffuse the considered measure, and we obtain that we shall call plane asymptotic density of an arithmetic function f.

Definition 2.1. *Let $f : \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{R}^+$ be an arithmetic function. Consider, for all $m \geq 1$, for all $n \geq 1$, the expression defined by*

$$\boldsymbol{\nu}_{n,\ m}(f) := \frac{1}{n\ m}\sum_{j,\ k=1}^{m,\ n} f(j,\ k).$$

We say that an arithmetic function f admits a plane asymptotic density ℓ, ($\ell \geq 0$) if the limit, $\lim\limits_{(n,\ m)} \boldsymbol{\nu}_{n,\ m}(f)$ exists and equal to ℓ, when (n, m) tends to $+\infty$. If this case, we shall denote this limit by $D(f)$.

Let E be subset of $\mathbb{N}^* \times \mathbb{N}^*$. As a corollary of this definition we find the following definition by taking $I_E = f$ where,

$$I_E(j,k) = \begin{cases} 1 \ \text{ if } \ (j,k) \in E, \\ \\ 0, \ \text{else} \end{cases}$$

is the indicator function of the subset E and

$$N_{n,m}(E) := Card(E \cap (\{1, ..., n\} \times \{1, ..., m\})).$$

Definition 2.2. *(Definition 2 of ([2]))Le E be a subset of $\mathbb{N}^* \times \mathbb{N}^*$. We say that E admits a plane asymptotic density ℓ, if the numerical sequence $(\boldsymbol{\nu}_{n,m}(I_E))_{n\geq 1,\ m\geq 1}$ admits a limit equal to ℓ, (necessarily this limit belongs to $[0,\ 1]$) when (n, m) tends to $+\infty$. If this the case, we shall denote this limit by $D(E)$, and we shall say that $D(E)$ is a plane asymptotic density of the set E.*

Among the positive arithmetical functions, there are functions which admit plane asymptotic densities and there is others which do not admit plane asymptotic densities.

Even so, if we make an other restriction, by considering only the positive and bounded arithmetical functions, as shows in ([3], p. 338) in the Propositions 1.1 and 1.2 by bringing a second argument in the proof.

3. Some Applications

Lemma 3.1. *For all $a \geq 1,\ b \geq 1$, one has*

$$2^{1-\alpha}(a+b)^\alpha \leq a^\alpha + b^\alpha < 2(a+b)^\alpha, \quad \text{for } \alpha < 1.$$

Proposition 3.2. *Let us consider the arithmetic function $f : \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{R}^+$, defined by*

$$f(j,k) = \frac{1}{j^\alpha + k^\alpha}.$$

If $\alpha > 1$, then, f admits a plane asymptotic density $D(f)$ and $D(f) = 0$.

Proof. Let us put

$$S_{j,\ k} = \sum_{j\geq 1}\sum_{k\geq 1} \frac{1}{j^\alpha + k^\alpha}$$

Let us show that $S_{j,\ k}$ converges if $\alpha > 2$.

According to the Theorem1.1, we show that $S_{j,\ k}$ converges for $\alpha > 2$ means showing that

$$\sum_{j,\ k\geq 1} \frac{1}{j^\alpha + k^\alpha}$$

converges for $\alpha > 2$, and, especially,

$$\sum_{s\geq 1} (\sum_{j+k=s} \frac{1}{j^\alpha + k^\alpha}).$$

According to the previous Lemma 3.1, if $\alpha < 1,\ j + k = s$, we have

$$2(\frac{1}{2}s)^\alpha \leq j^\alpha + k^\alpha < 2s^\alpha,$$

thus

$$\frac{1}{2s^\alpha} < \frac{1}{j^\alpha + k^\alpha} \leq \frac{1}{2(\frac{1}{2}s)^\alpha}$$

and

$$\frac{s-1}{2s^\alpha} < \sum_{j+k=s} \frac{1}{j^\alpha + k^\alpha} \leq \frac{s-1}{2(\frac{1}{2}s)^\alpha}.$$

Consequently, it results that the double series converges or diverges as $\sum\limits_{s\geq 1} \frac{1}{s^{\alpha-1}}$ converges or diverges.

Then, the "double" series and consequently the "repeated" series is convergent if $\alpha > 2$ and divergent if $\alpha \leq 2$.

