CRM 3357 (2016)

Semiclassical asymptotics of quantum weighted Hurwitz numbers

J. Harnad[1,2] and Janosch Ortmann[1,2]

[1] *Centre de recherches mathématiques, Université de Montréal*
C. P. 6128, succ. centre ville, Montréal, QC, Canada H3C 3J7
[2] *Department of Mathematics and Statistics, Concordia University*
1455 de Maisonneuve Blvd. W. Montréal, QC, Canada H3G 1M8

Abstract

This work concerns the semiclassical asymptotics of quantum weighted double Hurwitz numbers. We compute the leading term of the partition function for three versions of the quantum weighted Hurwitz numbers, as well as lower order semiclassical corrections. The classical limit $\hbar \to 0$ is shown to reproduce the simple Hurwitz numbers studied by Pandharipande and Okounkov [19, 21]. The KP-Toda τ-function that acts as generating function for the quantum Hurwitz numbers is shown to have their generating function as limit as do, with suitable scaling, the partition function, the weights and expectations of Hurwitz numbers.

Contents

1 Introduction: weighted Hurwitz numbers and their generating functions 2
1.1 Hurwitz numbers . . . 2
1.2 Weighted Hurwitz numbers . . . 4
1.3 The τ-function as generating function . . . 5
1.4 Quantum Hurwitz numbers . . . 6
1.5 Classical limit of the generating function for quantum Hurwitz numbers . . . 9

2 Probabilistic approach to quantum Hurwitz numbers 10

3 Classical limits and asymptotic expansion 12
3.1 Classical limit . . . 13
3.2 Semiclassical corrections . . . 14

4 Proofs 15

1 Introduction: weighted Hurwitz numbers and their generating functions

1.1 Hurwitz numbers

Multiparametric weighted Hurwitz numbers were introduced in [8–10, 12] as generalizations of the notion of simple Hurwitz numbers [13, 14, 19, 21] and other special cases [1–4, 7, 15, 23] previously studied. In general, parametric families of KP or $2D$ Toda τ-functions of *hypergeometric type* [16, 20] serve as generating functions for the weighted Hurwitz numbers, which appear as coefficients in an expansion over the basis of power sum symmetric functions in an auxiliary set of variables. The weights are determined by a parametric family of weight generating functions $G(z, \mathbf{c})$, with parameters $\mathbf{c} = (c_1, c_2, \dots)$ which can either be expressed as a formal sum

$$G(z) = 1 + \sum_{i=1}^{\infty} g_i z^i \tag{1.1}$$

or an infinite product

$$G(z) = \prod_{i=1}^{\infty} (1 + z c_i), \tag{1.2}$$

or some limit thereof. Comparing the two formulae, $G(z)$ can be interpreted as the generating function for elementary symmetric functions in the variables $\mathbf{c} = (c_1, c_2, \dots)$.

$$g_i = e_i(\mathbf{c}). \tag{1.3}$$

Another parametrization considered in [8–10, 12], consists of weight generating functions of the form

$$\tilde{G}(z) = \prod_{i=1}^{\infty} (1 - z c_i)^{-1}. \tag{1.4}$$

The corresponding power series expansions

$$\tilde{G}(z) = 1 + \sum_{i=1}^{\infty} \tilde{g}_i z^i \tag{1.5}$$

can similarly be interpreted as defining the complete symmetric functions

$$\tilde{g}_i = h_i(\mathbf{c}). \tag{1.6}$$

Hurwitz numbers $H(\mu^{(1)}, \cdots, \mu^{(k)})$ may be defined in one of two equivalent ways: geometrical and combinatorial. The geometrical definition is:

Definition 1.1. For a set of k partitions $(\mu^{(1)}, \cdots, \mu^{(k)})$ of n, $H(\mu^{(1)}, \cdots, \mu^{(k)})$ is the number of distinct n-sheeted branched coverings $\Gamma \to \mathbf{P}^1$ of the Riemann sphere having k branch points $(p_1, \ldots, p_k)$ with ramification profiles $\{\mu_i\}_{i=1,\ldots,k}$, divided by the order $\mathrm{aut}(\Gamma)$ of the automorphism group of Γ.

The combinatorial definition is:

Definition 1.2. $H(\mu^{(1)}, \cdots, \mu^{(k)})$ is the number of distinct factorization of the identity element $\mathbf{I} \in S_n$ of the symmetric group as an ordered product

$$\mathbf{I} = h_1, \ldots h_k, \quad h_i \in S_n, \quad i = 1, \ldots, k \tag{1.7}$$

where h_i belongs to the conjugacy class with cycle lengths equal to the parts of $\mu^{(i)}$, divided by $n!$.

The fact that these coincide follows from the monodromy representation of the fundamental group of the sphere minus the branch points into S_n, defined by lifting any closed loop to the branched cover, evaluating the lift of the simple loop surrounding all the branch points, and decomposing the homotopy class into an ordered product of those consisting of simple loops around each successive branch point.

Let $\mathcal{P}_n$ denote the set of integer partitions of n and $p(n)$ its cardinality. The Frobenius-Schur formula [5, 6, 17, 22] expresses the Hurwitz numbers in terms of the irreducible characters of S_n

$$H(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{\lambda \in \mathcal{P}_n} h_\lambda^{k-2} \prod_{i=1}^{k} z_{\mu^{(i)}}^{-1} \chi_\lambda(\mu^{(i)}), \tag{1.8}$$

where $\chi_\lambda(\mu^{(i)})$ is the irreducible character of the representation with Young symmetry class λ evaluated on the conjugacy class with cycle lengths equal to the parts of μ; h_λ is the product of hook lengths of the Young diagram of partition λ and

$$z_\mu = \prod_{i=1}^{\ell(\mu} m_i(\mu)! i^{m_i(\mu)} \tag{1.9}$$

is the order of the stabilizer of any element of the conjugacy class μ, with $m_i(\mu)$ equal to the number of times i appears as a part of μ. We denote the weight of a partition $|\mu|$, its length $\ell(\mu)$ and define its *colength* as

$$\ell^*(\mu) := |\mu| - \ell(\mu). \tag{1.10}$$

1.2 Weighted Hurwitz numbers

Following [8–10, 12] we define, for each positive integer d and every pair of ramification profiles (μ, ν) (i.e. partitions of n), the weighted double Hurwitz number

$$H_G^d(\mu,\nu) := \sum_{k=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}}' m_\lambda(\mathbf{c}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu), \tag{1.11}$$

where

$$m_\lambda(\mathbf{c}) := \frac{1}{|\mathrm{aut}(\lambda)|} \sum_{\sigma\in S_k} \sum_{1\le i_1<\cdots<i_k} c_{i_\sigma(1)}^{\lambda_1} \cdots c_{i_\sigma(k)}^{\lambda_k}, \tag{1.12}$$

is the monomial sum symmetric function [18] corresponding to a partition λ of weight

$$|\lambda| = d = \sum_{i=1}^{k} \ell^*(\mu^{(i)}) \tag{1.13}$$

whose parts $\{\lambda_i\}$ are the colengths $\ell^*(\mu^{(i)})$ in weakly descending order,

$$|\,\mathrm{aut}(\lambda)| := \prod_{i=1}^{\ell(\lambda)} m(\lambda_i)!, \tag{1.14}$$

and $\sum'$ denotes the sum over all k-tuples of partitions $(\mu^{(1)},\dots,\mu^{(k)})$ satisfying condition (1.13) other than the cycle type of the identity element.

