SPECTRAL APPROXIMATIONS OF DIRICHLET PROBLEM FOR POLYHARMONIC OPERATOR AND DECOMPOSITION TECHNIQUES

BENAISSA LAKHDAR AND DAILI NOUREDDINE

ABSTRACT. In this work, we describe and generalize a spectral method coupled with a variational decomposition technique introduced in ([2], [3]). We prove that a solution of Dirichlet problem for polyharmonic operator can be reduced to a solution of a system of Dirichlet problem for an harmonic operator. The linear system, also obtained, can be solved by spectral method. To decompose this problem, we use Babuška method ([1]) introduced in the case of approximation of second-order Dirichlet problem. By this formulation we obtain optimal error estimates.

1. INTRODUCTION

Let Ω be an open bounded domain of $\mathbb{R}^d$ ($d = 1, 2, 3$ in practice) of smooth boundary $Fr(\Omega)$. Consider the following problem :

$$(P_0) \qquad \begin{cases} \Delta^p u = f & \text{in } \Omega, \\ \Delta^j u = 0 & \text{on } Fr(\Omega),\ j = 0, 1, ..., \left[\frac{p-1}{2}\right], \\ \dfrac{\partial \Delta^j u}{\partial \eta} = 0 & \text{on } Fr(\Omega),\ j = 0, 1, ..., \left[\frac{p-2}{2}\right]. \end{cases}$$

Date: August 15, 2015.

1991 *Mathematics Subject Classification.* (2010) Primary 31B30, 31B20 ; Secondary 65N35, 65N30 .

Key words and phrases. Polyharmonic problem, spectral approximations, error estimations and approximations.

This paper is in final form and no version of it will be submitted for publication elsewhere.

The problem (P_0) can be written in the form

$$(P_1) \qquad \begin{cases} \Delta^p u = f \quad \text{in } \Omega, \\ \\ \dfrac{\partial^j \Delta^{\left[\frac{i}{2}\right]} u}{\partial \eta^j} = 0 \quad \text{on } Fr(\Omega),\ 0 \leq i \leq p-1, \end{cases}$$

where [.] denotes the integer part, $f \in L^2(\Omega)$, $\dfrac{\partial^j \Delta^{\left[\frac{i}{2}\right]} u}{\partial \eta^j} \in H^{p-i-\frac{1}{2}}(Fr(\Omega))$ and

$$j = \begin{cases} 0 \text{ if } i \text{ is even} \\ \\ 1 \text{ if } i \text{ is odd.} \end{cases}$$

For $p = 1$, (P_0) can be reduced to Dirichlet problem for harmonic operator (see [2]).

For $p = 2$, (P_0) can be reduced to Dirichlet problem for biharmonic operator (see [3]).

In this work, we describe and generalize a spectral method coupled with a variational decomposition technique introduced in ([2], [3]). We prove that a solution of Dirichlet problem for polyharmonic operator can be reduced to a solution of a system of Dirichlet problem for an harmonic operator. The linear system, also obtained, can be solved by spectral method. To decompose this problem, we use Babuška method ([1]) introduced in the case of approximation of second-order Dirichlet problem. The idea is to introduce a family of solutions $u(\sigma)$, where σ is a function defined on $Fr(\Omega)$ by

$$\sigma = \frac{\partial u}{\partial \eta} + \alpha u.$$

By this formulation we obtain optimal error estimates. Numerically, an approximation σ_M of σ is obtained as a fast limit of convergent sequence of solutions of problems with fixed conditions.

To study a problem (P_0) we shall require some definitions and preliminary results.

2. Study of Problem $(\mathbf{P}_0)$

Proposition 2.1. *For all real s, a mapping* $|||.||| : H^s(\Omega) \longrightarrow \mathbb{R}^+$ *defined for all function in* $H^s(\Omega)$ *and its trace in* $H^{s-\frac{1}{2}}(Fr(\Omega))$ *by*

$$|||u|||_{H^s(\Omega)} = \|u\|_{H^s(\Omega)} + |u|_{H^{s-\frac{1}{2}}(Fr(\Omega))}$$

is a norm on $H^s(\Omega)$.

Proposition 2.2. *For all real s, a mapping* $|.|_{\overrightarrow{H^s}(Fr(\Omega))} : \overrightarrow{H^s}(Fr(\Omega)) \longrightarrow \mathbb{R}^+$ *defined by*

$$|\overrightarrow{\sigma}|_{\overrightarrow{H^s}(Fr(\Omega))} = \sum_{j=1}^{\left[\frac{p+1}{2}\right]} |\sigma_j|_{H^{s+2(j-1)}(Fr(\Omega))},$$

where

$$\overrightarrow{H^s}(Fr(\Omega)) = H^s(Fr(\Omega)) \times H^{s+2}(Fr(\Omega)) \times ... \times H^{s+2\left[\frac{p+1}{2}\right]-2}(Fr(\Omega))$$

is a norm on $\overrightarrow{H^s}(Fr(\Omega))$*.*

Proposition 2.3. *We have*

- *for* $p = 2m$*,*

$$\int_\Omega \Delta^{2m} uvdx = \int_\Omega \Delta^m u \Delta^m vdx$$

$$+\sum_{k=1}^{m} \int_{Fr(\Omega)} \left(\frac{\partial \Delta^{2m-k} u}{\partial \eta} \Delta^{k-1} v - \Delta^{2m-k} u \frac{\partial \Delta^{k-1} v}{\partial \eta}\right) d\sigma;$$

- *for* $p = 2m+1$*,*

$$\int_\Omega \Delta^{2m+1} uvdx = -\int_\Omega \nabla \Delta^m u \nabla \Delta^m vdx$$

$$+\sum_{k=1}^{m} \int_{Fr(\Omega)} \left(\frac{\partial \Delta^{2m-k+1} u}{\partial \eta} \Delta^{k-1} v - \Delta^{2m-k+1} u \frac{\partial \Delta^{k-1} v}{\partial \eta}\right) d\sigma + \int_{Fr(\Omega)} \frac{\partial \Delta^m u}{\partial \eta} \Delta^m vd\sigma.$$

Proof. It is a simple recurrence. □

- **Variational Formulation:**

Define a space in which we search a solution of our problem (P_1) as follows :

$$H^p(\Omega; \Delta^p) := \{u \in H^p(\Omega),\ \ \Delta^p u = f \ \text{ in } \Omega\}.$$

Consider next a Hilbert space of functions of $H^p(\Omega)$ which satisfy in meaning traces corresponding boundary data, namely,

$$V := \left\{ v \in H^p(\Omega),\ \frac{\partial^j \Delta^{\left[\frac{i}{2}\right]} v}{\partial \eta^j} = 0 \ \text{ on } \ Fr(\Omega),\ \ 0 \le i \le p-1 \right\} \tag{2.1}$$

A characterization of a space V is given by the following lemma :

Lemma 2.4. *([10]) A space V is identical with* $H_0^p(\Omega)$*. In other words* $H_0^p(\Omega)$ *is characterized by expression* (2.1)*.*

Proof. By trace theorems we have

$$H_0^p(\Omega) := \left\{ v \in H^p(\Omega) : v = \frac{\partial v}{\partial \eta} = ... = \frac{\partial^{p-1} v}{\partial \eta^{p-1}} = 0 \text{ on } Fr(\Omega) \right\}.$$

Then it is enough to prove the equivalence of the following conditions :

(c_1) $\frac{\partial^j \Delta^{\left[\frac{i}{2}\right]} v}{\partial \eta^j} = 0$ on $Fr(\Omega)$, $0 \leq i \leq p-1$;

(c_2) $v = Dv = ... = D^{p-1} v = 0$ on $Fr(\Omega)$;

(c_3) $v = \frac{\partial v}{\partial \eta} = ... = \frac{\partial^{p-1} v}{\partial \eta^{p-1}} = 0$ on $Fr(\Omega)$.

Now, let $v \in V = H_0^p(\Omega)$, multiplying first equation in (P_1) by v and integrating on Ω we obtain, by Proposition 2.3, the following bilinear and linear forms :

i) for $p = 2m$: by Proposition 2.3, one has

$$\int_\Omega \Delta^{2m} u v dx = \int_\Omega \Delta^m u \Delta^m v dx$$

$$+ \sum_{k=1}^{m} \int_{Fr(\Omega)} \left(\frac{\partial \Delta^{2m-k} u}{\partial \eta} \Delta^{k-1} v - \Delta^{2m-k} u \frac{\partial \Delta^{k-1}}{\partial \eta} \right) d\sigma;$$

but $v \in V$, one takes then

$$\begin{cases} \mathfrak{a}_1(u, v) = \int_\Omega \Delta^m u \Delta^m v dx \\ l(v) = \int_\Omega f v dx. \end{cases}$$

So, a problem (P_1) confines oneself to the following variational problem :

$$(P_2) \quad \begin{cases} \text{Find } u \in H^p(\Omega;\ \Delta^p) \text{ with } \frac{\partial^j \Delta^{\left[\frac{i}{2}\right]} u}{\partial \eta^j} = 0 \text{ on } Fr(\Omega), \\ \qquad\qquad\qquad\qquad 0 \leq i \leq p-1 \\ \text{such that } \mathfrak{a}_1(u, v) = l(v), \quad \forall v \in H_0^p(\Omega). \end{cases}$$

ii) For $p = 2m + 1$,

$$(P_3) \quad \begin{cases} \text{Find } u \in H^p(\Omega;\ \Delta^p) \ \text{ with } \ \frac{\partial^j \Delta^{\left[\frac{i}{2}\right]} u}{\partial \eta^j} = 0 \ \text{ on } Fr(\Omega), \\ \hfill 0 \le i \le p-1 \\ \text{such that } \mathfrak{a}_2(u,v) = l(v), \quad \forall v \in H_0^p(\Omega), \end{cases}$$

where

$$\mathfrak{a}_2(u,v) = \int_\Omega \nabla \Delta^m u \nabla \Delta^m v dx.$$

To deal problems (P_2) and (P_3) simultaneously, we rewrite our bilinear form as follows :

$$\mathfrak{a}(u,v) = \begin{cases} \int_\Omega \Delta^m u \Delta^m v dx & \text{if } \ p = 2m \\ \int_\Omega \nabla \Delta^m u \nabla \Delta^m v dx & \text{if } \ p = 2m+1. \end{cases}$$

Then, in the two cases, a problem (P_1) confines oneself to the following variational problem :

$$(P_V) \quad \begin{cases} \text{Find } u \in V \ \text{ such that} \\ \mathfrak{a}(u,v) = l(v), \quad \forall v \in V. \end{cases}$$

□

Definition 2.1. (**a**) *If* $u \in H^p(\Omega)$ *with* $\Delta^p u \in L^2(\Omega)$, *then data* $(w_i)_{i=0}^{p-1}$ *belonging to* $\prod_{i=0}^{p-1} H^{p-i-\frac{1}{2}}(Fr(\Omega))$ *are defined as traces, that is to say*

$$w_i = (\frac{\partial^j \Delta^{\frac{i}{2}} u}{\partial \eta^j})_{|Fr(\Omega)}, \ 0 \le i \le p-1.$$

(**b**) *u is called a weak solution of* (P_0) *if*
(*i*) $(\frac{\partial^j \Delta^{\frac{i}{2}} u}{\partial \eta^j})_{|Fr(\Omega)} = 0, \ 0 \le i \le p-1;$
(*ii*) $\mathfrak{a}(u,v) = l(v), \ \ \forall v \in H^p(\Omega; \Delta^p).$

We can now state the following theorem :

Theorem 2.5. *We have*
(**a**) $\mathfrak{a}(.,.)$ *is a continuous bilinear form on* $V \times V$;
(**b**) $l(.)$ *is a continuous linear form on* V;
(**c**) $\mathfrak{a}(.,.)$ *is a* V*-elliptic form.*

Proof. (**a**) and (**b**) are obvious.
(**c**) $\forall v \in \mathfrak{D}(\Omega)$, we have

$$\sum_{|\lambda|=2m} \left\| D^{\lambda} v \right\|_{L^2(\Omega)}^2 = \left\| \Delta^m v \right\|_{L^2(\Omega)}^2$$

$$\sum_{|\lambda|=2m+1} \left\| D^{\lambda} v \right\|_{L^2(\Omega)}^2 = \left\| \nabla \Delta^m v \right\|_{L^2(\Omega)}^2 .$$

But, we have

$$\mathfrak{a}(v, v) = \begin{cases} \left\| \Delta^m v \right\|_{L^2(\Omega)}^2 & \text{if } p = 2m, \\ \left\| \nabla \Delta^m v \right\|_{L^2(\Omega)}^2 & \text{if } p = 2m+1; \end{cases}$$

so a norm in $H_0^p(\Omega)$

$$|v|_{H^p(\Omega)}^2 = \sum_{|\lambda|=p} \left\| D^{\lambda} v \right\|_{L^2(\Omega)}^2$$

therefore,

$$\mathfrak{a}(v, v) = |v|_{H^p(\Omega)}^2 \geq c \left\| v \right\|_V^2 .$$

□

This theorem allows to prove existence and uniqueness of a solution of (P_0). It holds the following corollary :

Corollary 2.6. *By Lax-Milgram Lemma, there exist one and only one solution to the problem* (P_1)*.*

Theorem 2.7. *Associated problem to a variational problem given by* (P_V) *is equivalent to problem* (P_1)*.*

Proof. Consider a weak solution of the problem (P_1) and equation

$$\mathfrak{a}(u, v) = l(v), \quad \forall v \in V \tag{2.2}$$

applying the Proposition 2.3, we obtain
(2.3)

$$\mathfrak{a}(u,v)=\begin{cases}\int_\Omega \Delta^{2m}uvdx-\sum_{k=1}^{m}\int_{Fr(\Omega)}(\frac{\partial\Delta^{2m-k}u}{\partial\eta}\Delta^{k-1}v-\Delta^{2m-k}u\frac{\partial\Delta^{k-1}v}{\partial\eta})d\sigma, \\ \qquad\qquad \text{if } p=2m \\ \int_\Omega \Delta^{2m+1}uvdx \\ -\sum_{k=1}^{m}\int_{Fr(\Omega)}(\frac{\partial\Delta^{2m-k+1}u}{\partial\eta}\Delta^{k-1}v-\Delta^{2m-k+1}u\frac{\partial\Delta^{k-1}v}{\partial\eta})d\sigma \\ \qquad +\int_{Fr(\Omega)}\frac{\partial\Delta^{m}u}{\partial\eta}\Delta^{m}vd\sigma, \quad \text{if } p=2m\end{cases}$$

Using the fact that

$$\frac{\partial^j\Delta^{\frac{i}{2}}u}{\partial\eta^j}=0 \quad \text{on} \quad Fr(\Omega)$$

and substitute expression (2.2) in (2.3) we shall have

$$(\Delta^p u,v)=(f,v), \quad \forall v\in\mathfrak{D}(\Omega)$$

namely

$$\Delta^p u=f \ \text{ in } \mathfrak{D}^{'}(\Omega).$$

But as $\Delta^p u\in L^2(\Omega)$, then we have

$$\Delta^p u=f \quad \text{almost every where in } \Omega,$$

this is the first equation of (P_1). Lastly by a Definition 2.1, we have

$$(\frac{\partial^j\Delta^{\frac{i}{2}}u}{\partial\eta^j})_{|_{Fr(\Omega)}}=0, \ \ 0\le i\le p-1$$

which is the second equation of (P_1). □

3. Decomposition and Spectral Approximation of $(\mathbf{P}_0)$

3.1. Decomposition of (P_0) for $\mathbf{p=1}$.

Consider nonhomogeneous problem (P_0) for $p=1$ denoted (P_2) :

$$(P_2) \quad \begin{cases}\Delta u=f \ \text{ in } \Omega, \\ u=g \ \text{ on } Fr(\Omega).\end{cases}$$

According to ([5]), we have the following proposition :

Proposition 3.1. *([5]) We have*

$$\|u\|_{H^s(\Omega)} \leq c(\|\Delta u\|_{H^{s-2}(\Omega)} + |u|_{H^{s-\frac{1}{2}}(Fr(\Omega))}).$$

Now, consider the following two problems :

$$(P_3) \qquad \begin{cases} \Delta v = 0 \ \text{ in } \Omega, \\ \\ \alpha v + \frac{\partial v}{\partial \eta} = \sigma \ \text{ on } Fr(\Omega) \end{cases}$$

and

$$(P_4) \qquad \begin{cases} \Delta w = f \ \text{ in } \Omega, \\ \\ \alpha w + \frac{\partial w}{\partial \eta} = 0 \ \text{ on } Fr(\Omega), \end{cases}$$

where $\alpha > 0$ is a constant.

Remark 3.1. *A choice of $\alpha > 0$ ensures the existence of solutions of problems (P_3) and (P_4).*

Define a bilinear form

$$A_\alpha(\phi, \psi) := \int_\Omega \nabla\phi\nabla\psi dx + \alpha < \phi, \psi >,$$

where $< .,. >$ is a bilinear form of duality between $H^{-s}(Fr(\Omega))$ and $H^s(Fr(\Omega))$.

