Configurations of singularities for quadratic differential systems with total finite multiplicity $m_f = 2^*$

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Abstract

In this work we consider the problem of classifying all configurations of singularities, both finite and infinite of quadratic differential systems, with respect to the *geometric equivalence relation* defined in [3]. This relation is finer than the *topological equivalence relation* which does not distinguish between a focus and a node or between a strong and a weak focus or between foci of different orders. Such distinctions are however important in the production of limit cycles close to the foci in perturbations of the systems. The notion of *geometric equivalence relation* of configurations of singularities allows us to incorporates all these important geometric features which can be expressed in purely algebraic terms. This equivalence relation is also finer than the *qualitative equivalence relation* introduced in [**?**]. The *geometric classification* of all configurations of singularities, finite and infinite, of quadratic systems was initiated in [4] where the classification was done for systems with total multiplicity m_f of finite singularities less than or equal to one. In this article we continue the work initiated in [4] and obtain the *geometric classification* of singularities, finite and infinite, for the subclass of quadratic differential systems possessing finite singularities of total multiplicity $m_f = 2$. We obtain 197 *geometrically distinct* configurations of singularities for this family. We also give here the global bifurcation diagram of configurations of singularities, both finite and infinite, with respect to the *geometric equivalence relation*, for this class of systems. The bifurcation set of this diagram is algebraic. The bifurcation diagram is done in the 12-dimensional space of parameters and it is expressed in terms of polynomial invariants. The results can therefore be applied for any family of quadratic systems in this class, given in any normal form. Determining the geometric configurations of singularities for any such family, becomes thus a simple task using computer algebra calculations.

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Résumé

Dans ce travail nous considérons le problème de classifier toutes les configurations de singularités, finies et infinies, des systèmes quadratiques, par rapport à la relation *d'équivalence géométrique* définie dans [3]. Cette relation d'équivalence est plus fine que la relation *d'équivalence topologique* qui ne fait aucune distinction entre un foyer et un neud, ni entre un foyer fort et un foyer faible, ni entre foyers de différents ordres. Ces distinctions sont pourtant importantes dans la production de cycles limites proches des foyers dans les perturbations des systèmes. La notion d'équivalence géométrique de configurations de singularités nous permet d'incorporer tous ces traits géométriques qui s'expriment algébriquement. Cette relation d 'equivalence est aussi plus fine que la *relation d'equivalence qualitative* introduite en [**?**]. La *classification géométrique* de toutes les configurations de singularités, finie et infinies, des systèmes quadratiques a été initié dans [4] où la classification est faite pour les systèmes ayant la multiplicité totale m_f des singularités finies, plus petite ou egale a un. Dans cet article nous continuons le travail commencé dans [4] et nous obtenons la *classification géométrique* des configurations de singularités, finies et infinies, pour la sous-classe des systèmes différentiels quadratiques formée de systèmes possédant la multiplicité totale $m_f = 2$ des singularités finies. Nous obtenons 197 *configurations geométriques* distinctes pour cette famille. Nous donnons aussi le diagramme global de bifurcation par rapport a la relation *d'équivalence géométrique* des configurations de singularités, finies et infinies, de cette classe de systèmes. L'ensemble de bifurcation est algébrique. Le diagramme de bifurcation est réalisé dans l'espace de 12 paramètres et il s'exprime en termes de polynômes invariants. Les resultats peuvent donc s'appliquer à toute famille de systèmes quadratiques de cette classe, présentée par rapport a n'importe quelle forme normale. Déterminer la configuartion géométrique des singularités d'une telle famille se ramène alors a de simples calculs à l'ordinateur.

1 Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x, y), \qquad \frac{dy}{dt} = q(x, y), \tag{1}$$

where p, $q \in \mathbb{R}[x, y]$, i.e. p, q are polynomials in x, y over $\mathbb{R}$. We call *degree* of a system (1) the integer $m = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

The study of the class **QS** has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. It is expected that we have a finite number of phase portraits in **QS**. Although we have phase portraits for several subclasses of **QS**, the complete list of phase portraits of this class is not known and attempting to topologically classify these systems, which occur rather often in applications, is a very complex task. This is partly due to the elusive nature of limit cycles and partly to the rather large number of parameters involved. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time homotheties, the class ultimately depends on five parameters which still a rather large number. For the moment only subclasses depending on at most three parameters were studied globally, including global bifurcation diagrams (for example [2]) . On the other hand we can restrict the study of the whole quadratic class by focusing on specific global features of the systems in this family. We may thus focus on the global study of singularities and their bifurcation diagram. The singularities are of two kinds: finite and infinite. The infinite singularities are obtained by compactifying the differential systems on the sphere or on the Poincaré disk as defined in Section 2 (see also [18]).

The global study of quadratic vector fields in the neighborhood of infinity was initiated by Nikolaev and Vulpe in [**?**] where they classified topologically the singularities at infinity in terms of invariant polynomials. Schlomiuk and Vulpe used geometrical concepts defined in [**?**], and also introduced some new geometrical concepts in [**?**] in order to simplify the invariant polynomials and the classification. To reduce the number of phase portraits in half, in both cases the *topological equivalence relation* was taken to mean the existence of a homeomorphism of the plane carrying orbits to orbits and *preserving or reversing* the orientation. In [5] the authors classified topologically (adding also the distinction between nodes and foci) the whole quadratic class, according to configurations of their finite singularities.

In the topological classification no distinction was made among the various types of foci or saddles, strong or weak of various orders. However these distinctions of an algebraic nature are very important in the study of perturbations of systems possessing such singularities. Indeed, the maximum number of limit cycles which can be produced close to the weak foci in perturbations depends on the orders of the foci.

The distinction among weak saddles is also important since for example when a loop is formed using two separatrices of one weak saddle, the maximum number of limit cycles that can be obtained close to the loop in perturbations is the order of weak saddle.

There are also three kinds of simple nodes as we can see in FIGURE 1 below where the local phase portraits around the singularities are given.

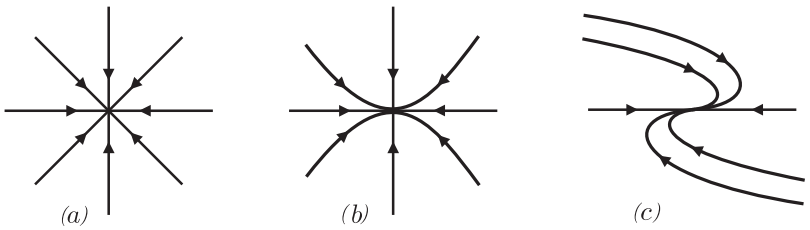


Figure 1: Different types of nodes

In the three phase portraits of FIGURE 1 the corresponding three singularities are stable nodes. These portraits are topologically equivalent but the solution curves do not arrive at the nodes in the same way. In the first case, any two distinct non-trivial phase curves arrive at the node with distinct slopes. Such a node is called a star node. In the second picture all non-trivial solution curves excepting two of them arrive at the node with the same slope but the two exception curves arrive at the node with a different slope. This is the generic node with two directions. In the third phase portrait all phase curves arrive at the node with the same slope. Algebraic means, i.e. the linearization matrices at these nodes and their eigenvalues, distinguish the nodes in FIGURE 1.

We recall that the first and the third types of nodes could produce foci in perturbations and the first type of nodes is also involved in the existence of invariant straight lines of differential systems. For example it can easily be shown that if a quadratic differential system has two finite star nodes then necessarily the system possesses invariant straight lines of total multiplicity 6.

Furthermore, a generic node at infinity may or may not have the two exceptional curves lying on the line at infinity. This leads to two different situations for the phase portraits. For this reason we split the generic nodes at infinity in two types.

The distinctions among the nilpotent and linearly zero singularities finite or infinite can also be refined, as done in [4, Section 4]. Such singularities are usually called *degenerate* singularities so here too we call them *degenerate*.

The geometric equivalence relation for finite or infinite singularities, introduced in [3] and used in [4], takes into account such distinctions.

This equivalence relation is finer than the *qualitative equivalence relation* introduced by Jiang and Llibre in [**?**] since it distinguishes among the foci of different orders and among the various types of nodes. This equivalence relation also induces a finer distinction among the more complicated degenerate singularities.

To distinguish among the foci (or saddles) of various orders we use the algebraic concept of Poincaré-Lyapunov constants. We call strong focus (or strong saddle) a focus with non–zero trace of the linearization matrix at this point. Such a focus (or saddle) will be considered to have the order zero. A focus (or saddle) with trace zero is called a weak focus (weak saddle). For details on Poincaré-Lyapunov constants and weak foci we refer to [**?**], [**?**].

Algebraic information may not be significant for the local (topological) phase portrait around a singularity. For example, topologically there is no distinction between a focus and a node or between a weak and a strong focus. However, as indicated before, algebraic information plays a fundamental role in the study of perturbations of systems possessing such singularities.

The following is a legitimate question:

How far can we go in the global theory of quadratic (or more generally polynomial) vector fields by using mainly algebraic means?

For certain subclasses of quadratic vector fields the full description of the phase portraits as well as of the bifurcation diagrams can be obtained using only algebraic tools. Examples of such classes are:

- the quadratic vector fields possessing a center [**?**, **?**, **?**, **?**];
- the quadratic Hamiltonian vector fields [1, 6];
- the quadratic vector fields with invariant straight lines of total multiplicity at least four [**?**, **?**];
- the planar quadratic differential systems possessing a line of singularities at infinity [**?**];
- the quadratic vector fields possessing an integrable saddle [7].
- the family of Lotka-Volterra systems [**?**, **?**], once we assume Bautin's analytic result saying that such systems have no limit cycles;

In the case of other subclasses of the quadratic class **QS**, such as the subclass of systems with a weak focus of order 3 or 2 (see [**?**, 2]) the bifurcation diagrams were obtained by using an interplay of algebraic, analytic and numerical methods. These subclasses were of dimensions 2 and 3 modulo the action of the affine group and time rescaling. So far no 4-dimensional subclasses of **QS** were studied globally so as to produce also bifurcation diagrams and such problems are very difficult due to the number of parameters as well as the increased complexities of these classes.

Although we now know that in trying to understand these systems, there is a limit to the power of algebraic methods, these methods have not been used far enough. For example the global classification of singularities, finite and infinite, using the *geometric equivalence relation*, *can* be done by using only algebraic methods. The first step in this direction was done in [3] where the study of *the whole class* **QS**, according to the configurations of the singularities at infinity was obtained by using only algebraic methods. This classification was done with respect to the *geometric equivalence relation* of configurations of singularities. Our work in [3] can be extended for the whole class **QS** by incorporating also the finite singularities. To obtain the global *geometric classification* of all possible configurations of singularities, finite and infinite, of the class **QS**, by purely algebraic means is a long term goal since we expect to finally obtain over 800 distinct configurations of singularities. In [4] we initiated the work on this project by studying the configurations of singularities for the subclass of **QS** for which the total multiplicity m_f of finite singularities is less than or equal to one.

Our goal here is to continue this work by *geometrically* classifying the configurations of all singularities with total multiplicity $m_f = 2$ for systems in **QS**.

We recall here below the notion of *geometric configuration of singularities* defined in [4] for both finite and infinite singularities. We distinguish two cases:

1) If we have a finite number of infinite singular points and a finite number of finite singularities we call *geometric configuration of singularities*, finite and infinite, the set of all these singularities each endowed with its own multiplicity together with their local phase portraits endowed with additional geometric structure involving the concepts of tangent, order and blow–up equivalences defined in Section 4 of [4] and using the notations described here in Section 4.

2) If the line at infinity $Z = 0$ is filled up with singularities, in each one of the charts at infinity $X \neq 0$ and $Y \neq 0$, the system is degenerate and we need to do a rescaling of an appropriate degree of the system, so that the degeneracy be removed. The resulting systems have only a finite number of singularities on the line $Z = 0$. In this case we call *geometric configuration of singularities*, finite and infinite, the union of the set of all points at infinity (they are all singularities) with the set of finite singularities - taking care to single out the singularities at infinity of the "reduced" system, taken together with the local phase portraits of finite singularities endowed with additional geometric structure as above and the local phase portraits of the infinite singularities of the reduced system.

We define the following affine invariants: Let Σ_C be the sum of the finite orders of weak singularities (foci or weak saddles) in a configuration C of a quadratic system and let M_C be the maximum finite order of a weak singularity in a configuration C of a quadratic system. Clearly Σ_C and M_C are affine invariants. Let Σ_2 (respectively M_2) be the maximum of all Σ_C (respectively M_C) for the subclass of (QS) with $m_f = 2$.

In stating our theorem we take care to include the results about the configurations containing centers and integrable saddles or containing weak singularities which are foci or saddles, since these singularities are especially important having the potential of producing limit cycles in perturbations. We use the notation introduced in [4] denoting by $f^{(1)}$, $s^{(1)}$, the weak foci and the weak saddles of order i and by c and \$ the centers and integrable saddles.

Our results are stated in the following theorem.

Main Theorem. (*A*) *We consider here all configurations of singularities, finite and infinite, of quadratic vector fields with finite singularities of total multiplicity $m_f = 2$. These configurations are classified in* Diagrams *1–3 according to the geometric equivalence relation. We have 197 geometric distinct configurations of singularities, finite and infinite. More precisely 16 configurations with two complex distinct finite singularities; 151 configurations with two real distinct finite singularities and 30 with one real finite singularity which is double.*

(*B*) *For the subclass of **QS** with $m_f = 2$ we have $\Sigma_2 = 2 = M_2$.*

There are only 6 configurations of singularities with finite weak singular points with $\Sigma_C = 2$. These have the following combinations of finite singularities: $f^{(1)}, f^{(1)}$; $s^{(1)}, s^{(1)}$; $s^{(2)}, n$; $s^{(2)}, n^d$; $s^{(2)}, f$; $f^{(2)}, s$.

There are 7 configurations of singularities with finite weak singular points with $\Sigma_C = 1$. These have the following combinations of finite singularities: $f^{(1)}, n$; $f^{(1)}, n^d$; $f^{(1)}, s$; $f^{(1)}, f$; $s^{(1)}, n$; $s^{(1)}, n^d$; $s^{(1)}, f$.

There are 19 configurations containing a center or an integrable saddle, only 6 of which with a center. There are 8 distinct couples of finite singularities occurring in these configurations. They are: c,\$; c, s; \$,\$; \$,s; \$,n; \$,n∗; \$,n^d; \$,f.

(*C*) *Necessary and sufficient conditions for each one of the 197 different equivalence classes can be assembled from these diagrams in terms of 31 invariant polynomials with respect to the action of the affine group and time rescaling, given in Section 5.*

(*D*) *The* Diagrams *1–3 actually contain the global bifurcation diagram in the 12-dimensional space of parameters, of the global configurations of singularities, finite and infinite, of this family of quadratic differential systems.*

(*E*) *Of all the phase portraits in the neighborhood of the line at infinity, which are here given in* Figure *2, 6 are are not realized in the family of systems with $m_f = 2$. They are:*

Configs -17; 19; 30; 32; 43; 44. (see Figure *2)*

The invariants and comitants of differential equations used for proving our main results are obtained following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his disciples (see for instance [**?**, **?**, **?**, 8, 13]).

Remark 1 *We note that the geometric equivalence relation for configurations is much finer than the topological equivalence. Indeed, for example the topological equivalence does not distinguish between the following three configurations which are geometrically non-equivalent: n, f, SN, C, C; n, $f^{(1)}$, SN, C, C and n^d, $f^{(1)}$, SN, C, C where n means a singularity which is a node, capital letters indicate points at infinity, C in case of a complex point and SN a saddle at infinity.*

2 Compactifications associated to planar polynomial differential systems

2.0.1 Compactification on the sphere and on the Poincaré disk

Planar polynomial differential systems (1) can be compactified on the sphere. For this we consider the affine plane of coordinates (x, y) as being the plane $Z = 1$ in $\mathbb{R}^3$ with the origin located at $(0, 0, 1)$, the x–axis parallel with the X–axis in $\mathbb{R}^3$, and the y–axis parallel to the Y–axis. We use central projection to project this plane on the sphere as follows: for each point $(x, y, 1)$ we consider the line joining the origin with $(x, y, 1)$. This line intersects the sphere in two points $P_1 = (X, Y, Z)$ and $P_2 = (-X, -Y, -Z)$ where $(X, Y, Z) = (1/\sqrt{x^2 + y^2 + 1})(x, y, 1)$. The applications $(x, y) \mapsto P_1$ and $(x, y) \mapsto P_2$ are bianalytic and associate to a vector field on the plane (x, y) an analytic vector field Ψ on the upper hemisphere and also an analytic vector field Ψ'on the lower hemisphere. A theorem stated by Poincaré

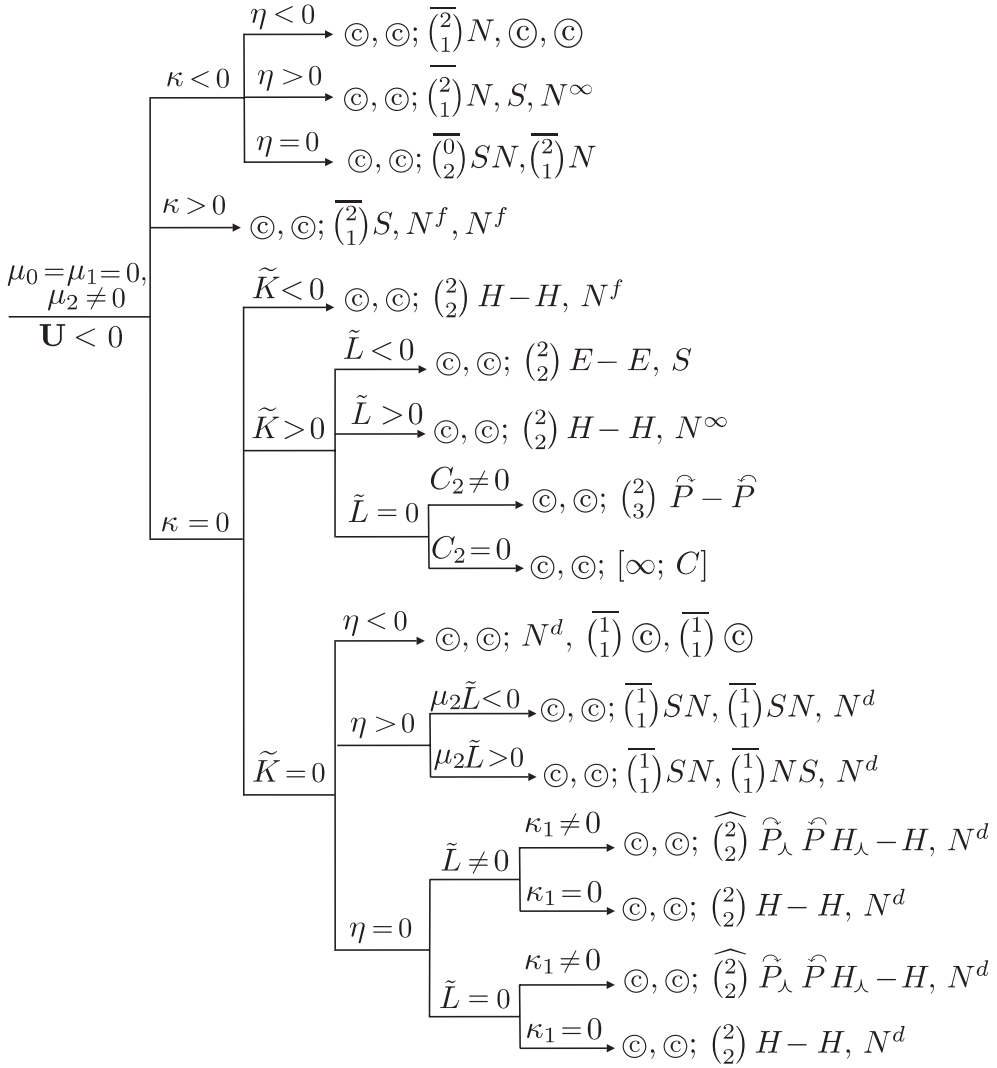


DIAGRAM 1: **Global configurations: the case $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$, $U < 0$**

and proved in [19] says that there exists an analytic vector field Θ on the whole sphere which simultaneously extends the vector fields on the two hemispheres. By the *Poincaré compactification on the sphere* of a planar polynomial vector field we mean the restriction $\bar{\Psi}$ of the vector field Θ to the union of the upper hemisphere with the equator. For more details we refer to [**?**]. The vertical projection of $\bar{\Psi}$ on the plane $Z = 0$ gives rise to an analytic vector field Φ on the unit disk of this plane. By the *compactification on the Poincaré disk* of a planar polynomial vector field we understand the vector field Φ. By a *singular point at infinity* of a planar polynomial vector field we mean a singular point of the vector field $\bar{\Psi}$ which is located on the equator of the sphere, respectively a singular point of the vector field Φ located on the circumference of the Poincaré disk.

2.0.2 Compactification on the projective plane

To a polynomial system (1) we can associate a differential equation $\omega_1 = q(x, y)dx - p(x, y)dy = 0$. Assuming the differential system (1) is with real coefficients, we may associate to it a foliation with singularities on the real, respectively complex, projective plane as indicated below. The equation $\omega_1 = 0$ defines a foliation with singularities on the real or complex plane depending if we consider the equation as being defined over the real or complex affine plane. It is known that we can compactify these foliations with singularities on the real respectively complex projective plane. In the study of real planar polynomial vector fields, their associated complex vector fields and their singularities play an important role. In particular such a vector field could have complex, non-real singularities, by this meaning singularities of the associated complex vector field. We briefly recall below how these foliations with singularities are defined.