By the Riemann-Hurwitz formula, the Euler characteristic of the covering surface is

$$\chi = 2 - 2g = 2n - d. \tag{1.15}$$

For weight generating functions of the form (1.4), the weighted double Hurwitz number is defined as:

$$H_{\tilde{G}}^d(\mu,\nu) := \sum_{k=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}}' f_\lambda(\mathbf{c}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu) \tag{1.16}$$

where

$$f_\lambda(\mathbf{c}) := \frac{(-1)^{\ell^*(\lambda)}}{|\mathrm{aut}(\lambda)|} \sum_{\sigma\in S_k} \sum_{1\le i_1\le\cdots\le i_k} c_{i_\sigma(1)}^{\lambda_1}, \cdots c_{i_\sigma(k)}^{\lambda_k}, \tag{1.17}$$

is the "forgotten" symmetric function [18].

The particular case where all the μ_i's represent simple branching (i.e. where they are all 2-cycles) was studied in [19, 21] and corresponds to the exponential weight generating function

$$G(z) = e^z = \lim_{k\to\infty} (1 + z/k)^k \tag{1.18}$$

The evaluation of the monomial sum symmetric function in this limit is

$$\lim_{k\to\infty} m_\lambda\left(\underbrace{\frac{1}{m}\cdots,\frac{1}{m}}_{k\text{ times}},0,0,\cdots\right)=\delta_{\lambda,(2,(1)^{n-2})}, \tag{1.19}$$

so the weight is uniform on all k-tuples $(\mu^{(1)},\cdots,\mu^{(k)})$ of partitions corresponding to simple branching

$$\mu^{(i)}=(2,(1)^{n-2}) \tag{1.20}$$

and vanishes on all others. This is what we view as the "classical" weighted (double) Hurwitz numbers.

1.3 The τ-function as generating function

Choosing a small parameter β, the following double Schur function expansion defines a $2D$-Toda τ function of hypergeometric type (at the lattice point $N=0$).

$$\tau^{(G,\beta)}(\mathbf{t},\mathbf{s})=\sum_\lambda r_\lambda^{(G,\beta)}s_\lambda(\mathbf{t})s_\lambda(\mathbf{s}), \tag{1.21}$$

where the coefficients $r_\lambda^{(G,\beta)}$ are defined in terms of the weight generating function G by the following *content product* formula

$$r_\lambda^{(G,\beta)}:=\prod_{(i,j)\in\lambda}G(\beta(j-i)), \tag{1.22}$$

The same formulae apply *mutatis mutandis* with the replacement $G\to\tilde{G}$ in the case of the second type of weight generating function $\tilde{G}$ defined by (1.4).

By changing the expansion basis from Schur functions [18] to the power sum symmetric functions $p_\mu(\mathbf{t})p_\nu(\mathbf{s})$ it follows [8–10, 12], that $\tau^{(G,\beta)}(\mathbf{t},\mathbf{s}))$ is interpretable as a generating function for the weighted double Hurwitz numbers $H_G^d(\mu,\nu)$.

THEOREM 1.1. *The 2D Toda τ-function $\tau^{(G,\beta)}(\mathbf{t},\mathbf{s})$ can be expressed as*

$$\tau^{(G,\beta)}(\mathbf{t},\mathbf{s})=\sum_{d=0}^\infty\beta^d\sum_{\substack{\mu,\nu\\|\mu|=|\nu|}}H_G^d(\mu,\nu)p_\mu(\mathbf{t})p_\nu(\mathbf{s}). \tag{1.23}$$

and the same formula holds under the replacement $G\to\tilde{G}$.

The case of the classical weight generating function (1.18) gives the following content product coefficient in the τ-function expansion (1.21)

$$r_\lambda^{(\exp,\beta)}=e^{\frac{\beta}{2}\sum_{i=1}^{\ell(\lambda)}\lambda_i(\lambda_i-2i+1)}, \tag{1.24}$$

as in [19], and the generating function expansion (1.21) becomes

$$\tau^{(\exp,\beta)}(\mathbf{t},\mathbf{s}) = \sum_{k=0}^{\infty} \frac{\beta^d}{d!} \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} H^d_{\exp}(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}), \tag{1.25}$$

where

$$H^d_{\exp}(\mu,\nu) := H((2,(1)^{n-2}),\dots(\ d \text{ times}),\dots,(2,(1)^{n-2})). \tag{1.26}$$

1.4 Quantum Hurwitz numbers

A special case consists of pure quantum Hurwitz numbers [9, 11] which are obtained by choosing the parameters c_i as

$$c_i = q^i, \quad i = 1, 2 \dots \tag{1.27}$$

where q is a real parameter between 0 and 1.

Remark 1.1. The parameter q may be interpreted as $q = e^{-\epsilon}$ for a small parameter

$$\epsilon = \beta\hbar\omega_0, \quad \beta = 1/kT, \tag{1.28}$$

where $\hbar\omega_0$ is the ground state energy, while the higher levels are integer multiples proportional to the colength of the partition representing the ramification type of a branch point; i.e., the degree of degeneration of the sheets

$$\epsilon(\mu) = \ell^*(\mu)\epsilon_0. \tag{1.29}$$

The corresponding weight generating function is

$$G(z) = E'(q,z) := \prod_{i=1}^{\infty}(1+q^i z) = (-zq;q)_\infty := 1 + \sum_{i=0}^{\infty} E'_i(q) z^i, \tag{1.30}$$

$$E'_i(q) := \frac{q^{\frac{1}{2}i(i+1)}}{\prod_{j=1}^{i}(1-q^j)} = \frac{q^{\frac{1}{2}i(i+1)}}{(q;q)_{i-1}}, \quad i \geq 1, \tag{1.31}$$

where

$$(z;q)_k := \prod_{j=0}^{k-1}((1-zq^j), \quad (z;q)_\infty := \prod_{j=0}^{\infty}(1-zq^j) \tag{1.32}$$

is the quantum Pochhammer symbol. This is related to the quantum dilogarithm function by

$$(1+z)E'(q,z) = e^{-\operatorname{Li}_2(q,-z)}, \quad \operatorname{Li}_2(q,z) := \sum_{k=1}^{\infty} \frac{z^k}{k(1-q^k)}. \tag{1.33}$$