Let us put

$$\boldsymbol{\nu}_{n,\ m}(f) := \frac{1}{n\ m} \sum_{j,\ k=1}^{m,\ n} f(j,k),$$

if $\alpha > 2$, then,

$$0 \leq \boldsymbol{\nu}_{n,\ m}(f) := \frac{1}{n\ m} \sum_{j,\ k=1}^{m,\ n} f(j,k) \leq \frac{C}{nm} \longrightarrow 0,$$

thus, $D(f)$ exists and $D(f) = 0$. □

Proposition 3.3. *Let $f(j,k) = I_{\mathbb{N}^*\times\mathbb{N}^*}(j,\ k)$. Then, f admits a plane asymptotic density $D(f)$ and $D(f) = 1$.*

Proposition 3.4. *Let $n, m \in \mathbb{N}^*$, let us denote by $N(n,m)$ the number of couples $(k,\ell) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $k \leq n,\ \ell \leq m$ and $(k,m) = 1$. Let us put*

$$\boldsymbol{\nu}_{n,\ m}(N) = \frac{N(n,m)}{n\ m}.$$

Then N admits a plane asymptotic density $D(N)$ and $D(N) = \frac{6}{\pi^2}$.

For the proof of this proposition, we need to the following proposition :

Proposition 3.5. *Let us given n and m two integers of $\mathbb{N}^*$. Let us denote by $N(n,m)$ the number of couples $(k,\ell) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $k \leq n,\ \ell \leq m$ and $(k,m) = 1$. Then, one has*

$$N(n,m) = \sum_{d \leq Inf(n,\ m)} \mu(d) \left[\frac{n}{d}\right] \left[\frac{m}{d}\right]$$

and

$$\boldsymbol{\nu}_{n,\ m}(N) = \frac{N(n,\ m)}{n\ m} = \frac{6}{\pi^2} + O(\frac{LogInf(n,\ m)}{Inf(n,\ m)}).$$

For the proof of this proposition, we need to the following Lemma :

Lemma 3.6. *One has*

(**a**) *all integer $n \in \mathbb{N}^*$ spells, in a unique way, in the form*

$$n = m^2 q, \text{ with } m \text{ and } q \in \mathbb{N}^*,\ q \text{ without square factor ;}$$

(**b**) *given an integer $n \in \mathbb{N}^*$ and let μ be the Möbius arithmetical function, then, one has*

$$\sum_{d^2|n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is without square factor,} \\ 0 & \text{otherwise ;} \end{cases}$$

(**c**) *given integers m and $n \in \mathbb{N}^*$, let μ be the Möbius arithmetical function, then, one has*

$$\sum_{d|m,\ d|n} \mu(d) = \begin{cases} 1 & \text{if } (m,n) = 1 \\ 0 & \text{if } (m,n) > 1. \end{cases}$$

Proof. Indeed,

(**a**) one has

- if $n = 1$, it is clear that there is an only solution $m = q = 1$;
- if $n > 1$, let us $n = \prod_{k=1}^{r} p_k^{\alpha_k}$, where $p_1, p_2 ..., p_r$ are distinct prime numbers and $\alpha_1,\ \alpha_2,\ ...,\ \alpha_r$ are integers ≥ 1. m and q, divisors of n, have to be of forms

$$m = \prod_{k=1}^{r} p_k^{\beta_k} \quad \text{and} \quad q = \prod_{k=1}^{r} p_k^{\gamma_k},$$

with β_k and $\gamma_k \geq 0$.

m and q being given by these formulae, we have

$$n = m^2 q, \text{ where } q \text{ without square factor,}$$
$$\Updownarrow$$
$$\text{for every } k,\ 2\beta_k + \gamma_k = \alpha_k \ \text{ and } \ \gamma_k = 0 \ \text{ or } \ 1.$$

There is thus an unique solution, obtained by setting β_k and γ_k equals respectively to the quotient and to the rest of the division of α_k by 2.

(**b**) We see that $d^2 \mid n$ amounts to $d \mid m$. Indeed ; if $n = 1$, the only one d possible is 1 and $m = 1$.

If $n = \prod_{k=1}^{r} p_k^{\alpha_k}$, d has to be of the shape $d = \prod_{k=1}^{r} p_k^{\delta_k}$, with $\delta_k \geq 0$.

But the square of this number divides n, if and only, if $2\delta_k \leq \alpha_k$; namely, $\delta_k \leq \beta_k$ as indicated in (**a**). We thus have

$$\sum_{d^2|n} \mu(d) = \sum_{d|m} \mu(d) = \begin{cases} 1 & \text{if } \ (m,n) = 1, \\ 0 & \text{if } \ (m,n) > 1. \end{cases}$$

But, one has $m = 1$, if and only, if n is without square factor.