The application $\Upsilon : \mathbb{K}^2 \longrightarrow P_2(\mathbb{K})$ defined by $(x, y) \mapsto [x : y : 1]$ is an injection of the plane $\mathbb{K}^2$ over the field $\mathbb{K}$ into the projective plane $P_2(\mathbb{K})$ whose image is the set of $[X : Y : Z]$ with $Z \neq 0$. If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ this application is an analytic injection. If $Z \neq 0$ then $(\Upsilon)^{-1}([X : Y : Z]) = (x, y)$ where $(x, y) = (X/Z, Y/Z)$. We obtain a map

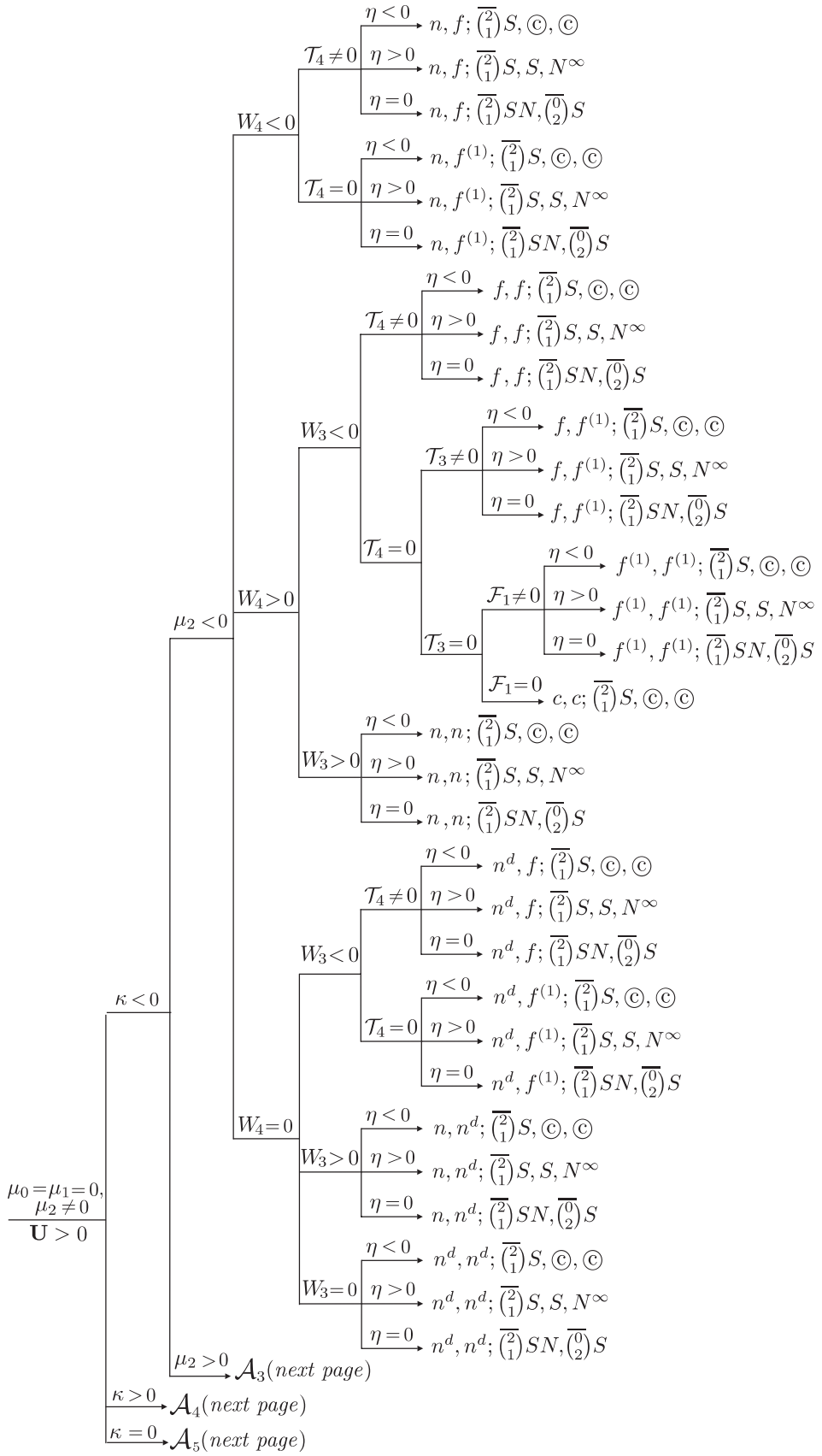


DIAGRAM 2: **Global configurations: the case $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$, $U > 0$**

$i : \mathbb{K}^3 - \{Z = 0\} \longrightarrow \mathbb{K}^2$ defined by $[X : Y : Z] \mapsto (X/Z, Y/Z)$.

Considering that $dx = d(X/Z) = (ZdX - XdZ)/Z^2$ and $dy = (ZdY - YdZ)/Z^2$, the pull-back of the form ω_1 via the map i yields the form $i*(\omega_1) = q(X/Z, Y/Z)(ZdX - XdZ)/Z^2 - p(X/Z, Y/Z)(ZdY - YdZ)/Z^2$ which has poles

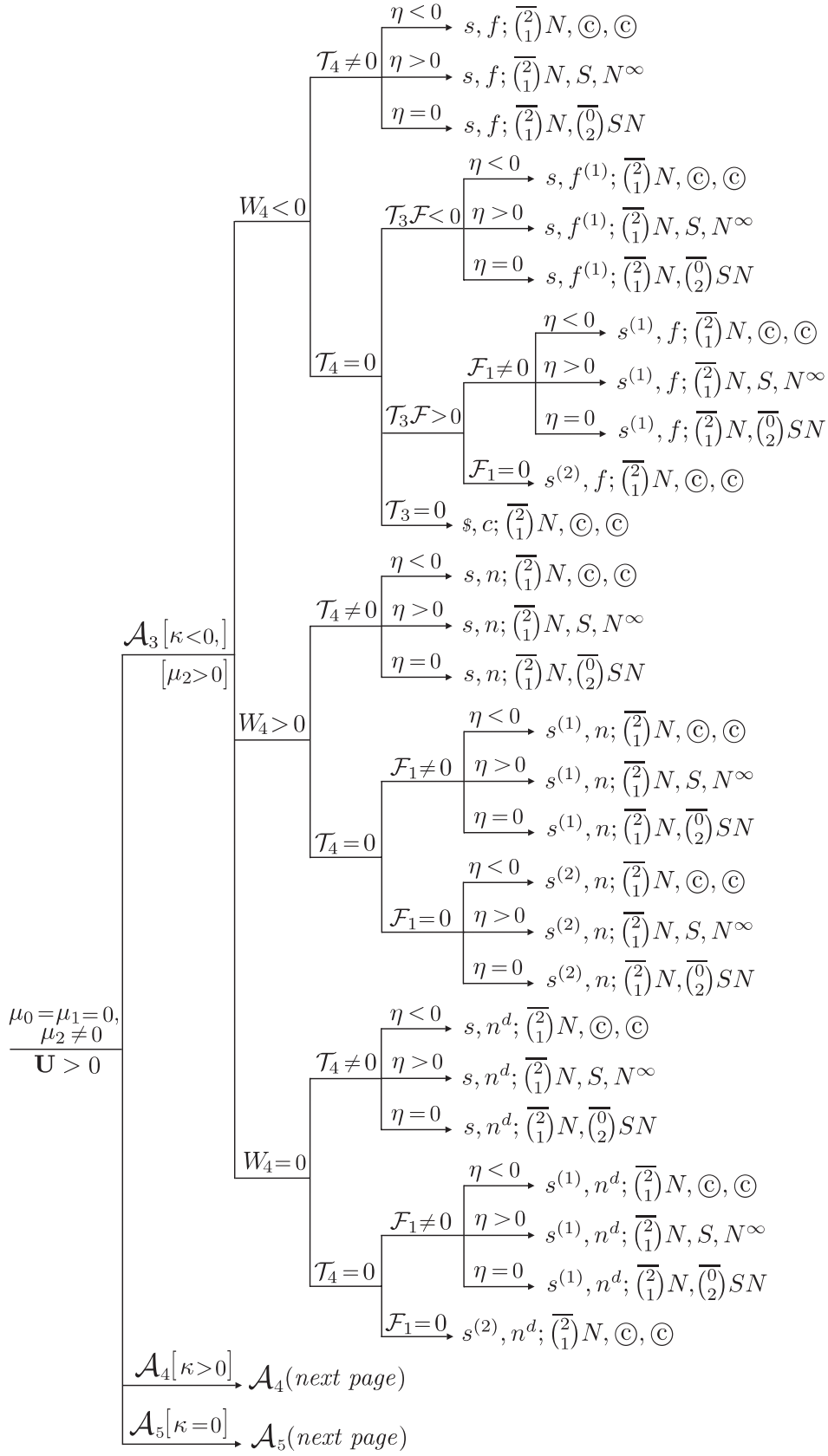


DIAGRAM 2 (*continued*). **Global configurations: the case $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$, $U > 0$**

on $Z = 0$. Then the form $\omega = Z^{m+2} i * (\omega_1)$ on $K^3 - \{Z = 0\}$, K being $\mathbb{R}$ or $\mathbb{C}$ and m being the degree of systems (1) yields the equation $\omega = 0$:

$$A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ = 0$$

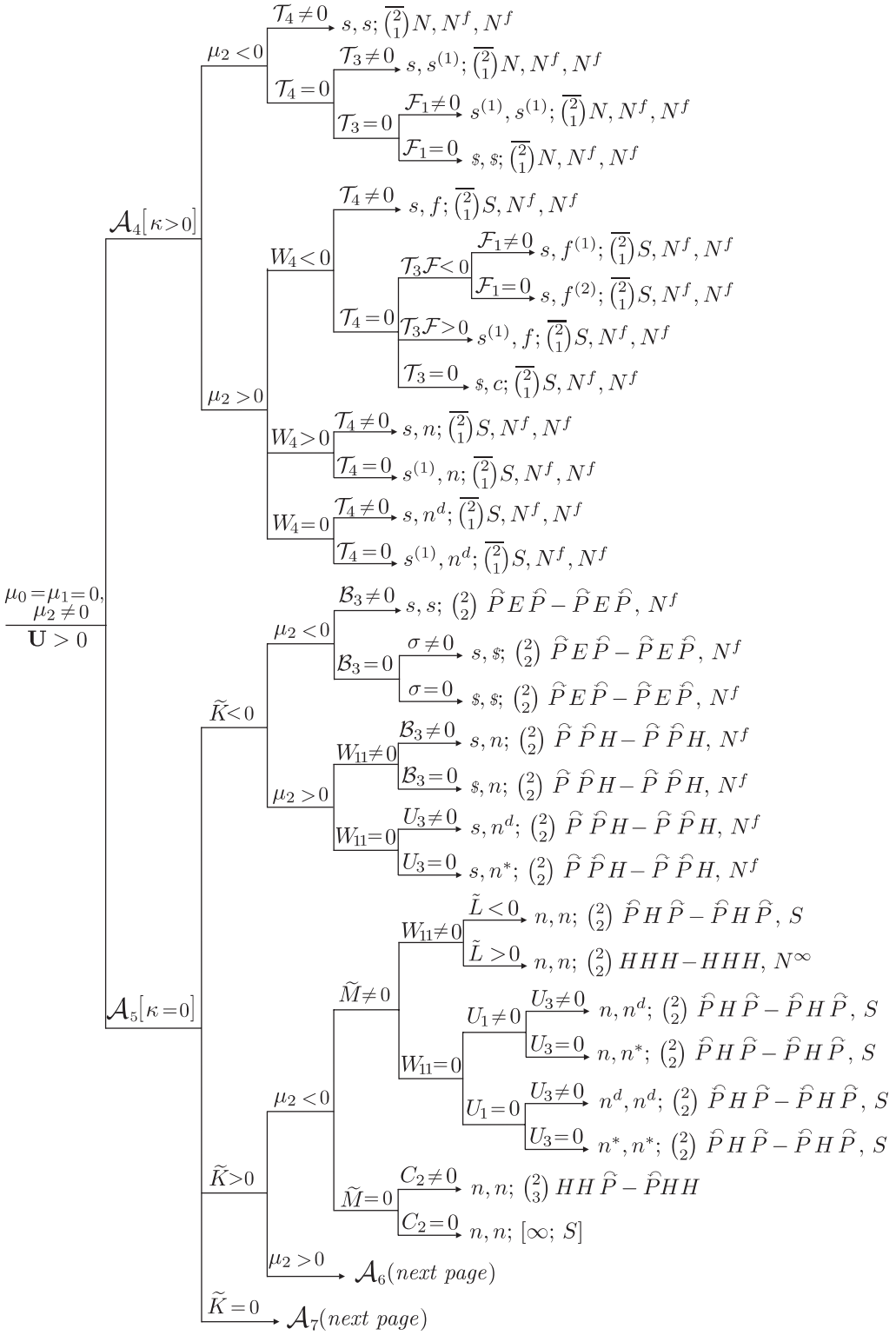


DIAGRAM 2 (*continued*). **Global configurations: the case $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$, $U > 0$**

on $K^3-\{Z=0\}$ where A, B, C are homogeneous polynomials over K whith $A(X,Y,Z) = ZQ(X,Y,Z)$, $Q(X,Y,Z) = Z^m q(X/Z, Y/Z)$, $B(X,Y,Z) = ZP(X,Y,Z)$, $P(X,Y,Z) = Z^m p(X/Z, Y/Z)$ and $C(X,Y,Z) = YP(X,Y,Z) - XQ(X,Y,Z)$.

The equation $AdX + BdY + CdZ = 0$ defines a foliation F with singularities on the projective plane over K with K either $\mathbb{R}$ or $\mathbb{C}$. The *points at infinity* of the foliation defined by $\omega_1 = 0$ on the affine plane are the points $[X : Y : 0]$ and the line $Z = 0$ is called the *line at infinity* of the foliation with singularities generated by $\omega_1 = 0$.

The singular points of the foliation F are the solutions of the three equations $A = 0$, $B = 0$, $C = 0$. In view of the definitions of A, B, C it is clear that the singular points at infinity are the points of intersection of $Z = 0$ with $C = 0$.

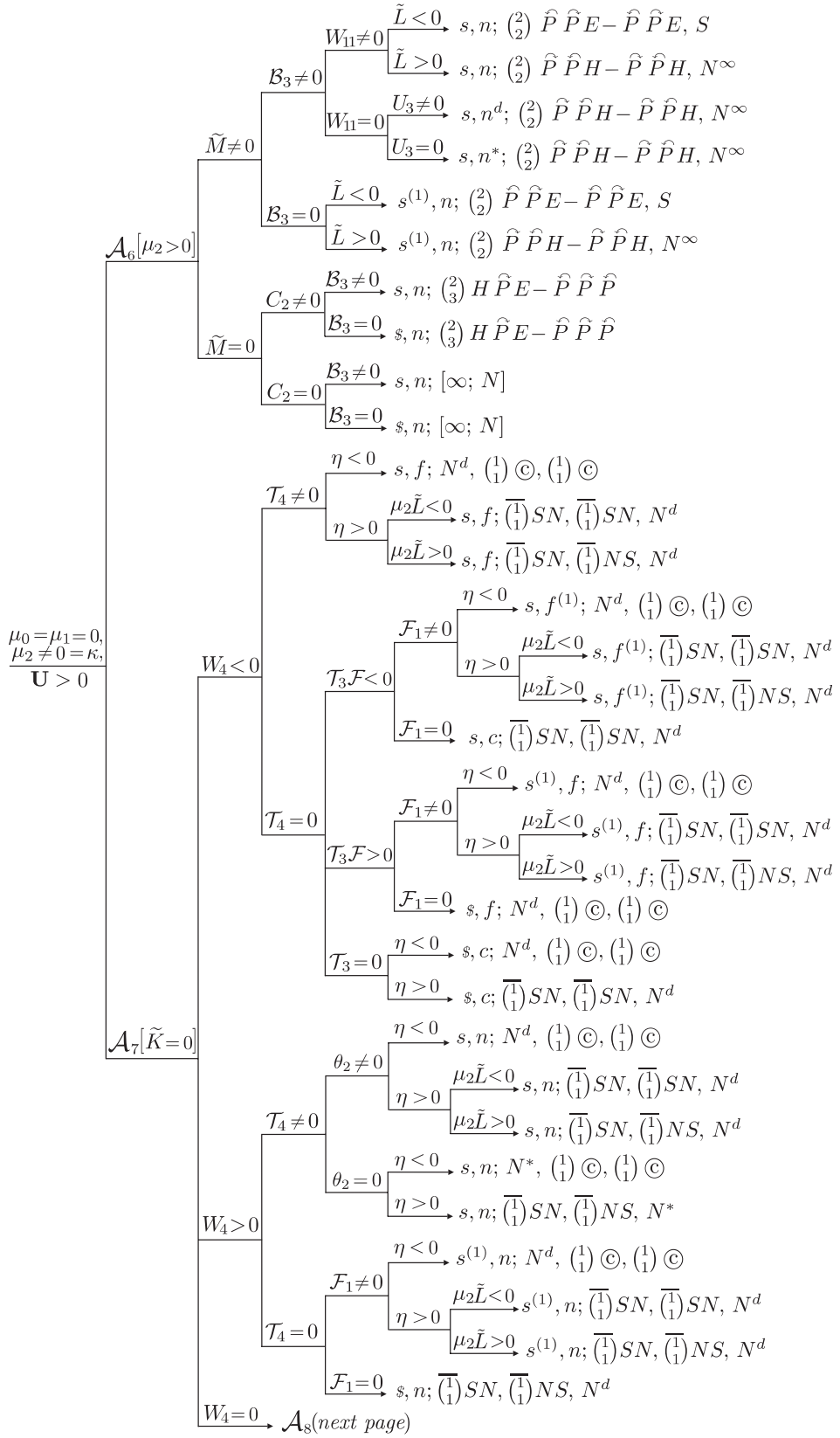


DIAGRAM 2 (*continued*). **Global configurations: the case $\boldsymbol{\mu_0 = \mu_1 = 0}$, $\boldsymbol{\mu_2 \neq 0}$, $\boldsymbol{U > 0}$**

2.1 Assembling data on infinite singularities in divisors of the line at infinity

In the previous sections we have seen that there are two types of multiplicities for a singular point p at infinity: one expresses the maximum number m of infinite singularities which can split from p, in small perturbations of the system and the other expresses the maximum number m' of finite singularities which can split from p, in small perturbations of the system. In Section 2 we mentioned that we shall use a column $(m, m')^t$ to indicate this situation.

We are interested in the global picture which includes *all* singularities at infinity. Therefore we need to assemble

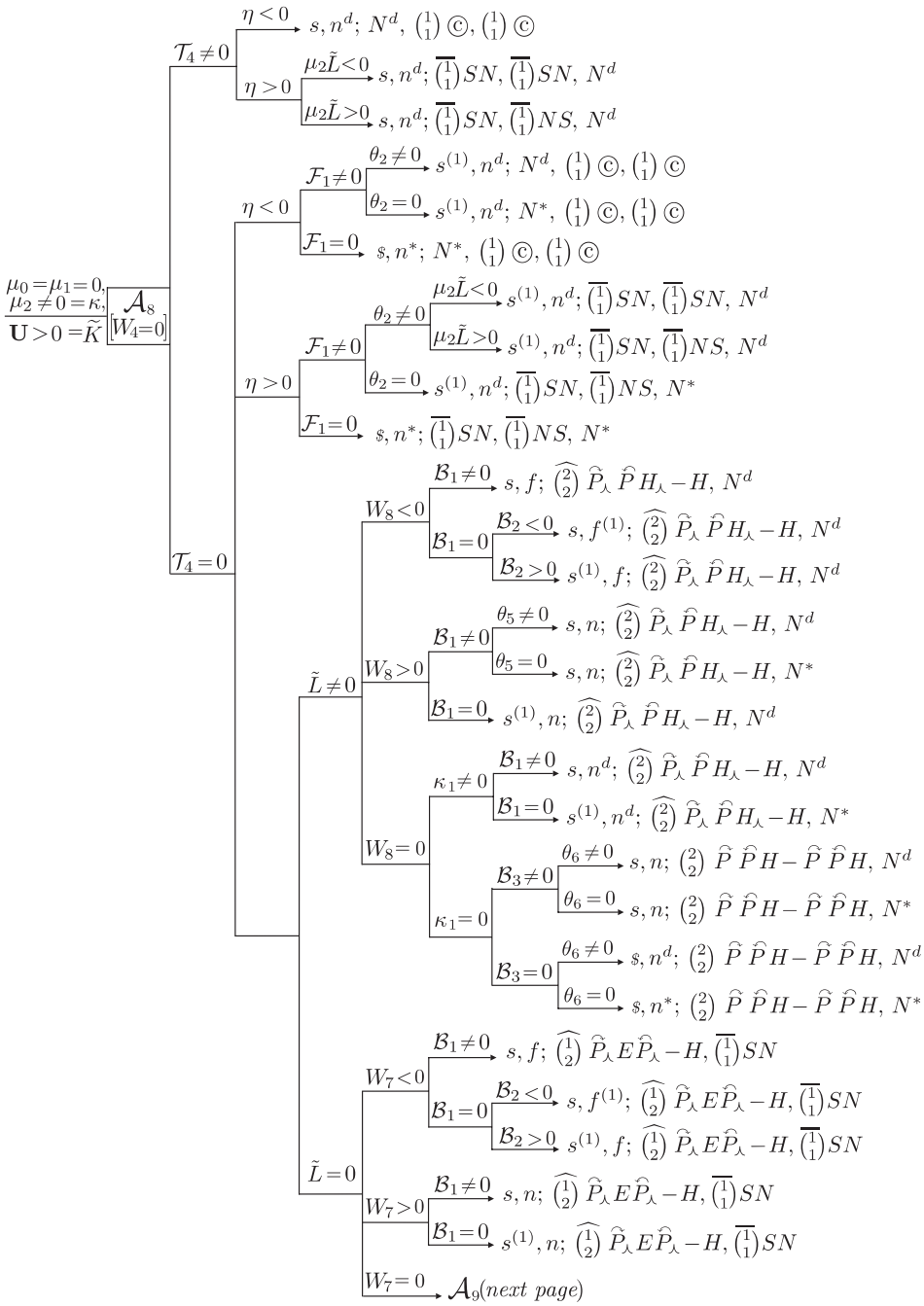


DIAGRAM 2 (*continued*). **Global configurations: the case $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$, $U > 0$**

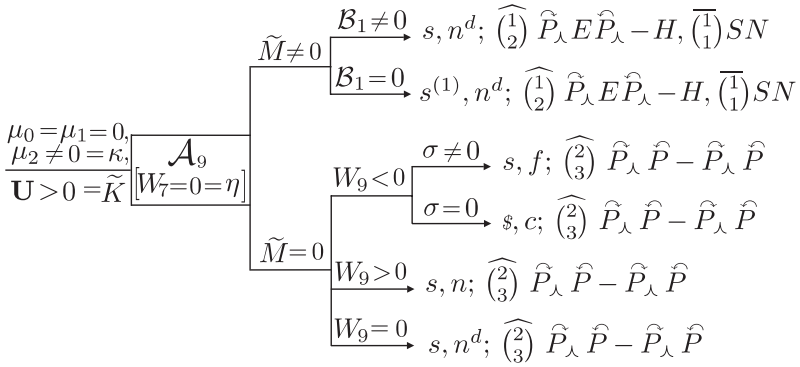


DIAGRAM 2 (*continued*). **Global configurations: the case $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$, $U > 0$**

the data for individual singularities in a convenient, precise way. To do this we use for this situation the notion of *cycle* on an algebraic variety as indicated in [**?**] and which was used in [**?**] as well as in [**?**].

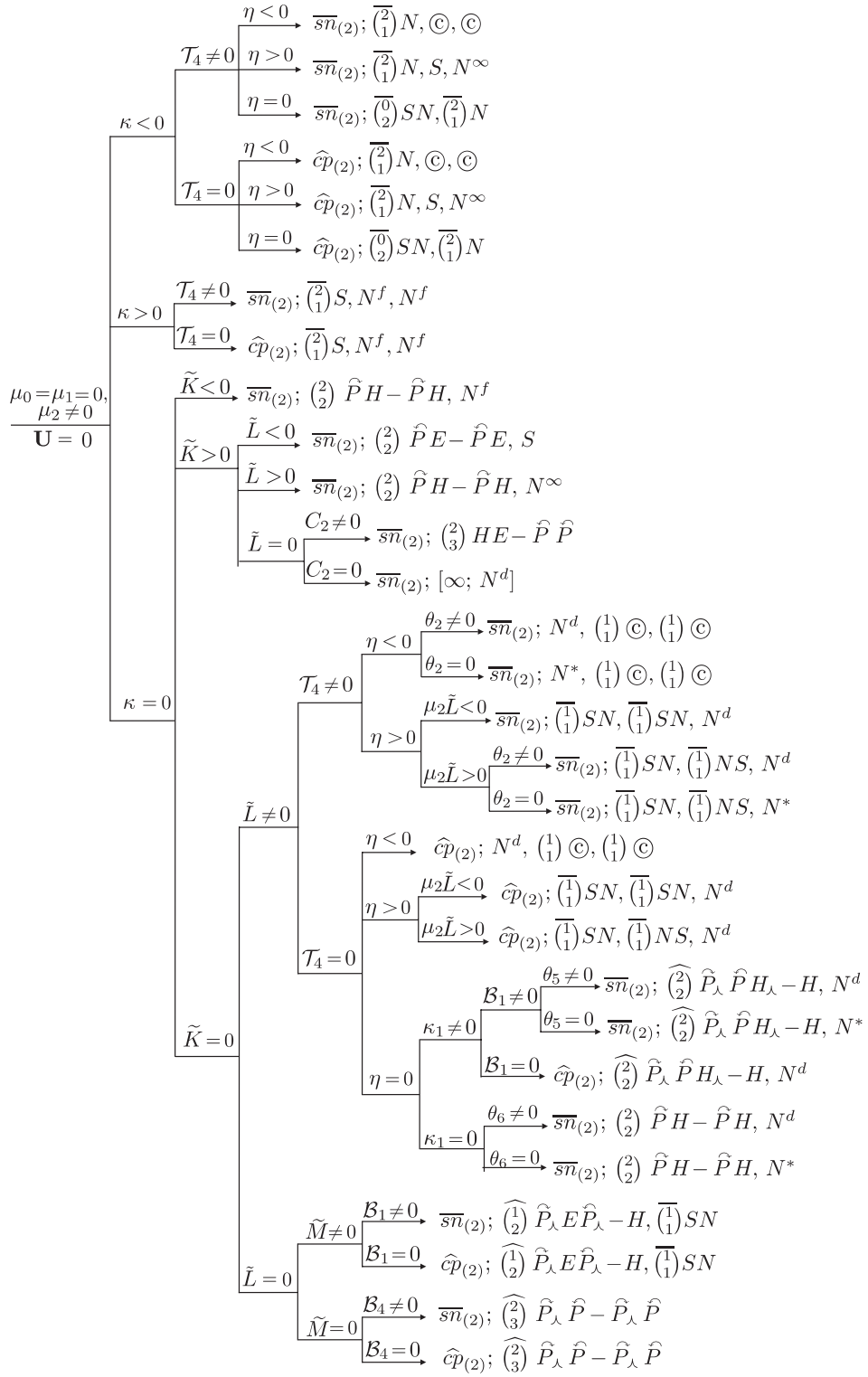


DIAGRAM 3: **Global configurations: the case $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$, $U = 0$**

We briefly recall here the definition of this notion. Let V be an irreducible algebraic variety over a field $\mathbb{K}$. A *cycle* of dimension r or $r-cycle$ on V is a formal sum $\sum_W n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of the coefficients n_W are non-zero. The degree $\deg(J)$ of a cycle J is defined by $\sum_W n_W$. An $(n-1)$-cycle is called a divisor on V. These notions were used for classification purposes of planar quadratic differential systems in [**?**, **?**, **?**].