We thus have

$$e_\lambda(\mathbf{c}) =: E'_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \frac{q^{\frac{1}{2}\lambda_i(\lambda_i+1)}}{\prod_{j=1}^{\lambda_i}(1-q^j)} = \prod_{i=1}^{\ell(\lambda)} \frac{q^{\frac{1}{2}\lambda_i(\lambda_i+1)}}{(q;q)_{\lambda_i-1}}. \tag{1.34}$$

The content product coefficient entering in the τ-function (1.21) for this case is

$$r_j^{(E'(q),\beta)} = \prod_{k=1}^{\infty}(1+q^k\beta j) = (-q\beta j;q)_\infty, \tag{1.35}$$

$$\begin{aligned} r_\lambda^{(E'(q),\beta)}(z) &= \prod_{k=1}^{\infty}\prod_{(i,j)\in\lambda}(1+q^k\beta(j-i)) = \prod_{(i,j)\in\lambda}(-q\beta(j-i);q)_\infty \\ &= \prod_{k=1}^{\infty}(\beta q^k)^{|\lambda|}(1/(\beta q^k))_\lambda, \end{aligned} \tag{1.36}$$

where $(x)_k$ denotes the rising Pochhammer symbol

$$(x)_k := \prod_{j=1}^{k}(x+j-1) \tag{1.37}$$

and

$$(x)_\lambda := \prod_{i=1}^{\ell(\lambda)}(x-i+1)_{\lambda_i} = \prod_{i=1}^{\ell(\mu)}\prod_{j=1}^{\lambda_i}(x+j-i). \tag{1.38}$$

Making the substitutions (1.27), the weights entering in (1.11) evaluate to

$$\begin{aligned} W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}) &:= m_\lambda(q,q^q,\dots) \\ &= \frac{1}{|\operatorname{aut}(\lambda)|}\sum_{\sigma\in S_k}\frac{q^{k\ell^*(\mu^{(\sigma(1))})}\cdots q^{\ell^*(\mu^{(\sigma(k))})}}{(1-q^{\ell^*(\mu^{(\sigma(1))})})\cdots(1-q^{\ell^*(\mu^{(\sigma(1))}}\cdots q^{\ell^*(\mu^{(\sigma(k))})})} \\ &= \frac{1}{|\operatorname{aut}(\lambda)|}\sum_{\sigma\in S_k}\frac{1}{(q^{-\ell^*(\mu^{(\sigma(1))})}-1)\cdots(q^{-\ell^*(\mu^{(\sigma(1))})}\cdots q^{-\ell^*(\mu^{(\sigma(k))})}-1)}, \end{aligned} \tag{1.39}$$

The (unnormalized) weighted Hurwitz numbers therefore become

$$H^d_{E'(q)}(\mu,\nu) := \sum_{k=0}^{\infty}\sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(k)}\\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}} W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)})H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu). \tag{1.40}$$

Another variant on the weight generating function for quantum Hurwitz numbers consists of choosing the parameters $\mathbf{c}=(c_1,c_2,\dots)$ in (1.2) to be

$$c_i := q^{i-1}, \tag{1.41}$$

which gives

$$G(z) = E(q,z) := (-qz;q)_\infty = \sum_{i=0}^{\infty}E_i(q)z^i, \tag{1.42}$$

$$E_i(q) := \frac{q^{\frac{1}{2}i(i-1)}}{(q;q)_{i-1}}, \quad i\geq 1. \tag{1.43}$$

This is related to the quantum dilogarithm function by

$$E(q,z) = e^{-\operatorname{Li}_2(q,-z)}, \quad \operatorname{Li}_2(q,z) := \sum_{k=1}^{\infty} \frac{z^k}{k(1-q^k)}. \tag{1.44}$$

We thus have

$$E_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \frac{q^{\frac{1}{2}\lambda_i(\lambda_i-1)}}{(q;q)_{\lambda_i-1}} \tag{1.45}$$

The content product coefficient entering in the τ-function (1.21) for this case is

$$r_j^{E(q)}(z) = \prod_{k=0}^{\infty}(1+q^k zj) = (-zj;q)_\infty, \tag{1.46}$$

$$\begin{aligned} r_\lambda^{E(q)}(z) &= \prod_{k=0}^{\infty} \prod_{(i,j)\in\lambda} (1+q^k z(j-i)) = \prod_{(i,j)\in\lambda} (-z(j-i);q)_\infty \\ &= \prod_{k=0}^{\infty} (zq^k)^{|\lambda|}(1/(zq^k))_\lambda \end{aligned} \tag{1.47}$$

The weights entering in (1.11) evaluate to

$$\begin{aligned} W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)}) &:= \frac{1}{|\operatorname{aut}(\lambda)|} \sum_{\sigma\in S_k} \sum_{0\le i_1<\dots<i_k}^{\infty} q^{i_1\ell^*(\mu^{(\sigma(1))})}\cdots q^{i_k\ell^*(\mu^{(\sigma(k))})} \\ &= \frac{1}{|\operatorname{aut}(\lambda)|} \sum_{\sigma\in S_k} \frac{q^{(k-1)\ell^*(\mu^{(\sigma(1))})}\cdots q^{\ell^*(\mu^{(\sigma(k-1))})}}{(1-q^{\ell^*(\mu^{(\sigma(1))})})\cdots(1-q^{\ell^*(\mu^{(\sigma(1))})}\cdots q^{\ell^*(\mu^{(\sigma(k))})})}, \end{aligned} \tag{1.48}$$

and the weighted Hurwitz numbers therefore become

$$H^d_{E(q)}(\mu,\nu) := \sum_{k=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}}' W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu). \tag{1.49}$$

A third variant on the weight generating function for quantum Hurwitz numbers consists of choosing it of the form (1.4) with parameters $\mathbf{c} = (c_1, c_2, \dots)$ again chosen as in (1.41).