Proposition 3.2. *A problem (P_3) is equivalent to*

$$\begin{cases} \text{Find } v \in H^1(\Omega) \text{ such that} \\ \\ A_\alpha(v, \chi) =< \sigma, \chi >, \ \forall \chi \in H^1(\Omega). \end{cases}$$

Put $v = G\sigma$, where $G : H^s(Fr(\Omega)) \longrightarrow H^{s+\frac{3}{2}}(\Omega)$, namely

$$A_\alpha(G\sigma, \chi) =< \sigma, \chi >, \ \forall \chi \in H^1(\Omega).$$

We have then

Proposition 3.3. *([5]) Operator G is bounded in $H^s(Fr(\Omega))$. Namely*

$$\|G\theta\|_{H^{s+\frac{3}{2}}(\Omega)} \leq c\,|\theta|_{H^s(Fr(\Omega))}\,, \ \forall \theta \in H^s(Fr(\Omega)).$$

Lemma 3.4. *([5])There exist two positive constants c_0 and c_1 such that*

$$c_0\,|\theta|^2_{H^{-\frac{1}{2}}(Fr(\Omega))} \leq < G\theta, \theta > \leq c_1\,|\theta|^2_{H^{-\frac{1}{2}}(Fr(\Omega))}\,, \ \ \forall \theta \in H^{-\frac{1}{2}}(Fr(\Omega)).$$

Lemma 3.5. *([5]) For all real s, there exist two positive constants c_0 and c_1 such that*

$$c_0\,|\theta|_{H^s(Fr(\Omega))} \leq |G\theta|_{H^{s+1}(Fr(\Omega))} \leq c_1\,|\theta|_{H^s(Fr(\Omega))}\,, \ \ \forall \theta \in H^s(Fr(\Omega)).$$

Let Tf be a solution of problem (P_4), where T is an operator defined from $H^s(\Omega)$ into $H^{s+2}(\Omega)$ by

$$A_\alpha(Tf, v) = (f, v), \ \ \forall v \in H^1(\Omega).$$

Then we have

Proposition 3.6. *([5]) Operator T is bounded, namely*

$$\|Tf\|_{H^{s+2}(\Omega)} \leq c \|f\|_{H^s(\Omega)}, \ \ \forall f \in H^s(\Omega).$$

According these notations a solution u of problem (P_2) is given by

$$u = G\sigma + Tf, \tag{3.1}$$

where $f \in H^{s-2}(\Omega)$, $\sigma \in H^{s-\frac{3}{2}}(Fr(\Omega))$ and $u \in H^s(\Omega)$.

Reciprocally, let us given $f \in H^{s-2}(\Omega)$ and $g \in H^{s-\frac{1}{2}}(Fr(\Omega))$, there exists an unique $\sigma \in H^{s-\frac{3}{2}}(Fr(\Omega))$ such that

$$G\sigma = g - Tf \quad \text{on } Fr(\Omega) \tag{3.2}$$

and hence u is given by (3.1).

Remark 3.2. *There exists an unique $\sigma = \alpha u + \frac{\partial u}{\partial \eta}$.*

3.2. **Approximation of $(\mathbf{P_0})$ by Spectral Method (case $\mathbf{p = 1}$).**

Consider the following finite dimensional space :

$$S_N^d(\Omega) := Span\left\{L_K, \ K \in \mathbb{N}^d, \ |K|_\infty \leq N\right\},$$

where $\Omega =]-1, +1[^d$ and let $d \geq 2$ be an integer. Define operators

$$T_N : H^{-1}(\Omega) \longrightarrow S_N^d(\Omega),$$

$$G_N : H^{-\frac{1}{2}}(Fr(\Omega)) \longrightarrow S_N^d(\Omega)$$

by

$$A_\alpha(T_N f, \ v_N) = (f, v_N), \ \ \forall v_N \in S_N^d(\Omega),$$

$$A_\alpha(G_N \theta, \ v_N) = <\theta, v_N>, \ \ \forall v_N \in S_N^d(\Omega).$$

Definition 3.1. *Define P_1 as an orthogonal projection on $S_N^d(\Omega)$ introduced by $A_\alpha(.,.)$. Namely*

$$A_\alpha(\phi - P_1\phi, \ \chi) = 0, \ \ \forall \chi \in S_N^d(\Omega).$$

Then we have immediately the following :

Corollary 3.7. *We have*

$$G_N\sigma = P_1 G\sigma$$

$$T_N f = P_1 Tf.$$

Lemma 3.8. *([7]) For $2-r \leq j \leq 1 \leq l \leq r$, $\sigma \in H^{l-\frac{3}{2}}(Fr(\Omega))$ and $f \in H^{l-2}(\Omega)$ we have*

$$|(T-T_N)f|_{H^{j-\frac{1}{2}}(Fr(\Omega))} + \|(T-T_N)f\|_{H^j(\Omega)} \leq cN^{j-l}\|Tf\|_{H^l(\Omega)}$$

and

$$|(G-G_N)\sigma|_{H^{j-\frac{1}{2}}(Fr(\Omega))} + \|(G-G_N)\sigma\|_{H^j(\Omega)} \leq cN^{j-l}\|G\sigma\|_{H^l(\Omega)}.$$

Now, let $\widehat{S}^d_M(\Omega)$ be a subspace of $H^n(Fr(\Omega))$, $n \geq 0$ and $M \geq 1$ is an integer. Let $\widehat{r} \geq 1$ be an integer. Then we have

Proposition 3.9. *([15]) For all $j \leq i \leq n$, we have*

$$|\phi|_{H^i(Fr(\Omega))} \leq cM^{i-j}|\phi|_{H^j(Fr(\Omega))}, \quad \forall \phi \in \widehat{S}^d_M(\Omega).$$

Proposition 3.10. *For any $i \leq j \leq l \leq \widehat{r}$, there exists an operator Π_M from $H^i(Fr(\Omega))$ into $\widehat{S}^d_M(\Omega)$ such that*

$$|\phi - \Pi_M\phi|_{H^j(Fr(\Omega))} \leq cM^{j-l}|\phi|_{H^l(Fr(\Omega))}.$$

Definition 3.2. *Define an orthogonal projection P_0 from $L^2(Fr(\Omega))$ into $\widehat{S}^d_M(\Omega)$ by*

$$< P_0\phi - \phi, \theta >= 0, \ \forall \theta \in \widehat{S}^d_M(\Omega).$$

Using ([6]) we can prove the following :

Lemma 3.11. *For all $-\widehat{r} \leq j \leq n$ and $\max(-n, j) \leq l \leq \widehat{r}$, there exists a positive constant c such that*

$$|(I-P_0)\phi|_{H^j(Fr(\Omega))} \leq cM^{j-l}|\phi|_{H^l(Fr(\Omega))}, \qquad \forall \phi \in H^l(Fr(\Omega))$$

and

$$|P_0\phi|_{H^j(Fr(\Omega))} \leq c(|\phi|_{H^j(Fr(\Omega))} + M^{j-l}|\phi|_{H^l(Fr(\Omega))}), \qquad \forall \phi \in H^l(Fr(\Omega)).$$

Lemma 3.12. *An operator*

$$P_0T_N^mG_N : \widehat{S}^d_M(\Omega) \longrightarrow \widehat{S}^d_M(\Omega), \quad m \geq 0$$

is selfadjoint. Namely

$$< P_0T_N^mG_N\phi, \psi >=< \phi, P_0T_N^mG_N\psi >, \ \ \forall \phi, \psi \in \widehat{S}^d_M(\Omega).$$

Proof. We have

$$A_\alpha(T_Nu_N, v_N) = (u_N, v_N) = A_\alpha(u_N, Tv_N), \ \ \forall u_N, \ v_N \in S^d_N(\Omega),$$

hence

$$A_\alpha(T_N^mG_N\phi, G_N\psi) = A_\alpha(G_N\phi, T_N^mG_N\psi).$$

According to definitions of operators $T_N,\ G_N$ and P_0 we have

$$< P_0 T_N^m G_N \phi, \psi >=< T_N^m G_N \phi, \psi >= A_\alpha(T_N^m G_N \phi, G_N \psi)$$

$$= A_\alpha(G_N \phi, T_N^m G_N \psi) =< \phi, T_N^m G_N \psi >=< \phi, P_0 T_N^m G_N \psi > .$$

□

According to expressions (3.1) and (3.2) we can approximate u by

$$u_{NM} = G_N \sigma_M + T_N f,$$

where σ_M is a solution of the problem

$$\begin{cases} \text{Find } \sigma_M \in \widehat{S}_M^d(\Omega) \text{ such that} \\ P_0 G_N \sigma_M = P_0(g - T_N f). \end{cases}$$

According to definitions of operators G_N and T_N we have

$$A_\alpha(u_{NM}, v_N) =< \sigma_M, v_N > +(f, v_N), \quad \forall v_N \in S_M^d(\Omega)$$

and

$$< u_{NM} - g, \theta >= 0, \quad \forall \theta \in \widehat{S}_M^d(\Omega).$$

To prove the existence and uniqueness of σ_M we have the following lemma :

Lemma 3.13. *For $M \le \varepsilon N$, with ε enough small, there exist two constants c_0 and c_1 such that*

$$c_0 \left|\theta\right|^2_{H^{-\frac{1}{2}}(Fr(\Omega))} \le< G_N \theta, \theta >\le c_1 \left|\theta\right|^2_{H^{-\frac{1}{2}}(Fr(\Omega))}, \ \forall \theta \in \widehat{S}_M^d(\Omega).$$

Proof. According to Lemma 3.4, we have

$$c_0 \left|\theta\right|^2_{H^{-\frac{1}{2}}(Fr(\Omega))} \le< G\theta, \theta >\le c_1 \left|\theta\right|^2_{H^{-\frac{1}{2}}(Fr(\Omega))}, \ \forall \theta \in H^{-\frac{1}{2}}(Fr(\Omega)).$$

Or

$$< G\theta, \theta >=< G_N \theta + G\theta - G_N \theta, \theta >$$

$$=< G_N \theta, \theta > + < G\theta - G_N \theta, \theta >\le c_1 \left|\theta\right|^2_{H^{-\frac{1}{2}}(Fr(\Omega))},$$

hence it is enough to prove

$$|< (G - G_N)\theta, \theta >| \le \delta \left|\theta\right|^2_{H^{-\frac{1}{2}}(Fr(\Omega))}, \ \forall \theta \in \widehat{S}_M^d(\Omega),$$

where δ is a positive constant. By definition, we have

$$< (G - G_N)\theta, \theta >= A_\alpha((G - G_N)\theta, G\theta)$$

$$= A_\alpha((G - G_N)\theta, (G - G_N)\theta) \leq c \left\|(G - G_N)\theta\right\|^2_{H^1(\Omega)}$$

$$\leq cN^{-2\beta} \left\|G\theta\right\|^2_{H^{1+\beta}(\Omega)},$$

where $0 < \beta < \frac{1}{2}$. According to Proposition 3.3, it holds that

$$\left\|G\theta\right\|_{H^{1+\beta}(\Omega)} \leq c \left|\theta\right|_{H^{\beta-\frac{1}{2}}(Fr(\Omega))}.$$

Using a Proposition 3.9, we obtain

$$\left|\theta\right|_{H^{\beta-\frac{1}{2}}(Fr(\Omega))} \leq cM^{\beta} \left|\theta\right|_{H^{-\frac{1}{2}}(Fr(\Omega))},$$

then we have

$$|< (G - G_N)\theta, \theta >| \leq cN^{-2\beta}M^{2\beta} \left|\theta\right|^2_{H^{-\frac{1}{2}}(Fr(\Omega))}$$

$$= c(\tfrac{M}{N})^{2\beta} \left|\theta\right|^2_{H^{-\frac{1}{2}}(Fr(\Omega))} \leq c\varepsilon^{2\beta} \left|\theta\right|^2_{H^{-\frac{1}{2}}(Fr(\Omega))}.$$

For ε enough small we obtain a result. □

Lemma 3.14. *Suppose $M \leq \varepsilon N$, where ε is enough small. Let $m = \min(r - \frac{3}{2}, \widehat{r})$ and $-m \leq s + 1 \leq 1$. Suppose $\widehat{S}^d_M(\Omega) \subset L^2(Fr(\Omega))$ and $\widehat{S}^d_M(\Omega) \subset H^1(Fr(\Omega))$ if $0 < s + 1 \leq 1$. Then there exist two positive constants c_0 and c_1 such that*

$$c_0 \left|\theta\right|_{H^s(Fr(\Omega))} \leq \left|P_0G_N\theta\right|_{H^{s+1}(Fr(\Omega))} \leq c_1 \left|\theta\right|_{H^s(Fr(\Omega))}, \quad \forall\theta \in \widehat{S}^d_M(\Omega).$$

Proof. We prove that Lemma 3.14 is verified for $-m \leq s + 1 \leq 0$ and for $s = 0$ and we obtain a result by interpolation.

- If $-m \leq s + 1 \leq 0$, according to Lemma 3.5, we have

$$c_0 \left|\theta\right|_{H^s(Fr(\Omega))} \leq \left|(G - G_N)\theta\right|_{H^{s+1}(Fr(\Omega))}$$

$$+ \left|(I - P_0)G_N\theta\right|_{H^{s+1}(Fr(\Omega))} + \left|P_0G_N\theta\right|_{H^{s+1}(Fr(\Omega))},$$

and

$$\left|P_0G_N\theta\right|_{H^{s+1}(Fr(\Omega))} \leq \left|(G - G_N)\theta\right|_{H^{s+1}(Fr(\Omega))}$$

$$+ \left|(I - P_0)G_N\theta\right|_{H^{s+1}(Fr(\Omega))} + c_1 \left|\theta\right|_{H^s(Fr(\Omega))}.$$

It is enough to prove that for all $\theta \in \widehat{S}^d_M(\Omega)$, we have

$$\left|(G - G_N)\theta\right|_{H^{s+1}(Fr(\Omega))} \leq c\varepsilon \left|\theta\right|_{H^s(Fr(\Omega))}, \quad \frac{3}{2} - r \leq s + 1 \leq 0 \tag{3.3}$$

and

$$(3.4)\quad \begin{aligned} &\left|(I-P_0)G_N\theta\right|^2_{H^{s+1}(Fr(\Omega))} \leq c\left|P_0G_N\theta\right|_{H^{s+1}(Fr(\Omega))}\left|\theta\right|_{H^s(Fr(\Omega))}, \\ &\qquad -\widehat{r} \leq s+1 \leq 0. \end{aligned}$$

To prove expression (3.3) using a Lemma 3.8 with $j-\frac{1}{2}=s+1$ and $l=\frac{3}{2}$, we obtain

$$\left|(G-G_N)\theta\right|_{H^{s+1}(Fr(\Omega))} \leq cN^s\left\|G\theta\right\|_{H^{\frac{3}{2}}(\Omega)}.$$

According to Proposition 3.3 and Proposition 3.9, we have for $-s \geq 1$

$$N^s\left\|G\theta\right\|_{H^{\frac{3}{2}}(\Omega)} \leq cN^s\left|\theta\right|_{L^2(Fr(\Omega))} \leq c(\frac{N}{M})^s\left|\theta\right|_{H^s(Fr(\Omega))} \leq c\varepsilon\left|\theta\right|_{H^s(Fr(\Omega))}$$

which implies

$$\left|(G-G_N)\theta\right|_{H^{s+1}(Fr(\Omega))} \leq c\varepsilon\left|\theta\right|_{H^s(Fr(\Omega))}.$$

To prove expression (3.4) we have, according to Lemma 3.11,

$$\left|(I-P_0)G_N\theta\right|^2_{H^{s+1}(Fr(\Omega))} \leq cM^{1+2s}\left|G_N\theta\right|_{H^{\frac{1}{2}}(Fr(\Omega))}.$$

By a mapping trace continuity, we have

$$\left|G_N\theta\right|^2_{H^{\frac{1}{2}}(Fr(\Omega))} \leq c\left\|G_N\theta\right\|^2_{H^1(\Omega)}.$$

Using a definition of G_N it holds

$$\left\|G_N\theta\right\|^2_{H^1(\Omega)} \leq cA_\alpha(G_NA, G_NA) = c < \theta, P_0G_N\theta >,$$

therefore

$$\begin{aligned} \left|(I-P_0)G_N\theta\right|^2_{H^{s+1}(Fr(\Omega))} &\leq cM^{1+2s} < \theta, P_0G_N\theta > \\ &\leq cM^{1+2s}\left|\theta\right|_{L^2(Fr(\Omega))}\left|P_0G_N\theta\right|_{L^2(Fr(\Omega))} = \\ &cM^{1+s}\left|P_0G_N\theta\right|_{L^2(Fr(\Omega))}M^s\left|\theta\right|_{L^2(Fr(\Omega))}. \end{aligned}$$

Using a Proposition 3.9, we obtain

$$\left|(I-P_0)G_N\theta\right|^2_{H^{s+1}(Fr(\Omega))} \leq c\left|P_0G_N\theta\right|_{H^{s+1}(Fr(\Omega))}\left|\theta\right|_{H^s(Fr(\Omega))},$$

then we obtain a result for $-m \leq s+1 \leq 0$.

- For $0 < s+1 \leq 1$ we have $\widehat{S}^d_M(\Omega) \subset H^1(Fr(\Omega))$. If $s=0$, let ψ be the unique solution in $\widehat{S}^d_M(\Omega)$ of the problem

$$\begin{cases} P_0G_N\psi = \theta \\ \left|\psi\right|_{H^{-1}(Fr(\Omega))} \leq c\left|\theta\right|_{L^2(Fr(\Omega))}. \end{cases}$$

Then we have

$$|\theta|^2_{L^2(Fr(\Omega))} =< \theta, P_0 G_N \psi >=< P_0 G_N \theta, \psi >$$

$$\leq c\,|P_0 G_N \theta|_{H^1(Fr(\Omega))}\,|\theta|_{L^2(Fr(\Omega))}$$

$$\Longrightarrow |\theta|_{L^2(Fr(\Omega))} \leq c\,|P_0 G_N \theta|_{H^1(Fr(\Omega))}\,.$$

But

$$|P_0 G_N \theta|_{H^1(Fr(\Omega))} \leq |P_0(G - G_N)\theta|_{H^1(Fr(\Omega))} + |P_0 G_N \theta|_{H^1(Fr(\Omega))}$$

$$\leq |P_0(G - G_N)\theta|_{H^1(Fr(\Omega))} + c\,|G\theta|_{H^1(Fr(\Omega))}\,.$$

According to Lemma 3.11, it holds that

$$|P_0 G_N \theta|_{H^1(Fr(\Omega))} \leq cM\,|(G - G_N)\theta|_{L^2(Fr(\Omega))} + c\,|G\theta|_{H^1(Fr(\Omega))}\,.$$

Using the Lemma 3.8, the Proposition 3.3 and the Lemma 3.5, we get

$$|P_0 G_N \theta|_{H^1(Fr(\Omega))} \leq c(\frac{M}{N})\,\|G\theta\|_{H^{\frac{3}{2}}(\Omega)} + c\,|G\theta|_{H^1(Fr(\Omega))} \leq c\,|\theta|_{L^2(Fr(\Omega))}\,.$$

Lastly, we obtain a result by interpolation.