To a system (1) we can associate two divisors on the line at infinity $Z = 0$ of the complex projective plane: $D_S(P, Q; Z) = \sum_w I_w(P, Q)w$ and $D_S(C, Z) = \sum_w I_w(C, Z)w$ where $w \in \{Z = 0\}$ and where by $I_w(F, G)$ we mean the intersection multiplicity at w of the curves $F(X, Y, Z) = 0$ and $G(X, Y, Z) = 0$, with F and G homogeneous polynomials in X, Y, Z over $\mathbb{C}$. For more details see [**?**].

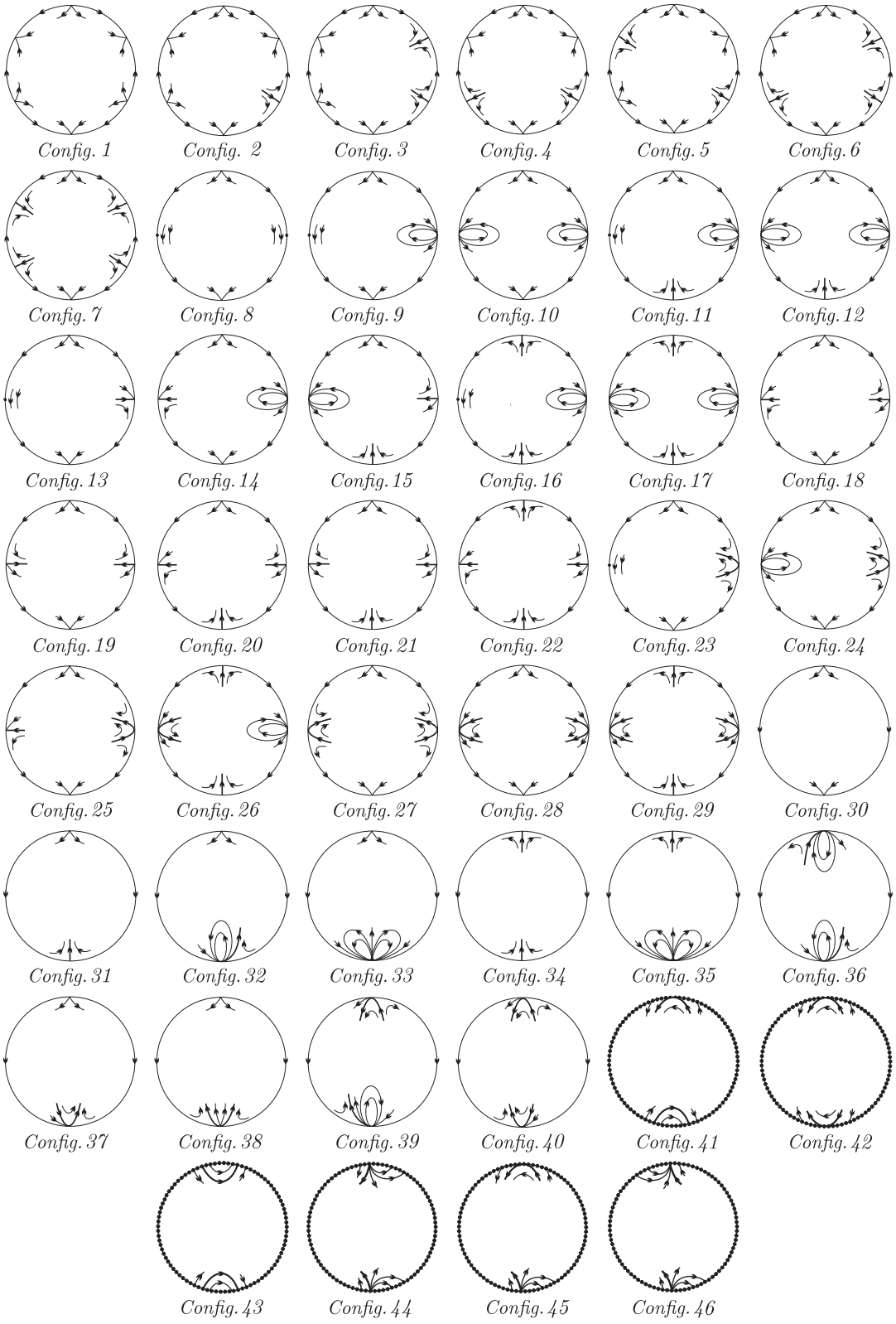


Figure 2: Topologically distinct local configurations of **ISP**s ([**?**],[**?**])

Following [**?**] we assemble the above two divisors on the line at infinity into just one but with values in the ring $\mathbb{Z}^2$:

$$D_S = \sum_{\omega \in \{Z=0\}} \begin{pmatrix} I_w(P,Q) \\ I_w(C,Z) \end{pmatrix} w.$$

This divisor encodes for us the total number of singularities at infinity of a system (1) as well as the two kinds of multiplicities which each singularity has. The meaning of these two kinds of multiplicities are described in the

definition of the two divisors $D_S(P,Q;Z)$ and $D_S(C,Z)$ on the line at infinity.

3 Some geometrical concepts

In this section we use the same concepts we considered in [3] and [4] such as *orbit γ tangent to a semi–line L at p*, *well defined angle at p*, *characteristic orbit at a singular point p*, *characteristic angle at a singular point*, *characteristic direction at p*. Since these are basic concepts for the notion of *geometric equivalence relation* we recall here these notions as well as a few others.

We assume that we have an isolated singularity p. Suppose that in a neighborhood U of p there is no other singularity. Consider an orbit γ in U defined by a solution $\Gamma(t) = (x(t), y(t))$ such that $\lim_{t\to+\infty}\Gamma(t) = p$ (or $\lim_{t\to-\infty}\Gamma(t) = p$). For a fixed t consider the unit vector $C(t) = (\overrightarrow{\Gamma(t) - p})/\|\overrightarrow{\Gamma(t) - p}\|$. Let L be a semi–line ending at p. We shall say that *the orbit γ is tangent to a semi–line L at p* if $\lim_{t\to+\infty} C(t)$ (or $\lim_{t\to-\infty} C(t)$) exists and L contains this limit point on the unit circle centered at p. In this case we call *well defined angle of Γ at p* the angle between the positive x–axis and the semi–line L measured in the counter–clockwise sense. We may also say that *the solution curve $\Gamma(t)$ tends to p with a well defined angle.* A *characteristic orbit at a singular point p* is the orbit of a solution curve $\Gamma(t)$ which tends to p with a well defined angle. We call *characteristic angle at the singular point p* a well defined angle of a solution curve $\Gamma(t)$. The line through p extending the semi-line L is called a *characteristic direction*.

If a singular point has an infinite number of characteristic directions, we will call it a *star–like* point.

It is known that the neighborhood of any isolated singular point of a polynomial vector field, which is not a focus, a center or a star-like point, is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [18]). It is also known that any degenerate singular point can be desingularized by means of a finite number of changes of variables, called blow–up's, into elementary singular points (for more details see the Section on blow–up in [3] or [18]).

Consider the three singular points given in FIGURE 3. All three are topologically equivalent and their neighborhoods can be described as having two elliptic sectors and two parabolic ones. But we can easily detect some geometric features which distinguish them. For example (a) and (b) have three characteristic directions and (c) has only two. Moreover in (a) the solution curves of the parabolic sectors are tangent to only one characteristic direction and in (b) they are tangent to two characteristic directions. All these properties can be determined algebraically.

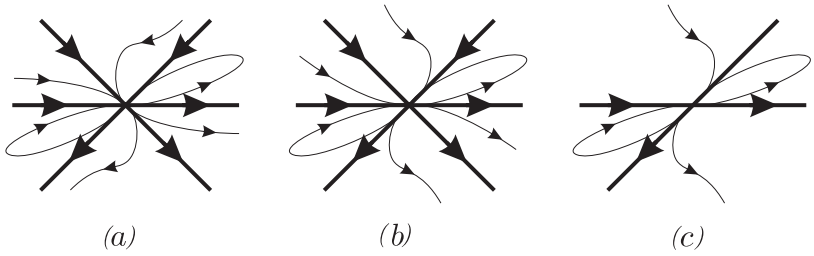


Figure 3: Some topologically equivalent singular points

The usual definition of a sector is of a topological nature and it is local with respect to a neighborhood around the singular point. We work with a new notion, namely of *geometric local sector*, introduced in [3] which distinguishes the systems of FIGURE 3. This notion is characterized by algebraic means.

We call *borsec* (contraction of border and sector) any orbit of the original system which carried through consecutive stages of the desingularization ends up as an orbit of the phase portrait in the final stage which is either a separatrix or a representative orbit of a characteristic angle of a node or a of saddle–node in the final desingularized phase portrait.

We call *geometric local sector* of a singular point p with respect to a neighborhood V as a region in V delimited by two consecutive borsecs.

To illustrate this notion consider for example, a semi–elemental saddle–node (see definition on the page 13). This can be topologically described as a singular point having two hyperbolic sectors and a single parabolic one. But if we add a borsec which is any orbit of the parabolic sector, then the description would consist of two hyperbolic sectors and two parabolic ones. This distinction will be significant when trying to describe a singular point like the one in FIGURE 4 which topologically is a saddle–node but qualitatively (in the sense of [**?**] is different from a semi–elemental saddle–node.

Generically, a *geometric local sector* is defined by two borsecs arriving at the singular point with two different well defined angles and which are consecutive. If this sector is parabolic, then the solutions can arrive at the singular

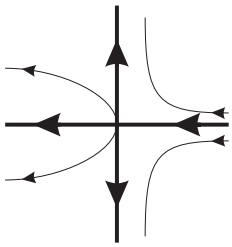
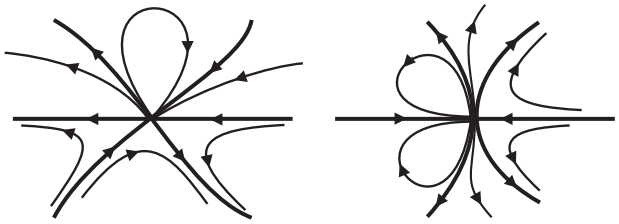


Figure 4: Local phase portrait of a saddle-node.

point with one of the two characteristic angles and this is a geometrical information than can be revealed with the blow–up.

There is also the possibility that two borsecs defining a geometric local sector tend to the singular point with the same well defined angle. Such a sector will be called a *cusp–like sector* which can either be hyperbolic, elliptic or parabolic respectively denoted by $H_\curlywedge$, $E_\curlywedge$ and $P_\curlywedge$.

Moreover, in the case of parabolic sectors we want to include the information as to whether the orbits arrive tangent to one or to the other borsec. We distinguish the two cases writing by $\widehat{P}$ if they arrive tangent to the borsec limiting the previous sector in clock–wise sense or $\widehat{\widehat{P}}$ if they arrive tangent to the borsec limiting the next sector. In the case of a cusp–like parabolic sector, all orbits must arrive with only one well determined angle, but the distinction between $\widehat{P}$ and $\widehat{\widehat{P}}$ is still valid because it occurs at some stage of the desingularization and this can be algebraically determined. Thus, complicated degenerate singular points like the two we see in FIGURE 5 may be described as $\widehat{\widehat{P}}E\,\widehat{P}\,HHH$ (case (a)) and $E\widehat{\widehat{P}}_\curlywedge HH\widehat{\widehat{P}}_\curlywedge E$ (case (b)), respectively.



(a) *(b)*

Figure 5: Two phase portraits of degenerate singular points.

A star–like point can either be a node or something much more complicated with elliptic and hyperbolic sectors included. In case there are hyperbolic sectors, they must be cusp–like. Elliptic sectors can either be cusp–like or star–like. We call *special characteristic angle* any well defined angle of a star-like point, in which either none or more than one solution curve tends to p within this well defined angle. We will call *special characteristic direction* any line such that at least one of the two angles defining it, is a special characteristic angle.

4 Notations for singularities of polynomial differential systems

In this work we limit ourselves to the class $m_f = 2$. Of the 197 *geometric configurations* of singularities we encounter in the whole quadratic class, only 6 do not appear in the subclass with $m_f = 2$. In [3] we introduced convenient notations which we also used in [4] and which we are also using here. These notations can easily be extended to general polynomial systems. Firstly let us recall some terminology we explained and used in these articles.

We call *elemental* a singular point with its both eigenvalues not zero;

We call *semi–elemental* a singular point with exactly one of its eigenvalues equal to zero;

We call *nilpotent* a singular point with both its eigenvalues zero but with its Jacobian matrix at that point not identically zero;

We call *intricate* a singular point with its Jacobian matrix identically zero.

The *intricate* singularities are usually called in the literature *linearly zero*. We use here the term *intricate* to indicate the rather complicated behavior of phase curves around such a singularity.

We describe the finite and infinite singularities, denoting the first ones with lower case letters and the second with capital letters. When describing in a sequence both finite and infinite singular points, we will always place first the finite ones and only later the infinite ones, separating them by a semicolon';'.

Elemental points: We use the letters 's','S' for "saddles"; 'n', 'N' for "nodes"; 'f' for "foci"; 'c' for "centers" and © (respectively Ⓒ) for complex finite (respectively infinite) singularities. In order to augment the level of precision we distinguish the finite nodes as follows:

- 'n' for a node with two distinct eigenvalues (generic node);
- 'n^d' (a one–direction node) for a node with two identical eigenvalues whose Jacobian matrix is not diagonal;
- 'n^*' (a star–node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

Moreover, in the case of an elemental infinite generic node, we want to distinguish whether the eigenvalue associated to the eigenvector directed towards the affine plane is, in absolute value, greater or lower than the eigenvalue associated to the eigenvector tangent to the line at infinity. This is relevant because this determines if all the orbits except one on the Poincaré disk arrive at infinity tangent to the line at infinity or transversal to this line. We will denote them as 'N^∞' and 'N^f' respectively.

Finite elemental foci and saddles are classified as strong or weak foci, respectively strong or weak saddles. When the trace of the Jacobian matrix evaluated at those singular points is not zero, we call them strong saddles and strong foci and we maintain the standard notations 's' and 'f.' But when the trace is zero, except for centers and saddles of infinite order (i.e. saddles with all their Poincaré-Lyapounov constants equal to zero), it is known that the foci and saddles, in the quadratic case, may have up to 3 orders. We denote them by '$s^{(i)}$' and '$f^{(i)}$' where $i = 1, 2, 3$ is the order. In addition we have the centers which we denote by 'c' and saddles of infinite order (integrable saddles) which we denote by '$\mathscr{s}$'.

Foci and centers cannot appear as singular points at infinity and hence there is no need to introduce their order in this case. In case of saddles, we can have weak saddles at infinity but the maximum order of weak singularities in cubic systems is not yet known. For this reason, a complete study of weak saddles at infinity cannot be done at this stage. Due to this, in [3] and in [4] and here we chose not even to distinguish between a saddle and a weak saddle at infinity.

All non–elemental singular points are multiple points, in the sense that there are perturbations which have at least two elemental singular points as close as we wish to the multiple point. For finite singular points we denote with a subindex their multiplicity as in '$\overline{s}_{(5)}$' or in '$\widehat{es}_{(3)}$' (the notation '$\overline{\ }$' indicates that the saddle is semi–elemental and '$\widehat{es}_{(3)}$' indicates that the singular point is nilpotent). In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [**?**]. Thus we denote by '$\binom{a}{b}$...' the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example '$\overline{\binom{1}{1}}SN$' means a saddle–node at infinity produced by the collision of one finite singularity with an infinite one; '$\overline{\binom{0}{3}}S$' means a saddle produced by the collision of 3 infinite singularities.

Semi–elemental points: They can either be nodes, saddles or saddle–nodes, finite or infinite. We will denote the semi–elemental ones always with an overline, for example '$\overline{sn}$', '$\overline{s}$' and '$\overline{n}$' with the corresponding multiplicity. In the case of infinite points we will put '$\overline{\ }$' on top of the parenthesis with multiplicities.

Moreover, in cases that will be explained later (see page 15), an infinite saddle–node may be denoted by '$\overline{\binom{1}{1}}NS$' instead of '$\overline{\binom{1}{1}}SN$'. Semi–elemental nodes could never be 'n^d' or 'n^*' since their eigenvalues are always different. In case of an infinite semi–elemental node, the type of collision determines whether the point is denoted by 'N^f' or by 'N^∞' where '$\overline{\binom{2}{1}}N$' is an 'N^f' and '$\overline{\binom{0}{3}}N$' is an 'N^∞'.

Nilpotent points: They can either be saddles, nodes, saddle–nodes, elliptic–saddles, cusps, foci or centers. The first four of these could be at infinity. We denote the nilpotent singular points with a hat '$\widehat{\ }$' as in $\widehat{es}_{(3)}$ for a finite nilpotent elliptic–saddle of multiplicity 3 and $\widehat{cp}_{(2)}$ for a finite nilpotent cusp point of multiplicity 2. In the case of nilpotent infinite points, we will put the '$\widehat{\ }$' on top of the parenthesis with multiplicity, for example $\widehat{\binom{1}{2}}\,PEP - H$ (the meaning of $PEP - H$ will be explained in next paragraph). The relative position of the sectors of an infinite nilpotent point, with respect to the line at infinity, can produce topologically different phase portraits. This forces us to use a notation for these points similar to the notation which we will use for the intricate points.

Intricate points: It is known that the neighborhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [18]). Then, a reasonable way to describe intricate and nilpotent points is to use a sequence formed by the types of their sectors. The description we give is the one which appears in the clock–wise direction (starting anywhere) once the blow–down of the desingularization is done. Thus in non degenerate quadratic systems, we have just seven possibilities for finite intricate singular points of multiplicity four (see [5]) which are the following ones: $phpphp_{(4)}$; $phph_{(4)}$; $hh_{(4)}$; $hhhhhh_{(4)}$; $peppep_{(4)}$; $pepe_{(4)}$; $ee_{(4)}$.

The lower case letters used here indicate that we have finite singularities and subindex (4) indicates the multiplicity 4 of the singularities.

For infinite intricate and nilpotent singular points, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. In this way we will distinguish between $\binom{2}{2}PHP - PHP$ and $\binom{2}{2}PPH - PPH$.

Whenever we have an infinite nilpotent or intricate singular point, we will always start with a sector bordering the infinity (to avoid using two dashes). When one needs to describe a configuration of singular points at infinity, then in some cases the relative positions of the points, is relevant. In [3] this situation only occurs once for systems having two semi–elemental saddle–nodes at infinity and a third singular point which is elemental. In this case we need to write NS instead of SN for one of the semi–elemental points in order to have coherence of the positions of the parabolic (nodal) sector of one point with respect to the hyperbolic (saddle) of the other semi–elemental point. More concretely, *Figure* 3 from [**?**] (which corresponds to *Config. 3* in FIGURE 2) must be described as $\overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}SN, N$ since the elemental node lies always between the hyperbolic sectors of one saddle–node and the parabolic ones of the other. However, *Figure* 4 from [**?**] (which corresponds to *Config. 4* in FIGURE 2) must be described as $\overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}NS, N$ since the hyperbolic sectors of each saddle–node lie between the elemental node and the parabolic sectors of the other saddle–node. These two configurations have exactly the same description of singular points but their relative position produces topologically (and geometrically) different portraits.

For the description of the topological phase portraits around the isolated singular points the information described above is sufficient. However we are interested in additional geometrical features such as the number of characteristic directions which figure in the final global picture of the desingularization. In order to add this information we need to introduce more notation. If two borsecs (the limiting orbits of a sector) arrive at the singular point with the same direction, then the sector will be denoted by $H_\curlywedge$, $E_\curlywedge$ or $P_\curlywedge$. The index in this notation refers to the cusp–like form of limiting trajectories of the sectors. Moreover, in the case of parabolic sectors we want to make precise whether the orbits arrive tangent to one borsec or to the other. We distinguish the two cases by $\widehat{P}$ if they arrive tangent to the borsec limiting the previous sector in clock–wise sense or $\widehat{P}$ if they arrive tangent to the borsec limiting the next sector. A parabolic sectorwill be P^* when all orbits orbits arrive with all possible slopes between the two consecutive borsecs. In the case of a cusp–like parabolic sector, all orbits must arrive with only one direction, but the distinction between $\widehat{P}$ and $\widehat{P}$ is still valid if we consider the different desingularizations we obtain from them. Thus, complicated intricate singular points like the two we see in FIGURE 5 may be described as $\binom{4}{2}\ \widehat{P}E\widehat{P} - HHH$ (case (a)) and $\binom{4}{3}\ E\widehat{P}_\curlywedge H - H\widehat{P}_\curlywedge E$ (case (b)), respectively.

Finally there is also the possibility that we have an infinite number of infinite singular points.

Line at infinity filled up with singularities: It is known that any such system has in a sufficiently small neighborhood of infinity one of 6 topological distinct phase portraits (see [**?**]). The way to determine these portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. In case a singular point still remains on the line at infinity we study such a point. In [**?**] the tangential behavior of the solution curves was not considered in the case of a node. If after the removal of the degeneracy in the local charts at infinity a node remains, this could either be of the type N^d, N and $N^\star$ (this last case does not occur in quadratic systems as it was shown in [3]). Since no eigenvector of such a node N (for quadratic systems) will have the direction of the line at infinity we do not need to distinguish N^f and N^∞. Other types of singular points at infinity of quadratic systems, after removal of the degeneracy, can be saddles, centers, semi–elemental saddle–nodes or nilpotent elliptic–saddles. We also have the possibility of no singularities after the removal of the degeneracy. To convey the way these singularities were obtained as well as their nature, we use the notation $[\infty;\ \emptyset]$, $[\infty;\ N]$, $[\infty;\ N^d]$, $[\infty;\ S]$, $[\infty;\ C]$, $[\infty; \overline{\binom{1}{0}}SN]$ or $[\infty; \widehat{\binom{3}{0}}ES]$.