This gives

$$\tilde{G}(z) = H(q,z) := \prod_{k=0}^{\infty}(1-q^k z)^{-1} = \frac{1}{(-z;q)_\infty} = e^{\mathrm{Li}_2(q,z)} = \sum_{i=0}^{\infty} H_i(q) z^i, \tag{1.50}$$

$$H_i(q) := \frac{1}{(q;q)_{i-1}}, \quad H_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \frac{1}{(q;q)_{\lambda_i - 1}} \tag{1.51}$$

$$H(q,\mathcal{J}) = \prod_{k=0}^{\infty}\prod_{a=1}^{n}(1-q^k z \mathcal{J}_a)^{-1}, \tag{1.52}$$

$$r_j^{H(q)}(z) = \prod_{k=0}^{\infty}(1-q^k z j)^{-1} = \frac{1}{(-z;q)_\infty}, \tag{1.53}$$

$$\begin{aligned} r_\lambda^{H(q)}(z) &= \prod_{k=0}^{\infty}\prod_{(i,j)\in\lambda}(1-q^k z(j-i))^{-1} = \prod_{(i,j)\in\lambda}\frac{1}{(-z(j-i);q)_\infty} \\ &= \prod_{k=0}^{\infty}(-1/(zq^k))^{-|\lambda|}(-1/(zq^k))_\lambda^{-1}. \end{aligned} \tag{1.54}$$

The weights entering in (1.11) then evaluate to

$$\begin{aligned} W_{H(q)}(\mu^{(1)},\ldots,\mu^{(k)}) &:= \frac{(-1)^{\ell^*(\lambda)}}{|\operatorname{aut}(\lambda)|}\sum_{\sigma\in S_k}\sum_{0\le i_1\le\cdots\le i_k}^{\infty} q^{i_1\ell^*(\mu^{(\sigma(1))})}\cdots q^{i_k\ell^*(\mu^{(\sigma(k))})} \\ &= \frac{(-1)^{\ell^*(\lambda)}}{|\operatorname{aut}(\lambda)|}\sum_{\sigma\in S_k}\frac{1}{(1-q^{\ell^*(\mu^{(\sigma(1))})})\cdots(1-q^{\ell^*(\mu^{(\sigma(1))})}\cdots q^{\ell^*(\mu^{(\sigma(k))})})} \end{aligned} \tag{1.55}$$

and the weighted Hurwitz numbers become

$$H^d_{H(q)}(\mu,\nu) := \sum_{k=0}^{\infty}\sum_{\substack{\mu^{(1)},\ldots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}}{}' W_{H(q)}(\mu^{(1)},\ldots,\mu^{(k)}) H(\mu^{(1)},\ldots,\mu^{(k)},\mu,\nu). \tag{1.56}$$

1.5 Classical limit of the generating function for quantum Hurwitz numbers

Choosing

$$q = e^{-\epsilon} \tag{1.57}$$

with ϵ a small positive number, and taking the limit $\epsilon \to 0^+$ of the scaled quantum dilogarithm function $Li_2(q,\epsilon z)$ gives

$$\lim_{\epsilon\to 0^+} Li_2(q,\epsilon z) = z. \tag{1.58}$$

It follows that all three generating functions $E(q,z), E'(q,z)$ and $H(q,z)$ have as scaled limits the generating function for the Okounkov-Pandharipande simple (single and double) Hurwitz numbers

$$\lim_{\epsilon\to 0^+} E(q,z) = \lim_{\epsilon\to 0^+} E'(q,z) = \lim_{\epsilon\to 0^+} H(q,z) = e^z \tag{1.59}$$

The corresponding scaled limit of the generating τ-functions for all three versions of quantum weighted Hurwitz numbers therefore coincides with the generating function for simple Hurwitz numbers considered in [19, 21]

$$\lim_{\epsilon\to 0^+} \tau^{E(q),\epsilon\beta)}(\mathbf{t},\mathbf{s}) = \lim_{\epsilon\to 0^+} \tau^{E'(q),\epsilon\beta)}(\mathbf{t},\mathbf{s}) = \lim_{\epsilon\to 0^+} \tau^{H(q),\epsilon\beta)}(\mathbf{t},\mathbf{s}) = \tau^{(\exp,\beta)}(\mathbf{t},\mathbf{s}). \tag{1.60}$$

Equivalently, this implies the limit

$$\lim_{\epsilon\to 0^+} \epsilon^d H^d_{E'(q=e^{-\epsilon})} = H^d_{\exp}(\mu,\nu). \tag{1.61}$$

(cf. Theorem 3.5 and Remark 3.2.)

2 Probabilistic approach to quantum Hurwitz numbers

Since $W_{E'(q)}\left(\mu^{(1)},\ldots,\mu^{(k)}\right)$ is always real, positive and normalizable, we can interpret $H^d_{E'(q)}$ in terms of an expectation. For $k\in\{1,\ldots,d\}$ consider the (finite) set of k-tuples

$$\mathfrak{M}^{(n)}_{d,k} = \left\{ \left(\mu^{(1)},\ldots,\mu^{(k)}\right) \in (\mathcal{P}_n)^k : \sum_{j=1}^{k} \ell^*\left(\mu^{(j)}\right) = d \right\} \tag{2.1}$$

and their disjoint union

$$\mathfrak{M}^{(n)}_{d} = \coprod_{k=1}^{d} \mathfrak{M}^{(n)}_{d,k}. \tag{2.2}$$

Define a measure $\theta^{(n,d)}_{E'(q)}$ on $\mathfrak{M}^{(n)}_d$ by

$$\theta^{(n,d)}_{E'(q)}\left(\left(\mu^{(1)},\ldots,\mu^{(k)}\right)\right) = \frac{1}{\widetilde{Z}^{(n,d)}_{E'(q)}} W_{E'(q)}\left(\mu^{(1)},\ldots,\mu^{(k)}\right), \tag{2.3}$$

where the *partition function* $\widetilde{Z}^{(n,d)}_{E'(q)}$ is defined so that $\theta^{(n,d)}_{E'(q)}$ is a probability measure; that is,

$$\widetilde{Z}^{(n,d)}_{E'(q)} = \sum_{k=1}^{d} \sum_{\mathfrak{M}^{(n)}_{d,k}} W_{E'(q)}\left(\mu^{(1)},\ldots,\mu^{(k)}\right). \tag{2.4}$$

We then have the expectation value

$$\langle H(\cdot,\ldots,\cdot,\mu,\nu)\rangle_{\theta^{(n,d)}_{E'(q)}} = \frac{1}{\widetilde{Z}^{(n,d)}_{E'(q)}} H^d_{E'(q)}(\mu,\nu), \tag{2.5}$$

where $\langle\cdot\rangle_{\theta^{(n,d)}_{E'(q)}}$ denotes integration with respect to the measure $\theta^{(n,d)}_q$.