(**c**) As $d \mid m$ and $d \mid n$ amounts to $d \mid (m,\ n)$, we have :

$$\sum_{d|m,\ d|n} \mu(d) = \sum_{d|(m,\ n)} \mu(d) = \begin{cases} 1 & \text{if } \ (m,\ n) = 1, \\ 0 & \text{if } \ (m,\ n) > 1. \end{cases}$$

□

Proof. (**Proposition 3.5**) According to Lemma 3.6, we can write, for n and $m \geq 1$,

$$N(n,m) = \sum_{k\leq n,\ \ell\leq m} \Big(\sum_{d|k,\ d|\ell} \mu(d) \Big) = \sum_{d\leq Inf(n,\ m)} \mu(d) \sum_{k\leq n,\ \ell\leq m,\ d|k,\ d|\ell} 1$$

$$= \sum_{d\leq Inf(n,\ m)} \mu(d) \left[\tfrac{n}{d}\right] \left[\tfrac{m}{d}\right]$$

But, if L and $M \geq 1$, we have :

$$LM - L - M + 1 = (L-1)(M-1) \leq [L]\,[M] \leq LM$$

and hence

$$[L]\,[M] = LM - h(L,\ M), \text{ with } \ 0 \leq h(L,\ M) < L + M.$$

So, putting $\alpha = Inf(n,m)$, we obtain for n and $m \geq 1$

$$N(n,m) = \sum_{d\leq\alpha}\mu(d)(\tfrac{n\ m}{d^2} - h(\tfrac{n}{d},\ \tfrac{m}{d})) = \sum_{d\leq\alpha}\tfrac{\mu(d)}{d^2} - \sum_{d\leq\alpha}\mu(d)h(\tfrac{n}{d},\ \tfrac{m}{d})$$

$$= \tfrac{6}{\pi^2}nm - nm\sum_{d>\alpha}\tfrac{\mu(d)}{d^2} - \sum_{d\leq\alpha}\mu(d)h(\tfrac{n}{d},\ \tfrac{m}{d})$$

hence

$$\left|\tfrac{N(n,\ m)}{n\ m} - \tfrac{6}{\pi^2}\right| \leq \left|\sum_{d>\alpha}\tfrac{\mu(d)}{d^2}\right| + \tfrac{1}{n\ m}\left|\sum_{d\leq\alpha}\mu(d)h(\tfrac{n}{d},\tfrac{m}{d})\right|$$

$$\leq \sum_{d>\alpha}\tfrac{1}{d^2} + \tfrac{1}{n\ m}\sum_{d\leq\alpha}(\tfrac{n}{d}+\tfrac{m}{d}) \leq \sum_{d>\alpha}\tfrac{1}{d^2} + (\tfrac{1}{n}+\tfrac{1}{m})\sum_{d\leq\alpha}\tfrac{1}{d}$$

$$\leq \sum_{d>\alpha}\tfrac{1}{d^2} + \tfrac{2}{\alpha}\sum_{d\leq\alpha}\tfrac{1}{d} = O(\tfrac{Log\alpha}{\alpha}).$$

Let us apply the previous proposition, then $D(N) = \frac{6}{\pi^2}$. □

Proposition 3.7. *Let f be a positive arithmetical function such that $\lim\limits_{(n,\ m)} f(n,\ m) = \ell$. Let us put, for all $m,\ n \geq 1$,*

$$\boldsymbol{\nu}_{n,\ m}(f) := \frac{1}{n\ m}\sum_{j,\ k=1}^{m,\ n} f(j,\ k).$$

Then, f admits a plane asymptotic density $D(f)$ and $D(f) = \ell$.

Proposition 3.8. *Let f be a positive arithmetical function such that*

$$\sum_{n,\ m\geq 1}\frac{f(nm)}{nm} < +\infty.$$

Then, f admits a plane asymptotic density $D(f)$ and $D(f) = 0$.

It results the following corollary :

Corollary 3.9. *Let E be a subset of $\mathbb{N}^* \times \mathbb{N}^*$ such that $\sum\limits_{n,\ m\geq 1}\frac{1}{n\ m} < +\infty$. Then, E admits a plane asymptotic density $D(E)$ and $D(E) = 0$.*

Theorem 3.10. *Let $f : \mathbb{N}^* \longrightarrow \mathbb{R}^+$ be a positive arithmetical function such that $\sum\limits_{k\geq 1} f^2(k) < +\infty$. Let us put*

$$h(j,k) = \frac{f(j)f(k)}{j+k}.$$

(**a**) *Under the previous conditions, we have*

$$\sum_{j\geq 1}\sum_{k\geq 1} h(j,k) < +\infty;$$

(**b**) *h admits a plane asymptotic density $D(h)$ and $D(h) = 0$.*

Before to proof this theorem, we need to the following proposition :

Proposition 3.11. *Let $f : \mathbb{N}^* \longrightarrow \mathbb{R}^+$ be a positive arithmetical function such that $\sum\limits_{n\geq 1} f^2(n) < +\infty$. Let us put*

$$g(n) := \frac{1}{n}\sum_{k=1}^{n} f(k) \ ;$$

and $h(n) = f(n)g(n)$.