5 Invariant polynomials and preliminary results

Consider real quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{2}$$

with homogeneous polynomials p_i and q_i $(i = 0, 1, 2)$ of degree i in x, y:

$$\begin{aligned} p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20},\ b_{11}, b_{02})$ be the 12-tuple of the coefficients of systems (2) and denote $\mathbb{R}[\tilde{a}, x, y] = \mathbb{R}[a_{00}, \ldots, b_{02}, x, y]$.

5.1 Affine invariant polynomials associated to infinite singularities

It is known that on the set **QS** of all quadratic differential systems (2) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [**?**]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (2) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (2) the 12–tuple $(a_{00},\ldots,b_{02})$ of its coefficients.

For the definitions of a GL–comitant and invariant as well as for the definitions of a T–comitant and a CT–comitant we refer the reader to the paper [**?**] (see also [**?**]). Here we shall only construct the necessary T–comitants and CT–comitants associated to configurations of infinite singularities (including multiplicities) of quadratic systems (2). All polynomials constructed here are GL–comitants. But some are also affine comitants or even affine invariants.

Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P^* + \beta Q^* \in \mathbb{R}[\tilde{a}, X, Y, Z, \alpha, \beta]$ where $P^* = Z^2 P(X/Z, Y/Z)$, $Q^* = Z^2 Q(X/Z, Y/Z)$, P, $Q \in \mathbb{R}[\tilde{a}, x, y]$ and $\max(\deg_{(x,y)} P, \deg_{(x,y)} Q) = 2$. Then

$$\Phi_{\alpha,\beta} = s_{11}(\tilde{a},\alpha,\beta)X^2 + 2s_{12}(\tilde{a},\alpha,\beta)XY + s_{22}(\tilde{a},\alpha,\beta)Y^2 + 2s_{13}(\tilde{a},\alpha,\beta)XZ + 2s_{23}(\tilde{a},\alpha,\beta)YZ + s_{33}(\tilde{a},\alpha,\beta)Z^2$$

and we denote

$$\begin{aligned}\widetilde{D}(\tilde{a},x,y) =& 4\det||s_{ij}(\tilde{a},y,-x)||_{i,j\in\{1,2,3\}},\\ \widetilde{H}(\tilde{a},x,y) =& 4\det||s_{ij}(\tilde{a},y,-x)||_{i,j\in\{1,2\}}.\end{aligned}$$

We consider the polynomials

$$\begin{aligned}C_i(\tilde{a},x,y) &= yp_i(\tilde{a},x,y) - xq_i(\tilde{a},x,y),\\ D_i(\tilde{a},x,y) &= \frac{\partial}{\partial x}p_i(\tilde{a},x,y) + \frac{\partial}{\partial y}q_i(\tilde{a},x,y),\end{aligned} \tag{3}$$

in $\mathbb{R}[\tilde{a},x,y]$ for $i = 0,1,2$ and $i = 1,2$ respectively. The polynomials C_2 and D_2 are trivial T-comitants. Using the so–called *transvectant of order k* (see [**?**], [**?**]) of two polynomials f, $g \in \mathbb{R}[\tilde{a},x,y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}},$$

we construct the following GL—comitants of the second degree with the coefficients of the initial system

$$\begin{array}{lll} T_1 = (C_0,C_1)^{(1)}, & T_2 = (C_0,C_2)^{(1)}, & T_3 = (C_0,D_2)^{(1)},\\ T_4 = (C_1,C_1)^{(2)}, & T_5 = (C_1,C_2)^{(1)}, & T_6 = (C_1,C_2)^{(2)},\\ T_7 = (C_1,D_2)^{(1)}, & T_8 = (C_2,C_2)^{(2)}, & T_9 = (C_2,D_2)^{(1)}. \end{array} \tag{4}$$

$T_i = T_i(\tilde{a})$ for $i = 1,...,4$ and $T_i = T_i(\tilde{a},x,y)$ for $i = 5,...,9$. Using these GL—comitants as well as the polynomials (3) we construct the additional invariant polynomials (see also [**?**])

$$\begin{aligned}\widetilde{M}(\tilde{a},x,y) =& (C_2,C_2)^{(2)} \equiv 2\mathrm{Hess}\big(C_2(\tilde{a},x,y)\big);\\ \eta(\tilde{a}) =& (\widetilde{M},\widetilde{M})^{(2)}/384 \equiv \mathrm{Discrim}\big(C_2(\tilde{a},x,y)\big);\\ \widetilde{K}(\tilde{a},x,y) =& \mathrm{Jacob}\big(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\big);\\ K_1(\tilde{a},x,y) =& p_1(\tilde{a},x,y)q_2(\tilde{a},x,y) - p_2(\tilde{a},x,y)q_1(\tilde{a},x,y);\\ K_2(\tilde{a},x,y) =& 4(T_2,\widetilde{M}-2\widetilde{K})^{(1)} + 3D_1(C_1,\widetilde{M}-2\widetilde{K})^{(1)} - (\widetilde{M}-2\widetilde{K})\big(16T_3 - 3T_4/2 + 3D_1^2\big);\\ K_3(\tilde{a},x,y) =& C_2^2(4T_3+3T_4) + C_2(3C_0\widetilde{K} - 2C_1T_7) + 2K_1(3K_1 - C_1D_2);\\ \tilde{L}(\tilde{a},x,y) =& 4\widetilde{K} + 8\widetilde{H} - \widetilde{M};\\ L_1(\tilde{a},x,y) =& (C_2,\widetilde{D})^{(2)};\end{aligned}$$

$$\begin{aligned}\widetilde{R}(\tilde{a},x,y) =& \tilde{L} + 8\widetilde{K};\\ \kappa(\tilde{a}) =& (\widetilde{M},\widetilde{K})^{(2)}/4;\\ \kappa_1(\tilde{a}) =& (\widetilde{M},C_1)^{(2)};\\ \widetilde{N}(\tilde{a},x,y) =& \widetilde{K}(\tilde{a},x,y) + \widetilde{H}(\tilde{a},x,y);\\ \theta_6(\tilde{a},x,y) =& C_1T_8 - 2C_2T_6.\end{aligned}$$

The geometrical meaning of the invariant polynomials C_2, $\widetilde{M}$ and η is revealed in the next lemma (see [**?**]).

Table 1

Case	Form of $D_S(C,Z)$	Necessary and sufficient conditions on the comitants
1	$w_1+w_2+w_3$	$\eta>0$
2	$w_1^c+w_2^c+w_3$	$\eta<0$
3	$2w_1+w_2$	$\eta=0, \quad \widetilde{M}\neq 0$
4	$3w$	$\widetilde{M}=0, \quad C_2\neq 0$
5	$D_S(C,Z)$ undefined	$C_2=0$

Lemma 1 *The form of the divisor $D_S(C,Z)$ for systems (2) is determined by the corresponding conditions indicated in Table 1, where we write $w_1^c+w_2^c+w_3$ if two of the points, i.e. w_1^c, w_2^c, are complex but not real. Moreover, for each form of the divisor $D_S(C,Z)$ given in Table 1 the quadratic systems (2) can be brought via a linear transformation to one of the following canonical systems $(\mathbf{S}_I)-(\mathbf{S}_V)$ corresponding to their behavior at infinity.*

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+(h-1)xy, \\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+(h+1)xy, \\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+hxy, \\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+hxy, \\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2, \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} &= a+cx+dy+x^2, \\ \dot{y} &= b+ex+fy+xy. \end{cases} \qquad (\mathbf{S}_V)$$

5.2 Affine invariant polynomials associated to finite singularities

Consider the differential operator $\mathcal{L}=x\cdot\mathbf{L}_2-y\cdot\mathbf{L}_1$ acting on $\mathbb{R}[a,x,y]$ constructed in [10], where

$$\begin{aligned} \mathbf{L}_1 = \;& 2a_{00}\tfrac{\partial}{\partial a_{10}}+a_{10}\tfrac{\partial}{\partial a_{20}}+\tfrac{1}{2}a_{01}\tfrac{\partial}{\partial a_{11}}+2b_{00}\tfrac{\partial}{\partial b_{10}}+b_{10}\tfrac{\partial}{\partial b_{20}}+\tfrac{1}{2}b_{01}\tfrac{\partial}{\partial b_{11}}, \\ \mathbf{L}_2 = \;& 2a_{00}\tfrac{\partial}{\partial a_{01}}+a_{01}\tfrac{\partial}{\partial a_{02}}+\tfrac{1}{2}a_{10}\tfrac{\partial}{\partial a_{11}}+2b_{00}\tfrac{\partial}{\partial b_{01}}+b_{01}\tfrac{\partial}{\partial b_{02}}+\tfrac{1}{2}b_{10}\tfrac{\partial}{\partial b_{11}}. \end{aligned}$$

Using this operator and the affine invariant $\mu_0=\operatorname{Res}_x\big(p_2(\tilde{a},x,y),q_2(\tilde{a},x,y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y)=\frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i=1,..,4,$$

where $\mathcal{L}^{(i)}(\mu_0)=\mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$.

These polynomials are in fact comitants of systems (2) with respect to the group $GL(2,\mathbb{R})$ (see [10]). Their geometrical meaning is revealed in Lemmas 2 and 3 below.

Lemma 2 ([9]) *The total multiplicity of all finite singularities of a quadratic system (2) equals k if and only if for every $i\in\{0,1,\dots,k-1\}$ we have $\mu_i(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ and $\mu_k(\tilde{a},x,y)\neq 0$. Moreover a system (2) is degenerate (i.e. $\gcd(P,Q)\neq$ constant) if and only if $\mu_i(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ for every $i=0,1,2,3,4$.*

Lemma 3 ([10]) *The point $M_0(0,0)$ is a singular point of multiplicity k $(1\leq k\leq 4)$ for a quadratic system (2) if and only if for every $i\in\{0,1,\dots,k-1\}$ we have $\mu_{4-i}(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ and $\mu_{4-k}(\tilde{a},x,y)\neq 0$.*

We denote

$$\sigma(\tilde{a},x,y)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=\sigma_0(\tilde{a})+\sigma_1(\tilde{a},x,y)\ (\equiv D_1(\tilde{a})+D_2(\tilde{a},x,y))$$

and observe that the polynomial $\sigma(\tilde{a},x,y)$ is an affine comitant of systems (2). It is known, that if (x_i,y_i) is a singular point of a system (2) then for the trace of its respective linear matrix we have $\rho_i=\sigma(x_i,y_i)$.

Applying the differential operators $\mathcal{L}$ and $(*,*)^{(k)}$ (i.e. transvectant of index k) we define the following polynomial function which governs the values of the traces for finite singularities of systems (2).

Definition 1 ([?]) We call *trace polynomial* $\mathfrak{T}(w)$ over the ring $\mathbb{R}[\tilde{a}]$ the polynomial defined as follows:

$$\mathfrak{T}(w)=\sum_{i=0}^{4}\frac{1}{(i!)^2}\left(\sigma_1^i,\ \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)\right)^{(i)}\ w^{4-i}=\sum_{i=0}^{4}\mathcal{G}_i(\tilde{a})\,w^{4-i}, \tag{5}$$

where the coefficients $\mathcal{G}_i(\tilde{a})=\dfrac{1}{(i!)^2}(\sigma_1^i,\ \mu_i)^{(i)}\in\mathbb{R}[\tilde{a}],\quad i=0,1,2,3,4\ \big(\mathcal{G}_0(\tilde{a})\equiv\mu_0(\tilde{a})\big)$ are GL–invariants.

Using the polynomial $\mathfrak{T}(w)$ we could construct the following four affine invariants $\mathcal{T}_4,\ \mathcal{T}_3,\ \mathcal{T}_2,\ \mathcal{T}_1$, which are responsible for the weak singularities:

$$\mathcal{T}_{4-i}(\tilde{a})=\frac{1}{i!}\,\frac{d^i\mathfrak{T}}{dw^i}\Big|_{w=\sigma_0}\in\mathbb{R}[\tilde{a}],\quad i=0,1,2,3\quad\big(\mathcal{T}_4\equiv\mathfrak{T}(\sigma_0)\big).$$

The geometric meaning of these invariants is revealed by the next lemma (see [?]).

Lemma 4 *Consider a non-degenerate system* (2) *and let* $\boldsymbol{a}\in\mathbb{R}^{12}$ *be its 12-tuple of coefficients. Denote by* ρ_s *the trace of the linear part of this system at a finite singular point* M_s, $1\le s\le 4$ *(real or complex, simple or multiple). Then the following relations hold, respectively:*
(i) For $\mu_0(\boldsymbol{a})\neq 0$ *(total multiplicity 4):*

$$\begin{aligned}
&\mathcal{T}_4(\boldsymbol{a})=\mathcal{G}_0(\boldsymbol{a})\rho_1\rho_2\rho_3\rho_4,\\
&\mathcal{T}_3(\boldsymbol{a})=\mathcal{G}_0(\boldsymbol{a})(\rho_1\rho_2\rho_3+\rho_1\rho_2\rho_4+\rho_1\rho_3\rho_4+\rho_2\rho_3\rho_4),\\
&\mathcal{T}_2(\boldsymbol{a})=\mathcal{G}_0(\boldsymbol{a})(\rho_1\rho_2+\rho_1\rho_3+\rho_1\rho_4+\rho_2\rho_3+\rho_2\rho_4+\rho_3\rho_4),\\
&\mathcal{T}_1(\boldsymbol{a})=\mathcal{G}_0(\boldsymbol{a})(\rho_1+\rho_2+\rho_3+\rho_4);
\end{aligned} \tag{6}$$

(ii) For $\mu_0(\boldsymbol{a})=0,\ \mu_1(\boldsymbol{a},x,y)\neq 0$ *(total multiplicity 3):*

$$\begin{aligned}
&\mathcal{T}_4(\boldsymbol{a})=\mathcal{G}_1(\boldsymbol{a})\rho_1\rho_2\rho_3, && \mathcal{T}_3(\boldsymbol{a})=\mathcal{G}_1(\boldsymbol{a})(\rho_1\rho_2+\rho_1\rho_3+\rho_2\rho_3),\\
&\mathcal{T}_2(\boldsymbol{a})=\mathcal{G}_1(\boldsymbol{a})(\rho_1+\rho_2+\rho_3), && \mathcal{T}_1(\boldsymbol{a})=\mathcal{G}_1(\boldsymbol{a});
\end{aligned} \tag{7}$$

(iii) For $\mu_0(\boldsymbol{a})=\mu_1(\boldsymbol{a},x,y)=0,\ \mu_2(\boldsymbol{a},x,y)\neq 0$ *(total multiplicity 2):*

$$\begin{aligned}
&\mathcal{T}_4(\boldsymbol{a})=\mathcal{G}_2(\boldsymbol{a})\rho_1\rho_2, && \mathcal{T}_3(\boldsymbol{a})=\mathcal{G}_2(\boldsymbol{a})(\rho_1+\rho_2),\\
&\mathcal{T}_2(\boldsymbol{a})=\mathcal{G}_2(\boldsymbol{a}), && \mathcal{T}_1(\boldsymbol{a})=0;
\end{aligned} \tag{8}$$

(iv) For $\mu_0(\boldsymbol{a})=\mu_1(\boldsymbol{a},x,y)=\mu_2(\boldsymbol{a},x,y)=0,\ \mu_3(\boldsymbol{a},x,y)\neq 0$ *(one elemental singularity):*

$$\mathcal{T}_4(\boldsymbol{a})=\mathcal{G}_3(\boldsymbol{a})\rho_1,\quad \mathcal{T}_3(\boldsymbol{a})=\mathcal{G}_3(\boldsymbol{a}),\quad \mathcal{T}_2(\boldsymbol{a})=\mathcal{T}_1(\boldsymbol{a})=0. \tag{9}$$

In order to be able to calculate the values of the invariant polynomials we need, directly for every canonical system, we define here a family of T–comitants (see [?] for detailed definitions) expressed through C_i $(i=0,1,2)$ and D_j $(j=1,2)$:

$$\begin{aligned}
\hat{A}&=\left(C_1,T_8-2T_9+D_2^2\right)^{(2)}/144,\\
\widehat{D}&=\Big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2\Big]/36,\\
\widehat{E}&=\Big[D_1(2T_9-T_8)-3\,(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\Big]/72,\\
\widehat{F}&=\big[6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)\ +48C_0\,(D_2,T_9)^{(1)}-9D_2^2T_4+288D_1\widehat{E}\\
&\quad-24\left(C_2,\widehat{D}\right)^{(2)}+120\left(D_2,\widehat{D}\right)^{(1)}-36C_1\,(D_2,T_7)^{(1)}+8D_1\,(D_2,T_5)^{(1)}\Big]/144,\\
\widehat{B}&=\Big\{16D_1\,(D_2,T_8)^{(1)}\,(3C_1D_1-2C_0D_2+4T_2)\ +32C_0\,(D_2,T_9)^{(1)}\,(3D_1D_2-5T_6+9T_7)\\
&\quad+2\,(D_2,T_9)^{(1)}\left(27C_1T_4-18C_1D_1^2\ -32D_1T_2+32\,(C_0,T_5)^{(1)}\ \right)\\
&\quad+6\,(D_2,T_7)^{(1)}\,[8C_0(T_8-12T_9)\ -12C_1(D_1D_2+T_7)+D_1(26C_2D_1+32T_5)\ +C_2(9T_4+96T_3)]\\
&\quad+6\,(D_2,T_6)^{(1)}\,[32C_0T_9-C_1(12T_7+52D_1D_2)\ -32C_2D_1^2]+48D_2\,(D_2,T_1)^{(1)}\left(2D_2^2-T_8\right)\\
&\quad-32D_1T_8\,(D_2,T_2)^{(1)}+9D_2^2T_4\,(T_6-2T_7)-16D_1\,(C_2,T_8)^{(1)}\left(D_1^2+4T_3\right)\\
&\quad+12D_1\,(C_1,T_8)^{(2)}\,(C_1D_2-2C_2D_1)+6D_1D_2T_4\left(T_8-7D_2^2-42T_9\right)\\
&\quad+12D_1\,(C_1,T_8)^{(1)}\,(T_7+2D_1D_2)+96D_2^2\left[D_1\,(C_1,T_6)^{(1)}+D_2\,(C_0,T_6)^{(1)}\right]\\
&\quad-16D_1D_2T_3\left(2D_2^2+3T_8\right)-4D_1^3D_2\left(D_2^2+3T_8+6T_9\right)+6D_1^2D_2^2\,(7T_6+2T_7)\\
&\quad-252D_1D_2T_4T_9\big\}/(2^83^3).
\end{aligned}$$

$$\begin{aligned}
\widehat{K} =&(T_8+4T_9+4D_2^2)/72 \quad \equiv \widetilde{K}/4,\\
\widehat{H} =&(8T_9-T_8+2D_2^2)/72 \equiv -\widetilde{H}/4,\\
\widehat{M} =&T_8.
\end{aligned}$$

These polynomials in addition to (3) and (4) will serve as bricks in constructing affine invariant polynomials for systems (2).

The following 42 affine invariants $A_1,\dots,A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [12] by constructing $A_1,\dots,A_{42}$ using the above bricks.

$$\begin{aligned}
A_1 &= \hat{A}, & A_{22} &= \tfrac{1}{1152}\big[C_2,\widehat{D})^{(1)},D_2\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}D_2\big)^{(1)},\\
A_2 &= (C_2,\widehat{D})^{(3)}/12, & A_{23} &= \big[\widehat{F},\widehat{H})^{(1)},\widehat{K}\big)^{(2)}/8,\\
A_3 &= \big[C_2,D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_{24} &= \big[C_2,\widehat{D})^{(2)},\widehat{K}\big)^{(1)},\widehat{H}\big)^{(2)}/32,\\
A_4 &= (\widehat{H},\widehat{H})^{(2)}, & A_{25} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(2)}/16,\\
A_5 &= (\widehat{H},\widehat{K})^{(2)}/2, & A_{26} &= (\widehat{B},\widehat{D})^{(3)}/36,\\
A_6 &= (\widehat{E},\widehat{H})^{(2)}/2, & A_{27} &= \big[\widehat{B},D_2)^{(1)},\widehat{H}\big)^{(2)}/24,\\
A_7 &= \big[C_2,\widehat{E})^{(2)},D_2\big)^{(1)}/8, & A_{28} &= \big[C_2,\widehat{K})^{(2)},\widehat{D}\big)^{(1)},\widehat{E}\big)^{(2)}/16,\\
A_8 &= \big[\widehat{D},\widehat{H})^{(2)},D_2\big)^{(1)}/8, & A_{29} &= \big[\widehat{D},\widehat{F})^{(1)},\widehat{D}\big)^{(3)}/96,\\
A_9 &= \big[\widehat{D},D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_{30} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/288,\\
A_{10} &= \big[\widehat{D},\widehat{K})^{(2)},D_2\big)^{(1)}/8, & A_{31} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{11} &= (\widehat{F},\widehat{K})^{(2)}/4, & A_{32} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{H}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{12} &= (\widehat{F},\widehat{H})^{(2)}/4, & A_{33} &= \big[\widehat{D},D_2)^{(1)},\widehat{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{13} &= \big[C_2,\widehat{H})^{(1)},\widehat{H}\big)^{(2)},D_2\big)^{(1)}/24, & A_{34} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{K}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{14} &= (\widehat{B},C_2)^{(3)}/36, & A_{35} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{15} &= (\widehat{E},\widehat{F})^{(2)}/4, & A_{36} &= \big[\widehat{D},\widehat{E})^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/16,\\
A_{16} &= \big[\widehat{E},D_2)^{(1)},C_2\big)^{(1)},\widehat{K}\big)^{(2)}/16, & A_{37} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/576,\\
A_{17} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}/64, & A_{38} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{18} &= \big[\widehat{D},\widehat{F})^{(2)},D_2\big)^{(1)}/16, & A_{39} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{19} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{H}\big)^{(2)}/16, & A_{40} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{K}\big)^{(2)}/64,\\
A_{20} &= \big[C_2,\widehat{D})^{(2)},\widehat{F}\big)^{(2)}/16, & A_{41} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{21} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(2)}/16, & A_{42} &= \big[\widehat{D},\widehat{F})^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/16.
\end{aligned}$$

In the above list, the bracket "[" is used in order to avoid placing the otherwise necessary up to five parenthesizes "(".

Using the elements of the minimal polynomial basis given above we construct the affine invariants

$$\begin{aligned}
\mathcal{F}_1(\tilde{a}) =&A_2,\\
\mathcal{F}_2(\tilde{a}) =&-2A_1^2A_3+2A_5(5A_8+3A_9)+A_3(A_8-3A_{10}+3A_{11}+A_{12})-\\
&-A_4(10A_8-3A_9+5A_{10}+5A_{11}+5A_{12}),\\
\mathcal{F}_3(\tilde{a}) =&-10A_1^2A_3+2A_5(A_8-A_9)-A_4(2A_8+A_9+A_{10}+A_{11}+A_{12})+\\
&+A_3(5A_8+A_{10}-A_{11}+5A_{12}),\\
\mathcal{F}_4(\tilde{a}) =&\,20A_1^2A_2-A_2(7A_8-4A_9+A_{10}+A_{11}+7A_{12})+A_1(6A_{14}-22A_{15})-4A_{33}+4A_{34},\\
\mathcal{F}(\tilde{a}) =&A_7,\\
\mathcal{B}(\tilde{a}) =&-(3A_8+2A_9+A_{10}+A_{11}+A_{12}),\\
\mathcal{H}(\tilde{a}) =&-(A_4+2A_5),
\end{aligned}$$

as well as the GL-comitants:

$$\begin{aligned}
\mathcal{B}_1(\tilde{a}) =&\Big\{\big(T_7,D_2\big)^{(1)}\big[12D_1T_3+2D_1^3+9D_1T_4+36\big(T_1,D_2\big)^{(1)}\big]\\
&-2D_1\big(T_6,D_2\big)^{(1)}\big[D_1^2+12T_3\big]+D_1^2\big[D_1\big(T_8,C_1\big)^{(2)}+6\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]\Big\}/144,\\
\mathcal{B}_2(\tilde{a}) =&\Big\{\big(T_7,D_2\big)^{(1)}\big[8T_3\big(T_6,D_2\big)^{(1)}-D_1^2\big(T_8,C_1\big)^{(2)}-4D_1\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]+\\
&+\Big[\big(T_7,D_2\big)^{(1)}\Big]^2\big(8T_3-3T_4+2D_1^2\big)\Big\}/384,
\end{aligned}$$

$$\mathcal{B}_3(\tilde{a},x,y) = -D_1^2(4D_2^2+T_8+4T_9)+3D_1D_2(T_6+4T_7)-24T_3(D_2^2-T_9),$$
$$\mathcal{B}_4(\tilde{a},x,y) = D_1(T_5+2D_2C_1)-3C_2(D_1^2+2T_3).$$

We note that the invariant polynomials $\mathcal{T}_i$, $\mathcal{F}_i$, $\mathcal{B}_i$ (i=1,2,3,4), and $\mathcal{B}$, $\mathcal{F}$, $\mathcal{H}$ and σ are responsible for weak singularities of the family of quadratic systems (see [**?**, Main Theorem]).