Definition 2.1. For $n,d\in\mathbb{Z}_{>0}$ define the function $\Lambda^{(n)}_d\colon \mathfrak{M}^{(n)}_d \longrightarrow \mathcal{P}_d$ as follows:

$$\Lambda^{(n)}_d\colon \left(\mu^{(1)},\ldots,\mu^{(k)}\right) \longmapsto \lambda \tag{2.6}$$

where λ is the unique partition of d such that

$$\{\lambda_1,\ldots,\lambda_k\} = \left\{\ell^*\left(\mu^1\right),\ldots,\ell^*\left(\mu^{(k)}\right)\right\}. \tag{2.7}$$

The weight of the partition $\Lambda^{(n)}_d\left(\mu^{(1)},\ldots,\mu^{(k)}\right)$ is thus the sum of colengths of $\mu^{(1)},\ldots,\mu^{(k)}$. Letting $\mathcal{P}_{n,k}$ denote the set of integer partitions, the image of $\mathfrak{M}^{(n)}_{d,k}$ under $\Lambda^{(n)}_d$ is thus $\mathcal{P}_{d,k}$.

Since $W_{E'(q)}\left(\mu^{(1)},\ldots,\mu^{(k)}\right)$ depends on the partitions $\mu^{(1)},\ldots,\mu^{(k)}$ only through their colength it makes sense to consider the push-forward

$$\widetilde{\xi}^{(n,d)}_{E'(q)} = \left(\Lambda^{(n)}_d\right)_* \theta^{(n,d)}_{E'(q)} \tag{2.8}$$

of $\theta^{(n,d)}_{E'(q)}$ under $\Lambda^{(n)}_d$ (as a measure on $\mathcal{P}_d$). Let $p(n,k) := |\mathcal{P}_{n,k}|$ denote the cardinality of $\mathcal{P}_{n,k}$ and observe that, for any $\lambda\in\mathcal{P}_d$,

$$\left|\left(\Lambda^{(n)}_d\right)^{-1}(\lambda)\right| = \prod_{j=1}^{\ell(\lambda)} p\left(n,n-\lambda_j\right). \tag{2.9}$$

Therefore

$$\widetilde{\xi}^{(n,d)}_{E'(q)}(\lambda) = \frac{1}{\widetilde{Z}^{(n,d)}_{E'(q)}}\left(\prod_{j=1}^{\ell(\lambda)} p\left(n,n-\lambda_j\right)\right) w_{E'(q)}(\lambda) \tag{2.10}$$

where $w_{E'(q)}$ is defined as the weight function $w_{E'(q)}\colon \mathcal{P}_d \longrightarrow [0,\infty)$ satisfying

$$w_{E'(q)}(\lambda) = \frac{\Phi_{E'(q)}(\lambda_1,\ldots,\lambda_{\ell(\lambda)})}{|\mathrm{aut}(\lambda)|} \tag{2.11}$$

with $\Phi\colon \coprod_{m\in\mathbb{N}}\mathbb{R}^m \longrightarrow \mathbb{R}$ defined by

$$\Phi_{E'(q)}\left(x_1,\ldots,x_m\right) = \sum_{\sigma\in S_m}\prod_{j=1}^{m}\left(q^{-\sum_{i=1}^{j} x_{\sigma(i)}}-1\right)^{-1}. \tag{2.12}$$

Lemma 2.1. *For any $n, \ell \in \mathbb{N}$ with $n \geq 2\ell$ we have*

$$p(n, n-\ell) = p(\ell). \tag{2.13}$$

The proof of this lemma is given in Section 4. From now on we always assume that $n \geq 2d$. We also denote

$$p(\lambda) = \prod_{j=1}^{\ell(\lambda)} p\left(\lambda_j\right). \tag{2.14}$$

From the above discussion and Lemma 2.1 we have the following result. For $d \in \mathbb{Z}_{\geq 0}$ and $q \in (0,1)$ let

$$Z^{(d)}_{E'(q)} := \sum_{\lambda \in \mathcal{P}_d} p(\lambda)\, w_{E'(q)}(\lambda) \tag{2.15}$$

and define a probability measure on $\mathcal{P}_d$ by

$$\xi^{(d)}_{E'(q)}(\lambda) := \frac{1}{Z^{(d)}_{E'(q)}}\, p\left(\lambda\right) w_{E'(q)}(\lambda) \qquad \forall\, \lambda \in \mathcal{P}_d. \tag{2.16}$$

Proposition 2.2. *Let $n, d \in \mathbb{Z}_{>0}$ with $n \geq 2d$. Then*

1. *The partition function $\widetilde{Z}^{(n,d)}_{E'(q)}$ does not depend on n:*

$$\widetilde{Z}^{(n,d)}_{E'(q)} = Z^{(d)}_{E'(q)} \tag{2.17}$$

2. *The probability measure $\widetilde{\xi}^{(n,d)}_{E'(q)}$ does not depend on n: for any $\lambda \in \mathcal{P}_d$,*

$$\widetilde{\xi}^{(n,d)}_{E'(q)}(\lambda) = \xi^{(d)}_{E'(q)}(\lambda) \tag{2.18}$$

We conclude this section by explaining how this transfers to the other two functions $E(q)$ and $H(q)$.

Definition 2.2. For $G(q) \in \{E(q), H(q)\}$ define the following:

1. $w_{G(q)}(\lambda) = \frac{1}{\mathrm{aut}(\lambda)}$

3 Classical limits and asymptotic expansion

In this section we state our asymptotic results for $q \longrightarrow 1^-$; all proofs are given in the following section.

3.1 Classical limit

We begin by stating the classical limits.

Definition 3.1. The *Dirac measure* δ_x at $x \in S$ on a measurable space (S, Σ) is defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

for all $A \in \Sigma$.

Recall that $m_k(\lambda)$ denotes the number of blocks of size k in a partition λ. We will use the following notation:

Definition 3.2. For $\lambda \in \mathcal{P}_d$,

$$\lambda = \left(1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \ldots\right). \tag{3.2}$$

We will also use the following notation for partitions with at most two different part lengths, namely $\ell \in \mathbb{Z}_{>0}$ and 1: we write

$$\boldsymbol{\ell}_n^m = \left(1^{n-m}, \ell^m\right), \tag{3.3}$$

When the weight of the partition is clear from context we simply write $\boldsymbol{\ell}^m := \boldsymbol{\ell}_n^m$. When $m = 1$ we write $\boldsymbol{\ell}_n := \boldsymbol{\ell}_n^1$, or simply $\boldsymbol{\ell}$.