□

Theorem 3.15. *Let $r \leq \widehat{r} + \frac{3}{2}$ and $M \leq \varepsilon N$, where ε is enough small. Then we have*

$$|\sigma - \sigma_M|_{H^{-\frac{1}{2}}(Fr(\Omega))} + \|u - u_{NM}\|_{H^1(\Omega)} \leq c \begin{pmatrix} N^{1-r}\,\|u\|_{H^r(\Omega)} + \\ \\ M^{-\widehat{r}-\frac{1}{2}}\,|\sigma|_{H^{\widehat{r}}(Fr(\Omega))} \end{pmatrix}.$$

Proof. We have

$$u - u_{NM} = (G - G_N)\sigma + (T - T_N)f + G_N(\sigma - \sigma_M).$$

Now we estimate $\|u - u_{NM}\|_{H^1(\Omega)}$. According to Lemma 3.8, we have

$$\|(G - G_N)\sigma\|_{H^1(\Omega)} \leq cN^{1-r}\,\|u\|_{H^r(\Omega)}$$

$$\|(T - T_N)f\|_{H^1(\Omega)} \leq cN^{1-r}\,\|u\|_{H^r(\Omega)}\,,$$

then it is enough to estimate $G_N(\sigma - \sigma_M)$. Using a Lemma 3.8 and triangular inequality we obtain

$$\|G_N(\sigma - \sigma_M)\|_{H^1(\Omega)} \leq c\,\|G(\sigma - \sigma_M)\|_{H^1(\Omega)} \leq c\,|\sigma - \sigma_M|_{H^{-\frac{1}{2}}(Fr(\Omega))}\,.$$

According to Proposition 3.10, it holds

$$|\sigma-\sigma_M|_{H^{-\frac{1}{2}}(Fr(\Omega))}\leq c\left(\begin{array}{l}|(I-\Pi_M)\sigma|_{H^{-\frac{1}{2}}(Fr(\Omega))}+\\ \\ |\Pi_M\sigma-\sigma_M|_{H^{-\frac{1}{2}}(Fr(\Omega))}\end{array}\right)$$

$$\leq c(M^{-\frac{1}{2}-\widehat{r}}\,|\sigma|_{H^{\widehat{r}}(Fr(\Omega))}+|\Pi_M\sigma-\sigma_M|_{H^{-\frac{1}{2}}(Fr(\Omega))}).$$

By a Lemma 3.13, we obtain

$$(3.5)\quad |\Pi_M\sigma-\sigma_M|^2_{H^{-\frac{1}{2}}(Fr(\Omega))}\leq c<G_N(\Pi_M\sigma-\sigma_M),\Pi_M\sigma-\sigma_M>.$$

But we have

$$P_0G_N\sigma_M=P_0(g-T_Nf)\quad \text{on } Fr(\Omega),$$

$$G\sigma=g-Tf\qquad\qquad \text{on } Fr(\Omega),$$

therefore

$$(3.6)\ \ P_0G_N(\Pi_M\sigma-\sigma_M)=P_0(G_N(\Pi_M\sigma-\sigma)+(G_N-G)\sigma+(T_N-T)f).$$

According to (3.6) it holds that

$$|\Pi_M\sigma-\sigma_M|_{H^{-\frac{1}{2}}(Fr(\Omega))}\leq$$

$$c\left(\begin{array}{r}|G_N(\Pi_M\sigma-\sigma)|_{H^{\frac{1}{2}}(Fr(\Omega))}+|(G_N-G)\sigma|_{H^{\frac{1}{2}}(Fr(\Omega))}\\ \\ +|(T_N-T)f|_{H^{\frac{1}{2}}(Fr(\Omega))}\end{array}\right).$$

Using a Lemma 3.8, we obtain

$$|\Pi_M\sigma-\sigma_M|_{H^{-\frac{1}{2}}(Fr(\Omega))}\leq c(N^{1-r}\,\|u\|_{H^r(\Omega)}+\|G_N(\Pi_M\sigma-\sigma)\|_{H^1(\Omega)}),$$

but

$$\|G_N(\Pi_M\sigma-\sigma)\|_{H^1(\Omega)}\leq c\,\|G(\Pi_M\sigma-\sigma)\|_{H^1(\Omega)}\leq c\,|\Pi_M\sigma-\sigma|_{H^{-\frac{1}{2}}(Fr(\Omega))},$$

by a Proposition 3.10, we obtain

$$|\Pi_M\sigma-\sigma|_{H^{-\frac{1}{2}}(Fr(\Omega))}\leq cM^{-\frac{1}{2}-\widehat{r}}\,|\sigma|_{H^{\widehat{r}}(Fr(\Omega))},$$

therefore

$$\|G_N(\Pi_M\sigma-\sigma)\|_{H^1(\Omega)}\leq cM^{-\frac{1}{2}-\widehat{r}}\,|\sigma|_{H^{\widehat{r}}(Fr(\Omega))}$$

$$\Longrightarrow |\sigma-\sigma_M|_{H^{-\frac{1}{2}}(Fr(\Omega))}\leq c(N^{1-r}\,\|u\|_{H^r(\Omega)}+M^{-\widehat{r}-\frac{1}{2}}\,|\sigma|_{H^{\widehat{r}}(Fr(\Omega))}),$$

then we have a result. □

Theorem 3.16. *Let* $\widehat{S}^d_M(\Omega) \subset H^n(Fr(\Omega))$, $0 \leq n \leq r - \frac{3}{2}$, $r \leq \widehat{r} + \frac{3}{2}$ *and* $M \leq \varepsilon N$, *where* ε *is enough small. Then for* $0 \leq i \leq r-2$, *we have*

$$\|u - u_{NM}\|_{H^{-i}(\Omega)} \leq \begin{cases} c(N^{-r-i}\|u\|_{H^r(\Omega)} + M^{-\widehat{r}-\frac{3}{2}-i}|\sigma|_{H^{\widehat{r}}(Fr(\Omega))}), \\ \qquad\qquad \text{if } \ 0 \leq i \leq n - \frac{1}{2} \\ c((\frac{N}{M})^{i+\frac{1}{2}-n} N^{-r-i}\|u\|_{H^r(\Omega)} + M^{-\widehat{r}-\frac{3}{2}-i}|\sigma|_{H^{\widehat{r}}(Fr(\Omega))}), \\ \qquad\qquad \text{if } \ n - \frac{1}{2} \leq i \leq r-2 \end{cases}$$

Proof. According to proof of Theorem 3.15, it is enough to estimate $\|G_N(\sigma - \sigma_M)\|_{H^{-i}(\Omega)}$.

Using a Lemma 3.8, we get

$$\|G_N(\sigma - \sigma_M)\|_{H^{-i}(\Omega)} \leq \|G(\sigma - \sigma_M)\|_{H^{-i}(\Omega)} + cN^{-1-i}\|G(\sigma - \sigma_M)\|_{H^1(\Omega)}.$$

Now using a Proposition 3.3 with $s = -\frac{1}{2}$ and a Theorem 3.15, it holds that

$$\|G(\sigma - \sigma_M)\|_{H^1(\Omega)} \leq c|\sigma - \sigma_M|_{H^{-\frac{1}{2}}(Fr(\Omega))}$$
$$\leq c(N^{1-r}\|u\|_{H^1(\Omega)} + M^{-\widehat{r}-\frac{1}{2}}|\sigma|_{H^{\widehat{r}}(Fr(\Omega))}).$$

But

$$\|G(\sigma - \sigma_M)\|_{H^{-i}(\Omega)} \leq c|\sigma - \sigma_M|_{H^{-i-\frac{3}{2}}(Fr(\Omega))}$$
$$\leq c(|(I - \Pi_M)\sigma|_{H^{-i-\frac{3}{2}}(Fr(\Omega))} + |\Pi_M\sigma - \sigma_M|_{H^{-i-\frac{3}{2}}(Fr(\Omega))}).$$

According to Proposition 3.10, we have

$$\|G(\sigma - \sigma_M)\|_{H^{-i}(\Omega)} \leq c(M^{-\widehat{r}-\frac{3}{2}-i}|\sigma|_{H^{\widehat{r}}(Fr(\Omega))} + |\Pi_M\sigma - \sigma_M|_{H^{-i-\frac{3}{2}}(Fr(\Omega))}).$$

Now, we estimate $|\Pi_M\sigma - \sigma_M|_{H^{-i-\frac{3}{2}}(Fr(\Omega))}$. By a Lemma 3.14 and expression (3.6) it holds

(3.7)
$$|\Pi_M\sigma - \sigma_M|_{H^{-i-\frac{3}{2}}(Fr(\Omega))} \leq$$
$$c\begin{pmatrix} |P_0(G_N - G)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} + |P_0(T_N - T)f|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \\ + |P_0(G_N(\Pi_M\sigma - \sigma))|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \end{pmatrix}.$$

For $0 \leq i \leq n - \frac{1}{2}$, the operator P_0 is bounded in $H^{-i-\frac{1}{2}}(Fr(\Omega))$.

According to Lemma 3.8, we have

$$|P_0(G_N - G)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \leq cN^{-r-i}\|u\|_{H^r(\Omega)}$$

$$|P_0(T_N - T)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \leq cN^{-r-i}\|u\|_{H^r(\Omega)}.$$

In other way,

$$|G_N(\Pi_M\sigma - \sigma)|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \leq |G(\Pi_M - I)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))}$$

$$+ |(G_N - G)(\Pi_M - I)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))}$$

$$\leq c(|(I - \Pi_M)\sigma|_{H^{-i-\frac{3}{2}}(Fr(\Omega))} + N^{-i-\frac{3}{2}}|G(I - \Pi_M)\sigma|_{H^{\frac{3}{2}}(Fr(\Omega))})$$

$$\leq c(|(I - \Pi_M)\sigma|_{H^{-i-\frac{3}{2}}(Fr(\Omega))} + N^{-i-\frac{3}{2}}|(I - \Pi_M)\sigma|_{L^2(Fr(\Omega))})$$

$$\leq cM^{-\widehat{r}-\frac{3}{2}-i}|\sigma|_{H^{\widehat{r}}(Fr(\Omega))}$$

therefore

$$|G_N(\sigma - \sigma_M)|_{H^{-i}(\Omega)} \leq cN^{-r-i}\|u\|_{H^r(\Omega)} + cM^{-\widehat{r}-\frac{3}{2}-i}|\sigma|_{H^{\widehat{r}}(Fr(\Omega))}.$$

For $n - \frac{1}{2} < i \leq r - 2$, operator P_0 is not bounded in $H^{-i-\frac{1}{2}}(Fr())$, by expression (3.7) we obtain

$$|\Pi_M\sigma - \sigma_M|_{H^{-i-\frac{3}{2}}(Fr(\Omega))} \leq$$

$$c\begin{pmatrix} |(I - P_0)(G_N - G)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} + |(G_N - G)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \\ + |(I - P_0)(T_N - T)f|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} + |(T_N - T)f|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \\ + |(I - P_0)G_N(\Pi_M\sigma - \sigma)|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} + \\ |G_N(\Pi_M\sigma - \sigma)|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \end{pmatrix}.$$

According to Lemma 3.8, we obtain

$$|(T_N - T)f|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \leq cN^{-i-r}\|u\|_{H^r(\Omega)},$$

$$|(G_N - G)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \leq cN^{-i-r}\|u\|_{H^r(\Omega)}.$$

And by a Lemma 3.11, we have

$$|(I - P_0)(G_N - G)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \leq cM^{-i-\frac{1}{2}+n}|(G_N - G)\sigma|_{H^{-n}(Fr(\Omega))}.$$

Using a Lemma 3.8, we obtain

$$|(G_N - G)\sigma|_{H^{-n}(Fr(\Omega))} \le cN^{-n+\frac{1}{2}-r} \|u\|_{H^r(\Omega)},$$

therefore

$$|(I-P_0)(G_N - G)\sigma|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \le cM^{-i-\frac{1}{2}+n} N^{-n+\frac{1}{2}-r} \|u\|_{H^r(\Omega)}$$

$$= c(\tfrac{N}{M})^{i+\frac{1}{2}-n} N^{-i-r} \|u\|_{H^r(\Omega)}$$

In the same way we obtain

$$|(I-P_0)(T_N - T)f|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \le c(\frac{N}{M})^{i+\frac{1}{2}-n} N^{-i-r} \|u\|_{H^r(\Omega)}.$$

In other way

(3.8)

$$|(I-P_0)G_N(\Pi_M\sigma - \sigma)|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \le |G_N(\Pi_M\sigma - \sigma)|_{H^{-i-\frac{1}{2}}(Fr(\Omega))},$$

according to expression (3.8), it holds

$$|(I-P_0)G_N(\Pi_M\sigma - \sigma)|_{H^{-i-\frac{1}{2}}(Fr(\Omega))} \le cM^{-\widehat{r}-\frac{3}{2}-i} \|\sigma\|_{H^{\widehat{r}}(Fr(\Omega))},$$

which gives a result. □

Theorem 3.17. *Under assumptions of Theorem 3.16, we have*

$$|\sigma - \sigma_M|_{H^{-i-\frac{3}{2}}(Fr(\Omega))} \le \begin{cases} c(N^{-r-i}\|u\|_{H^r(\Omega)} + M^{-\widehat{r}-\frac{3}{2}-i}|\sigma|_{H^{\widehat{r}}(Fr(\Omega))}), \\ \qquad \text{if } \ 0 \le i \le n - \frac{1}{2} \\ c\begin{pmatrix} (\frac{N}{M})^{i+\frac{1}{2}-n} N^{-r-i}\|u\|_{H^r(\Omega)} \\ +M^{-\widehat{r}-\frac{3}{2}-i}|\sigma|_{H^{\widehat{r}}(Fr(\Omega))} \end{pmatrix}, \\ \qquad \text{if } \ n - \frac{1}{2} \le i \le r-2. \end{cases}$$

Corollary 3.18. *Under assumptions of Theorem 3.16, we have*

$$\left|\frac{\partial u}{\partial \eta} - (\sigma_M - \alpha g)\right|_{L^2(Fr(\Omega))} \le c(M^{n+1}N^{-r-i+\frac{1}{2}}\|u\|_{H^r(\Omega)} + M^{-\widehat{r}}|\sigma|_{H^{\widehat{r}}(\Omega)}).$$

Proof. We have

$$\left|\tfrac{\partial u}{\partial \eta} - (\sigma_M - \alpha g)\right|_{L^2(Fr(\Omega))} = |\sigma - \sigma_M|_{L^2(Fr(\Omega))}$$

$$= |\sigma - \Pi_M\sigma + \Pi_M\sigma - \sigma_M|_{L^2(Fr(\Omega))}$$

$$\le |\Pi_M\sigma - \sigma|_{L^2(Fr(\Omega))} + |\Pi_M\sigma - \sigma_M|_{L^2(Fr(\Omega))}.$$

According to Proposition 3.10, it holds

$$\left|\frac{\partial u}{\partial \eta} - (\sigma_M - \alpha g)\right|_{L^2(Fr(\Omega))} \leq |\Pi_M \sigma - \sigma|_{L^2(Fr(\Omega))}$$

$$+cM^{n+1} |\Pi_M \sigma - \sigma|_{H^{-n-1}(Fr(\Omega))} + cM^{n+1} |\sigma - \sigma_M|_{H^{-n-1}(Fr(\Omega))}$$

$$\leq c(M^{-\widehat{r}} |\sigma|_{H^{\widehat{r}}(Fr(\Omega))} + M^{n+1} |\sigma - \sigma_M|_{H^{-n-1}(Fr(\Omega))}).$$

Using a Theorem 3.17, we obtain a result. □

3.3. Decomposition of (P_0) for $p \geq 2$.

Theorem 3.19. *([6])Let $\overrightarrow{w} = (w_0, ..., w_{p-1})$ and $\overrightarrow{\sigma} = (\sigma_1, ..., \sigma_{\left[\frac{p+1}{2}\right]})$. Consider the following problem*

$$(P_5) \quad \begin{cases} \text{Find } (\overrightarrow{w}, \overrightarrow{\sigma}) \in (H^1(\Omega))^p \times (H^{\frac{-1}{2}}(Fr(\Omega)))^{\left[\frac{p+1}{2}\right]} \text{ such that} \\ A_\alpha(w_{p-1}, v) = -(f, v) + < \sigma_1, v >, \quad \forall v \in H^1(\Omega), \\ A_\alpha(w_{p-j}, v) = -(w_{p+1-j}, v) + < \sigma_j, v >, \quad \forall v \in H^1(\Omega), \\ \qquad j = 2, 3, ..., \left[\frac{p+1}{2}\right], \\ A_\alpha(w_{j-1}, v) = -(w_j, v), \quad \forall v \in H^1(\Omega), \ j = 1, 2, ..., \left[\frac{p}{2}\right], \\ < w_{j-1}, \theta >, \quad \forall \theta \in H^{-\frac{1}{2}}(Fr(\Omega)), \ j = 1, 2, ..., \left[\frac{p+1}{2}\right]. \end{cases}$$

Then a problem (P_0) amounts to (P_5).

We use definitions of operators G and T to prove the following :

Proposition 3.20. *We have*

$$w_{p-1} = -Tf + G\sigma_1;$$

$$w_{p-j} = -Tw_{p+1-j} + G\sigma_j, \quad j = 2, 3, ..., \left[\frac{p+1}{2}\right];$$

$$w_{j-1} = -Tw_j, \quad j = 1, 2, ..., \left[\frac{p}{2}\right].$$

Corollary 3.21. *We have*

$$u = w_0 = (-1)^p T^p f + \sum_{j=1}^{\left[\frac{p+1}{2}\right]} (-1)^{p-j} T^{p-j} G\sigma_j.$$

Define

$$w_i(\overrightarrow{\sigma}) = \sum_{j=1}^{m}(-1)^{p-i-j}T^{p-i-j}G\sigma_j, \quad i = 0, 1, ..., p-1,$$

where $m = \min(p-i, \left[\frac{p+1}{2}\right])$.