Now we need also the invariant polynomials which are responsible for the types of the finite singularities. These were constructed in [5]. Here we need only the following ones (we keep the notations from [5]):

$$\begin{aligned}
W_4(\tilde{a}) =&\big[1512A_1^2(A_{30}-2A_{29})-648A_{15}A_{26}+72A_1A_2(49A_{25}+39A_{26})\\
&+6A_2^2(23A_{21}-1093A_{19})-87A_2^4+4A_2^2(61A_{17}+52A_{18}+11A_{20})\\
&-6A_{37}(352A_3+939A_4-1578A_5)-36A_8(396A_{29}+265A_{30})\\
&+72A_{29}(17A_{12}-38A_9-109A_{11})+12A_{30}(76A_9-189A_{10}-273A_{11}-651A_{12})\\
&-648A_{14}(23A_{25}+5A_{26})-24A_{18}(3A_{20}+31A_{17})+36A_{19}(63A_{20}+478A_{21})\\
&+18A_{21}(2A_{20}+137A_{21})-4A_{17}(158A_{17}+30A_{20}+87A_{21})\\
&-18A_{19}(238A_{17}+669A_{19})\big]/81,\\
W_7(\tilde{a}) =&12A_{26}(A_{26}-2A_{25})+(2A_{29}-A_{30})(A_2^2-20A_{17}-12A_{18}+6A_{19}+6A_{21})\\
&+48A_{37}(A_1^2-A_8-A_{12}),\\
W_8(\tilde{a}) =&64D_1\big[\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big]^2\big[16\big(C_0,T_6\big)^{(1)}-37\big(D_2,T_1\big)^{(1)}+12D_1T_3\big]+4(108D_1^4-3T_4^2\\
&-128T_3T_4+42D_1^2T_4)\big[\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big]^2+36D_1\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big[4D_1\big(C_0,T_6\big)^{(1)}\\
&-D_1^2(4T_3+T_4)++24T_3^2\big]\big(C_1,T_8\big)^{(2)}+64\big[\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big]^2\big[27T_3^2+\\
&+16\big((T_6,C_1)^{(1)},C_0\big)^{(1)}\big]-54\big[8D_1^4+D_1^2T_4-8D_1\big(C_0,T_6\big)^{(1)}+8D_1^2T_3+8T_3^2\big]\\
&\times\big((T_6,C_1)^{(1)},T_6\big)^{(1)}\big(C_1,T_8\big)^{(2)}+108D_1T_3\big[\big(C_1,T_8\big)^{(2)}\big]^2\big[D_1T_3-2\big(C_0,T_6\big)^{(1)}\big]\\
&+576\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big((T_6,C_1)^{(1)},T_6\big)^{(1)}\big[2\big(D_2,T_1\big)^{(1)}-5D_1T_3\big]\\
&-27\big[\big(C_1,T_8\big)^{(2)}\big]^2\big[T_4^4/8+\big(C_0,T_1\big)^{(1)}\big],\\
F_4(\tilde{a},x,y) =&\ \mu_3(\tilde{a},x,y),\\
F_5(\tilde{a},x,y) =&\ T_5+2C_1D_2-3C_2D_1,\\
G_3(\tilde{a}) =&\ A_2.
\end{aligned}$$

Finally we need the invariant polynomials which are responsible for the existence of one (or two) star node(s) arbitrarily located on the phase plane of a system (2). We have the following lemma (see [**?**]):

Lemma 5 *A quadratic system* (2) *possesses one star node if and only if one of the following set of conditions holds:*

$$\begin{array}{ll}(i) & U_1\neq 0,\ U_2\neq 0,\ U_3=Y_1=0;\\ (ii) & U_1=U_4=U_5=U_6=0,\quad Y_2\neq 0;\end{array}$$

and it possesses two star nodes if and only if

$$(iii)\quad U_1=U_4=U_5=0,\quad U_6\neq 0,\quad Y_2>0,$$

where

$$\begin{aligned}
&U_1(\tilde{a},x,y)=\widetilde{N},\quad U_2(\tilde{a},x,y)=(C_1,\widetilde{H}-\widetilde{K})^{(1)}-2D_1\widetilde{N},\\
&U_3(\tilde{a},x,y)=3\widetilde{D}(D_2^2-16\widetilde{K})+C_2\big[(C_2,\widetilde{D})^{(2)}-5(D_2,\widetilde{D})^{(1)}+6\,\widetilde{F}\big],\\
&U_4(\tilde{a},x,y)=2T_5+C_1D_2,\quad U_5(\tilde{a},x,y)=3C_1D_1+4T_2-2C_0D_1,\\
&U_6(\tilde{a},x,y)=\widetilde{H},\quad Y_1(\tilde{a})=A_1,\quad Y_2(\tilde{a},x,y)=2D_1^2+8T_3-T_4.
\end{aligned}$$

We base our work here on results obtained in [3] and [5]. Here U_1, U_2, U_6 are T-comitants.

6 The proof of the Main Theorem

According to [**?**] for the quadratic systems having the finite singularities of total multiplicity 2 the conditions $\mu_0 = \mu_1 = 0$ and $\mu_2 \neq 0$ must be satisfied. So by [3] the following lemma is valid.

Lemma 6 *The configurations of singularities at infinity of the family of quadratic systems possessing finite singularities (real or complex) of total multiplicity 2 (i.e. $\mu_0 = \mu_1 = 0$ and $\mu_2 \neq 0$) are classified in* Diagram *4 according to the geometric equivalence relation. Necessary and sufficient conditions for each one of the 22 different equivalence classes can be assembled from this diagrams in terms of 14 invariant polynomials with respect to the action of the affine group and time rescaling, given in Section 5.*

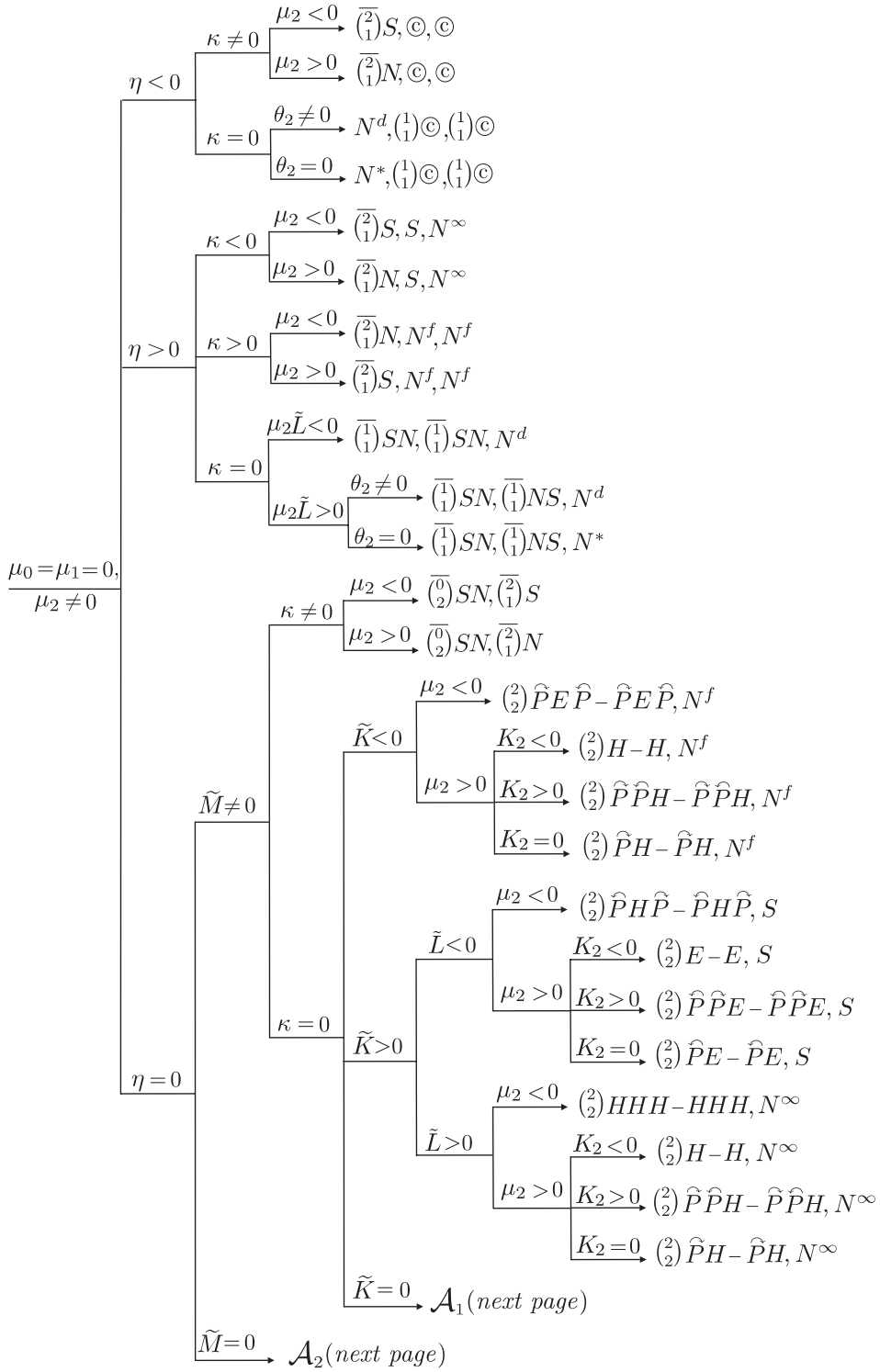


Diagram 4: **The case $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$**

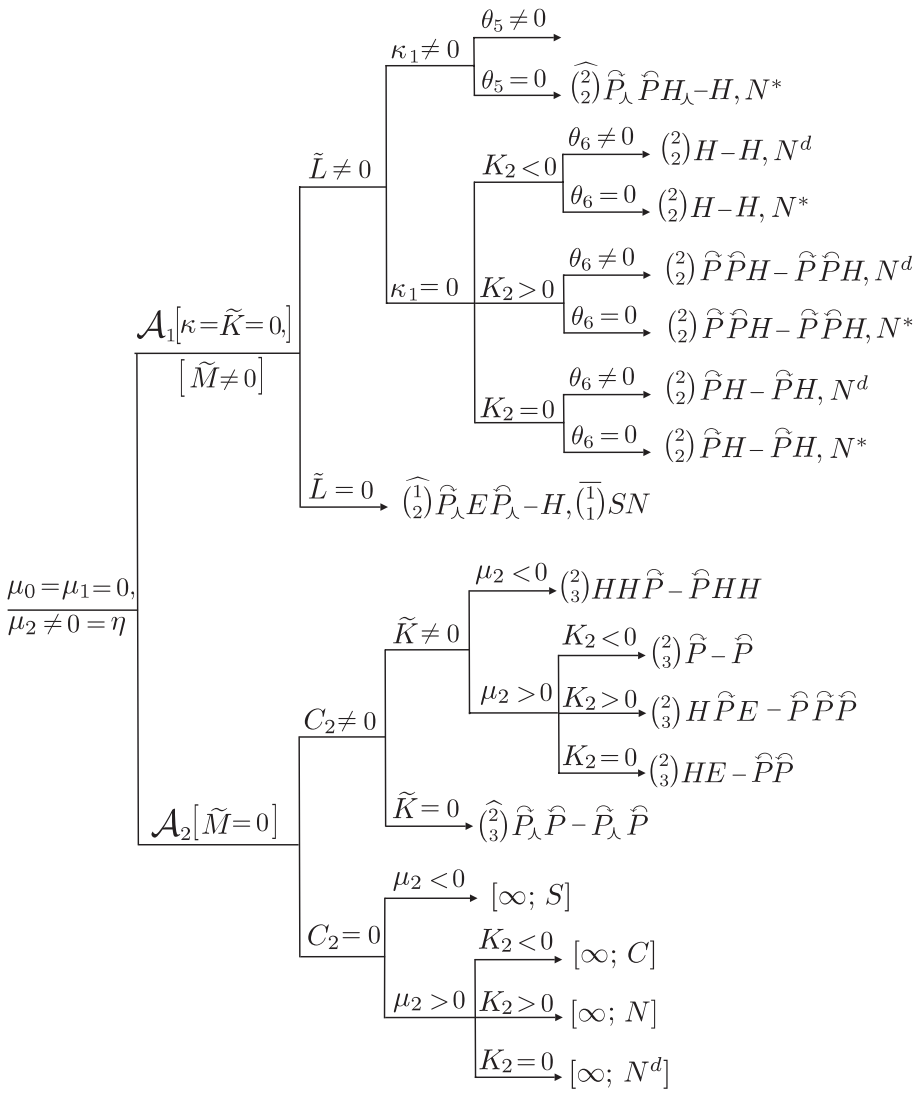


DIAGRAM 4 (*continued*). **The case $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$**

6.1 The family of quadratic differential systems with only two distinct complex finite singularities

Assuming that quadratic systems (2) possess two finite complex singular points, according to [**?**] we have to consider two cases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

6.1.1 Systems with $\widetilde{K} \neq 0$

In this case according to [**?**] we shall consider the following family of systems

$$\dot{x} = a + hux + 2hxy + ay^2, \quad \dot{y} = b + mux + 2mxy + by^2 \tag{10}$$

possessing the singular points $M_{1,2}(0, \pm i)$. For these systems calculations yield

$$\begin{gathered}\mu_0 = \mu_1 = 0, \ \ \mu_2 = (bh - am)^2(4 + u^2)y^2, \ \ \widetilde{K} = 4(bh - am)y^2, \ \ \kappa = -128m^2(bh - am), \\ \eta = 4m^2\big[(b + 2h)^2 - 8(bh - am)\big], \ \ \widetilde{M} = -32m^2x^2 - 16(b - 2h)mxy - 8\big[(b - 2h)^2 + 6am\big]y^2.\end{gathered} \tag{11}$$

Remark 2 *We observe that $\mu_2 > 0$ and if $\kappa \neq 0$ then $\widetilde{M} \neq 0$. Moreover the condition $\kappa > 0$ implies $\eta > 0$.*

Remark 3 *The family of systems* (10) *depends on five parameters. However due to a rescaling we can reduce the number of the parameters to three. More precisely since by the condition $\widetilde{K} \neq 0$ (i.e. $bh - am \neq 0$) we have $m^2 + h^2 \neq 0$, then we may assume $(m, h) \in \{(1, 1), (1, 0), (0, 1)\}$ due to the rescaling: (i) $(x, y, t) \mapsto (hx/m, y, t/h)$ if $mh \neq 0$; (ii) $(x, y, t) \mapsto (x/m, y, t)$ if $h = 0$ and (iii) $(x, y, t) \mapsto (x, y, t/h)$ if $m = 0$.*

Considering (11) and $\widetilde{K} \neq 0$ we deduce that the condition $m \neq 0$ is equivalent to $\kappa \neq 0$.

6.1.1.1 The case $\kappa \neq 0$ Then considering Remark 3 we shall examine the subfamilies of systems (10) with $(m, h) = (1, 1)$ and $(m, h) = (1, 0)$.

A. Systems with $m = h = 1$

So we consider the 3-parameter family of systems:

$$\dot{x} = a + ux + 2xy + ay^2, \quad \dot{y} = b + ux + 2xy + by^2 \tag{12}$$

for which calculations yield

$$\begin{gathered} \mu_0 = \mu_1 = 0, \quad \mu_2 = (a-b)^2(4+u^2)y^2, \quad \widetilde{K} = 4(b-a)y^2, \\ \kappa = 128(a-b), \quad \eta = 4\big[(b-2)^2 + 8a\big], \quad \theta = 64(b-a). \end{gathered} \tag{13}$$

6.1.1.1.1 The subcase $\kappa < 0$ Since $\mu_2 > 0$ and $\widetilde{M} \neq 0$ (see Remark 2) according to Lemma 6 we get the following three global configurations of singularities:

©, ©; $\overline{\binom{2}{1}} N$, ©, © : *Example* $\Rightarrow$ $(a = -1,\ b = 0,\ u = 0)$ (if $\eta < 0$);

©, ©; $\overline{\binom{2}{1}} N, S, N^{\infty}$: *Example* $\Rightarrow$ $(a = 0,\ b = 1,\ u = 0)$ (if $\eta > 0$);

©, ©; $\overline{\binom{0}{2}} SN, \overline{\binom{2}{1}} N$: *Example* $\Rightarrow$ $(a = 0,\ b = 2,\ u = 0)$ (if $\eta = 0$);

6.1.1.1.2 The subcase $\kappa > 0$ By Remark 2 we have $\eta > 0$ and considering Lemma 6 we arrive at the global configuration of singularities

©, ©; $\overline{\binom{2}{1}} S, N^f, N^f$: *Example* $\Rightarrow$ $(a = 1,\ b = 0,\ u = 0)$.

B. Systems with $m = 1, h = 0$

So we consider the 3-parameter family of systems:

$$\dot{x} = a + ay^2, \quad \dot{y} = b + ux + 2xy + by^2, \tag{14}$$

where we may assume $b \in \{0, 1\}$ due to the rescaling $(x, y, t) \mapsto (bx,\ y,\ t/b)$ (if $b \neq 0$). For these systems calculations yield

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = a^2(4+u^2)y^2, \quad \widetilde{K} = -4ay^2, \quad \kappa = 128a, \quad \eta = 4(8a + b^2). \tag{15}$$

Considering Remark 2 we conclude that the above systems could not possess new configurations different from the configurations of systems (12).

6.1.1.2 The case $\kappa = 0$ Considering (11), due to $\widetilde{K} \neq 0$ we obtain $m = 0$ and then by Remark 3 we may assume $h = 1$. Thus we arrive at the following systems

$$\dot{x} = a + ux + 2xy + ay^2, \quad \dot{y} = b + by^2, \tag{16}$$

where due to the rescaling $(x, y, t) \mapsto (ax, y, t)$ (if $a \neq 0$) we can assume $a \in \{0, 1\}$.

For these systems we calculate

$$\begin{gathered} \mu_0 = \mu_1 = \kappa = \kappa_1 = 0, \quad \mu_2 = b^2(4+u^2)y^2, \quad \widetilde{K} = 4by^2, \\ \tilde{L} = 8b(b-2)y^2, \quad K_2 = -384b^2(4 - 3b + b^2)y^2, \\ \eta = 0, \quad \widetilde{M} = -8(b-2)^2y^2, \quad C_2 = (2-b)xy^2 + ay^3. \end{gathered} \tag{17}$$

6.1.1.2.1 The subcase $\widetilde{K} < 0$ Then $b < 0$ and this implies $\widetilde{M} \neq 0$. We observe that $K_2 < 0$, $\mu_2 > 0$ and considering Lemma 6 we obtain the configuration

©, ©; $\binom{2}{2} H - H,\ N^f$: *Example* $\Rightarrow$ $(a = 0,\ b = -2,\ u = 0)$.

6.1.1.2.2 The subcase $\widetilde{K} > 0$ We consider two possibilities: $\tilde{L} \neq 0$ and $\tilde{L} = 0$.

1) *The possibility* $\tilde{L} \neq 0$. In this case $\widetilde{M} \neq 0$ and taking into account the conditions $K_2 < 0$ and $\mu_2 > 0$ by Lemma 6 we arrive at the following two global configurations of singularities:

©, ©; $\binom{2}{2} E - E,\ S$: *Example* $\Rightarrow$ $(a = 0,\ b = 1,\ u = 0)$ (if $\tilde{L} < 0$);

©, ©; $\binom{2}{2} H - H,\ N^{\infty}$: *Example* $\Rightarrow$ $(a = 0,\ b = 3,\ u = 0)$ (if $\tilde{L} > 0$).

2) *The possibility* $\tilde{L} = 0$. Since $b \neq 0$ (due to $\mu_2 \neq 0$) we have $b = 2$ and then $\widetilde{M} = 0$ and $C_2 = ay^3$. So considering Lemma 6 we obtain the following two configurations:

©, ©; $\binom{2}{3} \widehat{P} - \widehat{P}$: *Example* $\Rightarrow$ $(a = 1,\ b = 2,\ u = 0)$ (if $C_2 \neq 0$);

©, ©; $[\infty;\ C]$: *Example* $\Rightarrow$ $(a = 0,\ b = 2,\ u = 0)$ (if $C_2 = 0$);

6.1.2 Systems with $\widetilde{K} = 0$

In this case according to [**?**] we consider the following family of systems

$$\dot{x} = a + cx + gx^2 + 2hxy + ay^2, \quad \dot{y} = x, \quad (a \neq 0) \tag{18}$$

for which we calculate

$$\begin{gathered} \mu_0 = \mu_1 = \kappa = 0, \quad \mu_2 = a(gx^2 + 2hxy + ay^2)x^2, \quad \widetilde{K} = 0, \\ \tilde{L} = 8g(gx^2 + 2hxy + ay^2), \quad \eta = 4g^2(h^2 - ag), \\ \theta_2 = h^2 - ag, \quad \widetilde{M} = -8g^2x^2 - 16ghxy + 8(3ag - 4h^2)y^2. \end{gathered} \tag{19}$$

As $\mu_2 \neq 0$ we have $\operatorname{sign}(\mu_2 \tilde{L}) = \operatorname{sign}(ag)$.

6.1.2.1 The case $\eta < 0$ Then $\theta_2 \neq 0$ and considering Lemma 6 we arrive at the configuration

©, ©; N^d, $\binom{1}{1}$ ©, $\binom{1}{1}$ © : *Example* $\Rightarrow$ $(a = 2,\ c = 0,\ g = 1,\ h = 1)$.

6.1.2.2 The case $\eta > 0$ As $\theta_2 \neq 0$ considering Lemma 6 we obtain the following two global configurations of singularities:

©, ©; $\overline{\binom{1}{1}} SN$, $\overline{\binom{1}{1}} SN$, N^d : *Example* $\Rightarrow$ $(a = 1,\ c = 0,\ g = -1,\ h = 1)$ (if $\mu_2 \tilde{L} < 0$);

©, ©; $\overline{\binom{1}{1}} SN$, $\overline{\binom{1}{1}} NS$, N^d : *Example* $\Rightarrow$ $(a = 1,\ c = 0,\ g = 1/2,\ h = 1)$ (if $\mu_2 \tilde{L} > 0$).

6.1.2.3 The case $\eta = 0$

6.1.2.3.1 The subcase $\tilde{L} \neq 0$ Then $g \neq 0$ and we obtain $h^2 - ag = 0$ and as $a \neq 0$ we get $g = h^2/a$. Calculations yield

$$\begin{gathered} \tilde{L} = 8h^2(hx + ay)^2/a^2, \quad \widetilde{M} = -8h^2(hx + ay)^2/a^2, \\ \kappa_1 = 32h^2(a + ch)/a, \quad \theta_5 = -96h(a + ch)(hx + ay)^3/a^2 \end{gathered}$$

As we observe the condition $\tilde{L} \neq 0$ implies $\widetilde{M} \neq 0$, i.e. at infinity we have two distinct singularities.