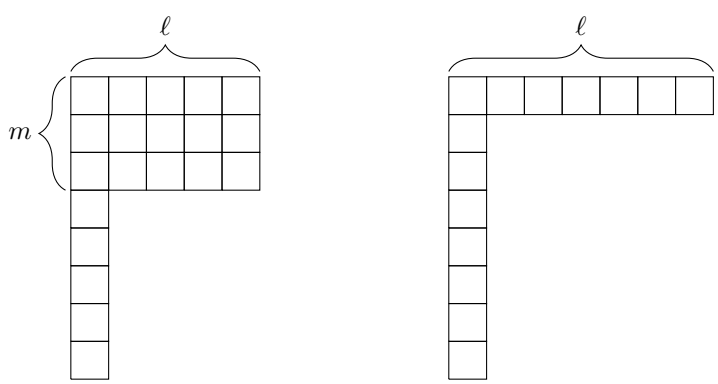

Figure 1: The partitions $\boldsymbol{\ell}_n^m = \mathbf{5}_{20}^3$ (left) and $\boldsymbol{\ell} = \mathbf{7}$ (with $m = 1$ and $n = 14$ suppressed from the notation, right)

THEOREM 3.1. *Let $d \in \mathbb{Z}_{>0}$. As $q \longrightarrow 1^-$, the sequence of measures $\xi_q^{(d)}$ on $\mathcal{P}_d$ converges weakly to the Dirac measure $\delta_{(1^d)}$ at $(1^d) \in \mathcal{P}_d$.*

By the discussion in Section 2 this translates to a convergence result on $\mathfrak{M}_d^{(n)}$:

COROLLARY 3.2. *If $d \geq 2n$. Then the sequence of measures $\theta^{(n,d)}_{E'(q)}$ on $\mathfrak{M}_d^{(n)}$ converges weakly to the Dirac measure at $(\underbrace{\mathbf{2},\ldots,\mathbf{2}}_{d \text{ terms}})$ (in the notation of* (3.3)*)*

Remark 3.1. *Observe that the limiting measure in Corollary 3.2 corresponds to the Okounkov / Pandharipande measure from* (1.19)*.*

3.2 Semiclassical corrections

We now turn to semiclassical asymptotics. Throughout we set $q = e^{-\epsilon}$ and let $\epsilon \longrightarrow 0^+$. We begin by giving the asymptotic expansion for each weight.

THEOREM 3.3. *For any $\lambda \in \mathcal{P}_d$ we have*

$$\epsilon^{-\ell(\lambda)} w_{E'(e^{-\epsilon})}(\lambda) = \sum_{\sigma \in S_{\ell(\lambda)}} \frac{1}{\prod_{j=1}^{\ell(\lambda)} \sum_{i=1}^{j} \lambda_{\sigma(i)}} - \frac{\epsilon}{2} \sum_{\sigma \in S_{\ell(\lambda)}} \sum_{r=1}^{\ell(\lambda)} \frac{\sum_{i=1}^{r} \lambda_{\sigma(i)}}{\prod_{j=1}^{\ell(\lambda)} \sum_{i=1}^{j} \lambda_{\sigma(i)}} + O\left(\epsilon^2\right). \quad (3.4)$$

From this result it oe can deduce the following semiclassical expansion for the partition function:

THEOREM 3.4. *For $d \in \mathbb{Z}_{\geq 0}$ and $q = e^{-\epsilon}$ we have*

$$\epsilon^d \, Z^{(d)}_{E'(e^{-\epsilon})} = \frac{1}{d!} + \epsilon \, \frac{3-d}{4(d-1)!} + O\left(\epsilon^2\right). \quad (3.5)$$

We also obtain a convergence result for the weighted Hurwitz numbers. (Recall our notation for partitions from Definition 3.2.)

THEOREM 3.5. *For any $\mu, \nu \in \mathcal{P}_n$ we have*

$$\epsilon^d \; H^d_{E'(q)}(\mu,\nu) = \frac{1}{d!} H(\underbrace{\mathbf{2},\ldots,\mathbf{2}}_{d \text{ times}}, \mu, \nu) \quad (3.6)$$

$$+ \frac{\epsilon}{(d-1)!} \Bigg[H(\underbrace{\mathbf{2},\ldots,\mathbf{2}}_{d-1 \text{ times}}, \mathbf{3}, \mu, \nu)) + H(\underbrace{\mathbf{2},\ldots,\mathbf{2}}_{d-1 \text{ times}}, \mathbf{2^2}, \mu, \nu)) \quad (3.7)$$

$$- \frac{(d+1)}{4} H(\underbrace{\mathbf{2},\ldots,\mathbf{2}}_{d \text{ times}}, \mu, \nu) \Bigg] + O\left(\epsilon^2\right). \quad (3.8)$$

Remark 3.2. *In particular we have*

$$\lim_{\epsilon \to 0} H_{E'(e^{-\epsilon})}(\mu,\nu) = \frac{1}{d!} H_{\exp}(\mu,\nu), \quad (3.9)$$

which agrees with (1.61)*.*

4 Proofs

Proof of Lemma 2.1. Consider the function $f\colon \mathcal{P}_{n,n-\ell} \longrightarrow \mathcal{P}_\ell$ defined as follows. Let $\lambda \in \mathcal{P}_{n,n-\ell}$, then the first column of the Young diagram of λ has $n-\ell$ boxes. Remove these to obtain a partition $\nu := f(\lambda)$ of ℓ. This function has an inverse: for $\nu \in \mathcal{P}_\ell$ simply add a new column with $n-\ell$ to the left of the Young diagram of ν. Since $n-\ell \geq \ell$ by assumption the result is the Young diagram of an integer partition $\lambda := f^{-1}(\nu)$: it is easy to see that $\lambda \in \mathcal{P}_n$ and that $\ell(\lambda) = n-\ell$. □

The proofs of the results stated in Section 3 all rely on the following asymptotic expansion of $\Phi_{e^{-\epsilon}}$ as $\epsilon \longrightarrow 0$:

LEMMA 4.1. *let $x_1, \ldots, x_m \in \mathbb{Z}_{>0}$. Then, as $\epsilon \longrightarrow 0$*

$$\Phi_{E'(e^{-\epsilon})}(x_1, \ldots, x_m) = \epsilon^{-m} \sum_{\sigma \in S_m} \left[\frac{1}{\prod_{j=1}^m \sum_{i=1}^j x_{\sigma(i)}} - \frac{\epsilon}{2} \sum_{r=1}^m \frac{\sum_{i=1}^r x_{\sigma(i)}}{\prod_{j=1}^m \sum_{i=1}^j x_{\sigma(i)}} \right] + O\left(\epsilon^{2-m}\right) \tag{4.1}$$