Then we have the following proposition :

Proposition 3.22. *We have*

$$w_i = w_i(\overrightarrow{\sigma}) + (-1)^{p-i}T^{p-i}f, \quad i = 0, 1, ..., p-1.$$

Proposition 3.23. *For any function $v \in H^1(\Omega)$, $w_i(\overrightarrow{\sigma})$ verify the following variational system :*

$$A_\alpha(w_{p-1}(\overrightarrow{\sigma}), v) = < \sigma_1, v >;$$

$$A_\alpha(w_{p-1}(\overrightarrow{\sigma}), v) = -(w_{p+1-j}(\overrightarrow{\sigma}), v) = < \sigma_j, v >, \quad j = 2, 3, ..., \left[\tfrac{p+1}{2}\right];$$

$$A_\alpha(w_{j-1}(\overrightarrow{\sigma}), v) = -(w_j(\overrightarrow{\sigma}), v), \quad j = 1, 2, 3, ..., \left[\tfrac{p}{2}\right].$$

Corollary 3.24. *We have*

$$w_j(\overrightarrow{\sigma}) = \Delta^j w_0(\overrightarrow{\sigma}), \; j = 1, 2, ..., p-1;$$

$$\sigma_j = (\tfrac{\partial}{\partial\eta} + \alpha)w_{p-j}(\overrightarrow{\sigma}) = (\tfrac{\partial}{\partial\eta} + \alpha)\Delta^{p-j}w_0(\overrightarrow{\sigma}), \; j = 1, 2, ..., \left[\tfrac{p+1}{2}\right]$$

$$(\tfrac{\partial}{\partial\eta} + \alpha)w_{j-1}(\overrightarrow{\sigma}) = (\tfrac{\partial}{\partial\eta} + \alpha)\Delta^{j-1}w_0(\overrightarrow{\sigma}) = 0, \; j = 1, 2, ..., \left[\tfrac{p}{2}\right].$$

Proposition 3.25. *([6]) We can reduce a problem (P_5) to the following equivalent variational problem :*

$$(Q) \qquad \begin{cases} \text{Find } \overrightarrow{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_{\left[\frac{p+1}{2}\right]}) \in (H^{-\frac{1}{2}}(Fr(\Omega)))^{\left[\frac{p+1}{2}\right]} \\ \text{such that} \\ w_i(\overrightarrow{\sigma}) = (-1)^{p-i+1}T^{p-i}f, \text{ on } Fr(\Omega), \; i = 0, 1, ..., \left[\frac{p-1}{2}\right]. \end{cases}$$

where

$$w_i(\overrightarrow{\sigma}) = \sum_{j=1}^{m}(-1)^{p-i-j}T^{p-i-j}G\sigma_j, \quad i = 0, 1, ..., p-1,$$

and $\qquad m = \min(p-i, \left[\frac{p-1}{2}\right]).$

Lemma 3.26. *([6]) We have*

$$\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) >= \begin{cases} -(\Delta^{\frac{p}{2}} w_0(\overrightarrow{\sigma}), \Delta^{\frac{p}{2}} w_0(\overrightarrow{\sigma})) & \text{if } p \text{ is even} \\ \\ A_\alpha(\Delta^{\frac{p-1}{2}} w_0(\overrightarrow{\sigma}), \Delta^{\frac{p-1}{2}} w_0(\overrightarrow{\sigma})) & \text{if } p \text{ is old.} \end{cases}$$

Lemma 3.27. *([12]) Let z be a solution of equation $\Delta^p z = 0$ in Ω. Then there exists a positive constant c independent of z such that for any real s, we have*

$$\|z\|_{H^s(\Omega)} \leq c(\sum_{j=0}^{\left[\frac{p-1}{2}\right]} \left|\Delta^j z\right|_{H^{s-2j-\frac{1}{2}}(Fr(\Omega))} + \sum_{j=0}^{\left[\frac{p-2}{2}\right]} \left|\frac{\partial}{\partial \eta}\Delta^j z\right|_{H^{s-\frac{3}{2}-2j}(Fr(\Omega))}).$$

Lemma 3.28. *([6]) There exist two positive constants c_0 and c_1 such that*

$$c_0 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} \leq \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) >\right| \leq c_1 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

To prove the existence and uniqueness of a solution of problem (Q) we need the following result :

Lemma 3.29. *([6]) For all real s, There exist two constants c_0 and c_1 such that*

$$c_0 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|w_{i-1}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$\leq c_1 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

3.4. **Approximation of (P_0) by Spectral Method for $p \geq 2$.**
Consider the following finite dimensional space :

$$V_N^d(\Omega) := \left\{\overrightarrow{w}_N : w_{n,i} \in S_N^d(\Omega), \quad i = 0, 1, ..., p-1\right\}$$

where $\Omega :=]-1, 1[^d$.

Theorem 3.30. *For $l \geq 1$, $-1 \leq t \leq -\min(2l-r, j)$ and $2-r \leq j \leq 1$, there exists a positive constant c such that*

$$\left|(T^l - T_N^l)f\right|_{H^{j-\frac{1}{2}}(Fr(\Omega))} + \left\|(T^l - T_N^l)f\right\|_{H^j(\Omega)} \leq cN^{-t-2l+\max(2l-r,j)} \|f\|_{H^t(\Omega)}.$$

Proof. We prove this result by recurrence.

- For $l = 1$, a Theorem 3.30 is a Lemma 3.8.

- Suppose a Theorem 3.30 is verified for any $l \geq 1$, and we prove for all $-1 \leq t \leq -\min(2(l+1)-r, j),\ 2-r \leq j \leq 1$ we have

$$\left|(T^{l+1}-T_N^{l+1})f\right|_{H^{j-\frac{1}{2}}(Fr(\Omega))} + \left\|(T^{l+1}-T_N^{l+1})f\right\|_{H^j(\Omega)} \leq$$

$$cN^{-t-2(l+1)+\max(2(l+1)-r,j)} \left\|f\right\|_{H^t(\Omega)},$$

so

$$T^{l+1}-T_N^{l+1} = T^l(T-T_N) + (T^l-T_N^l)(T_N-T) + (T^l-T_N^l)T.$$

But

$$\left\|T^l(T-T_N)f\right\|_{H^j(\Omega)} \leq c\left\|(T-T_N)f\right\|_{H^{j-2l}(\Omega)}$$

$$\leq cN^{-t-2+\max(2-r,j-2l)} \left\|Tf\right\|_{H^{t+2}(\Omega)} \leq cN^{-t-2+\max(2-r,j-2l)} \left\|f\right\|_{H^t(\Omega)}$$

$$= cN^{-t-2(l+1)+\max(2(l+1)-r,j)} \left\|f\right\|_{H^t(\Omega)}.$$

By hypothesis, for any $-1 \leq \widetilde{t} \leq -\min(2l-r, j)$, we have

$$\left\|\left|(T^l-T_N^l)Tf\right|\right\|_{H^j(\Omega)} \leq cN^{-\widetilde{t}-2l+\max(2l-r,j)} \left\|Tf\right\|_{H^{\widetilde{t}}(\Omega)}$$

(3.9)

$$\leq cN^{-\widetilde{t}-2l+\max(2l-r,j)} \left\|f\right\|_{H^{\widetilde{t}-2}(\Omega)}.$$

Now, we prove that for all values of j, expression (3.9) implies

$$\left\|\left|(T^l-T_N^l)Tf\right|\right\|_{H^j(\Omega)} \leq cN^{-t-2(l+1)+\max(2(l+1)-r,j)} \left\|f\right\|_{H^t(\Omega)}.$$

Distinguish three cases :

Case 1 : if $j \geq 2l+2-r$, we get for $-1 \leq t \leq -\min(2(l+1)-r, j) = r-2l-2$ and $\widetilde{t} = t+2$,

$$\left\|\left|(T^l-T_N^l)Tf\right|\right\|_{H^j(\Omega)} \leq cN^{-t-2(l+1)+\max(2l-r,j)} \left\|f\right\|_{H^t(\Omega)}.$$

But $\max(2l-r, j) = \max(2(l+1)-r, j)$, hence

$$\left\|\left|(T^l-T_N^l)Tf\right|\right\|_{H^j(\Omega)} \leq cN^{-t-2(l+1)+\max(2(l+1)-r,j)} \left\|f\right\|_{H^t(\Omega)}.$$

Case 2 : if $2-r \leq j \leq 2l-r$, we get for $-1 \leq t \leq -\min(2(l+1)-r, j) = -j$ and $\widetilde{t} = t$,

$$\left\|\left|(T^l-T_N^l)Tf\right|\right\|_{H^j(\Omega)} \leq cN^{-t-2l+\max(2l-r,j)} \left\|f\right\|_{H^{t-2}(\Omega)}$$

$$= cN^{-t-2(l+1)-r+2(l+1)} \left\|f\right\|_{H^{t-2}(\Omega)} = cN^{-t-2(l+1)+\max(2(l+1)-r,j)} \left\|f\right\|_{H^{t-2}(\Omega)}$$

$$\leq cN^{-t-2(l+1)+\max(2(l+1)-r,j)} \left\|f\right\|_{H^t(\Omega)}.$$

Case 3 : if $2l-r \leq j \leq 2l+2-r$, we get for $-1 \leq t \leq -\min(2(l+1)-r, j) = -j$ and $\widetilde{t} = t+r-2l+j$,

$$\left|\left|\left|(T^l - T_N^l)Tf\right|\right|\right|_{H^j(\Omega)} \leq cN^{-t-r+2l-j-2l+\max(2l-r,j)} \left\|f\right\|_{H^{\widetilde{t}-2}(\Omega)}$$

$$= cN^{-t-r-j+\max(2l-r,j)} \left\|f\right\|_{H^{\widetilde{t}-2}(\Omega)}.$$

But $\max(2l-r,j) = j$, then

$$\left|\left|\left|(T^l - T_N^l)Tf\right|\right|\right|_{H^j(\Omega)} \leq cN^{-t-r} \left\|f\right\|_{H^{\widetilde{t}-2}(\Omega)}.$$

We have

$$\widetilde{t}-2 = t+r-2l-2+j \ ; \quad -j \geq r-2l-2$$

$$\Longrightarrow r-2l-2+j \leq 0 \Longrightarrow \widetilde{t}-2 \leq t.$$

Therefore it holds

$$\left|\left|\left|(T^l - T_N^l)Tf\right|\right|\right|_{H^j(\Omega)} \leq cN^{-t-r} \left\|f\right\|_{H^t(\Omega)}.$$

But $\max(2(l+1)-r, j) = 2(l+1)-r$, hence we get

$$\left|\left|\left|(T^l - T_N^l)Tf\right|\right|\right|_{H^j(\Omega)} \leq cN^{-t-2(l+1)+\max(2(l+1)-r,j)} \left\|f\right\|_{H^t(\Omega)}.$$

In other, we have

$$2l-r < j < 2l+2-r \quad \text{and} \quad \widetilde{t} = t+r-2l+j,$$

where $-1 \leq t \leq -\min(2(l+1)-r, j)$, then

$$-1+r-2l+j \leq \widetilde{t} \leq r-2l = -\min(2l-r, j).$$

Now, estimate the second term. According to recurrence hypothesis and Lemma 3.8, we have for all $-1 \leq t \leq r-2,\ r \geq 3$ and $2-r \leq j \leq 1$,

$$\left|\left|\left|(T^l - T_N^l)(T_N - T)f\right|\right|\right|_{H^j(\Omega)} \leq cN^{1-2l+\max(2l-r,j)} \left\|(T-T_N)f\right\|_{H^{-1}(\Omega)}$$

$$\leq cN^{1-2l+\max(2l-r,j)-t-3} \left\|Tf\right\|_{H^{t+2}(\Omega)}$$

$$\leq cN^{-t-2-2l+\max(2(l+1)-r,j)} \left\|f\right\|_{H^t(\Omega)} = cN^{-t-2(l+1)+\max(2(l+1)-r,j)} \left\|f\right\|_{H^t(\Omega)}$$

and for $-1 \leq t \leq r-2,\quad r=2$ and $0 \leq j \leq 1$ we have

$$\left|\left|\left|(T^l - T_N^l)(T_N - T)f\right|\right|\right|_{H^j(\Omega)} \leq cN^{j-2l+\max(2l-2,j)} \left\|(T-T_N)f\right\|_{H^{-j}(\Omega)}.$$

According to Lemma 3.8, we have

$$\left|\left|\left|(T_N - T)f\right|\right|\right|_{H^{-j}(\Omega)} \leq cN^{-t-2-j} \left\|Tf\right\|_{H^{t+2}(\Omega)} \leq cN^{-t-2-j} \left\|f\right\|_{H^t(\Omega)}.$$

Then

$$\left|\left|\left|(T^l - T_N^l)(T_N - T)f\right|\right|\right|_{H^j(\Omega)} \leq cN^{j-2l+\max(2l-2,j)-t-2-j} \|f\|_{H^t(\Omega)}$$

$$= cN^{-t-2(l+1)+\max(2l-2,j)} \|f\|_{H^t(\Omega)}$$

For $0 \leq j \leq 1$ we have

$$\max(2l-2,j) = \begin{cases} j \text{ if } l = 1, \\ 2l-2 \text{ if } l \geq 2, \end{cases}$$

and

$$\max(2(l+1)-2,j) = \begin{cases} 2 \text{ if } l = 1, \\ 2l \text{ if } l \geq 2. \end{cases}$$

It holds that

$$\left|\left|\left|(T^l - T_N^l)(T_N - T)f\right|\right|\right|_{H^j(\Omega)} \leq cN^{-t-2(l+1)+\max(2(l+1)-2,j)} \|f\|_{H^t(\Omega)}.$$

□

Corollary 3.31. *For all* $2l - r \leq j \leq 1$ *and* $-1 \leq t \leq r - 2l$, *there exists a positive constant* c *such that*

$$\left|(T^l - T_N^l)f\right|_{H^{j-\frac{1}{2}}(Fr(\Omega))} + \left\|(T^l - T_N^l)f\right\|_{H^j(Fr(\Omega))} \leq cN^{-t-2l+j} \|f\|_{H^t(\Omega)}.$$

Theorem 3.32. *For all integer* $l \geq 1$, $-1 \leq t \leq -\min(2l-r,j)$ *and* $2 - r \leq j \leq 1$, *there exists a positive constant* c *independent of* N *and* σ *such that*

$$\left|\left|\left|T^{l-1}G\sigma - T_N^{l-1}G_N\sigma\right|\right|\right|_{H^j(\Omega)} \leq cN^{-t-2l+\max(2l-r,j)} |\sigma|_{H^{t+\frac{1}{2}}(Fr(\Omega))}.$$

Proof. Indeed,

- for $l = 1$, a Theorem 3.32 is a Lemma 3.8 ;
- suppose a Theorem 3.32 is true for all $l \geq 1$, and prove

$$\left|\left|\left|T^l G\sigma - T_N^l G_N\sigma\right|\right|\right|_{H^j(\Omega)} \leq cN^{-t-2(l+1)+\max(2(l+1)-r,j)} |\sigma|_{H^{t+\frac{1}{2}}(Fr(\Omega))}.$$

We have

$$T^l G\sigma - T_N^l G_N\sigma = T^l G\sigma - T_N^l G_N\sigma + T_N^l G\sigma - T_N^l G\sigma$$

$$= T^l G\sigma - T_N^l G\sigma + T_N^l G_N\sigma - T_N^l G_N\sigma$$

$$= (T^l - T_N^l)G\sigma + (T_N^l G\sigma - T_N^l G_N\sigma)$$

$$= (T^l - T_N^l)G\sigma + (T_N^l - T^l)(G\sigma - G_N\sigma) + T^l(G\sigma - G_N\sigma)$$

and the proof takes end by the same method as in a Theorem 3.30.

□

Corollary 3.33. *For $2l-r \leq j \leq 1$ and $-1 \leq t \leq r-2l$, there exists a positive constant c independent of σ and N such that*

$$\left\|\left|T^{l-1}G\sigma - T_N^{l-1}G_N\sigma\right|\right\|_{H^j(\Omega)} \leq cN^{-t-2l+j}\left|\sigma\right|_{H^{t+\frac{1}{2}}(Fr(\Omega))}.$$

Now, consider the following finite space :

$$\widehat{V}_M^d(\Omega) := \left\{\overrightarrow{\sigma}_M : \sigma_{M,i} \in \widehat{S}_M^d(\Omega),\ i = 0, 1, ..., \left[\frac{p-1}{2}\right]\right\},$$

where $\widehat{S}_M^d(\Omega)$ is a finite dimensional subspace of $H^n(Fr(\Omega))$, $n \geq \max(0, 2\left[\frac{p-1}{2}\right] - \frac{1}{2})$.

Remark 3.3. *We can approximate a problem (Q) by the following problem :*

$$(Q_N^M) \quad \begin{cases} \text{Find} \quad \overrightarrow{\sigma}_M = (\sigma_{M,1}, \sigma_{M,2}, ..., \sigma_{M,\left[\frac{p+1}{2}\right]}) \in \widehat{V}_M^d(\Omega) \\ \text{such that} \\ P_0 w_{N,i}(\overrightarrow{\sigma}) = P_0((-1)^{p-i+1}T_N^{p-i}f) \text{ on } Fr(\Omega), \\ \qquad\qquad i = 0, 1, ..., \left[\frac{p-1}{2}\right]. \end{cases}$$

Now, let

$\overrightarrow{w}_N = (w_{N,0}, w_{N,1}, ..., w_{N,p-1})$ and $\overrightarrow{\sigma}_M = (\sigma_{M,1}, \sigma_{M,2}, ..., \sigma_{M,\left[\frac{p+1}{2}\right]})$.