1) *The possibility* $\kappa_1 \neq 0$. Then $\theta_5 \neq 0$ and considering Lemma 6 we obtain the configuration

©, ©; $\widehat{\binom{2}{2}}\ \widehat{P}_{\curlywedge}\ \widehat{P} H_{\curlywedge} - H,\ N^d$: *Example* $\Rightarrow$ $(a = 1,\ c = 0,\ g = 1,\ h = 1)$.

2) *The possibility* $\kappa_1 = 0$. As $\tilde{L} \neq 0$ we get $a = -ch \neq 0$ and then we have

$$K_2 = -384h^4(x - cy)^2/c^2, \quad \theta_6 = 8h^2(x - cy)^4/c^2, \quad \mu_2 = h^2x^2(x - cy)^2 \neq 0.$$

So we obtain $K_2 < 0$, $\theta_6 \neq 0$ and considering Lemma 6 we obtain the configuration of singularities

©, ©; $\binom{2}{2} H - H,\ N^d$: *Example* $\Rightarrow$ $(a = -1,\ c = 1,\ g = -1,\ h = 1)$.

6.1.2.3.2 The subcase $\tilde{L} = 0$ In this case considering (19) we get $g = 0$ and then we calculate

$$\eta = \tilde{L} = 0, \quad \widetilde{M} = -32h^2y^2, \quad C_2 = y^2(2hx + ay), \quad \kappa_1 = 128h^2, \quad \mu_2 = ay(2hx + ay) \neq 0.$$

We observe that the condition $\mu_2 \neq 0$ implies $C_2 \neq 0$. Therefore since $\tilde{L} = 0$ according to Lemma 6 we obtain the following two configurations:

©, ©; $\widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H, \overline{\binom{1}{1}} SN$: *Example* $\Rightarrow$ $(a = 1,\ c = 0,\ g = 0,\ h = 1)$ (if $\widetilde{M} \neq 0$);

©, ©; $\widehat{\binom{2}{3}}\ \widehat{P}_{\curlywedge}\ \widehat{P} - \widehat{P}_{\curlywedge}\ \widehat{P}$: *Example* $\Rightarrow$ $(a = 1,\ c = 0,\ g = 0,\ h = 0)$ (if $\widetilde{M} = 0$).

As all the possible cases are examined, we have proved that the family of systems with two complex distinct finite singularities possess exactly 16 geometrically distinct global configurations of singularities.

6.2 The family of quadratic differential systems with two real distinct finite singularities which in additional are elemental

Assume that quadratic systems (2) possess two real finite singular points (and both are elemental), i.e. by [**?**] the conditions $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$ and $\mathbf{U} > 0$ hold. According to [**?**] we have to consider two cases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

6.2.1 Systems with $\widetilde{K} \neq 0$

In this case according to [**?**] we consider the family of systems

$$\dot{x} = cx + dy - cx^2 + 2duxy, \quad \dot{y} = ex + fy - ex^2 + 2fuxy, \tag{20}$$

which possess the singular points $M_1(0,0)$ and $M_2(1,0)$.

Remark 4 *Assume that we have obtained a normal form of a family of quadratic systems translating an elemental real singular point at the origin of coordinates* $(0,0)$ *and fixing the position of all other singular points, even depending on some parameters. Then any geometrical property of any real elemental finite singular point can be considered to belong to the singularity* $(0,0)$.

For systems (20) calculations yield

$$\mu_0 = \mu_1 = 0, \;\; \mu_2 = (cf - de)^2(1 + 2u)x^2, \;\; \widetilde{K} = 4(de - cf)ux^2, \;\; \kappa = 128d^2(cf - de)u^3, \tag{21}$$

We remark that for the above systems the condition $\widetilde{\mu}_2 K \neq 0$ hold. So in what follows we assume that the following condition is satisfied:

$$(cf - de)(1 + 2u)u \neq 0. \tag{22}$$

Remark 5 *We observe that the family of systems* (20) *depends on five parameters. However due to a rescaling we can reduce the number of the parameters to three. More precisely since according to the condition* (22) *we have* $d^2 + f^2 \neq 0$*, then we may assume* $(d, f) \in \{(1,1), (1,0), (0,1)\}$ *due to the rescaling:* (*i*) $(x, y, t) \mapsto (x, fy/d, t/f)$ *if* $df \neq 0$*;* (*ii*) $(x, y, t) \mapsto (x, y/d, t)$ *if* $f = 0$ *and* (*iii*) $(x, y, t) \mapsto (x, y, t/f)$ *if* $d = 0$.

Considering (21) and (22) the condition $d \neq 0$ is equivalent to $\kappa \neq 0$.

6.2.1.1 The case $\kappa \neq 0$ Then considering Remark 5 we examine the subfamilies of systems (20) with $(d, f) = (1,1)$ and $(d, f) = (1,0)$.

A. Systems with $d = f = 1$

So we consider the 3-parameter family of systems:

$$\begin{aligned} &\dot{x} = cx + y - cx^2 + 2uxy, \;\; (c - e)(1 + 2u)u \neq 0, \\ &\dot{y} = ex + y - ex^2 + 2uxy \end{aligned} \tag{23}$$

for which calculations yield

$$\begin{gathered} \mu_0 = \mu_1 = 0, \;\; \mu_2 = (c - e)^2(1 + 2u)x^2, \;\; \widetilde{K} = 4(e - c)ux^2, \;\; \kappa = 128(c - e)u^3, \\ \eta = 4u^2\big[(c - 2u)^2 + 8(c - e)u\big], \;\; F_2 = (c - e)^2(1 + 2u)x^2, \;\; G_8 = 2(c - e)^2u^2(1 + 2u), \\ W_4 = 16(c - e)^4u^4(1 + 2u)^2\big[(c - 1)^2 + 4e\big]\big[(1 - c + 2u)^2 + 4(c - e)(1 + 2u)\big], \\ W_3 = 32(c - e)^4u^4(1 + 2u)^2(1 + c^2 + 2u + 2cu - 4eu + 2u^2), \\ \widetilde{M} = -8\big[(c + 2u)^2 - 6eu\big])x^2 + 16u(c + 2u)xy - 32u^2y^2, \\ \mathcal{T}_4 = 4(1 + c)(c - e)^2(1 - c + 2u)u^2(1 + 2u). \end{gathered} \tag{24}$$

Remark 6 *We observe that the condition* $\mu_2\kappa \neq 0$ *gives* $\widetilde{M}F_2G_8 \neq 0$ *and the condition* $\kappa > 0$ *implies* $\eta > 0$. *Moreover we have*

$$\operatorname{sign}(\widetilde{K}) = -\operatorname{sign}(\kappa), \quad \operatorname{sign}(G_8) = \operatorname{sign}(\mu_2) = \operatorname{sign}(F_2).$$

6.2.1.1.1 The subcase $\kappa < 0$ Then by Remark 6 we obtain $\widetilde{K} > 0$.

1) *The possibility* $\mu_2 < 0$. Then $1 + 2u < 0$, i.e. $u < -1/2$ and considering (24) we obtain $G_8 < 0$. So since $\widetilde{K} > 0$, according to [5] (see Table 1, lines 165-170) both finite singularities are anti-saddles.

a) *Assume first* $W_4 < 0$. Then we have a node and a focus and whether the focus is a weak one or not depends on the invariant polynomial $\mathcal{T}_4$. On the other hand due to $W_4 \neq 0$ we have a generic node.

***a*.1)** *The case* $\mathcal{T}_4 \neq 0$. Then by [**?**] the focus is strong.

***α*)** *The subcase* $\boldsymbol{\eta < 0}$. Then at infinity we have one real and two complex singularities and as $\boldsymbol{\mu_2 < 0}$ and $\boldsymbol{\kappa \neq 0}$ considering Lemma 6 we get the global configuration of singularities

$\boldsymbol{n, f; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 5,\ e = -1,\ u = -2)}$.

***β*)** *The subcase* $\boldsymbol{\eta > 0}$ In this case at infinity we have three real singularities. As $\boldsymbol{\kappa < 0}$ and $\boldsymbol{\mu_2 < 0}$, by Lemma 6 we get the configuration

$\boldsymbol{n, f; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1/2,\ e = -1/5,\ u = -2)}$.

***γ*)** *The subcase* $\boldsymbol{\eta = 0}$ In this case considering Remark 6 we have $\boldsymbol{\widetilde{M} \neq 0}$. As $\boldsymbol{\kappa \neq 0}$ and $\boldsymbol{\mu_2 < 0}$, considering Lemma 6 we get the global configuration of singularities

$\boldsymbol{n, f; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 3,\ e = -1/16,\ u = -2)}$.

***a*.2)** *The case* $\boldsymbol{\mathcal{T}_4 = 0}$. Then by [**?**] the focus is weak and its order depends on the value of the invariant polynomial $\boldsymbol{\mathcal{F}_1}$. Considering (24) and the condition (22), the condition $\boldsymbol{\mathcal{T}_4 = 0}$ gives $\boldsymbol{(c + 1)(2u + 1 - c) = \rho_1\rho_2 = 0}$. By Remark 4 we may assume without loss of generality that $\boldsymbol{\rho_1 = 0}$, i.e. $\boldsymbol{c = -1}$. Then for systems (23) we calculate:

$$\begin{aligned} \boldsymbol{\mathcal{T}_3 = 8(1 + e)^2u^2(1 + u)(1 + 2u),} \quad & \boldsymbol{\mathcal{F}_1 = 2(1 + e)(u - 1)(1 + 2u),} \\ \boldsymbol{\mu_2 = (1 + e)^2(1 + 2u)x^2,} \quad & \boldsymbol{\kappa = -128(1 + e)u^3,} \\ \boldsymbol{W_4 = -256(1 + e)^5u^4(1 + 2u)^2(e + 2eu - u^2).} & \end{aligned} \tag{25}$$

We observe that the condition $\boldsymbol{\mu_2 < 0}$ implies $\boldsymbol{\mathcal{F}_1 \neq 0}$. Moreover as $\boldsymbol{W_4 < 0}$ we have $\boldsymbol{\mathcal{T}_3 \neq 0}$, otherwise we get $\boldsymbol{u = -1}$ and this gives $\boldsymbol{W_4 = 256(1+e)^6 \geq 0}$. Therefore by [**?**] the weak focus has order one. Therefore according to Lemma 6 we get the following three global configurations of singularities:

$\boldsymbol{n, f^{(1)}; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = -3/2,\ u = -6/10)}$ (if $\boldsymbol{\eta < 0}$);

$\boldsymbol{n, f^{(1)}; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = -51/50,\ u = -7/10)}$ (if $\boldsymbol{\eta > 0}$);

$\boldsymbol{n, f^{(1)}; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = -36/35,\ u = -7/10)}$ (if $\boldsymbol{\eta = 0}$).

b) *Suppose now* $\boldsymbol{W_4 > 0}$. In this case as $\boldsymbol{\widetilde{K} > 0}$ and $\boldsymbol{G_8 < 0}$ so according to [5] systems (23) possess two nodes if $\boldsymbol{W_3 > 0}$ and two foci or/and centers if $\boldsymbol{W_3 < 0}$.

***b*.1)** *The case* $\boldsymbol{W_3 < 0}$.

***α*)** *The subcase* $\boldsymbol{\mathcal{T}_4 \neq 0}$. Then by [**?**] both foci are strong. Thus considering the conditions $\boldsymbol{\mu_0 = \mu_1 = 0}$, $\boldsymbol{\mu_2 < 0}$, $\boldsymbol{\kappa \neq 0}$ and Lemma 6 we arrive at the following three global configurations of singularities:

$\boldsymbol{f, f; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1,\ e = -1,\ u = -2)}$ (if $\boldsymbol{\eta < 0}$);

$\boldsymbol{f, f; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1,\ e = -1/2,\ u = -2)}$ (if $\boldsymbol{\eta > 0}$);

$\boldsymbol{f, f; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1,\ e = -9/16,\ u = -2)}$ (if $\boldsymbol{\eta = 0}$).

***β*)** *The subcase* $\boldsymbol{\mathcal{T}_4 = 0}$. As it was mention earlier we may assume $\boldsymbol{c = -1}$. Then at least one focus is a weak one.

***β*.1)** *The possibility* $\boldsymbol{\mathcal{T}_3 \neq 0}$. Then only one focus is weak. Moreover considering (25) we observe that the condition $\boldsymbol{\mu_2 < 0}$ implies $\boldsymbol{\mathcal{F}_1 \neq 0}$. So the weak focus could only be of the first order. So in view of the arguments above and Lemma 6 we get three global configurations of singularities:

$\boldsymbol{f, f^{(1)}; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = -2,\ u = -3/2)}$ (if $\boldsymbol{\eta < 0}$);

$\boldsymbol{f, f^{(1)}; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = -5/4,\ u = -3/2)}$ (if $\boldsymbol{\eta > 0}$);

$\boldsymbol{f, f^{(1)}; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = -4/3,\ u = -3/2)}$ (if $\boldsymbol{\eta = 0}$).

***β*.2)** *The possibility* $\boldsymbol{\mathcal{T}_3 = 0}$. Considering (25) we get $\boldsymbol{u = -1}$ and then we calculate:

$$\begin{aligned} \boldsymbol{\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{F} = 0,} \quad & \boldsymbol{\mathcal{T}_2 = -4(1 + e)^2,} \quad \boldsymbol{\mathcal{F}_1 = 4(1 + e),} \\ \boldsymbol{\mu_2 = -(1 + e)^2x^2,} \quad & \boldsymbol{\kappa = 128(1 + e),} \quad \boldsymbol{\eta = 4(9 + 8e).} \end{aligned}$$

So the condition $\boldsymbol{\mu_2 \neq 0}$ gives $\boldsymbol{\mathcal{F}_1 \neq 0}$ and according to [**?**] we have two first order weak singularities, which in this case are foci.

Thus we considering Lemma 6 we obtain the following three global configurations of singularities

$\boldsymbol{f^{(1)}, f^{(1)}; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=-2,\ u=-1)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{f^{(1)}, f^{(1)}; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=-17/16,\ u=-1)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{f^{(1)}, f^{(1)}; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=-9/8,\ u=-1)}$ (if $\boldsymbol{\eta=0}$).

b.2) *The case* $\boldsymbol{W_3>0}$. According to [5] (see Table 1, line 165) we have two nodes and both are generic (due to $\boldsymbol{W_4\neq 0}$). According to Lemma 6 we arrive at the following three global configurations of singularities:

$\boldsymbol{n, n; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=6,\ e=-1/3,\ u=-2)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{n, n; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=6,\ e=-1/5,\ u=-2)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{n, n; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c=6,\ e=-1/4,\ u=-2)}$ (if $\boldsymbol{\eta=0}$).

c) *Admit finally* $\boldsymbol{W_4=0}$. Then we have a node with coinciding eigenvalues and by Remark 4 without loss of generality we may assume that $\boldsymbol{M_1(0,0)}$ is such a point. So $\boldsymbol{\delta_1=(c-1)^2+4e=0}$ and we obtain $\boldsymbol{e=-(c-1)^2/4}$ and in this case calculations yield:

$$\begin{gathered}\boldsymbol{\mu_2=(1+c)^4(1+2u)x^2/16,\ \kappa=32(1+c)^2u^3,\ \eta=4u^2(1+2u)(c^2+2u),}\\ \boldsymbol{W_4=0,\ W_3=(1+c)^8u^4(1+u)(1+2u)^2(1+c^2+2u)/8,}\\ \boldsymbol{\mathcal{T}_4=-(1+c)^5(c-1-2u)u^2(1+2u)/4.}\end{gathered}\tag{26}$$

c.1) *The case* $\boldsymbol{W_3<0}$. According to [5] (see Table 1, line 168) we have one focus and one node which is $\boldsymbol{n^d}$ (as the Jacobian matrix is not diagonal).

α) *The subcase* $\boldsymbol{\mathcal{T}_4\neq 0}$. Then the focus is strong and we get the following three global configurations of singularities

$\boldsymbol{n^d, f; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=2,\ e=-1/4,\ u=-3/2)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{n^d, f; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=2,\ e=-1/4,\ u=-9/4)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{n^d, f; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c=2,\ e=-1/4,\ u=-2)}$ (if $\boldsymbol{\eta=0}$).

β) *The subcase* $\boldsymbol{\mathcal{T}_4=0}$. Considering (26) and the condition $\boldsymbol{\mu_2\kappa\neq 0}$ we obtain $\boldsymbol{(c-1-2u)=0}$, i.e. $\boldsymbol{c=2u+1}$. Then we calculate

$$\begin{gathered}\boldsymbol{\mathcal{T}_3=8u^2(1+u)^5(1+2u),\ \ \mathcal{F}_1=-2(1+u)^2(1+2u)(1+3u),}\\ \boldsymbol{\eta=4u^2(1+2u)(1+6u+4u^2),\ \ \mu_2=(1+u)^4(1+2u)x^2,}\\ \boldsymbol{\kappa=128u^3(1+u)^2,\ \ W_4=0,\ \ W_3=64u^4(1+u)^{10}(1+2u)^3}\end{gathered}\tag{27}$$

So the conditions $\boldsymbol{W_3\neq 0}$ and $\boldsymbol{\mu_2<0}$ implies $\boldsymbol{\mathcal{T}_3\mathcal{F}_1\neq 0}$ and according to [?] the focus $\boldsymbol{M_2(1,0)}$ is a weak focus of the first order. Therefore considering Lemma 6 we arrive at the following three global configurations of singularities :

$\boldsymbol{n^d, f^{(1)}; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-2,\ e=-9/4,\ u=-3/2)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{n^d, f^{(1)}; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-7/5,\ e=-36/25,\ u=-6/5)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{n^d, f^{(1)}; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c=2u+1,\ e=-(c-1)^2/4,\ u=-(3+\sqrt{5})/4)}$ (if $\boldsymbol{\eta=0}$).

c.2) *The case* $\boldsymbol{W_3>0}$. According to [5] (see Table 1, line 166) we have two nodes (one of them being $\boldsymbol{n^d}$).

Therefore considering Lemma 6 we get the following three global configurations of singularities:

$\boldsymbol{n, n^d; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=4/3,\ e=-1/36,\ u=-8/10)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{n, n^d; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=4/3,\ e=-1/36,\ u=-17/18)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{n, n^d; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c=4/3,\ e=-1/36,\ u=-8/9)}$ (if $\boldsymbol{\eta=0}$).

c.3) *The case* $\boldsymbol{W_3=0}$. Considering (26) and the condition $\boldsymbol{\mu_2<0}$ we obtain $\boldsymbol{(1+u)(1+c^2+2u)=0}$ and in this case we have two nodes $\boldsymbol{n^d}$ (as no one of the Jacobian matrices is diagonal).

α) *The subcase* $\boldsymbol{u=-1}$. We have

$$\boldsymbol{\eta=4(2-c^2),\ \ \mu_2=-(1+c)^4x^2/16<0,\ \ \kappa=-32(1+c)^2<0.}$$

So considering Lemma 6 we obtain the following three global configurations of singularities:

$\boldsymbol{n^d, n^d; \overline{\binom{2}{1}}S, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=2,\ e=-1/4,\ u=-1)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{n^d, n^d; \overline{\binom{2}{1}}S, S, N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1/2,\ e=-1/16,\ u=-1)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{n^d, n^d; \overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S}$: *Example* $\Rightarrow$ $\boldsymbol{(c=\sqrt{2},\ e=-(c-1)^2/4,\ u=-1)}$ (if $\boldsymbol{\eta=0}$).

$\boldsymbol{\beta}$) *The subcase* $\boldsymbol{u=-(c^2+1)/2}$. We calculate

$$\boldsymbol{\eta=c^2(1+c^2)^2,\ \ \mu_2=-c^2(1+c)^4x^2/16<0,\ \ \kappa=-4(1+c)^2(1+c^2)^3<0.}$$

So in this case we could only have $\boldsymbol{\eta>0}$ and such a configuration was detected above.

2) *The possibility* $\boldsymbol{\mu_2>0}$. Then by (24) we have $\boldsymbol{1+2u>0}$ and this implies $\boldsymbol{G_8>0}$. As $\boldsymbol{\widetilde{K}>0}$ according to [5] systems (23) possess a saddle and a focus (or a center) if $\boldsymbol{W_4<0}$ and a saddle and a node if $\boldsymbol{W_4\geq 0}$.

$\boldsymbol{a}$) *Assume first* $\boldsymbol{W_4<0}$, i.e. we have a saddle and a focus.

$\boldsymbol{a.1}$) *The subcase* $\boldsymbol{\mathcal{T}_4\neq 0}$. Then by [**?**] we could not have weak singularities. So considering Remark 6 and Lemma 6 we get the following three global configurations of singularities:

$\boldsymbol{s,f;\overline{\binom{2}{1}}N,©,©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-8,\ e=2,\ u=2)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{s,f;\overline{\binom{2}{1}}N,S,N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-8,\ e=1/2,\ u=2)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{s,f;\overline{\binom{0}{2}}SN,\overline{\binom{2}{1}}N}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-8,\ e=1,\ u=2)}$ (if $\boldsymbol{\eta=0}$).

$\boldsymbol{a.2}$) *The subcase* $\boldsymbol{\mathcal{T}_4=0}$. As it was shown earlier we can assume $\boldsymbol{c=-1}$ and we calculate

$$\boldsymbol{\mathcal{T}_3\mathcal{F}=8(1+e)^3u^4(1+u)^2(1+2u)^2.}$$

Therefore considering (25) we conclude that the condition $\boldsymbol{\kappa<0}$ and $\boldsymbol{\mu_2>0}$ imply $\boldsymbol{\mathcal{T}_3\mathcal{F}\neq 0}$. Moreover, as $\boldsymbol{(1+e)u>0}$ (due to $\boldsymbol{\kappa<0}$) from (25) we have

$$\boldsymbol{\operatorname{sign}(\mathcal{T}_3\mathcal{F})=\operatorname{sign}(u),\ \ \operatorname{sign}(W_4)=\operatorname{sign}(u(e+2eu-u^2)).} \tag{28}$$

$\boldsymbol{\alpha}$) *The possibility* $\boldsymbol{\mathcal{T}_3\mathcal{F}<0}$. Then $\boldsymbol{u<0}$ and according to [**?**] systems (23) with $\boldsymbol{c=-1}$ possess a weak focus. As $\boldsymbol{-1/2<u<0}$ then $\boldsymbol{\mathcal{F}_1\neq 0}$ and we have a weak first order focus. So considering Lemma 6 we get the following three global configurations of singularities:

$\boldsymbol{s,f^{(1)};\overline{\binom{2}{1}}N,©,©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=-2,\ u=-1/4)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{s,f^{(1)};\overline{\binom{2}{1}}N,S,N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=-17/16,\ u=-1/4)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{s,f^{(1)};\overline{\binom{0}{2}}SN,\overline{\binom{2}{1}}N}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=-9/8,\ u=-1/4)}$ (if $\boldsymbol{\eta=0}$).

$\boldsymbol{\beta}$) *The possibility* $\boldsymbol{\mathcal{T}_3\mathcal{F}>0}$. In this case $\boldsymbol{u>0}$ and according to [**?**] systems (23) with $\boldsymbol{c=-1}$ possess a weak saddle.

$\boldsymbol{\beta.1}$) *Assume first* $\boldsymbol{\mathcal{F}_1\neq 0}$. The weak saddle has order one and considering Lemma 6 we obtain the following three global configurations of singularities :

$\boldsymbol{s^{(1)},f;\overline{\binom{2}{1}}N,©,©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=1,\ u=1/5)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{s^{(1)},f;\overline{\binom{2}{1}}N,S,N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=1/5,\ u=1/5)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{s^{(1)},f;\overline{\binom{0}{2}}SN,\overline{\binom{2}{1}}N}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=9/40,\ u=1/5)}$ (if $\boldsymbol{\eta=0}$).