Proof. A direct computation yields

$$\begin{aligned}
\Phi_{E'(e^{-\epsilon})}(x_1, \ldots, x_m) &= \sum_{\sigma \in S_m} \prod_{j=1}^m \left(e^{\epsilon \sum_{i=1}^j x_{\sigma(i)}} - 1 \right)^{-1} && (4.2)\\
&= \sum_{\sigma \in S_m} \prod_{j=1}^m \frac{\epsilon^{-1}}{\sum_{i=1}^j x_{\sigma(i)}} \left(1 + \frac{\epsilon}{2} \sum_{i=1}^j x_{\sigma(i)} + O(\epsilon^2) \right)^{-1} && (4.3)\\
&= \epsilon^{-m} \sum_{\sigma \in S_m} \prod_{j=1}^m \left(\frac{1}{\sum_{i=1}^j x_{\sigma(i)}} - \frac{\epsilon}{2} + O(\epsilon^2) \right) && (4.4)\\
&= \epsilon^{-m} \sum_{\sigma \in S_m} \left(\prod_{j=1}^m \frac{1}{\sum_{i=1}^j x_{\sigma(i)}} - \frac{\epsilon}{2} \sum_{j=1}^m \frac{\sum_{i=1}^j x_{\sigma(i)}}{\prod_{r=1}^m \sum_{i=1}^r x_{\sigma(i)}} + O(\epsilon^2) \right) && (4.5)
\end{aligned}$$

as claimed. □

By considering the highest order terms it follows immediately that, letting $d = \sum_{r=1}^m x_r \geq m$,

$$\lim_{\epsilon \to 0} \epsilon^d \Phi_{E'(e^{-\epsilon})}(x_1, \ldots, x_m) = \begin{cases} \frac{1}{d!} & \text{if } d = m \\ 0 & \text{if } d > m. \end{cases} \tag{4.6}$$

This completes the proof of Theorem 3.1 and hence also Corollary 3.2. Setting $q = e^{-\epsilon}$ and considering additionally the terms of order ϵ^{1-d} gives Theorem 3.3.
Moreover we obtain the following intermediate result:

Proposition 4.2. *For any function* $f\colon \mathfrak{M}_d^{(n)} \longrightarrow \mathbb{R}$,

$$\epsilon^d \sum_{\mathfrak{M}_d^{(n)}} f\left(\mu^{(1)},\dots,\mu^{(k)}\right) W_{E'(q)}\left(\mu^{(1)},\dots,\mu^{(k)}\right) = \frac{1}{d!} f(\underbrace{\mathbf{2},\dots,\mathbf{2}}_{d\ times}) \tag{4.7}$$

$$+ \frac{\epsilon}{(d-1)!}\left[f\left(\underbrace{\mathbf{2},\dots,\mathbf{2}}_{d-1\ times},\mathbf{3}\right) + f\left(\underbrace{\mathbf{2},\dots,\mathbf{2}}_{d-1\ times},\mathbf{2^2}\right) - \frac{d+1}{4} f(\underbrace{\mathbf{2},\dots,\mathbf{2}}_{d\ times})\right] \tag{4.8}$$

$$+ O\left(\epsilon^2\right) \tag{4.9}$$

where we recall that $\mathbf{2} = (1^{n-1},2)$ *and* $\mathbf{3} = (1^{n-3},3)$ *and* $\mathbf{2^2} = (1^{n-4},2^2)$.

Proof. From Lemma 4.1 it follows that $w_{E'(q)}(\lambda)$ contributes terms of order $\epsilon^{-\ell(\lambda)}$ and lower. Thus the only terms in (4.7) that are not $o(\epsilon^{-d+1})$ correspond to elements $(\mu^{(1)},\dots,\mu^{(k)})$ of $\mathfrak{M}_d^{(n)}$ such that $\lambda = \Lambda_d^{(n)}(\mu^{(1)},\dots,\mu^{(k)})$ has length d or $d-1$, i.e. $\lambda \in \{\mathbf{1},\mathbf{2}\}$. (recalling once more the notation from Definition 3.2). Therefore,

$$\sum_{\mathfrak{M}_d^{(n)}} f\left(\mu^{(1)},\dots,\mu^{(k)}\right) W_{E'(q)}\left(\mu^{(1)},\dots,\mu^{(k)}\right) = \frac{p(\mathbf{1})}{|\mathrm{aut}(\mathbf{1})|}\Phi_{e^{-\epsilon}}(1,\dots,1) \sum_{\Lambda_d^{-1}(1^d)} f\left(\mu^{(1)},\dots,\mu^{(k)}\right) \tag{4.10}$$

$$+ \frac{p(\mathbf{2})}{|\mathrm{aut}(\mathbf{2})|}\Phi_{e^{-\epsilon}}(2,1,\dots,1) \sum_{\Lambda_d^{-1}(\mathbf{2})} f\left(\mu^{(1)},\dots,\mu^{(k)}\right) + O(\epsilon^{2-d}). \tag{4.11}$$

We first deal with the term in (4.10): $p(\mathbf{1}) = 1$ and $\mathrm{aut}(\mathbf{1}) = d!$. Further, by Lemma 4.1,

$$\Phi_{E'(q)}(\underbrace{1,\dots,1}_{d\ \text{times}}) = \epsilon^{-d}\sum_{\sigma\in S_d}\left(\frac{1}{d!} - \frac{\epsilon}{2}\frac{d(d+1)/2}{d!} + O\left(\epsilon^2\right)\right) = \epsilon^{-d}\left(1 - \epsilon\,\frac{d(d+1)}{4}\right) + O\left(\epsilon^{-d+2}\right). \tag{4.12}$$

For the terms in (4.11): $p(\mathbf{2}) = p(2) = 2$ and $\mathrm{aut}(\mathbf{2}) = (d-1)!$. This time we only need the first order approximation of Lemma 4.1, and we obtain

$$\Phi_{E'(e^{-\epsilon})}(2,\underbrace{1,\dots,1}_{d-2\ \text{times}}) = \epsilon^{-d+1}\sum_{\sigma\in S_{d-1}}\left(\prod_{j=1}^{d-1}\sum_{i=1}^{j} x_{\sigma(i)}\right)^{-1}\Bigg|_{x=(2,1,\dots,1)} + O(\epsilon^{-d+2}) \tag{4.13}$$

If $x = (2,1,\dots,1)$ then we have, for $j \in \{1,\dots,d-1\}$ and $\sigma \in S_{d-1}$,

$$\sum_{i=1}^{j} x_{\sigma(i)} = \begin{cases} j+1 & \text{if } j < \sigma^{-1}(1) \\ j & \text{otherwise,} \end{cases} \tag{4.14}$$

and therefore

$$\sum_{\sigma\in S_{d-1}}\left(\prod_{j=1}^{d-1}\sum_{i=1}^{j}x_{\sigma(i)}\right)^{-1}\Bigg|_{x=(2,1,\ldots,1)} = \sum_{\sigma\in S_{d-1}}\left(\prod_{j=1}^{\sigma^{-1}(1)-1} j\right)^{-1}\left(\prod_{j=\sigma^{-1}(1)}^{d-1}(j+1)\right)^{-1} \tag{4.15}$$

$$= \sum_{\sigma\in S_{d-1}}\frac{\sigma^{-1}(1)}{d!} = \frac{1}{d!}\sum_{r=1}^{d-1}\sum_{\sigma^{-1}(1)=r} r \tag{4.16}$$

$$= \frac{(d-2)!}{d!}\cdot\frac{d(d-1)}{2} = \frac{1}{2} \tag{4.17}$$

It follows that

$$\Phi_{E'(e^{-\epsilon})}(2,\underbrace{1,\ldots,1}_{d-2\text{ times}}) = \frac{1}{2}\epsilon^{-d+1} + O(\epsilon^{-d+2}). \tag{4.18}$$