Consider a problem

$$(P_6) \quad \begin{cases} \text{Find } (\overrightarrow{w}_N, \overrightarrow{\sigma}_M) \in V_N^d(\Omega) \times \widehat{V}_M^d(\Omega) \text{ such that} \\ A_\alpha(w_{N,p-1}, v_N) = -(f, v_N) + < \sigma_{M,1}, v_N >, \quad \forall v_N \in S_N^d(\Omega); \\ A_\alpha(w_{N,p-j}, v_N) = -(w_{N,p+1-j}, v_N) + < \sigma_{M,j}, v_N >, \\ \qquad \forall v_N \in S_N^d(\Omega),\ j = 2, 3, ..., \left[\frac{p+1}{2}\right]; \\ A_\alpha(w_{N,j-1}, v_N) = -(w_{N,j}, v_N), \forall v_N \in S_N^d(\Omega),\ j = 1, 2, ..., \left[\frac{p}{2}\right]; \\ < w_{N,j-1}, \theta_M) = 0, \quad \forall \theta_M \in \widehat{S}_M^d(\Omega),\ j = 1, 2, ..., \left[\frac{p+1}{2}\right], \end{cases}$$

where $w_{N,i}(u_{N,i})$ is an approximation of $w_j(u_j)$.

We use a definition of operator T_N and G_N to prove the following proposition :

Proposition 3.34. *We have*

$$w_{N,p-1} = -T_N f + G_N \sigma_{M,1};$$

$$w_{N,p-j} = -T_N w_{N,p+1-j} + G_N \sigma_{M,j}, \quad j = 2,3,...,\left[\tfrac{p+1}{2}\right];$$

$$w_{N,j-1} = -T_N w_{N,j}, \quad j = 1,2,...,\left[\tfrac{p}{2}\right].$$

Corollary 3.35. *We have*

$$u_N = w_{N,0} = (-1)^p T_N^p f + \sum_{j=1}^{\left[\frac{p+1}{2}\right]} (-1)^{p-j} T_N^{p-j} G_N \sigma_{M,j}.$$

Define

$$w_{N,i}(\overrightarrow{\sigma}) = \sum_{j=1}^{m} (-1)^{p-i-j} T_N^{p-i-j} G_N \sigma_{M,j}, \quad i = 0,1,...,p-1,$$

where $\qquad m = \min(p-i, \left[\tfrac{p+1}{2}\right]).$

Then we have the following proposition :

Proposition 3.36. *We have*

$$w_{N,i} = w_{N,i}(\overrightarrow{\sigma}_M) + (-1)^{p-i} T_N^{p-i} f, \quad i = 0,1,..., \ p-1.$$

Proposition 3.37. *For all $v_N \in S_N^d(\Omega)$, $w_{N,i}(\overrightarrow{\sigma})$ verify the following variational system :*

$$\begin{cases} A_\alpha(w_{N,p-1}(\overrightarrow{\sigma}), v_N) = <\sigma_{M,1}, v_N>, \\ \\ A_\alpha(w_{N,p-j}(\overrightarrow{\sigma}), v_N) = -(w_{N,p+1-j}(\overrightarrow{\sigma}), v_N) + <\sigma_{M,j}, v_N>, \\ \hfill j = 2,3,...,\left[\tfrac{p+1}{2}\right]; \\ \\ A_\alpha(w_{N,j-1}(\overrightarrow{\sigma}), v_N) = -(w_{N,j}(\overrightarrow{\sigma}), v_N), \ j = 1,2,...,\left[\tfrac{p}{2}\right]. \end{cases}$$

Lemma 3.38. *Let $\overrightarrow{\sigma} \in \overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))$, then for $-1 \le t \le 2i - \min(2p-r, s+2i)$ and $2-r \le s \le 1$, we have*

$$|||w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})|||_{H^s(\Omega)} \le cN^{-t-2p+\max(2p-r,s+2i)} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}.$$

Proof. We have, for $m = \min(p-i, \left[\tfrac{p+1}{2}\right])$,

$$w_i(\overrightarrow{\sigma}) = \sum_{j=1}^{m} (-1)^{p-i-j} T^{p-i-j} G \sigma_j, \ i = 0,1,...,p-1;$$

$$w_{N,i}(\overrightarrow{\sigma}) = \sum_{j=1}^{m} (-1)^{p-i-j} T_N^{p-i-j} G_N \sigma_j, \ i = 0,1,...,p-1,$$

then

$$|||w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})|||_{H^s(\Omega)} =$$

$$\left|\left|\left|\sum_{j=1}^{m}(-1)^{p-i-j}(T^{p-i-j}G - T_N^{p-i-j}G_N)\sigma_j\right|\right|\right|_{H^s(\Omega)}$$

$$\leq \sum_{j=1}^{m} |||(T^{p-i-j}G - T_N^{p-i-j}G_N)\sigma_j|||_{H^s(\Omega)}.$$

We have the following three cases :

Case 1 :

if $\min(2p-r, s+2i) = 2p-r$ and $2-r \leq s \leq 1$, then

$$-2i + 2p - r \leq s \leq 1, \; -1 \leq t \leq 2i + r - 2p.$$

Hence for $j \geq 1$ we have

$$-1 \leq t + 2(j-1) \leq 2i + r - 2p + 2(j-1) = -2(p-j-i+1) + r$$

and

$$-2(p-j-i+1) + r = -\min(2(p-j-i+1) - r, s).$$

Using a Theorem 3.32, we obtain

$$|||w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})|||_{H^s(\Omega)} \leq$$

$$\sum_{j=1}^{m} cN^{-t-2(j-1)-2(p-i-j+1)+\max(2(p-i-j+1)-r,s)} |\sigma_j|_{H^{t+2(j-1)+\frac{1}{2}}(Fr(\Omega))}$$

$$= \sum_{j=1}^{m} cN^{-t-2(p-i)+s} |\sigma_j|_{H^{t+2(j-1)+\frac{1}{2}}(Fr(\Omega))} \leq cN^{-t-2(p-i)+s} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}.$$

But

$$-2i + 2p - r \leq s \leq 1, \quad -1 \leq t \leq 2i + r - 2p,$$

then

$$|||w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})|||_{H^s(\Omega)} \leq cN^{-t-2p+\max(2p-r,s+2i)} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}.$$

Case 2 :

if $\min(2p-r, s+2i) = s+2i$ and $2-r \leq s \leq 1$, then we have

$$-1 \leq t \leq -s, \;\; 2-r \leq s \leq 2(p-i) - r.$$

To prove the second case we use a Theorem 3.32 with

$$l = p - i - j + 1, \;\; i \geq 0 \;\text{ and }\; j = 1, 2, ..., m.$$

To verify conditions of this Theorem, namely, if

$$\min(2l-r,s)=s, \qquad 2-r\leq s\leq 1,$$

then we have

$$-1\leq t\leq -s \quad \text{and} \quad 2-r\leq s\leq 2l-r,$$

we split the second case in two subcases as follows :

Subcase 2.1 : if

$$-1\leq t\leq -s \quad \text{and} \quad 2-r\leq s\leq 2(p-i-m+1)-r,$$

then

$$2(p-i-j+1)-r\geq s \;\text{ for }\; j=1,2,...,m.$$

We obtain, according to Theorem 3.32,

$$|||w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma})|||_{H^s(\Omega)}\leq$$

$$\sum_{j=1}^{m}cN^{-t-2(p-i-j+1)+2(p-i-j+1)-r}\left|\sigma_j\right|_{H^{t+\frac{1}{2}}(Fr(\Omega))}$$

$$=\sum_{j=1}^{m}cN^{-t-r}\left|\sigma_j\right|_{H^{t+\frac{1}{2}}(Fr(\Omega))}\leq\sum_{j=1}^{m}cN^{-t-r}\left|\sigma_j\right|_{H^{t+2(j-1)+\frac{1}{2}}(Fr(\Omega))}$$

$$\leq cN^{-t-r}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}$$

but $2-r\leq s\leq 2p-r-2i$ and $-1\leq t\leq -s$, then

$$|||w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma})|||_{H^s(\Omega)}\leq cN^{-t-2p+\max(2p-r,s+2i)}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}.$$

Subcase 2.2 : if

$$-1\leq t\leq -s \quad \text{and} \quad 2(p-q-i)-r\leq s\leq 2(p-q-i+1)-r,$$

$$q=1,...,m-1.$$

For

$$-1\leq t_j\leq -\min(2(p-i-j+1)-r,s) \;\text{ and }\; 2-r\leq s\leq 1,$$

by Theorem 3.32, we have

$$|||w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma})|||_{H^s(\Omega)}\leq\sum_{j=1}^{m}cN^{-t_j-2(p-i-j+1)+\max(2(p-i-j+1)-r,s)}\left|\sigma_j\right|_{H^{t_j+\frac{1}{2}}(Fr(\Omega))}.$$

In other, we have

$$\max(2(p-i-j+1)-r,s)=\begin{cases}2(p-i-j+1)-r, & j=1,...,q,\\ s, & j=q+1,...,m,\end{cases}$$

where $q=1,2,...,m-1$.

Then it holds

$$\left|\left|\left|w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma})\right|\right|\right|_{H^s(\Omega)} \leq \sum_{j=1}^{q} cN^{-t_j-2(p-i-j+1)+2(p-i-j+1)-r}\left|\sigma_j\right|_{H^{t_j+\frac{1}{2}}(Fr(\Omega))}$$

$$+\sum_{j=q+1}^{m} cN^{-t_j-2(p-i-j+1)+s}\left|\sigma_j\right|_{H^{t_j+\frac{1}{2}}(Fr(\Omega))}$$

$$=\sum_{j=1}^{q} cN^{-t_j-r}\left|\sigma_j\right|_{H^{t_j+\frac{1}{2}}(Fr(\Omega))}+\sum_{j=q+1}^{m} cN^{-t_j-2(p-i-j+1)+s}\left|\sigma_j\right|_{H^{t_j+\frac{1}{2}}(Fr(\Omega))}.$$

Now, we choose

$$t_j=t \quad \text{for} \quad j=1,...,q$$

and

$$t_j=t+s+r-2(p-i-j+1) \quad \text{for} \quad j=q+1,...,m.$$

Since

$$\min(2(p-i-j+1)-r,s)=\begin{cases} s, & \text{if} \quad j=1,...,q, \\ 2(p-i-j+1)-r, & \text{if} \quad j=q+1,...,m, \end{cases}$$

then we remark that for $-1\leq t\leq -s$, we have always

$$-1\leq t_j\leq -\min(2(p-i-j+1)-r,s).$$

It holds

$$\left|\left|\left|w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma})\right|\right|\right|_{H^s(\Omega)} \leq \sum_{j=1}^{q} cN^{-t-r}\left|\sigma_j\right|_{H^{t+\frac{1}{2}}(Fr(\Omega))}$$

$$+\sum_{j=q+1}^{m} cN^{-t-r}\left|\sigma_j\right|_{H^{t+r+s-2(p-i-j+1)+\frac{1}{2}}(Fr(\Omega))}.$$

For $s\leq 2(p-q-i+1)-r$, we have

$$t+r+s-2(p-i-j+1)+\tfrac{1}{2}\leq t+2(j-1)+\tfrac{1}{2}+2(1-q)$$

$$\leq t+2(j-1)+\tfrac{1}{2},$$

hence

$$\left|\sigma_j\right|_{H^{t+r+s-2(p-i-j+1)+\frac{1}{2}}(Fr(\Omega))}\leq\left|\sigma_j\right|_{H^{t+2(j-1)+\frac{1}{2}}(Fr(\Omega))}.$$

In other, $t+\frac{1}{2}\leq t+2(j-1)+\frac{1}{2}$, so

$$\left|\sigma_j\right|_{H^{t+\frac{1}{2}}(Fr(\Omega))}\leq\left|\sigma_j\right|_{H^{t+2(j-1)+\frac{1}{2}}(Fr(\Omega))}.$$

Consequently

$$|||w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})|||_{H^s(\Omega)} \leq \sum_{j=1}^{m} cN^{-t-r} \left|\sigma_j\right|_{H^{t+2(j-1)+\frac{1}{2}}(Fr(\Omega))}$$

$$\leq cN^{-t-r} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}.$$

But $2-r \leq s \leq 2p-r-2i, \quad -1 \leq t \leq -s$, it results that

$$|||w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})|||_{H^s(\Omega)} \leq cN^{-t-2p+\max(2p-r,s+2i)} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}.$$

□

Corollary 3.39. *Let* $\overrightarrow{\sigma} \in \overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))$, *then for* $-1 \leq t \leq r-2(p-i)$ *and* $2p-r \leq s+2i \leq 1+2i$, *we have*

$$\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}}(Fr(\Omega))} + \left\|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right\|_{H^s(\Omega)}$$

$$\leq cN^{-t-2p+s+2i} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}.$$

Lemma 3.40. *For* $2p-r \leq 1$ *and* $M \leq \varepsilon N$ *with* ε *is enough small, there exist two positive constants* c_0 *and* c_1 *such that for all* $\overrightarrow{\sigma} \in \widehat{V}_M^d(\Omega)$ *we have*

$$c_0 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} \leq \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{N,i-1}(\overrightarrow{\sigma}) > \right| \leq c_1 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

Proof. According to Lemma 3.28, we have

$$c_0 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} \leq \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) > \right| \leq c_1 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))},$$

so

$$c_0 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} - \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) > \right| \leq 0 \leq$$

$$c_1 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} - \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) > \right|.$$

But, we have

$$\left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) >\right| - \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{N,i-1}(\overrightarrow{\sigma}) >\right|$$

$$\leq \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) >\right|,$$

conséquently

$$(3.10)\quad \begin{aligned} & c_0 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} - \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) >\right| \\ & \leq \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{N,i-1}(\overrightarrow{\sigma}) >\right| \leq \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{N,i-1}(\overrightarrow{\sigma}) >\right| \\ & + c_1 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} - \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) >\right| \\ & \leq c_1 \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} + \left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) >\right| \end{aligned}$$

and

$$< \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \leq$$

$$\left|\sigma_i\right|_{H^{-\frac{1}{2}}(Fr(\Omega))} \left|w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma})\right|_{H^{\frac{1}{2}}(Fr(\Omega))}.$$

By a Lemma 3.38, it holds

$$< \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \leq$$

$$\left|\sigma_i\right|_{H^{-\frac{1}{2}}(Fr(\Omega))} cN^{1-2p+\max(2p-r,1+2(i-1))} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}$$

For $2p-r\leq 1$, we have $\max(2p-r,1+2(i-1))=1+2(i-1)$. Hence

$$< \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \leq$$

$$\left|\sigma_i\right|_{H^{-\frac{1}{2}}(Fr(\Omega))} cN^{1-2p+1+2(i-1)} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}$$

$$= c\left|\sigma_i\right|_{H^{-\frac{1}{2}}(Fr(\Omega))} N^{-2(p-i)} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}.$$

Using a Proposition 3.9, we obtain

$$\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))} \leq cM^{-1+p} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}$$

$$\left|\sigma_i\right|_{H^{-\frac{1}{2}}(Fr(\Omega))} \leq cM^{-1+p-2(i-1)} \left|\sigma_i\right|_{H^{\frac{1}{2}-p+2(i-1)}(Fr(\Omega))},$$

then

$$< \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \leq$$

$$cN^{-2(p-i)}M^{2(p-i)} \left|\sigma_i\right|_{H^{\frac{1}{2}-p+2(i-1)}(Fr(\Omega))} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}$$

$$= c\left(\frac{M}{N}\right)^{2(p-i)} \left|\sigma_i\right|_{H^{\frac{1}{2}-p+2(i-1)}(Fr(\Omega))} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

It holds that

$$\left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \right| \leq$$

$$\sum_{i=1}^{\left[\frac{p+1}{2}\right]} \left| < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \right|$$

$$\leq \sum_{i=1}^{\left[\frac{p+1}{2}\right]} c\left(\frac{M}{N}\right)^{2(p-i)} \left|\sigma_i\right|_{H^{\frac{1}{2}-p+2(i-1)}(Fr(\Omega))} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

If $M\leq \varepsilon N$, with ε is enough small then we have

$$\left|\sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \right| \leq c\varepsilon \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

□

Lemma 3.41. *For all* $-\widehat{r}+2\left[\frac{p-1}{2}\right]+\frac{1}{2}\leq s\leq \min(p,n+\frac{1}{2})$ *and* $\overrightarrow{\sigma}\in \overrightarrow{V}^d_M(\Omega)$, *there exist two positive constants* c_0 *and* c_1 *independent* *of $\overrightarrow{\sigma}$ such that*

$$c_0 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0 w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq c_1 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

Proof. By a Lemma 3.29, we obtain

$$c_0 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} - \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq 0 \leq$$

$$c_1 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} - \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))},$$

so

$$c_0 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} - \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|(I-P_0) w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$\sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0 w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq c_1 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} +$$

$$\sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|(I-P_0) w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}.$$

For $-\widehat{r}+2\left[\frac{p-1}{2}\right]+\frac{1}{2} \leq s \leq n+\frac{1}{2}$ and $0 \leq i \leq \left[\frac{p-1}{2}\right]$ we have

$$-\widehat{r} \leq s-2i-\frac{1}{2} \leq n,$$

then from a Lemma 3.11, we obtain

$$\left|(I-P_0) w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq c M^{s-p} \left|w_i(\overrightarrow{\sigma})\right|_{H^{p-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$\leq c M^{s-p} \left\|w_i(\overrightarrow{\sigma})\right\|_{H^{p-2i}(\Omega)} = c M^{s-p} \left\|\Delta^i w_0(\overrightarrow{\sigma})\right\|_{H^{p-2i}(\Omega)}$$

$$\leq c M^{s-p} \left\|w_0(\overrightarrow{\sigma})\right\|_{H^p(\Omega)}$$

But we have

$$w_0(\overrightarrow{\sigma}) = \sum_{j=1}^{\left[\frac{p+1}{2}\right]} (-1)^{p-j} T^{p-j} G \sigma_j$$

which implies

$$\left\|w_0(\overrightarrow{\sigma})\right\|_{H^p(\Omega)} \leq c \sum_{j=1}^{\left[\frac{p+1}{2}\right]} \left\|T^{p-j} G \sigma_j\right\|_{H^p(\Omega)}.$$