$\boldsymbol{\beta.2}$) *Suppose now that* $\boldsymbol{\mathcal{F}_1=0}$. Then $\boldsymbol{u=1}$ and we calculate

$$\begin{gathered}\boldsymbol{\mathcal{T}_3\mathcal{F}=288(1+e)^3,\ \ \mathcal{F}_1=0,\ \ \mathcal{F}_2=-432(1+e)^2,}\\ \boldsymbol{\kappa=-128(1+e),\ \ \eta=4(1-8e),\ \ W_4=2304(1+e)^5(1-3e)}\end{gathered} \tag{29}$$

and as $\boldsymbol{\kappa<0}$ we get $\boldsymbol{e+1>0}$ and hence $\boldsymbol{\mathcal{F}_2\neq 0}$. Therefore as $\boldsymbol{\mathcal{T}_3\mathcal{F}>0}$ and $\boldsymbol{\mathcal{F}_1=0}$ by [**?**] we obtain a second order weak saddle.

On the other hand the condition $\boldsymbol{W_4<0}$ due to $\boldsymbol{e+1>0}$ gives $\boldsymbol{e>1/3}$ and this implies $\boldsymbol{\eta<0}$.

Thus considering Lemma 6 we obtain the configuration

$\boldsymbol{s^{(2)},f;\overline{\binom{2}{1}}N,©,©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-1,\ e=1,\ u=1)}$.

$\boldsymbol{b}$) *Suppose now that* $\boldsymbol{W_4>0}$, i.e. by [5] we have a saddle and a node.

$\boldsymbol{b}$.1) *The subcase* $\boldsymbol{\mathcal{T}_4\neq 0}$. Then by [**?**] we could not have weak singularities, i.e. the saddle is strong.

So considering Lemma 6 we obtain the following three global configurations of singularities:

$\boldsymbol{s,n;\overline{\binom{2}{1}}N,©,©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-2,\ e=2/5,\ u=2)}$ (if $\boldsymbol{\eta<0}$);

$\boldsymbol{s,n;\overline{\binom{2}{1}}N,S,N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-2,\ e=1/5,\ u=2)}$ (if $\boldsymbol{\eta>0}$);

$\boldsymbol{s,n;\overline{\binom{0}{2}}SN,\overline{\binom{2}{1}}N}$: *Example* $\Rightarrow$ $\boldsymbol{(c=-2,\ e=1/4,\ u=2)}$ (if $\boldsymbol{\eta=0}$).

$\boldsymbol{b}$.2) *The subcase* $\boldsymbol{\mathcal{T}_4=0}$. As it was shown earlier we can assume $\boldsymbol{c=-1}$ and considering (25) we conclude that the conditions $\boldsymbol{\kappa<0}$ and $\boldsymbol{\mu_2>0}$ imply $\boldsymbol{\mathcal{T}_3\neq 0}$.

$\boldsymbol{\alpha}$) *Assume first* $\boldsymbol{\mathcal{F}_1\neq 0}$. Then the weak saddle has order one and considering Lemma 6 we obtain the following three global configurations of singularities:

$s^{(1)}, n; \binom{2}{1}N, ©, ©$: *Example* $\Rightarrow (c=-1,\ e=5/8,\ u=2)$ (if $\eta<0$);

$s^{(1)}, n; \binom{2}{1}N, S, N^{\infty}$: *Example* $\Rightarrow (c=-1,\ e=1/2,\ u=2)$ (if $\eta>0$);

$s^{(1)}, n; \binom{0}{2}SN, \binom{2}{1}N$: *Example* $\Rightarrow (c=-1,\ e=9/16,\ u=2)$ (if $\eta=0$).

β) *Suppose now* $\mathcal{F}_1=0$. In this case we get $u=1$ and considering (29) we deduce that the condition $\kappa<0$ implies $\mathcal{F}_2\neq 0$. Therefore according to [?] we have a second order weak saddle and we arrive at the following three global configurations of singularities:

$s^{(2)}, n; \binom{2}{1}N, ©, ©$: *Example* $\Rightarrow (c=-1,\ e=1/4,\ u=1)$ (if $\eta<0$);

$s^{(2)}, n; \binom{2}{1}N, S, N^{\infty}$: *Example* $\Rightarrow (c=-1,\ e=1/9,\ u=1)$ (if $\eta>0$);

$s^{(2)}, n; \binom{0}{2}SN, \binom{2}{1}N$: *Example* $\Rightarrow (c=-1,\ e=1/8,\ u=1)$ (if $\eta=0$).

c) *Admit finally* $W_4=0$. As it was mentioned above in this case we may assume $e=-(c-1)^2/4$. On the other hand as $G_8>0$ and $\widetilde{K}>0$, according to [5] systems (23) possess a saddle and a node, which is n^d (due to $W_4=0$). According to (26) the conditions $\kappa<0$ and $\mu_2>0$ yield $-1/2<u<0$.

c.1) *The subcase* $\mathcal{T}_4\neq 0$. Then by [?] we could not have weak singularities, i.e. the saddle is strong.

So considering Lemma 6 we obtain the following three global configurations of singularities:

$s, n^d; \binom{2}{1}N, ©, ©$: *Example* $\Rightarrow (c=1/2,\ e=-1/16,\ u=-1/5)$ (if $\eta<0$);

$s, n^d; \binom{2}{1}N, S, N^{\infty}$: *Example* $\Rightarrow (c=1/2,\ e=-1/16,\ u=-1/9)$ (if $\eta>0$);

$s, n^d; \binom{0}{2}SN, \binom{2}{1}N$: *Example* $\Rightarrow (c=1/2,\ e=-1/16,\ u=-1/8)$ (if $\eta=0$).

c.2) *The subcase* $\mathcal{T}_4=0$. Considering (26) and the condition $\mu_2\kappa\neq 0$ we obtain $c-1-2u=0$, i.e. $c=2u+1$. Then we obtain (27) and we can observe that $\mathcal{T}_3\neq 0$ (due to $\mu_2\kappa\neq 0$).

α) *The possibility* $\mathcal{F}_1\neq 0$. Then by [?] the weak saddle has order one and considering Lemma 6 we obtain the following three global configurations of singularities :

$s^{(1)}, n^d; \binom{2}{1}N, ©, ©$: *Example* $\Rightarrow (c=1/2,\ e=-1/16,\ u=-1/4)$ (if $\eta<0$);

$s^{(1)}, n^d; \binom{2}{1}N, S, N^{\infty}$: *Example* $\Rightarrow (c=2/3,\ e=-1/36,\ u=-1/6)$ (if $\eta>0$);

$s^{(1)}, n^d; \binom{0}{2}SN, \binom{2}{1}N$: *Example* $\Rightarrow (c=2u+1,\ e=-(c-1)^2/4,\ u=(\sqrt{5}-3)/4)$ (if $\eta=0$).

β) *The possibility* $\mathcal{F}_1=0$. Considering (27) we obtain $u=-1/3$ and then we have

$$\mathcal{T}_4=\mathcal{F}_1=0,\ \ \mathcal{T}_3=256/6561,\ \ \mathcal{F}_2=-256/19683,\ \ \eta=-20/243.$$

Hence by [?] the saddle is of the second order. As $\eta<0$ considering Lemma 6 we obtain the configuration

$s^{(2)}, n^d; \binom{2}{1}N, ©, ©$: *Example* $\Rightarrow (c=1/3,\ e=-1/9,\ u=-1/3)$.

6.2.1.1.2 The subcase $\kappa>0$ According to (24) and Remark 6 the condition $\kappa>0$ implies $\eta>0$ and $\widetilde{K}<0$, and we consider two possibilities: $\mu_2<0$ and $\mu_2>0$.

1) *The possibility* $\mu_2<0$. Then $u<-1/2$ and considering (24) we obtain $F_2<0$ and $G_8<0$. As $\widetilde{K}<0$ according to [5] (see Table 1, line 148) both finite singularities are saddles.

On the other hand according to Lemma 6 due to $\eta>0$, $\kappa>0$ and $\mu_2<0$ at infinity we have the configuration $\binom{2}{1}N, N^f, N^f$.

a) *Assume first* $\mathcal{T}_4\neq 0$, i.e. both saddles are strong and this leads to the global configuration of singularities

$s, s; \binom{2}{1}N, N^f, N^f$: *Example* $\Rightarrow (c=-2,\ e=0,\ u=-2)$.

b) *Suppose now* $\mathcal{T}_4=0$. As it was shown earlier we can assume $c=-1$ and we consider (25).

b.1) *The case* $\mathcal{T}_3\neq 0$. In this situation only one saddle is weak and as $\mathcal{F}_1\neq 0$ (due to $\mu_2<0$), according to [?] the order of the weak saddle is one and we get the configuration

$s, s^{(1)}; \binom{2}{1}N, N^f, N^f$: *Example* $\Rightarrow (c=-1,\ e=1,\ u=-2)$.

b.2) *The case* $\mathcal{T}_3=0$. Then $u=-1$ and we have

$$\mathcal{T}_4=\mathcal{T}_3=\mathcal{F}=0,\ \ \mathcal{T}_2=-4(1+e)^2,\ \ \mathcal{F}_1=4(1+e),\ \ \mu_2=-(1+e)^2x^2\neq 0.$$

We have $\mathcal{T}_2<0$ and as $\mathcal{F}_1\neq 0$ according to [?] we have two weak saddles each one of the first order:

$s^{(1)}, s^{(1)}; \binom{2}{1}N, N^f, N^f$: *Example* $\Rightarrow (c=-1,\ e=1,\ u=-1)$.

2) *The possibility* $\mu_2>0$. In this case we have $u>-1/2$ and considering (24) we obtain $F_2>0$ and $G_8>0$. As $\widetilde{K}<0$ according to [5] (see Table 1, lines 149,156,161) we have a saddle and an anti-saddle. The type of the anti-saddle is governed by invariant polynomial W_4.

On the other hand due to the condition $\eta > 0$, $\kappa > 0$, $\mu_2 > 0$ and considering Remark 6, according to Lemma 6 at infinity we have the configuration $\overline{\binom{2}{1}}S, N^f, N^f$.

a) *Assume first* $W_4 < 0$, i.e. we have a saddle and a focus or a center.

a.1) *The case* $\mathcal{T}_4 \neq 0$. Then by [?] we could not have weak singularities i.e. the saddle and the focus are both strong ones. Thus we get the global configuration of singularities

$s, f; \overline{\binom{2}{1}}S, N^f, N^f$: *Example* $\Rightarrow (c = 1,\ e = -1,\ u = 2)$.

a.2) *The subcase* $\mathcal{T}_4 = 0$. As it was shown earlier we can assume $c = -1$ and considering (25) we conclude that the condition $\kappa > 0$ and $\mu_2 > 0$ imply $\mathcal{T}_3 \neq 0$. Moreover as $(1 + e)u < 0$ (due to $\kappa > 0$) we have $\operatorname{sign}(\mathcal{T}_3\mathcal{F}) = -\operatorname{sign}(u)$.

α) *The possibility* $\mathcal{T}_3\mathcal{F} < 0$. Then $u > 0$ and in this case according to [?] systems (23) with $c = -1$ possess a weak focus.

α.1) *Assume first* $\mathcal{F}_1 \neq 0$. In this case by [?] the weak saddle is of order one and this leads to the configuration $s, f^{(1)}; \overline{\binom{2}{1}}S, N^f, N^f$: *Example* $\Rightarrow (c = -1,\ e = -2,\ u = 2)$.

α.2) *Admit now that* $\mathcal{F}_1 = 0$. Then $u = 1$, $e < -1$ and this implies $\mathcal{F}_2 = -432(1 + e)^2 \neq 0$. So by [?] we could have a weak focus of the order at most two and this leads to the configuration

$s, f^{(2)}; \overline{\binom{2}{1}}S, N^f, N^f$: *Example* $\Rightarrow (c = -1,\ e = -2,\ u = 1)$.

β) *The possibility* $\mathcal{T}_3\mathcal{F} > 0$. Then $u < 0$ and in this case according to [?] systems (23) with $c = -1$ possess a weak saddle. As $-1/2 < u < 0$ then $\mathcal{F}_1 \neq 0$ and we have a weak saddle of order one. Thus we get the configuration $s^{(1)}, f; \overline{\binom{2}{1}}S, N^f, N^f$: *Example* $\Rightarrow (c = -1,\ e = 1/2,\ u = -1/4)$.

b) *Suppose now* $W_4 > 0$. By [5] besides the saddle we have a node (which is generic due to $W_4 \neq 0$).

b.1) *The subcase* $\mathcal{T}_4 \neq 0$. Then by [?] we could not have weak singularities and this leads to the configuration of singularities $s, n; \overline{\binom{2}{1}}S, N^f, N^f$: *Example* $\Rightarrow (c = 2,\ e = 1,\ u = 2)$.

b.2) *The subcase* $\mathcal{T}_4 = 0$. As it was shown earlier we can assume $c = -1$ and considering (25) we conclude that the condition $\kappa > 0$ and $\mu_2 > 0$ imply $\mathcal{T}_3 \neq 0$. Moreover we claim that in this case the condition $\mathcal{F}_1 \neq 0$ holds. Indeed assuming $\mathcal{F}_1 = 0$ by (25) we obtain $u = 1$ and then the conditions

$$\kappa = -128(1 + e) > 0, \quad W_4 = 2304(1 + e)^5(1 - 3e) > 0$$

are incompatible. Thus $\mathcal{F}_1 \neq 0$ and we have a weak saddle of the first order:

$s^{(1)}, n; \overline{\binom{2}{1}}S, N^f, N^f$: *Example* $\Rightarrow (c = -1,\ e = -1/2,\ u = -1/3)$.

c) *Admit finally that* $W_4 = 0$. As it was mentioned above in this case we may assume $e = -(c - 1)^2/4$ (i.e. $\tau_1 = 0$). On the other hand as $G_8 > 0$, $F_2 > 0$ and $\widetilde{K} < 0$ according to [5] (see Table 1, line 151) systems (23) possess a saddle and a node, which is n^d (due to $W_4 = 0$ and the non-diagonal corresponding matrix of the linearization). According to (26) the condition $\kappa > 0$ yields $u > 0$.

c.1) *The subcase* $\mathcal{T}_4 \neq 0$. Then by [?] we could not have weak singularities, i.e. the saddle is strong. So we get the configuration $s, n^d; \overline{\binom{2}{1}}S, N^f, N^f$: *Example* $\Rightarrow (c = 1,\ e = 0,\ u = 1)$.

c.2) *The subcase* $\mathcal{T}_4 = 0$. Considering (26) and the condition $\mu_2\kappa \neq 0$ we get $c - 1 - 2u = 0$, i.e. $c = 2u + 1$. Then we obtain (27) and we observe that $\mathcal{T}_3 \neq 0$ (due to $\mu_2\kappa \neq 0$). Moreover in this case we obtain $\mathcal{F}_1 = -2(1 + u)^2(1 + 2u)(1 + 3u) \neq 0$ due to $u > 0$. Therefore we have a weak saddle of order one and this leads to the configuration $s^{(1)}, n^d; \overline{\binom{2}{1}}S, N^f, N^f$: *Example* $\Rightarrow (c = 3,\ e = -1,\ u = 1)$.

As all the cases have been examined the investigation of systems (23) is completed.

B. Systems with $d = 1$, $f = 0$

In this case for systems (20) with $d = 1$ and $f = 0$ we calculate

$$\kappa = -128eu^3, \quad \mu_2 = e^2(1 + 2u)x^2, \quad \mathcal{T}_4 = -4c^2e^2u^2(1 + 2u).$$

We observe that due to $\mu_2\kappa \neq 0$ the condition $c \neq 0$ is equivalent to $\mathcal{T}_4 \neq 0$.

6.2.1.1.3 The subcase $\mathcal{T}_4 \neq 0$ Then $c \neq 0$ and we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (x, cy, t/c)$. So we get the 2-parameter family of systems

$$\dot{x} = x + y - x^2 + 2uxy, \quad \dot{y} = ex - ex^2, \quad eu(1 + 2u) \neq 0, \tag{30}$$

for which we calculate

$$\begin{gathered}\boldsymbol{\mu_0=\mu_1=0,\ \ \mu_2=e^2(1+2u)x^2,\ \ \widetilde{K}=4eux^2,\ \ \kappa=-128eu^3,}\\ \boldsymbol{\eta=4u^2(1-8eu),\ \ F_2=e^2(1+2u)x^2,\ \ G_8=2e^2u^2(1+2u),}\\ \boldsymbol{W_4=16e^4(1+4e)u^4(1+2u)^2(1-4e-8eu),}\\ \boldsymbol{W_3=32e^4u^4(1+2u)^2(1-4eu),\ \ \widetilde{M}=8(6eu-1)x^2+16uxy-32u^2y^2.}\end{gathered}\tag{31}$$

Remark 7 *We observe that the condition* $\boldsymbol{\mu_2\kappa\neq 0}$ *gives* $\boldsymbol{\widetilde{M}F_2G_8\neq 0}$. *Moreover we have*

$$\boldsymbol{\operatorname{sign}(\widetilde{K})=-\operatorname{sign}(\kappa),\quad \operatorname{sign}(G_8)=\operatorname{sign}(\mu_2)=\operatorname{sign}(F_2).}$$

We observe that for systems (30) the same relations between the signs of invariant polynomials as in the case of systems (23) hold. Moreover as we shall use the same Table 1 of [5] and the same conditions for infinite singularities (i.e. Lemma 6), we only need to detect if for systems (30) we could obtain some configurations, which we have not obtained for systems (23).

For this goal we only need to detect if some logically possible configurations of singularities in the case $\boldsymbol{\mathcal{T}_4\neq 0}$ could not be realized for systems (23). And then to examine the respective case for systems (30) and to find out if such such detected configuration could be realized for systems (30).

1) *The possibility* $\boldsymbol{\kappa<0}$. We observe that all the logically possible configurations for systems (20) in the case $\boldsymbol{\kappa<0}$ and $\boldsymbol{\mathcal{T}_4\neq 0}$ are realized. More precisely we have the following number of configurations in the mentioned case:

$$\begin{array}{llll}\boldsymbol{\mu_2<0,\ W_4<0} & \boldsymbol{\Rightarrow} & \boldsymbol{3;} & \boldsymbol{\mu_2<0,\ W_4>0\ \Rightarrow\ 6;}\\ \boldsymbol{\mu_2<0,\ W_4=0} & \boldsymbol{\Rightarrow} & \boldsymbol{9;} & \boldsymbol{\mu_2>0,\ W_4<0\ \Rightarrow\ 3;}\\ \boldsymbol{\mu_2>0,\ W_4>0} & \boldsymbol{\Rightarrow} & \boldsymbol{3;} & \boldsymbol{\mu_2>0,\ W_4=0\ \Rightarrow\ 3.}\end{array}$$

2) *The possibility* $\boldsymbol{\kappa>0}$. In this case by (24) the condition $\boldsymbol{\kappa>0}$ implies $\boldsymbol{\eta>0}$ for systems (23), i.e. the configurations with $\boldsymbol{\eta\leq 0}$ do not exists for these systems.

The question is: *Could we obtain such configurations for systems* (30)*?*

The answer is *no*, as from (31) we observe that the condition $\boldsymbol{\kappa>0}$ gives $\boldsymbol{eu<0}$ and then clearly $\boldsymbol{\eta>0}$.

Considering Remark 7 and [5] we conclude that all logical possibilities in the case $\boldsymbol{\kappa>0}$ and $\boldsymbol{\mathcal{T}_4\neq 0}$ are realized for systems (23). More precisely we have the following number of configurations in the mentioned case:

$$\begin{array}{llll}\boldsymbol{\mu_2<0} & \boldsymbol{\Rightarrow} & \boldsymbol{1;} & \boldsymbol{\mu_2>0,\ W_4<0\ \Rightarrow\ 1;}\\ \boldsymbol{\mu_2>0,\ W_4>0} & \boldsymbol{\Rightarrow} & \boldsymbol{1;} & \boldsymbol{\mu_2>0,\ W_4=0\ \Rightarrow\ 1.}\end{array}$$

Thus we consider now the next possibility.

6.2.1.1.4 The subcase $\boldsymbol{\mathcal{T}_4=0}$ Then $\boldsymbol{c=0}$ and we get the systems

$$\boldsymbol{\dot{x}=(1+2ux)y,\ \ \dot{y}=ex-ex^2,\quad eu(1+2u)\neq 0.}\tag{32}$$

Moreover due to the rescaling $\boldsymbol{(x,y,t)\mapsto(x,\ |e|^{1/2}y,\ |e|^{-1/2}t)}$ we can assume $\boldsymbol{e\in\{-1,1\}}$.

For these systems we calculate

$$\begin{gathered}\boldsymbol{\mu_0=\mu_1=0,\ \ \mu_2=e^2(1+2u)x^2,\ \ \widetilde{K}=4eux^2,\ \ \kappa=-128eu^3,\ \ \eta=-32eu^3,}\\ \boldsymbol{\widetilde{M}=16u(3ex^2-2uy^2),\ \ F_2=e^2(1+2u)x^2,\ \ G_8=2e^2u^2(1+2u),}\\ \boldsymbol{W_4=-256e^6u^4(1+2u)^3,\ \ W_3=-128e^5u^5(1+2u)^2,}\\ \boldsymbol{\mathcal{T}_4=\mathcal{T}_3=\mathcal{F}_1=\mathcal{F}=0,\ \ \mathcal{T}_2=4e^2u^2(1+2u),\ \ \mathcal{B}=-2e^2u^4,\ \ \mathcal{H}=-4eu^3.}\end{gathered}\tag{33}$$

As $\boldsymbol{\mathcal{T}_4=\mathcal{T}_3=\mathcal{F}=0}$ according to [?] systems (32) possess two weak singularities. Moreover, since $\boldsymbol{\mathcal{F}_1=0}$ these singularities could be only centers and/or integrable saddles. We observe that due to $\boldsymbol{\mu_2\widetilde{K}\neq 0}$ we have

$$\boldsymbol{\operatorname{sign}(\eta)=\operatorname{sign}(\mathcal{H})=\operatorname{sign}(\kappa)=-\operatorname{sign}(\widetilde{K}),\quad \operatorname{sign}(G_8)=\operatorname{sign}(\mu_2)=\operatorname{sign}(\mathcal{T}_2)=\operatorname{sign}(F_2).}\tag{34}$$

1) *The possibility* $\boldsymbol{\kappa<0}$. In this case we get $\boldsymbol{\eta<0}$ and $\boldsymbol{\mathcal{H}<0}$.

a) *The case* $\boldsymbol{\mu_2<0}$. This implies $\boldsymbol{\mathcal{T}_2<0}$ and as $\boldsymbol{\mathcal{H}<0}$ and $\boldsymbol{\mathcal{B}<0}$, according to [?] we have two centers. Therefore considering Lemma 6 we get the global configuration of singularities

$\boldsymbol{c,c;\ \overline{\binom{2}{1}}S,}$ ©, © : *Example* $\boldsymbol{\Rightarrow\ (e=-1,\ u=-1)}$.

b) *The case* $\boldsymbol{\mu_2 > 0}$. Then $\boldsymbol{\mathcal{T}_2 > 0}$ and as $\boldsymbol{\mathcal{B} < 0}$, according to [?] we have one saddle and one center. At infinity we have the same configuration and we get $\boldsymbol{\$, c; \binom{2}{1}N, ©, ©}$: *Example* $\Rightarrow$ $\boldsymbol{(e = 1,\ u = 1)}$.