(**a**) *Under the previous conditions, we have*

$$\sum_{n\geq 1} h(n) < +\infty \quad \textit{and} \quad \sum_{n\geq 1} g^2(n) < +\infty.$$

(**b**) *We have*

- *the arithmetical function h admits an asymptotic density $d(h)$ and $d(h) = 0$;*
- *the arithmetical function g^2 admits an asymptotic density $d(g^2)$ and $d(g^2) = 0$.*

Proof. (**a**) Let us show that

$$\sum_{n\geq 1} h(n) < +\infty \ \text{ and } \ \sum_{n\geq 1} g^2(n) < +\infty.$$

We have

$$ng(n) = f(n) + (n-1)g(n-1),$$

hence

$$(1 - \tfrac{2}{n+1})g^2(n) \leq (4 - \tfrac{2}{n})f^2(n) + 2(G(n-1) - G(n))$$

$$< 4f^2(n) + 2(G(n-1) - G(n)),$$

where

$$G(n) = \frac{n^2 g^2(n)}{n+1}.$$

By summing on k, $k = 2, 3, ..., n$, then

$$\sum_{k=2}^{n}(1 - \frac{2}{k+1})g^2(k) < 4\sum_{k=2}^{n} f^2(k) + 2(G(1) - G(n)) < 4\sum_{k=1}^{n} f^2(k) + g^2(1).$$

By multiplying by 2, then

$$\sum_{k=1}^{n} g^2(k) < 8\sum_{k=1}^{n} f^2(k) + (3g^2(1) + \frac{1}{3}g^2(2)).$$

Which implies $\sum_{n\geq 1} g^2(n) < +\infty$.

The convergence of $\sum_{n\geq 1} h(n)$ results from the fact that

$$h(n) = f(n)g(n) \leq \frac{f^2(n) + g^2(n)}{2}.$$

(**b**) Indeed ;

• one has

$$\boldsymbol{\nu}_n(h) = \frac{1}{n}\sum_{k=1}^{n} h(k)$$

and then $0 \leq \boldsymbol{\nu}_n(h) \leq \frac{M}{n}$. Hence, when $(n \longrightarrow +\infty)$, $d(h)$ exists and $d(h) = 0$.

• One has

$$\boldsymbol{\nu}_n(g^2) = \frac{1}{n}\sum_{k=1}^{n} g^2(k)$$

and then $0 \leq \boldsymbol{\nu}_n(g^2) \leq \frac{C}{n}$. Hence, when $(n \longrightarrow +\infty)$, $d(g^2)$ exists and $d(g^2) = 0$. □

Proof. (**Théorème 3.10**) Indeed ;

(**a**) we limit our selves to a square $j = n$, $k = n$.

For reason of a symmetry, we consider only the terms below the diagonal $j = k$. By adding the terms on the diagonal, we obtain

$$\sum_{j=1}^{n} f(j)(\sum_{k=1}^{j} \frac{f(k)}{k+j}) < \sum_{j=1}^{k} f(j)g(j),$$

with

$$g(j) := \frac{1}{j}\sum_{k=1}^{j} f(k);$$

hence, indicating by σ_n the value of $S_{j,\ k}$, when $j = k = n$, then

$$\sigma_n < 2\sum_{j=1}^{n} f(j)g(j) < \sum_{j=1}^{n} f^2(j) + \sum_{j=1}^{n} g^2(j).$$

Using the previous proposition, then

$$\sum_{j\geq 1}\sum_{k\geq 1} h(j,k) < +\infty.$$

(**b**) According to (**a**) above, then

$$0 \leq \boldsymbol{\nu}_{n,\ m}(h) \leq \frac{M}{n\ m},$$

thus it holds that h admits a plane asymptotic density $D(h)$ and $D(h) = 0$. □

4. Conclusions

- The previous definition can be formulated while taking the limit on $(n^2 + m^2)$, namely, while making burst the ball of center $(0, 0)$.
- A good open problem which persists is to find criteria similar to those obtained in ([3], [4]) calculating the plane asymptotic density of arithmetical functions.

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