Using Propositions 3.3 and 3.6, we obtain

$$\|w_0(\overrightarrow{\sigma})\|_{H^p(\Omega)} \leq c \sum_{j=1}^{\left[\frac{p+1}{2}\right]} \|G\sigma_j\|_{H^{p-2(p-j)}(\Omega)}$$

$$\leq c \sum_{j=1}^{\left[\frac{p+1}{2}\right]} |\sigma_j|_{H^{p-2(p-j)-\frac{3}{2}}(Fr(\Omega))}$$

$$= c \sum_{j=1}^{\left[\frac{p+1}{2}\right]} |\sigma_j|_{H^{\frac{1}{2}-p+2(j-1)}(Fr(\Omega))} \leq c\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

Hence

$$|(I-P_0)w_i(\overrightarrow{\sigma})|^2_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq cM^{2(s-p)}\,|\overrightarrow{\sigma}|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

By a Lemma 3.28, we have

$$|(I-P_0)w_i(\overrightarrow{\sigma})|^2_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq cM^{2(s-p)} \left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) > \right|$$

and

$$|(I-P_0)w_i(\overrightarrow{\sigma})|^2_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq cM^{2(s-p)} \sum_{i=1}^{\left[\frac{p+1}{2}\right]} |< \sigma_i, P_0 w_{i-1}(\overrightarrow{\sigma}) >|$$

$$\leq cM^{2(s-p)} \sum_{i=1}^{\left[\frac{p+1}{2}\right]} |\sigma_i|_{H^{-\alpha_i}(Fr(\Omega))}\,|P_0 w_{i-1}(\overrightarrow{\sigma})|_{H^{\alpha_i}(Fr(\Omega))},$$

where $\alpha_i = \min(n, p-\frac{1}{2}) - 2(i-1)$. But

$$n \geq \max(0, 2\left[\tfrac{p-1}{2}\right] - \tfrac{1}{2}),$$

$$-\alpha_i = 2(i-1) - \min(n, p-\tfrac{1}{2}) \leq \begin{cases} \frac{1}{2}, & \text{if } p \geq 3, \\ 0, & \text{if } p = 2 \end{cases}$$

then $-\alpha_i \leq n$. For $s \leq \min(p, n+\frac{1}{2})$, $s - \frac{1}{2} - \min(n, p-\frac{1}{2}) \leq 0$, and

$$s + \tfrac{1}{2} - 2p + \min(n, p-\tfrac{1}{2}) = s - 2p + \min(n+\tfrac{1}{2}, p)$$

$$= 2(\min(n+\tfrac{1}{2}, p) - p) + s - \min(n+\tfrac{1}{2}, p) \leq 2(\min(n+\tfrac{1}{2}, p) - p) \leq 0$$

$$\Longrightarrow s + \tfrac{1}{2} - 2p + 2(i-1) \leq -\alpha_i$$

and

$$s-\frac{1}{2}-2(i-1)\leq\alpha_i,$$

hence, by a Proposition 3.9, we obtain

$$\left|\sigma_i\right|_{H^{-\alpha_i}(Fr(\Omega))}\leq cM^{-\alpha_i-s-\frac{1}{2}+2p-2(i-1)}\left|\sigma_i\right|_{H^{s+\frac{1}{2}-2p+2(i-1)}(Fr(\Omega))}$$

$$=cM^{-s-\frac{1}{2}+2p-\min(n,p-\frac{1}{2})}\left|\sigma_i\right|_{H^{s+\frac{1}{2}-2p+2(i-1)}(Fr(\Omega))}$$

and

$$|P_0w_{i-1}(\overrightarrow{\sigma})|_{H^{\alpha_i}(Fr(\Omega))}\leq cM^{-s+\frac{1}{2}+\min(n,p-\frac{1}{2})}\left|P_0w_{i-1}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2(i-1)}(Fr(\Omega))}.$$

Consequently

$$\left|(I-P_0)w_i(\overrightarrow{\sigma})\right|^2_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}\leq$$

$$c\sum_{i=1}^{\left[\frac{p+1}{2}\right]}\left|\sigma_i\right|_{H^{s+\frac{1}{2}-2p+2(i-1)}(Fr(\Omega))}\left|P_0w_{i-1}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2(i-1)}(Fr(\Omega))}$$

$$\leq c\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}\sum_{i=1}^{\left[\frac{p+1}{2}\right]}\left|P_0w_{i-1}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2(i-1)}(Fr(\Omega))}$$

$$=c\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}\sum_{i=0}^{\left[\frac{p-1}{2}\right]}\left|P_0w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))},$$

it holds

$$\sum_{i=0}^{\left[\frac{p-1}{2}\right]}\left|(I-P_0)w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}\leq\beta\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}$$

$$+\tfrac{c}{\beta}\left|P_0w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))},\quad\beta>0.$$

So it is enough to choose β enough small to obtain first inequality and to choose β enough large to obtain the second inequality. □

Theorem 3.42. *Let* $\max(2p-1,2p-n-\frac{1}{2})\leq r\leq\widehat{r}-2\left[\frac{p-1}{2}\right]-\frac{1}{2}+2p$. *Then for* $M\leq\varepsilon N$ *with* ε *enough small, there exist two positive constants* c_0 *and* c_1 *independent of* $\overrightarrow{\sigma}$, N *and* M *such that for all* $2p-r\leq s\leq\min(p,n+\frac{1}{2})$ *and* $\overrightarrow{\sigma}\in\widehat{V}^d_M(\Omega)$ *we have*

$$c_0\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}\leq\sum_{i=0}^{\left[\frac{p-1}{2}\right]}\left|P_0w_{N,i}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}\leq c_1\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

Proof. For $r \leq \widehat{r} - 2\left[\frac{p-1}{2}\right] - \frac{1}{2} + 2p$, we have $2p - r \geq -\widehat{r} + 2\left[\frac{p-1}{2}\right] + \frac{1}{2}$ so far

$$-\widehat{r} + 2\left[\frac{p-1}{2}\right] + \frac{1}{2} \leq 2p - r \leq s \leq \min(p, n + \frac{1}{2})$$

and by a Lemma 3.41, we obtain

$$c_0 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} - \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0 w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq 0 \leq$$

$$c_1 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} - \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0 w_i(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))},$$

hence

$$c_0 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} - \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$\leq \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0 w_{N,i}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$\leq c_1 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} + \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}.$$

By a Lemma 3.11, for $2p - r \leq s \leq \min(p, n + \frac{1}{2}, 1 + 2i)$, we have

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$c\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$+cM^{-1-2i+s}\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{\frac{1}{2}}(Fr(\Omega))}.$$

By a Corollary 3.39, we obtain for $2p - r \leq s \leq 1 + 2i$,

$$\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq cN^{1-2p+s}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}.$$

It results, from a Lemma 3.38, that

$$\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{\frac{1}{2}}(Fr(\Omega))} \leq cN^{1-2p+1+2i}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}$$

hence

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq cN^{1-2p+s}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}$$

$$+cM^{-1-2i+s}N^{2-2p+2i}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}.$$

And a Proposition 3.9 implies

$$\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))} \leq cM^{-s-1+2p}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))},$$

hence

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$c(N^{1-2p+s}M^{-s-1+2p} + M^{-1-2i+s}N^{2-2p+2i}M^{-s-1+2p})\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}$$

$$= c((\tfrac{M}{N})^{-1-s+2p} + (\tfrac{M}{N})^{-2+2p-2i})\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))},$$

so for $M \leq \varepsilon N$, with ε is enough small, it holds

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq c\varepsilon\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}. \tag{3.11}$$

But, for $1 + 2i \leq s \leq \min(p, n + \frac{1}{2})$, we obtain by a Proposition 3.9,

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$cM^{-1+s-2i}\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{\frac{1}{2}}(Fr(\Omega))}.$$

Using a Lemma 3.11, we obtain

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$cM^{-1+s-2i}\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{\frac{1}{2}}(Fr(\Omega))}.$$

And it holds from a Lemma 3.38 that

$$\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{\frac{1}{2}}(Fr(\Omega))} \leq cN^{1-2p+1+2i}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}$$

$$= cN^{2-2p+2i}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}$$

but we have

$$\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))} \leq cM^{-s-1+2p}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))},$$

so

$$\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{\frac{1}{2}}(Fr(\Omega))} \leq cN^{2-2p+2i}M^{-s-1+2p}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

Namely

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$cM^{-1+s-2i}(cN^{2-2p+2i}M^{-s-1+2p})\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}$$

$$\leq c^2(\tfrac{M}{N})^{-2+2p-2i}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))},$$

so

(3.12)

$$\sum_{i=0}^{\left[\frac{p-1}{2}\right]} |P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq c\varepsilon \, |\overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))},$$

with ε enough small. Using expressions (3.12) and (3.13), it hols that

$$c_0^{'} \, |\overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq \sum_{i=0}^{\left[\frac{p-1}{2}\right]} |P_0 w_{N,i}(\overrightarrow{\sigma})|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq c_1^{'} \, |\overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))},$$

where $c_0^{'}$ and $c_1^{'}$ are positive constants. □

Lemma 3.43. *Suppose $r = 2$, and let $\alpha = -\min(0, n - 2\left[\frac{p-1}{2}\right] - \frac{1}{2})$. Then for $M \leq \varepsilon N^{\frac{2-\alpha}{2(p-\alpha)}}$ with ε enough small, there exist two positive constants c_0 and c_1 such that for all $\overrightarrow{\sigma} \in \widehat{V}_M^d(\Omega)$ we have*

$$c_0 \, |\overrightarrow{\sigma}|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} \leq \left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{N,i-1}(\overrightarrow{\sigma}) > \right| \leq c_1 \, |\overrightarrow{\sigma}|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

Proof. Let $\alpha = -\min(0, n - 2\left[\frac{p-1}{2}\right] - \frac{1}{2})$, we have $n \geq \max(0, 2\left[\frac{p-1}{2}\right] - \frac{1}{2})$, so we have $0 \leq \alpha \leq 1$, and

$$< \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \leq$$
$$|\sigma_i|_{H^{\frac{1}{2}-\alpha}(Fr(\Omega))} \, |w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma})|_{\overrightarrow{H}^{\alpha-\frac{1}{2}}(Fr(\Omega))}$$

By a Lemma 3.38, with $t = -\alpha$, we have

$$|w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma})|_{H^{\alpha-\frac{1}{2}}(Fr(\Omega))} \leq cN^{\alpha-2} \, |\overrightarrow{\sigma}|_{\overrightarrow{H}^{-\alpha+\frac{1}{2}}(Fr(\Omega))},$$

so

$$< \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \leq c \, |\sigma_i|_{H^{\frac{1}{2}-\alpha}(Fr(\Omega))} \, N^{\alpha-2} \, |\overrightarrow{\sigma}|_{\overrightarrow{H}^{-\alpha+\frac{1}{2}}(Fr(\Omega))}.$$

By a Proposition 3.9, we obtain

$$|\sigma_i|_{H^{\frac{1}{2}-\alpha}(Fr(\Omega))} \leq cM^{p-2(i-1)-\alpha} \, |\sigma_i|_{H^{\frac{1}{2}-p+2(i-1)}(Fr(\Omega))}$$

and

$$|\overrightarrow{\sigma}|_{\overrightarrow{H}^{-\alpha+\frac{1}{2}}(Fr(\Omega))} \leq cM^{p-\alpha} \, |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))},$$

so

$$< \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \leq$$
$$cN^{\alpha-2} M^{p-2(i-1)-\alpha} \left|\sigma_i\right|_{H^{\frac{1}{2}-p+2(i-1)}(Fr(\Omega))} M^{p-\alpha} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}$$
$$= cN^{\alpha-2} M^{2(p-\alpha)-2(i-1)} \left|\sigma_i\right|_{H^{\frac{1}{2}-p+2(i-1)}(Fr(\Omega))} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}$$
$$= c\frac{M^{2(p-\alpha)}}{N^{2-\alpha}} M^{-2(i-1)} \left|\sigma_i\right|_{H^{\frac{1}{2}-p+2(i-1)}(Fr(\Omega))} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}$$
$$\leq c\frac{M^{2(p-\alpha)}}{N^{2-\alpha}} \left|\sigma_i\right|_{H^{\frac{1}{2}-p+2(i-1)}(Fr(\Omega))} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

Then

$$< \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \leq c\frac{M^{2(p-\alpha)}}{N^{2-\alpha}} \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

But we have $M \leq \varepsilon N^{\frac{2-\alpha}{2(p-\alpha)}}$ which implies

$$\left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} < \sigma_i, w_{i-1}(\overrightarrow{\sigma}) - w_{N,i-1}(\overrightarrow{\sigma}) > \right| \leq c\varepsilon \left|\overrightarrow{\sigma}\right|^2_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}.$$

Using expression (3.11), then we obtain wanted result. □

Theorem 3.44. *Suppose* $r = 2$, $\alpha = -\min(0, n - \frac{1}{2} - 2\left[\frac{p-1}{2}\right])$ *and*
(H_1) : $0 \leq s \leq \alpha$ *and* $M \leq \varepsilon N^{\frac{2-\alpha}{(2p-\alpha-s)}}$;
(H_2) : $\alpha \leq s \leq \min(p, n + \frac{1}{2})$ *and* $M \leq \varepsilon N^{\frac{2-\alpha}{2(p-\alpha)}}$, *with* ε *enough small.*

If one of assumptions (H_1) *and* (H_2) *is verified, then there exist two positive constants* c_0 *and* c_1 *such that for all* $\overrightarrow{\sigma} \in \widehat{V}^d_M(\Omega)$ *we have*

$$c_0 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0 w_{N,i}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$
$$\leq c_1 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

Proof. As $n \geq \max(0, 2\left[\frac{p-1}{2}\right] - \frac{1}{2})$, we have always $\widehat{r} \geq n+1$. By a Lemma 3.41, we have for $0 \leq s \leq \min(p, n+\frac{1}{2})$

(3.13)

$$c_0 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} - \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$\sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0 w_{N,i}(\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$c_1 \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} + \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))},$$

so for $0 \leq s \leq 2i+\alpha$, we have $s - \frac{1}{2} - 2i \leq \alpha - \frac{1}{2}$. It holds

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq \left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{\alpha-\frac{1}{2}}(Fr(\Omega))}.$$

But we have $\alpha = -\min(0, n - \frac{1}{2} - 2\left[\frac{p-1}{2}\right])$, so

$$\alpha - \tfrac{1}{2} = -\min(0, -\tfrac{1}{2} + n - 2\left[\tfrac{p-1}{2}\right]) - \tfrac{1}{2}$$

$$= \max(-\tfrac{1}{2}, 2\left[\tfrac{p-1}{2}\right] - n) \leq \begin{cases} \frac{1}{2} & \text{if } p \geq 3, \\ 0 & \text{if } p = 2. \end{cases}$$

But, $n \geq \max(0, 2\left[\frac{p-1}{2}\right] - \frac{1}{2})$, therefore $\left|\alpha - \frac{1}{2}\right| \leq n$ for all $p \geq 2$.

Using a Lemma 3.11, we obtain

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{\alpha-\frac{1}{2}}(Fr(\Omega))} \leq c\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{\alpha-\frac{1}{2}}(Fr(\Omega))}.$$

By a Lemma 3.38 and a Proposition 3.9 it holds that

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{\alpha-\frac{1}{2}}(Fr(\Omega))} \leq cN^{\alpha-2}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}$$

$$\leq cN^{\alpha-2}M^{-s+2p-\alpha}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

But for $0 \leq s \leq \alpha$ we have $\frac{M^{2p-\alpha-s}}{N^{2-\alpha}} \leq \varepsilon$ consequently

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{\alpha-\frac{1}{2}}(Fr(\Omega))} \leq c\varepsilon\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

For $2i+\alpha \leq s \leq \min(p, n+\frac{1}{2})$, it holds by a Proposition 3.9,

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$cM^{s-2i-\alpha}\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{\alpha-\frac{1}{2}}(Fr(\Omega))}.$$

A Lemma 3.11 implies

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$cM^{s-2i-\alpha}\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right|_{H^{\alpha-\frac{1}{2}}(Fr(\Omega))},$$

and by a Lemma 3.38, we obtain

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq$$

$$cN^{\alpha-2}M^{-2(\alpha-p)-2i}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))},$$

but for $\alpha \leq s \leq \min(p, n+\frac{1}{2})$ we obtain $\frac{M^{2(p-\alpha)}}{N^{2-\alpha}} \leq \varepsilon$, so

$$\left|P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}))\right|_{H^{\alpha-\frac{1}{2}}(Fr(\Omega))} \leq c\varepsilon\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

Using expression (3.13), we obtain a result. □

Lemma 3.45. *Let $\overrightarrow{\sigma}$ and $\overrightarrow{\sigma}_M$ be solutions of problems (Q) and (Q_N^M) respectively, and let $\Pi_M\overrightarrow{\sigma}$ be an approximation of $\overrightarrow{\sigma}$ satisfied a Proposition 3.10. Then for $M \leq N$ and $2p - r \leq s \leq 1 + 2i$ we have*

$$\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M\overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} + \left\|w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M\overrightarrow{\sigma})\right\|_{H^{s-2i}(\Omega)}$$

$$\leq c(N^{s-r}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))} + M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]-2p+s+\frac{1}{2}}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}).$$

and

$$\left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}_M)\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} + \left\|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}_M)\right\|_{H^{s-2i}(\Omega)}$$

$$\leq c(N^{s-r}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))} + M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]-2p+s+\frac{1}{2}}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))})$$

$$+c\left|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M\right|_{\overrightarrow{H}^{s-2p+\frac{1}{2}}(Fr(\Omega))}.$$

Proof. Let $\overrightarrow{\beta} \in \widehat{V}_M^d()$, then

$$\left|\left\|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\beta})\right\|\right|_{H^{s-2i}(\Omega)} \leq$$

$$\left|\left\|w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma})\right\|\right|_{H^{s-2i}(\Omega)} + \left|\left\|w_{N,i}(\overrightarrow{\sigma} - \overrightarrow{\beta}) - w_i(\overrightarrow{\sigma} - \overrightarrow{\beta})\right\|\right|_{H^{s-2i}(\Omega)}$$

$$+\left|\left\|w_i(\overrightarrow{\sigma}) - w_i(\overrightarrow{\beta})\right\|\right|_{H^{s-2i}(\Omega)}.$$