Substituting (4.12) and (4.18) into (4.10) and (4.11) gives

$$\sum_{\mathfrak{M}_d^{(n)}} f\left(\mu^{(1)},\ldots,\mu^{(k)}\right)W_{E'(q)}\left(\mu^{(1)},\ldots,\mu^{(k)}\right) = \frac{\epsilon^{-d}}{d!}\left(1-\epsilon\frac{d(d+1)}{4}\right)f\left(\underbrace{\mathbf{2},\ldots,\mathbf{2}}_{d\text{ times}}\right) \tag{4.19}$$

$$+\frac{\epsilon^{-d+1}}{(d-1)!}\left(f(\mathbf{3},\underbrace{\mathbf{2},\ldots,\mathbf{2}}_{d-2\text{ times}}) + f(\mathbf{2}^2,\underbrace{\mathbf{2},\ldots,\mathbf{2}}_{d-2\text{ times}})\right) + O(\epsilon^{-d+2}) \tag{4.20}$$

as required. □

Choosing $f\left(\mu^{(1)},\ldots,\mu^{(k)}\right) = H\left(\mu^{(1)},\ldots,\mu^{(k)},\mu,\nu\right)$ gives Theorem 3.5. On the other hand by setting $f\left(\mu^{(1)},\ldots,\mu^{(k)}\right) = 1$ we obtain

$$Z_{e^{-\epsilon}}^{(d)} = \frac{\epsilon^{-d}}{d!} + \epsilon^{1-d}\left(\frac{1}{(d-1)!} - \frac{d(d+1)}{4d!}\right) \tag{4.21}$$

$$= \frac{\epsilon^{-d}}{d!} + \epsilon^{1-d}\frac{3-d}{4(d-1)!} + O\left(\epsilon^{2-d}\right) \tag{4.22}$$

and we have proved Proposition 3.4.

Acknowledgements. The authors would like to thank G. Borot and A.Yu. Orlov for helpful discussions. The work of JH was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds de recherche du Québec, Nature et technologies (FRQNT). JO was partially supported by a CRM-ISM postdoctoral fellowship.

References

[1] J. Ambjørn and L. Chekhov, "The matrix model for dessins d'enfants", *Ann. Inst. Henri Poincaré, Comb. Phys. Interact.* **1**, 337-361 (2014). 2

[2] J. Ambjørn and L. Chekhov, "A matrix model for hypergeometric Hurwitz numbers", *Theor. Math. Phys.* **181**, 1486-1498 (2014). 2

[3] A. Alexandrov, A. Mironov, A. Morozov and S. Natanzon, "Integrability of Hurwitz Partition Functions. I. Summary," *J. Phys. A* **45**, 045209 (2012). 2

[4] A. Alexandrov, A. Mironov, A. Morozov and S. Natanzon, " On KP-integrable Hurwitz functions", arXiv:1405.1395; *JHEP* doi:10.1007/JHEP11(2014)080. 2

[5] G. Frobenius, "Über die Charaktere der symmetrischen Gruppe", *Sitzber. Akad. Wiss., Berlin*, 516-534 (1900). Gesammelte Abhandlung III, 148-166. 3

[6] G. Frobenius, "Über die Charakterische Einheiten der symmetrischen Gruppe", *Sitzber. Akad. Wiss., Berlin*, 328-358 (1903). Gesammelte Abhandlung III, 244-274. 3

[7] I. P. Goulden, M. Guay-Paquet and J. Novak, "Monotone Hurwitz numbers and the HCIZ Integral", *Ann. Math. Blaise Pascal* **21** 71-99 (2014). 2

[8] M. Guay-Paquet and J. Harnad, "2D Toda τ-functions as combinatorial generating functions", *Lett. Math. Phys.* **105** Page 827-852 (2015). 2, 4, 5

[9] M. Guay-Paquet and J. Harnad, "Generating functions for weighted Hurwitz numbers", arXiv:1408.6766. 2, 4, 5, 6

[10] J. Harnad, "Weighted Hurwitz numbers and hypergeometric τ-functions: an overview", *Proc. Symp. Pure Math.* **93**: 289-333 (2016). arXiv:1504.03408 . 2, 4, 5

[11] J. Harnad, "Quantum Hurwitz numbers and Macdonald polynomials", *J. Math. Phys.* (in press, 2016). arXiv:1504.03311. 6

[12] J. Harnad and A. Yu. Orlov, "Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths", *Commun. Math. Phys.* **338**, 267-284 (2015). 2, 4, 5

[13] A. Hurwitz, "Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten", *Math. Ann.* **39**, 1-61 (1891); Mathematische Werke I, 321-384. 2

[14] A. Hurwitz, "Über die Anzahl der Riemann'sche Flächen mit gegebenen Verzweigungspunkten", *Math. Ann.* **55**, 53-66 (1902); Mathematische Werke I, 42-505. 2

[15] M. Kazarian and P. Zograf, "Virasoro constraints and topological recursion for Grothendieck's dessin counting", *Lett. Math. Phys.* **105** 1057-1084 (2015). 2

[16] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, "Generalized Kazakov-Migdal-Kontsevich Model: group theory aspects", *Int. J. Mod. Phys.* **A10** 2015 (1995). 2

[17] S. K. Lando and A.K. Zvonkin *Graphs on Surfaces and their Applications*, Encyclopaedia of Mathematical Sciences, Volume **141**, with appendix by D. Zagier, Springer, N.Y. (2004). 3

[18] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, (1995). 4, 5

[19] A. Okounkov, "Toda equations for Hurwitz numbers", *Math. Res. Lett.* **7**, 447–453 (2000). 1, 2, 4, 6, 10

[20] A. Yu. Orlov and D. M. Scherbin, "Hypergeometric solutions of soliton equations", *Theoretical and Mathematical Physics* **128**, 906-926 (2001). 2

[21] R. Pandharipande, "The Toda Equations and the Gromov-Witten Theory of the Riemann Sphere", *Lett. Math. Phys.* **53**, 59-74 (2000). 1, 2, 4, 10

[22] I. Schur "Neue Begründung' der Theorie der Gruppencharaktere", *Sitzber. Akad. Wiss., Berlin*, 406-432 (1905). 3

[23] P. Zograf, "Enumeration of Grothendieck's dessins and KP hierarchy", *Int. Math. Res. Notices* **2015**, No. 24, 13533-13544 (2015). 2