By a Corollary 3.39, we obtain for $2p-r\leq s\leq 1+2i$,

$$|||w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma})|||_{H^{s-2i}(\Omega)}\leq cN^{-t-2p+s}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))},$$

$$\text{for } -1\leq t\leq 2i+r-2p$$

and

$$\left|\left|\left|w_i(\overrightarrow{\sigma}-\overrightarrow{\beta})-w_{N,i}(\overrightarrow{\sigma}-\overrightarrow{\beta})\right|\right|\right|_{H^{s-2i}(\Omega)}\leq$$

$$cN^{1-2p+s}\left|\overrightarrow{\sigma}-\overrightarrow{\beta}\right|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))}$$

using a Proposition 3.10 and triangular inequality we obtain

$$|||w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma})|||_{H^{s-2i}(\Omega)}\leq cN^{s-r}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))}.$$

But we have

$$w_i(\overrightarrow{\sigma})=\sum_{j=1}^{m}(-1)^{p-i-j}T^{p-i-j}G\sigma_j,\quad i=0,1,...,p-1,$$

$$w_{N,i}(\overrightarrow{\sigma})=\sum_{j=1}^{m}(-1)^{p-i-j}T_N^{p-i-j}G_N\sigma_j,\quad i=0,1,...,p-1.$$

By Propositions 3.3 and 3.6, it holds that

$$\left|\left|\left|w_i(\overrightarrow{\sigma})-w_i(\overrightarrow{\beta})\right|\right|\right|_{H^{s-2i}(\Omega)}\leq\sum_{j=1}^{m}|||T^{p-i-j}G(\sigma_j-\beta_j)|||_{H^{s-2i}(\Omega)}$$

$$\leq c\sum_{j=1}^{m}|||G(\sigma_j-\beta_j)|||_{H^{s-2i-2(p-i-j)}(\Omega)}\leq c\sum_{j=1}^{m}\left|\sigma_j-\beta_j\right|_{H^{s-2(p-j)-\frac{3}{2}}(Fr(\Omega))}$$

$$=c\sum_{j=1}^{m}\left|\sigma_j-\beta_j\right|_{H^{s-2p+\frac{1}{2}+2(j-1)}(Fr(\Omega))}\leq c\left|\overrightarrow{\sigma}-\overrightarrow{\beta}\right|_{\overrightarrow{H}^{s-2p+\frac{1}{2}}(Fr(\Omega))},$$

so

$$\left|\left|\left|w_i(\overrightarrow{\sigma})-w_i(\overrightarrow{\beta})\right|\right|\right|_{H^{s-2i}(\Omega)}\leq c\left|\overrightarrow{\sigma}-\overrightarrow{\beta}\right|_{\overrightarrow{H}^{s-2p+\frac{1}{2}}(Fr(\Omega))}. \tag{3.14}$$

We choose $\overrightarrow{\beta}=\Pi_M\overrightarrow{\sigma}$ and use a Proposition 3.10, we obtain

$$\left|\overrightarrow{\sigma}-\Pi_M\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s-2p+\frac{1}{2}}(Fr(\Omega))}\leq cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s-2p+\frac{1}{2}}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))},$$

namely,

$$|||w_i(\overrightarrow{\sigma})-w_i(\Pi_M\overrightarrow{\sigma})|||_{H^{s-2i}(\Omega)}\leq$$

$$<cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s-2p+\frac{1}{2}}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}.$$

But

$$|\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{-\frac{1}{2}}(Fr(\Omega))} \leq cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]-\frac{1}{2}} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))},$$

so

$$\left|\left|\left|w_i(\overrightarrow{\sigma} - \overrightarrow{\beta}) - w_{N,i}(\overrightarrow{\sigma} - \overrightarrow{\beta})\right|\right|\right|_{H^{s-2i}(\Omega)} \leq$$

$$cN^{1-2p+s} M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]-\frac{1}{2}} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}.$$

For $M \leq N$ we have

$$|||w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma})|||_{H^{s-2i}(\Omega)} \leq$$

$$cN^{s-r} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))} + c^2 M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s-2p+\frac{1}{2}} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}.$$

In the same way for a second inequality. □

Theorem 3.46. *Let* $\max(2p-1, 2p-n-\frac{1}{2}) \leq r \leq \widehat{r} - 2\left[\frac{p-1}{2}\right] - \frac{1}{2} + 2p$ *and* $M \leq \varepsilon N$, *with* ε *is enough small. Then for all* $2p - r \leq s \leq 1$ *and* $\alpha = \max(0, -n - s + \frac{1}{2} + 2\left[\frac{p-1}{2}\right])$,*we have*

$$|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq$$

$$cN^{s-r}(1 + (\tfrac{N}{M})^{\alpha})(\|f\|_{H^{r-2p}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))})$$

$$+cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s-2p+\frac{1}{2}} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}.$$

Proof. Let $\Pi_M \overrightarrow{\sigma} \in \widehat{V}_M^d(\Omega)$ be an approximation of $\overrightarrow{\sigma}$ which satisfied a Proposition 3.10, according to Theorem 3.42, we have for $2p - r \leq s \leq \min(1, n + \frac{1}{2})$

$$c_0 |\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq \sum_{i=0}^{\left[\frac{p-1}{2}\right]} |P_0 w_{N,i}(\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma})|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}.$$

By a definition of a problem (Q_N^M), we have

$$P_0 w_{N,i}(\overrightarrow{\sigma}_M) - P_0 w_{N,i}(\Pi_M \overrightarrow{\sigma}) = P_0(-1)^{p-i+1} T_N^{p-i} f - P_0 w_{N,i}(\Pi_M \overrightarrow{\sigma})$$

$$= (-1)^{p-i+1} P_0(T_N^{p-i} f - T^{p-i} f) + P_0 w_i(\overrightarrow{\sigma}) - P_0 w_{N,i}(\Pi_M \overrightarrow{\sigma}),$$

so

$$c_0 \left| \overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma} \right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left| P_0(T_N^{p-i} f - T^{p-i} f) \right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$+ \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left| P_0(w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma})) \right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}.$$

By a Lemma 3.11, it holds
(3.15)

$$c_0 \left| \overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma} \right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq$$

$$c \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left| T^{p-i} f - T_N^{p-i} f \right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$+ c \sum_{i=0}^{\left[\frac{p-1}{2}\right]} M^{-\alpha_i + s - 2i - \frac{1}{2}} \left| T^{p-i} f - T_N^{p-i} f \right|_{H^{\alpha_i}(Fr(\Omega))}$$

$$+ c \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left| w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma}) \right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$+ c \sum_{i=0}^{\left[\frac{p-1}{2}\right]} M^{-\alpha_i + s - 2i - \frac{1}{2}} \left| w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma}) \right|_{H^{\alpha_i}(Fr(\Omega))},$$

where $\alpha_i = \max(-n, s-2i-\frac{1}{2})$. Using a Corollary 3.31 for $l = p-i,\ j = s-2i$ and $t = r-2p$, we obtain

$$\left| T^{p-i} f - T_N^{p-i} f \right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq c N^{s-r} \left\| f \right\|_{H^{r-2p}(\Omega)}.$$

For $j = \alpha_i + \frac{1}{2}$ we have

$$\left| T^{p-i} f - T_N^{p-i} f \right|_{H^{\alpha_i}(Fr(\Omega))} \leq c N^{-r+2i+\alpha_i+\frac{1}{2}} \left\| f \right\|_{H^{r-2p}(\Omega)}.$$

We have $2(p-i) - r \leq \alpha_i + \frac{1}{2} \leq 1$, according to Lemma 3.45, it holds

$$\left| w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma}) \right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} \leq c N^{s-r} \left| \overrightarrow{\sigma} \right|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))}$$

$$+ c M^{-\widehat{r} + 2\left[\frac{p-1}{2}\right] + s - 2p + \frac{1}{2}} \left| \overrightarrow{\sigma} \right|_{\overrightarrow{H}^{\widehat{r} - 2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}$$

and

$$|w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma})|_{H^{\alpha_i}(Fr(\Omega))} \leq cN^{-r+\alpha_i+2i+\frac{1}{2}} \left|\overrightarrow{\sigma}\right|_{H^{r-2p+\frac{1}{2}}(Fr(\Omega))}$$

$$+cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+\alpha_i+2i-2p+1} \left|\overrightarrow{\sigma}\right|_{H^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}.$$

But,

$$1+2i \geq \alpha_i + \frac{1}{2} + 2i \geq 2p - r \quad \text{for} \quad 2p - r \leq s \leq 1,$$

then

$$|\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq$$

$$c\sum_{i=0}^{\left[\frac{p-1}{2}\right]} N^{s-r}(1+(\tfrac{N}{M})^{\alpha_i - s+\frac{1}{2}+2i})(\|f\|_{H^{r-2p}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))})$$

$$+c\sum_{i=0}^{\left[\frac{p-1}{2}\right]} M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s-2p+\frac{1}{2}} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}$$

$$\leq cN^{s-r}(1+(\tfrac{N}{M})^{\alpha})(\|f\|_{H^{r-2p}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))})$$

$$+M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s-2p+\frac{1}{2}} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))},$$

where $\alpha = \max(0, -n-s+\frac{1}{2}+2\left[\frac{p-1}{2}\right])$. By a Proposition 3.9, we have for $\min(1, n+\frac{1}{2}) \leq s \leq 1$,

$$|\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq cM^{-n-\frac{1}{2}+s} \left|\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}\right|_{\overrightarrow{H}^{n+1-2p}(Fr(\Omega))}.$$

For $s = n + \frac{1}{2}$ we have

$$|\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{n+1-2p}(Fr(\Omega))} \leq$$

$$cN^{-r+n+\frac{1}{2}}(1+(\tfrac{N}{M})^{\alpha})(\|f\|_{H^{r-2p}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))})$$

$$+cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+n-2p+1} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))},$$

and

$$|\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq$$

$$cM^{-n-\frac{1}{2}+s}N^{-r+n+\frac{1}{2}}(1+(\tfrac{N}{M})^{\alpha})(\|f\|_{H^{r-2p}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))})$$

$$+ c^2 N^{-n-\frac{1}{2}+s} M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+n-2p+1} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}.$$

In the same way, we have for $M \leq N$,

$$|\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq$$

$$cN^{s-r}(1+(\tfrac{N}{M})^{\alpha})(\|f\|_{H^{r-2p}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))})$$

$$+ c^2 M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s-2p+\frac{1}{2}} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}.$$

But we have

$$|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq |\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}$$

$$+ |\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

It holds by a Proposition 3.10, that

$$|\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s-2p+\frac{1}{2}} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))},$$

wich end the proof. □

Theorem 3.47. *Under assuption of Theorem 3.46, we have for all* $2p - r \leq s \leq 1 + 2i$,

$$|w_i - w_{N,i}|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} + \|w_i - w_{N,i}\|_{H^{s-2i}(\Omega)}$$

$$\leq cN^{s-r}(1+(\tfrac{N}{M})^{\alpha})(\|f\|_{H^{r-2p}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))})$$

$$+cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]-2p+s+\frac{1}{2}} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))},$$

where $\alpha = \max(0, -n - s + 2\left[\frac{p-1}{2}\right] + \frac{1}{2})$.

Proof. We have

$$w_i = w_i(\overrightarrow{\sigma}) + (-1)^{p-i} T^{p-i} f, \quad i = 0, 1, ..., p-1,$$

$$w_{N,i} = w_{N,i}(\overrightarrow{\sigma}_M) + (-1)^{p-i} T_N^{p-i} f, \quad i = 0, 1, ..., p-1,$$

so

$$|||w_i - w_{N,i}|||_{H^{s-2i}(\Omega)} \leq |||w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}_M)|||_{H^{s-2i}(\Omega)}$$
$$+ \left|\left|\left|T^{p-i}f - T_N^{p-i}f\right|\right|\right|_{H^{s-2i}(\Omega)}.$$

By a Corollary 3.31, it holds for $t = r - 2p$, $l = p - i$ and $j = -s - 2i$

$$\left|\left|\left|T^{p-i}f - T_N^{p-i}f\right|\right|\right|_{H^{s-2i}(\Omega)} \leq cN^{s-r} \|f\|_{H^{r-2p}(\Omega)}.$$

For $2p - r \leq s \leq 1$, by a Lemma 3.45 and a Theorem 3.46, we obtain directely a result.

For $1 \leq s \leq 1 + 2i$, we have

$$|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq |\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}$$
$$+ |\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}.$$

It holds according to Proposition 3.9 that

$$|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq |\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}$$
$$+cM^{s-1} |\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{3}{2}-2p}(Fr(\Omega))}$$
$$\leq |\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} + cM^{s-1} \begin{pmatrix} |\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{3}{2}-2p}(Fr(\Omega))} \\ + |\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\frac{3}{2}-2p}(Fr(\Omega))} \end{pmatrix}.$$

And by a Proposition 3.10, we obtain

$$|\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]-2p+s+\frac{1}{2}} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}$$

and

$$|\overrightarrow{\sigma} - \Pi \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\frac{3}{2}-2p}(Fr(\Omega))} \leq cM^{1-r} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))}$$

so

$$|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]-2p+s+\frac{1}{2}} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}$$
$$+cM^{s-1}(M^{1-r} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))} + |\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\frac{3}{2}-2p}(Fr(\Omega))}).$$

By a Theorem 3.46, we have

$$|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\frac{3}{2}-2p}(Fr(\Omega))}\leq$$

$$cN^{1-r}(1+(\tfrac{N}{M})^{\alpha})(\|f\|_{H^{r-2p}(\Omega)}+|\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))})$$

$$+cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]-2p+\frac{3}{2}}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))},$$

so

$$|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))}\leq$$

$$cN^{1-r}M^{s-1}(1+(\tfrac{N}{M})^{\alpha})(\|f\|_{H^{r-2p}(\Omega)}+|\overrightarrow{\sigma}|_{\overrightarrow{H}^{r-2p+\frac{1}{2}}(Fr(\Omega))})$$

$$+cM^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s-2p+\frac{1}{2}}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}.$$

Using a Lemma 3.45, we obtain a result. □

Lemma 3.48. *Under assumptions of a Lemma 3.45, we have for* $r=2$, $s\leq 1,\ -1\leq t\leq\min(0,-s),\ M\leq N$ *and* $\overrightarrow{\beta}\in\widehat{V}_M^d(\Omega)$

$$\left|\left|\left|w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\beta})\right|\right|\right|_{H^s(\Omega)}\leq cN^{-t-2}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}$$

$$+cN^{-\bar{t}-2}\left|\overrightarrow{\sigma}-\overrightarrow{\beta}\right|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))}+c\left|\overrightarrow{\sigma}-\overrightarrow{\beta}\right|_{\overrightarrow{H}^{s+2i-2p+\frac{1}{2}}(Fr(\Omega))},$$

where $-1\leq t\leq\min(0,-s)$ *and* $-1\leq\bar{t}\leq\min(0,-s,n-\frac{1}{2})$.

Proof. We have

$$\left|\left|\left|w_i(\overrightarrow{\sigma})-w_i(\overrightarrow{\beta})\right|\right|\right|_{H^s(\Omega)}=$$

$$\left|\left|\left|w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma})+w_{N,i}(\overrightarrow{\sigma}-\overrightarrow{\beta})-w_i(\overrightarrow{\sigma}-\overrightarrow{\beta})+w_i(\overrightarrow{\sigma})-w_i(\overrightarrow{\beta})\right|\right|\right|_{H^s(\Omega)}.$$

By expression (3.14) it holds

$$\left|\left|\left|w_i(\overrightarrow{\sigma})-w_i(\overrightarrow{\beta})\right|\right|\right|_{H^s(\Omega)}\leq c\left|\overrightarrow{\sigma}-\overrightarrow{\beta}\right|_{\overrightarrow{H}^{s+2i-2p+\frac{1}{2}}(Fr(\Omega))}.$$

And using a Lemma 3.38, we obtain for $0\leq\overline{s}\leq 1$

$$|||w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma})|||_{H^{\overline{s}}(\Omega)}\leq cN^{-t-2}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))},\ -1\leq t\leq-\overline{s}.$$

We choose $\overline{s} = \max(s, 0)$, then for $-1 \leq t \leq \min(0, -s)$ and $-1 \leq \overline{t} \leq \min(0, -s, n - \frac{1}{2})$ it holds

$$\left\|\left| w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\beta}) \right|\right\|_{H^{\overline{s}}(\Omega)} \leq cN^{-t-2} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))}$$

$$+cN^{-\overline{t}-2} \left|\overrightarrow{\sigma} - \overrightarrow{\beta}\right|_{\overrightarrow{H}^{\overline{t}+\frac{1}{2}}(Fr(\Omega))} + c\left|\overrightarrow{\sigma} - \overrightarrow{\beta}\right|_{\overrightarrow{H}^{s+2i-2p+\frac{1}{2}}(Fr(\Omega))}.$$

□

Theorem 3.49. *Let $\overrightarrow{\sigma}$ and $\overrightarrow{\sigma}_M$ be solutions of problems (Q) and (Q_N^M). Suppose $r = 2$, and let $\alpha = -\min(0, n - 2\left[\frac{p-1}{p}\right] - \frac{1}{2})$. Then we have for ε enough small, $0 \leq s \leq \alpha$, $-\frac{1}{2} \leq t \leq \widehat{r} - 2\left[\frac{p-1}{2}\right]$ and $M \leq \varepsilon N^{\frac{2-\alpha}{2p-\alpha-s}}$;*

$$\left|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq$$

$$cN^{s-2}(1 + (\tfrac{N}{M})^{\max(0,-n-s-\frac{1}{2})})(\|f\|_{H^{-s}(\Omega)} + \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))})$$

$$+cM^{-t+s-2p+\frac{1}{2}} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t}(Fr(\Omega))},$$

and we have for $\alpha \leq s \leq 2p-1$, $-\frac{1}{2} \leq t \leq \widehat{r} - 2\left[\frac{p-1}{2}\right]$ and $M \leq \varepsilon N^{\frac{2-\alpha}{2(p-\alpha)}}$

$$\left|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq c\left(\begin{array}{c} N^{\alpha-2}M^{s-\alpha}(\|f\|_{H^{-\alpha}(\Omega)} + \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}) \\ +M^{-t+s-2p+\frac{1}{2}} \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{t}(Fr(\Omega))} \end{array}\right).$$

Proof. Let $\Pi_M \overrightarrow{\sigma} \in \widehat{V}_M^d(\Omega)$ be an approximation of $\overrightarrow{\sigma}$ verified a Proposition 3.10, according to Theorem 3.44, we have for $0 \leq s \leq \alpha$

$$c_0 \left|\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|P_0 w_{N,i}(\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))},$$

using expression (3.15) it holds

$$\left|\overrightarrow{\sigma}_M - \Pi_M \overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq c \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|T^{p-i}f - T_N^{p-i}f\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$+c\sum_{i=0}^{\left[\frac{p-1}{2}\right]} M^{-\alpha_i+s-2i-\frac{1}{2}} \left|T^{p-i}f - T_N^{p-i}f\right|_{H^{\alpha_i}(Fr(\Omega))}$$

$$+c\sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$+c\sum_{i=0}^{\left[\frac{p-1}{2}\right]} M^{-\alpha_i+s-2i-\frac{1}{2}} \left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma})\right|_{H^{\alpha_i}(Fr(\Omega))},$$

where $\alpha_i = \max(-n, s - \frac{1}{2} - 2i)$. By a Theorem 3.30 and a Lemma 3.48, we have for $s - \frac{1}{2} - 2i \leq -\frac{1}{2},\ i \geq 1$,

$$\left|T^{p-i}f - T_N^{p-i}f\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} + \left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$\leq cN^{s-r}(\|f\|_{H^{-s}(\Omega)} + \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))})+$$

$$cN^{-2-\min(-s,n-\frac{1}{2})}\left|\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\min(\frac{1}{2}-s,n)}(Fr(\Omega))} + c\left|\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s-2p+\frac{1}{2}}(Fr(\Omega))}.$$

For $0 \leq s \leq \alpha,\ M \leq N$ and $n \geq 1$ we have

$$cN^{-2-\min(-s,n-\frac{1}{2})}\left|\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\min(\frac{1}{2}-s,n)}(Fr(\Omega))} \leq cN^{s-2}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))}$$

and for $n = 0$ (it is the case where $p = 2$), $0 \leq s \leq \alpha = \frac{1}{2}$ and $M \leq N$ we have

$$cN^{-2-\min(-s,n-\frac{1}{2})}\left|\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\min(\frac{1}{2}-s,n)}(Fr(\Omega))} \leq cN^{s-2}\left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))}.$$

Hence we obtain

$$\left|T^{p-i}f - T_N^{p-i}f\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))} + \left|w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M \overrightarrow{\sigma})\right|_{H^{s-\frac{1}{2}-2i}(Fr(\Omega))}$$

$$\leq cN^{s-2}(\|f\|_{H^{-s}(\Omega)} + \left|\overrightarrow{\sigma}\right|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))}) + c\left|\overrightarrow{\sigma} - \Pi_M \overrightarrow{\sigma}\right|_{\overrightarrow{H}^{s-2p+\frac{1}{2}}(Fr(\Omega))}.$$

But for $n \geq 1$ we have $\alpha_i \leq -\frac{1}{2}$ if $i \geq 1$ and $\alpha_i = s - \frac{1}{2}$ if $i = 0$, then it holds

$$\left|T^{p-i}f - T_N^{p-i}f\right|_{H^{\alpha_i}(Fr(\Omega))} + |w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M\overrightarrow{\sigma})|_{H^{\alpha_i}(Fr(\Omega))}$$

$$\leq cN^{s-2}(\|f\|_{H^{-s}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))}) + c\,|\overrightarrow{\sigma} - \Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{\alpha_i+2i-2p+1}(Fr(\Omega))}$$

$$+cN^{-2-\min(-s,n-\frac{1}{2})}\,|\overrightarrow{\sigma} - \Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{\min(\frac{1}{2}-s,n)}(Fr(\Omega))}$$

$$\leq cN^{s-2}(\|f\|_{H^{-s}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))}) + c\,|\overrightarrow{\sigma} - \Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{\alpha_i+2i-2p+1}(Fr(\Omega))}\,.$$

For $n = 0$ and $0 \leq s \leq \frac{1}{2} = \alpha$ we have $\alpha_i = 0$, hence

$$\left|T^{p-i}f - T_N^{p-i}f\right|_{H^{\alpha_i}(Fr(\Omega))} + |w_i(\overrightarrow{\sigma}) - w_{N,i}(\Pi_M\overrightarrow{\sigma})|_{H^{\alpha_i}(Fr(\Omega))} \leq$$

$$cN^{-\frac{3}{2}}(\|f\|_{H^{-\frac{1}{2}}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{0}(Fr(\Omega))}) + c\,|\overrightarrow{\sigma} - \Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{2i-2p+1}(Fr(\Omega))}\,.$$

Namely, for $n \geq 1$, $M \leq \varepsilon N^{\frac{2-\alpha}{2p-\alpha-s}}$, $0 \leq s \leq \alpha$ and $-\frac{1}{2} \leq t \leq \widehat{r} - 2\left[\frac{p-1}{2}\right]$, it holds

$$|\overrightarrow{\sigma}_M - \Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{s+\frac{1}{2}-2p}(Fr(\Omega))} \leq cN^{s-2}(\|f\|_{H^{-s}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))})$$

$$+cM^{-t+s-2p+\frac{1}{2}}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{t}(Fr(\Omega))}\,.$$

For $n = 0$, $M \leq \varepsilon N^{\frac{3}{7-2s}}$, $0 \leq s \leq \frac{1}{2} = \alpha$ and $-\frac{1}{2} \leq t \leq \widehat{r}$, we have

$$|\overrightarrow{\sigma}_M - \Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{s-\frac{7}{2}}(Fr(\Omega))} \leq$$

$$cN^{s-2}(1 + (\tfrac{N}{M})^{\frac{1}{2}-s})(\|f\|_{H^{-s}(\Omega)} + |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))})$$

$$+cM^{-t+s-\frac{7}{2}}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{t}(Fr(\Omega))}\,.$$

Using triangular inequality and a Proposition 3.10 we obtain first inequality.

In the same way we obtain the second inequality. □

Theorem 3.50. *Under assumptions of Theorems 3.49, we have for* $\alpha \le s \le 1$ *and* $M \le N^{\frac{2-\alpha}{2(p-\alpha)}}$

$$|w_i - w_{N,i}|_{H^{s-\frac{1}{2}}(Fr(\Omega))} + \|w_i - w_{N,i}\|_{H^s(\Omega)} \le$$

$$cN^{s-2}(|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))} + \|f\|_{H^{-s}(\Omega)}) +$$

$$cK_1(|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))} + \|f\|_{H^{-s}(\Omega)}) + cK_2\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))},$$

where

$$K_1 = N^{s-2}M^{\alpha-s} + N^{\alpha-2}M^{-\alpha+s+2i}\ ;$$

$$K_2 = M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s+2i-2p-\frac{1}{2}},\ i = 0, 1, ..., p-2.$$

For $i = p-1$ *we replace* $\|f\|_{H^{-s}(\Omega)}$ *by* $\|f\|_{L^2(\Omega)}$.

Proof. We have

$$w_i = w_i(\overrightarrow{\sigma}) + (-1)^{p-i}T^{p-i}f$$

$$w_{N,i} = w_{N,i}(\overrightarrow{\sigma}_M) + (-1)^{p-i}T_N^{p-i}f.$$

Then

$$|||w_i - w_{N,i}|||_{H^s(\Omega)} \le |||w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}_M)|||_{H^s(\Omega)} +$$

$$\left|\left|\left|T^{p-i}f - T_N^{p-i}f\right|\right|\right|_{H^s(\Omega)}.$$

Using a Lemma 3.48, we obtain for $0 \le s \le 1$, $-1 \le t \le -s$ and $-1 \le \bar{t} \le \min(-s, n-\frac{1}{2})$,

$$|||w_i(\overrightarrow{\sigma}) - w_{N,i}(\overrightarrow{\sigma}_M)|||_{H^s(\Omega)} \le cN^{-t-2}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{t+\frac{1}{2}}(Fr(\Omega))} +$$

$$cN^{-\bar{t}-2}\,|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))} + c\,|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{s+2i-2p+\frac{1}{2}}(Fr(\Omega))}.$$

But we have

$$|\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))} \le |\overrightarrow{\sigma} - \Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))} + |\Pi_M\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))}.$$

By a Proposition 3.9, we have

$$|\Pi_M\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))} \le cM^{\bar{t}+p}\,|\Pi_M\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}$$

$$\le cM^{\bar{t}+p}(|\overrightarrow{\sigma} - \Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} + |\overrightarrow{\sigma} - \overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}),$$

which implies

$$|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))} \leq |\overrightarrow{\sigma}-\Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))}$$

$$+cM^{\bar{t}+p}(|\overrightarrow{\sigma}-\Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}+|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}).$$

And by a Proposition 3.10, it holds

$$\begin{array}{l} |\overrightarrow{\sigma}-\Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))} \leq cM^{\bar{t}+\alpha}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))} \\ |\overrightarrow{\sigma}-\Pi_M\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))} \leq cM^{\alpha-p}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))} \end{array},$$

hence

$$|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))} \leq cM^{\bar{t}+\alpha}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}$$

$$+cM^{\bar{t}+p}(cM^{\bar{t}+\alpha}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}+|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))})$$

$$\leq cM^{\bar{t}+\alpha}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}+cM^{\bar{t}+p}\,|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\frac{1}{2}-p}(Fr(\Omega))}$$

Using a Theorem 3.49, we obtain for $t=-s$, $\bar{t}=\min(-s,n-\frac{1}{2})$

$$|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))} \leq cM^{\bar{t}+\alpha}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}$$

$$+cM^{\bar{t}+p}\left(\begin{array}{l} N^{\alpha-2}M^{p-\alpha}(\|f\|_{H^{-\alpha}(\Omega)}+|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}) \\ +M^{\alpha-p}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))} \end{array}\right)$$

and

$$cN^{-\bar{t}-2}\,|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{\bar{t}+\frac{1}{2}}(Fr(\Omega))} \leq$$

$$cN^{-\bar{t}-2}M^{\bar{t}+\alpha}\left(\begin{array}{l} |\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}+ \\ N^{\alpha-2}M^{2(p-\alpha)}(\|f\|_{H^{-\alpha}(\Omega)}+|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}) \end{array}\right).$$

For $\alpha\leq s\leq 1,\ -s\leq-\alpha\leq n-2\left[\frac{p-1}{2}\right]-\frac{1}{2}\leq n-\frac{1}{2}$, we have $\bar{t}=-s$.

Then for $\alpha\leq s\leq 1$ and $M\leq\varepsilon N^{\frac{2-\alpha}{2(p-\alpha)}}$, we have

$$|||w_i(\overrightarrow{\sigma})-w_{N,i}(\overrightarrow{\sigma}_M)|||_{H^s(\Omega)} \leq cN^{s-2}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}+$$

$$cN^{s-2}M^{\alpha-s}(\|f\|_{H^{-\alpha}(\Omega)}+|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))})+c\,|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{s+2i-2p+\frac{1}{2}}(Fr(\Omega))}.$$

For $\alpha \leq s \leq 1$ we have $\alpha \leq s+2i \leq 2p-1, \quad i=0,1,...,p-1$, therefore by a Theorem 3.49, it holds

$$|\overrightarrow{\sigma}-\overrightarrow{\sigma}_M|_{\overrightarrow{H}^{s+2i-2p+\frac{1}{2}}(Fr(\Omega))} \leq$$

$$c\begin{pmatrix} N^{\alpha-2}M^{s+2i-\alpha}(\|f\|_{H^{-\alpha}(\Omega)}+|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}) \\ +M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s+2i-2p-\frac{1}{2}}\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))} \end{pmatrix}.$$

Then for $\alpha \leq s \leq 1$ and $M \leq N^{\frac{2-\alpha}{2(p-\alpha)}}$, we have

$$N^{s-2}(\|f\|_{H^{-s}(\Omega)}+|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-s}(Fr(\Omega))})+\||w_i-w_{N,i}|\|_{H^s(\Omega)} \leq$$

$$c(K_1(|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\frac{1}{2}-\alpha}(Fr(\Omega))}+\|f\|_{H^{-s}(\Omega)}))+K_2\,|\overrightarrow{\sigma}|_{\overrightarrow{H}^{\widehat{r}-2\left[\frac{p-1}{2}\right]}(Fr(\Omega))}$$

where

$$K_1 = N^{s-2}M^{\alpha-s}+N^{\alpha-2}M^{-\alpha+s+2i}\ ;$$

$$K_2 = M^{-\widehat{r}+2\left[\frac{p-1}{2}\right]+s+2i-2p-\frac{1}{2}},\ i=0,1,...,p-2.$$

For $i=p-1$ we replace $\|f\|_{H^{-s}(\Omega)}$ by $\|f\|_{L^2(\Omega)}$. □

4. **Conclusion**

- In this work, we have proved that a solution of Dirichlet problem for biharmonic operator Δ^2 by Galerkin spectral method is equivalent to resolve a finite number of sequences of Dirichlet problems for Δ by the same method. This approach is being done on a space $\mathcal{M}_N$. We have proved some results of convergence which are effecient and improve previously obtained results ([2], [3]).
- We have proved that a solution of Dirichlet problem for polyharmonic operator Δ^p, $p \geq 2$, by Galerkin spectral method is equivalent to resolve a finite number of systems of Dirichlet problems of second-order. By this formulation we have obtained optimal error estimates.
- There exist two open problems related to a subject :

Problem 1. We shall study, by the same techniques, the following evolution problem :

$$(\Pi_1) \qquad \begin{cases} \frac{\partial u}{\partial t} + Lu = f \quad \text{in } Q, \\ \begin{cases} B_L u = 0 \quad \text{on } \Sigma, \\ u(x,0) = u_0(x) \quad \text{on } \Omega, \end{cases} \end{cases}$$

where

$$u(x,t): \quad \overline{\Omega} \times [0,\ +\infty[\longrightarrow \mathbb{R},$$

$$Q = \Omega \times]0,\ +\infty[\,, \quad \Sigma = Fr(\Omega) \times]0,\ +\infty[\,.$$

Problem 2. We shall study, again by the same techniques, the following perturbed evolution problem with uniqueness and without uniqueness solution :

$$(\Pi_2) \qquad \begin{cases} \frac{\partial u}{\partial t} + L_\varepsilon u = f \quad \text{in } Q, \\ \begin{cases} B_L u = 0 \quad \text{on } \Sigma, \\ u(x,0) = u_0(x) \quad \text{on } \Omega, \end{cases} \end{cases}$$

where

$$u(x,t): \quad \overline{\Omega} \times [0,\ +\infty[\longrightarrow \mathbb{R},$$

$$Q = \Omega \times]0,\ +\infty[\,, \quad \Sigma = Fr(\Omega) \times]0,\ +\infty[\,, \quad \varepsilon \geq 0.$$

References

[1] I. Babuška, The finite element method with Lagrangian multipliers, Numer. Math. 20(1973), pp. 179-192.

[2] L. Benaissa and N. Daili, Spectral Approximations of Dirichlet Problem for Harmonic Operator, AMO - Advanced Modeling and Optimization, Vol. 15, number 2 (2013), pp. 431-448.

[3] L. Benaissa and N. Daili, About Biharmonic Problem via a Spectral Approach and Decomposition Techniques, Acta Univ. Apul. 17(2009), pp. 157-182.

[4] C. Bernardi and Y. Maday, Approximations Spectrales de Problèmes aux Limites Elliptiques, Springer-Verlag, Berlin, 1992.

[5] J.H. Bramble, The Lagrange Multiplier Method for Dirichlet's Problem, Math. Comput., Vol. 37 (1981), pp. 1-11.

[6] J.H. Bramble and R.S. Falk, A mixed-Lagrange multiplier finite element method for the polyharmonic equation, M^2AN. 19 (4) (1985), pp. 519-557.

[7] J.H. Bramble and J.E. Osborn, Rate of convergence estimates for non-selfadjoint eigenvalue approximations, Math. Comput. Vol. 27 (1973), pp. 526-549.

[8] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, Spectral methods in Fluid Dynamic, Springer-Verlag, Berlin, Heidelberg, 1987.
[9] C. Canuto and A. Quarteroni, Approximation results for orthogonal polynomials in Sobolev spaces., Math. Comput., Vol. 38 (1982), pp. 67-86.
[10] J. Giroire, Formulation variationnelle par équations intégrales de problèmes aux limites extérieurs, Technical Report $N°$ 06, Ecole Polytechnique, Palaiseaux, Paris.
[11] D. Gottlieb and S.A. Orszag, Numerical Analysis of Spectral Methods, Theory and Application, *SIAM* Publications, Philadelphia, 1977.
[12] M. Schechter, On L_p estimates and regularity II, Math. Scand. 13 (1963), pp. 47-69.
[13] J. Shen, Efficient spectral-Galerkin method I. Direct solvers of second - and fourth-order using Legendre polynomials, SIAM. Jour. Sci. Comput. 5 (1994), pp. 1489-1505.
[14] J. Shen and R. Temam, Nonlinear Galerkin method using Chebyshev and Legendre polynomials I. The one-dimensional case, SIAM. Jour. 32 (1995), pp. 215-234.
[15] G. Szegö, Orthogonal Polynomials, Colloquim Publications, AMS Providence, 1978.

BP. 513 Bordj El Bahri, 16046, Alger - Algeria
E-mail address: l_benaissa@hotmail.fr

Current address: Department of Mathematics. Faculté des Sciences, Pôles 2 - El Bez. Université de Sétif 1. 19000 Sétif - Algeria
E-mail address: nourdaili_dz@yahoo.fr