2) *The possibility* $\boldsymbol{\kappa > 0}$. In this case by (34) we have $\boldsymbol{\eta > 0}$ and $\boldsymbol{\mathcal{H} > 0}$.

a) *The case* $\boldsymbol{\mu_2 < 0}$. Then $\boldsymbol{\mathcal{T}_2 < 0}$ and since $\boldsymbol{\mathcal{H} > 0}$ and $\boldsymbol{\mathcal{B} < 0}$, according to [?] we have two integrable saddles. On the other hand considering the signs of the invariant polynomials $\boldsymbol{\mu_2}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\eta}$ according to Lemma 6 we get the global configuration of singularities

$\boldsymbol{\$, \$; \binom{2}{1}N, N^f, N^f}$: *Example* $\Rightarrow$ $\boldsymbol{(e = 1,\ u = -1)}$.

a) *The case* $\boldsymbol{\mu_2 > 0}$. Then we obtain $\boldsymbol{\mathcal{T}_2 > 0}$ and as $\boldsymbol{\mathcal{B} < 0}$, according to [?] we have one saddle and one center. So considering Lemma 6 we get the configuration

$\boldsymbol{\$, c; \binom{2}{1}S, N^f, N^f}$: *Example* $\Rightarrow$ $\boldsymbol{(e = -1,\ u = 1)}$.

Thus in the case $\boldsymbol{\kappa \neq 0}$ all the possibilities are examined for systems (20).

We observe that the four configurations detected for systems (32) are not realizable for systems (23). In order to insert these configurations in the global diagram we use the next remark.

Remark 8 *For the four configurations above the following conditions are satisfied, respectively:*

$$
\begin{array}{lcl}
\boldsymbol{c, c; \binom{2}{1}S, ©, ©} & \Rightarrow & \boldsymbol{\kappa < 0, \mu_2 < 0, W_4 > 0, W_3 < 0, \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{F}_1 = 0;} \\
\boldsymbol{\$, c; \binom{2}{1}N, ©, ©} & \Rightarrow & \boldsymbol{\kappa < 0, \mu_2 > 0, W_4 < 0, \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{F}_1 = 0;} \\
\boldsymbol{\$, \$; \binom{2}{1}N, N^f, N^f} & \Rightarrow & \boldsymbol{\kappa > 0, \mu_2 < 0, W_4 > 0, W_3 > 0, \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{F}_1 = 0;} \\
\boldsymbol{\$, c; \binom{2}{1}S, N^f, N^f} & \Rightarrow & \boldsymbol{\kappa > 0, \mu_2 > 0, W_4 < 0, \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{F}_1 = 0;}
\end{array}
$$

6.2.1.2 The case $\boldsymbol{\kappa = 0}$ Considering (21) due to $\boldsymbol{\widetilde{K} \neq 0}$ we obtain $\boldsymbol{d = 0}$ and then by Remark 5 we may assume $\boldsymbol{f = 1}$. Thus we arrive at the following systems

$$
\begin{aligned}
&\boldsymbol{\dot{x} = cx(1 - x), \qquad cu(1 + 2u) \neq 0,} \\
&\boldsymbol{\dot{y} = ex + y - ex^2 + 2uxy,}
\end{aligned}
\tag{35}
$$

for which due to the rescaling $\boldsymbol{(x, y, t) \mapsto (x,\ ey,\ t)}$ (if $\boldsymbol{e \neq 0}$) we can assume $\boldsymbol{e \in \{0, 1\}}$.

For these systems we calculate

$$
\begin{gathered}
\boldsymbol{\mu_0 = \mu_1 = \kappa = \kappa_1 = 0,\ \ \mu_2 = c^2(1 + 2u)x^2,\ \ \widetilde{K} = -4cux^2,} \\
\boldsymbol{\tilde{L} = 8c(c + 2u)x^2,\ \ K_2 = 96c^2(c^2 + 3cu + 4u^2)x^2,} \\
\boldsymbol{\eta = 0,\ \ \widetilde{M} = -8(c + 2u)^2x^2,\ \ C_2 = ex^3 - (c + 2u)x^2y,\ \ G_8 = 0,} \\
\boldsymbol{F_2 = c^2(1 + 2u)x^2,\ \ \mathcal{T}_i = 0,\ (i = 1, 2, 3, 4),\ \ \sigma = 1 + c - 2(c - u)x,} \\
\boldsymbol{\mathcal{F}_1 = \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = 0,\ \ \mathcal{B}_3 = 72c^2(1 + c)(1 - c + 2u)x^2 \equiv 72c^2\rho_1\rho_2x^2.}
\end{gathered}
\tag{36}
$$

Considering these values of the invariant polynomials above according to [?] (see the Main Theorem) we arrive at the following remark.

Remark 9 *Systems* (35) *possess at least one weak singularity if and only if* $\boldsymbol{\mathcal{B}_3 = 0}$. *More exactly as* $\boldsymbol{\widetilde{K} \neq 0}$, *by [?] we have one integrable saddle in the case* $\boldsymbol{\sigma \neq 0}$ *(the statement* $\boldsymbol{(e_3)[\gamma]}$*), and we have two integrable saddles in the case* $\boldsymbol{\sigma = 0}$ *(the statement* $\boldsymbol{(f_4)[\gamma]}$*).*

6.2.1.2.1 The subcase $\boldsymbol{\widetilde{K} < 0}$ As $\boldsymbol{G_8 = 0}$ according to [5] (see Table 1, lines 148, 150) we have two saddles if $\boldsymbol{F_2 < 0}$ and a saddle and a node if $\boldsymbol{F_2 > 0}$. On the other hand from (36) it follows $\operatorname{sign}\boldsymbol{(F_2)} = \operatorname{sign}\boldsymbol{(\mu_2)}$.

1) *The possibility* $\boldsymbol{\mu_2 < 0}$. Then $\boldsymbol{F_2 < 0}$ and systems (36) possess two saddles.

On the other hand by (36) the condition $\boldsymbol{\widetilde{K} < 0}$ gives $\boldsymbol{cu > 0}$ and then $\boldsymbol{\widetilde{M} \neq 0}$. Therefore as $\boldsymbol{\widetilde{K} < 0}$ and $\boldsymbol{\mu_2 < 0}$ according to Lemma 6 and Remark 9 we arrive at the following three global configurations of singularities:

$\boldsymbol{s, s;\ \binom{2}{2}\ \widehat{P}E\widehat{P} - \widehat{P}E\widehat{P},\ N^f}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -2,\ e = 0,\ u = -1)}$ (if $\boldsymbol{\sigma\mathcal{B}_3 \neq 0}$);

$\boldsymbol{\$, s;\ \binom{2}{2}\ \widehat{P}E\widehat{P} - \widehat{P}E\widehat{P},\ N^f}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = 0,\ u = -2)}$ (if $\boldsymbol{\sigma \neq 0,\ \mathcal{B}_3 = 0}$);

$\boldsymbol{\$, \$;\ \binom{2}{2}\ \widehat{P}E\widehat{P} - \widehat{P}E\widehat{P},\ N^f}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = 0,\ u = -1)}$ (if $\boldsymbol{\sigma = 0}$).

2) *The possibility* $\boldsymbol{\mu_2 > 0}$. Then $\boldsymbol{F_2 > 0}$ and systems (36) possess one saddle and one node. We observe that the Jacobian matrices for the singularities $\boldsymbol{M_1(0, 0)}$ and $\boldsymbol{M_2(1, 0)}$ are:

$$
M_1 \Rightarrow \begin{pmatrix} c & 0 \\ e & 1 \end{pmatrix}; \qquad M_2 \Rightarrow \begin{pmatrix} -c & 0 \\ -e & 1 + 2u \end{pmatrix}.
$$

Therefore systems (35) possess a node with coinciding eigenvalues if and only if $(c-1)(c+2u+1)=0$, and this node is a star node if and only if $(c-1)(c+2u+1)=e=0$.

On the other hand for these systems we have

$$W_{11}=96cu^3(1-c)(1+c+2u)x^4,$$
$$U_3\big|_{c=1}=-24eu(1+u)^2x^5,\quad U_3\big|_{c=-1-2u}=24eu(1+u)^2(1+2u)x^5,$$

where W_{11} is a new invariant polynomial, constructed as follows:

$$W_{11}=2\widetilde{K}(C_2,\widetilde{D})^{(2)}+12\widetilde{F}(2\widetilde{H}+\widetilde{K})-D_2(\widetilde{D},\widetilde{K})^{(1)}.$$

Since the conditions $\widetilde{K}\neq 0$ and $\mu_2>0$ imply $u(1+u)(1+2u)\neq 0$ we have the next remark

Remark 10 *Systems* (35) *possess a node with coinciding eigenvalues if and only if* $W_{11}=0$. *Moreover this node is* n^d *if* $U_3\neq 0$ *and it is a star node if* $U_3=0$.

a) *The case* $W_{11}\neq 0$. Then the node is generic. We observe that in this case $\sigma\neq 0$, otherwise we get $c=u=-1$ and this contradicts $\mu_2>0$.

a.1) *The subcase* $\mathcal{B}_3\neq 0$. Then by Remark 9 the saddle is strong and we shall examine the infinite singularities. We have again $\widetilde{M}\neq 0$ (due to $cu>0$ and considering (36) we obtain $K_2>0$. So according to Lemma 6 we get the configuration $s,n;\ \binom{2}{2}\widehat{P}\widehat{P}H-\widehat{P}\widehat{P}H,\ N^f$: *Example* $\Rightarrow (c=2,\ e=0,\ u=1)$.

a.2) *The subcase* $\mathcal{B}_3=0$. Since $\sigma\neq 0$, by Remark 9 we have an integrable saddle and we arrive at the global configuration $\$,n;\ \binom{2}{2}\widehat{P}\widehat{P}H-\widehat{P}\widehat{P}H,\ N^f$: *Example* $\Rightarrow (c=-1,\ e=0,\ u=-1/3)$.

b) *The case* $W_{11}=0$. Then one of the finite singularities is a node with coinciding eigenvalues. Due to the Remark 4 without loss of generality we may assume that such a node is $M_1(0,0)$, i.e. the condition $c=1$ holds.

b.1) *The subcase* $U_3\neq 0$. Then $e\neq 0$ and by Remark 10 besides the saddle we have a node n^d. On the other hand if $c=1$ then $u>0$ (due to $\widetilde{K}<0$) and we obtain $\mathcal{B}_3\neq 0$, i.e. the saddle is strong. Thus we obtain the configuration $s,n^d;\ \binom{2}{2}\widehat{P}\widehat{P}H-\widehat{P}\widehat{P}H,\ N^f$: *Example* $\Rightarrow (c=1,\ e=1,\ u=1)$.

b.2) *The subcase* $U_3=0$. In this case we have a star node and a strong saddle and this leads to the global configuration of singularities $s,n^*;\ \binom{2}{2}\widehat{P}\widehat{P}H-\widehat{P}\widehat{P}H,\ N^f$: *Example* $\Rightarrow (c=1,\ e=0,\ u=1)$.

6.2.1.2.2 The subcase $\widetilde{K}>0$. Since $G_8=0$ according to [5] the types of the finite singularities are governed by the polynomial F_2.

1) *The possibility* $\mu_2<0$. Then $F_2<0$ and as $G_8=0$ according to [5] (see Table 1, line 164) systems (36) possess two nodes.

a) *The case* $\widetilde{M}\neq 0$. Then at infinity we have two real distinct singularities.

a.1) *The subcase* $W_{11}\neq 0$. Then by Remark 10 both nodes are generic. On the other hand according to Lemma 6 the configuration of the infinite singularities depends on the sign of the invariant polynomial $\tilde{L}$ (we note that $\tilde{L}\neq 0$ due to $\widetilde{M}\neq 0$). So we get the following two global configurations of singularities:

$n,n;\ \binom{2}{2}\widehat{P}H\widehat{P}-\widehat{P}H\widehat{P},\ S$: *Example* $\Rightarrow (c=2,\ e=0,\ u=-2)$ (if $\tilde{L}<0$);

$n,n;\ \binom{2}{2}HHH-HHH,\ N^\infty$: *Example* $\Rightarrow (c=2,\ e=0,\ u=-2/3)$ (if $\tilde{L}>0$).

a.2) *The subcase* $W_{11}=0$. Then we may assume $c=1$ and hence the singular point M_1 of systems (36) is a node n^d, whereas for the second singularity M_2 we have $\lambda_1=-1$ and $\lambda_2=2u+1$. Therefore the second node will be a node with coinciding eigenvalues if and only if $u=-1$ and due to $\widetilde{K}\neq 0$ this is equivalent to $U_1=-4u(1+u)x^2=0$.

On the other hand for $c=1$ we obtain $\tilde{L}=8(2u+1)$ and due to $\mu_2=(1+2u)x^2<0$ we get $\tilde{L}<0$. Thus considering Lemma 6 we arrive to the following four global configurations of singularities:

$n,n^d;\ \binom{2}{2}\widehat{P}H\widehat{P}-\widehat{P}H\widehat{P},\ S$: *Example* $\Rightarrow (c=1,\ e=1,\ u=-2)$ (if $U_1\neq 0$, $B_3\neq 0$);

$n,n^*;\ \binom{2}{2}\widehat{P}H\widehat{P}-\widehat{P}H\widehat{P},\ S$: *Example* $\Rightarrow (c=1,\ e=0,\ u=-2)$ (if $U_1\neq 0$, $B_3=0$);

$n^d,n^d;\ \binom{2}{2}\widehat{P}H\widehat{P}-\widehat{P}H\widehat{P},\ S$: *Example* $\Rightarrow (c=1,\ e=1,\ u=-1)$ (if $U_1=0$, $B_3\neq 0$);

$n^*,n^*;\ \binom{2}{2}\widehat{P}H\widehat{P}-\widehat{P}H\widehat{P},\ S$: *Example* $\Rightarrow (c=1,\ e=0,\ u=-1)$ (if $U_1=0$, $B_3=0$).

b) *The case* $\widetilde{M}=0$. Then we have $c=-2u$ and this gives $C_2=ex^3$ and $W_{11}=-192u^4(1+2u)x^4\neq 0$ (due to $\mu_2\neq 0$). Therefore at infinity we have one real singularity of multiplicity five if $C_2\neq 0$ and the infinite line is filled up with singularities if $C_2=0$. At the same time due to Remark 10 systems (35) possess two generic nodes.

$\boldsymbol{b}$.1) *The subcase* $\boldsymbol{C_2 \neq 0}$. As $\widetilde{\boldsymbol{K}} \neq \boldsymbol{0}$ and $\boldsymbol{\mu_2 < 0}$ according to Lemma 6 we arrive at the global configuration of singularities $\boldsymbol{n, n;\ \binom{2}{3}\ HH\widehat{P} - \widehat{P}HH}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 2,\ e = 1,\ u = -1)}$.

$\boldsymbol{b}$.2) *The subcase* $\boldsymbol{C_2 = 0}$. Then we have $\boldsymbol{c + 2u = e = 0}$ and as the nodes are generic, considering Lemma 6 we obtain the configuration $\boldsymbol{n, n;\ [\infty;\ S]}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 2,\ e = 0,\ u = -1)}$.

2) *The possibility* $\boldsymbol{\mu_2 > 0}$. Then $\boldsymbol{F_2 > 0}$ and as $\boldsymbol{G_8 = 0}$ according to [5] (see Table 1, line 150) systems (36) possess a saddle and a node. We observe that due to $\widetilde{\boldsymbol{K}} > \boldsymbol{0}$ and $\boldsymbol{\mu_2 > 0}$ in this case the conditions $\boldsymbol{2u + 1 > 0}$ and $\boldsymbol{cu < 0}$ hold.

$\boldsymbol{a}$) *The case* $\widetilde{\boldsymbol{M}} \neq \boldsymbol{0}$. Then at infinity we have two real distinct singularities

$\boldsymbol{a}$.1) *The subcase* $\boldsymbol{\mathcal{B}_3 \neq 0}$. Then by Remark 9 the saddle is strong.

$\boldsymbol{\alpha}$) *The possibility* $\boldsymbol{W_{11} \neq 0}$. Then the node is generic and as $\widetilde{\boldsymbol{K}} \neq \boldsymbol{0}$,$\boldsymbol{K_2 > 0}$ and $\boldsymbol{\mu_2 > 0}$ considering Lemma 6 we arrive at the following two global configurations of singularities:

$\boldsymbol{s, n;\ \binom{2}{2}\ \widehat{P}\widehat{P}E - \widehat{P}\widehat{P}E,\ S}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -2,\ e = 0,\ u = 2)}$ (if $\tilde{\boldsymbol{L}} < \boldsymbol{0}$);

$\boldsymbol{s, n;\ \binom{2}{2}\ \widehat{P}\widehat{P}H - \widehat{P}\widehat{P}H,\ N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -2,\ e = 0,\ u = 1/4)}$ (if $\tilde{\boldsymbol{L}} > \boldsymbol{0}$).

$\boldsymbol{\beta}$) *The possibility* $\boldsymbol{W_{11} = 0}$. Then we may assume $\boldsymbol{c = 1}$ and then the condition $\boldsymbol{\mu_2 > 0}$ implies $\tilde{\boldsymbol{L}} = \boldsymbol{8(1 + 2u)x^2 > 0}$. As a result we get the following two global configurations of singularities

$\boldsymbol{s, n^d;\ \binom{2}{2}\ \widehat{P}\widehat{P}H - \widehat{P}\widehat{P}H,\ N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1,\ e = 1,\ u = -1/4)}$ (if $\boldsymbol{B_3 \neq 0}$);

$\boldsymbol{s, n^*;\ \binom{2}{2}\ \widehat{P}\widehat{P}H - \widehat{P}\widehat{P}H,\ N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1,\ e = 0,\ u = -1/4)}$ (if $\boldsymbol{B_3 = 0}$).

$\boldsymbol{a}$.2) *The subcase* $\boldsymbol{\mathcal{B}_3 = 0}$. Then by Remark 9 we have an integrable saddle and we may assume that it is located at $\boldsymbol{M_1(0, 0)}$, i.e. $\boldsymbol{c = -1}$. Then we have $\boldsymbol{W_{11} = -384u^4x^4 \neq 0}$ and by Remark 10 the node is generic. Considering Lemma 6 we arrive at the following two global configurations of singularities :

$\boldsymbol{s^{(1)}, n;\ \binom{2}{2}\ \widehat{P}\widehat{P}E - \widehat{P}\widehat{P}E,\ S}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = 0,\ u = 2)}$ (if $\tilde{\boldsymbol{L}} < \boldsymbol{0}$);

$\boldsymbol{s^{(1)}, n;\ \binom{2}{2}\ \widehat{P}\widehat{P}H - \widehat{P}\widehat{P}H,\ N^{\infty}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = 0,\ u = 1/4)}$ (if $\tilde{\boldsymbol{L}} > \boldsymbol{0}$).

$\boldsymbol{b}$) *The case* $\widetilde{\boldsymbol{M}} = \boldsymbol{0}$. Then we have $\boldsymbol{c = -2u}$ and we obtain

$$\boldsymbol{C_2 = ex^3,\ \ \mathcal{B}_3 = -288u^2(-1 + 2u)(1 + 4u)x^2,\ \ W_{11} = -192u^4(1 + 2u)x^4.}$$

$\boldsymbol{b}$.1) *The subcase* $\boldsymbol{C_2 \neq 0}$. Due to $\boldsymbol{\mu_2 > 0}$ we have $\boldsymbol{u > -1/2}$ and this implies $\boldsymbol{W_{11} \neq 0}$, i.e. the node is generic. So since $\widetilde{\boldsymbol{K}} \neq \boldsymbol{0}$, $\boldsymbol{\mu_2 > 0}$ and $\boldsymbol{K_2 > 0}$ according to Lemma 6 we arrive to the following two global configurations of singularities:

$\boldsymbol{s, n;\ \binom{2}{3}\ H\widehat{P}E - \widehat{P}\widehat{P}\widehat{P}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -2,\ e = 1,\ u = 1)}$ (if $\boldsymbol{\mathcal{B}_3 \neq 0}$);

$\boldsymbol{\$, n;\ \binom{2}{3}\ H\widehat{P}E - \widehat{P}\widehat{P}\widehat{P}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = 1,\ u = 1/2)}$ (if $\boldsymbol{\mathcal{B}_3 = 0}$).

$\boldsymbol{b}$.2) *The subcase* $\boldsymbol{C_2 = 0}$. Then we have $\boldsymbol{c + 2u = e = 0}$ and as the node is generic, considering Lemma 6 we obtain the configurations

$\boldsymbol{s, n;\ [\infty;\ N]}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -2,\ e = 0,\ u = 1)}$ (if $\boldsymbol{\mathcal{B}_3 \neq 0}$);

$\boldsymbol{\$, n;\ [\infty;\ N]}$: *Example* $\Rightarrow$ $\boldsymbol{(c = -1,\ e = 0,\ u = 1/2)}$ (if $\boldsymbol{\mathcal{B}_3 = 0}$).

6.2.2 Systems with $\widetilde{\boldsymbol{K}} = \boldsymbol{0}$

Since $\widetilde{\boldsymbol{K}}(\tilde{\boldsymbol{a}}, \boldsymbol{x}, \boldsymbol{y}) = \mathrm{Jacob}\big(\boldsymbol{p_2}(\tilde{\boldsymbol{a}}, \boldsymbol{x}, \boldsymbol{y}), \boldsymbol{q_2}(\tilde{\boldsymbol{a}}, \boldsymbol{x}, \boldsymbol{y})\big)$ (see Section 5) the condition $\widetilde{\boldsymbol{K}} = \boldsymbol{0}$ means that the homogeneous quadratic parts of generic quadratic systems are proportional, say $\boldsymbol{q_2 = \lambda p_2}$. Therefore clearly the transformation $\boldsymbol{x_1 = x, y_1 = -\lambda x + y}$ leads to quadratic systems, of which the second equation is linear. Applying a translation (as these systems must have two finite real distinct singularities we get the systems

$$\boldsymbol{\dot{x} = cx + dy + gx^2 + 2hxy + ky^2,\quad \dot{y} = ex + fy,}$$

for which we calculate

$$\boldsymbol{\mathrm{U} = \alpha^2(ex + fy)^2(gx^2 + 2hxy + ky^2)^2,\ \ \mu_2 = \beta(gx^2 + 2hxy + ky^2),}$$

where $\boldsymbol{\alpha = cf - de}$ and $\boldsymbol{\beta = f^2g - 2efh + e^2k}$. These systems possess the singularities $\boldsymbol{M_1(0, 0)}$ and $\boldsymbol{M_2\,(-f\alpha/\beta,\ e\alpha/\beta)}$ which are distinct due to $\boldsymbol{\mathrm{U} > 0}$ and $\boldsymbol{\mu_2 \neq 0}$. We observe that the condition $\boldsymbol{e = 0}$ implies the existence of the invariant lines $\boldsymbol{y = 0}$ for these systems. So we consider two cases: $\boldsymbol{e \neq 0}$ and $\boldsymbol{e = 0}$.

In the first case we apply the transformation $\boldsymbol{x_1 = ex + fy}$, $\boldsymbol{y_1 = \beta y/(e\alpha)}$ and $\boldsymbol{t_1 = \beta t/(e\alpha)}$ which places the point $\boldsymbol{M_2}$ at the point $\boldsymbol{(0, 1)}$. This leads to the family of systems (we keep the old variables)

$$\boldsymbol{\dot{x} = cx + dy + gx^2 + 2hxy - dy^2,\quad \dot{y} = x.} \tag{37}$$

If $e = 0$ then $cfg \neq 0$ (as $\alpha\beta \neq 0$ and after the rescaling $(x, y, t) \mapsto (-cx/g, y, t/f)$ (which replaces $M_2(-c/g, 0)$ to the point $(1, 0)$) we arrive at the family of systems

$$\dot{x} = cx + dy - cx^2 + 2hxy + ky^2, \quad \dot{y} = y. \tag{38}$$

In what follows we consider each one of the families of systems we obtained.

A. Systems (37)

We observe that the Jacobian matrices for the singularities $M_1(0, 0)$ and $M_2(0, 1)$ of these systems are respectively $\begin{pmatrix} c & d \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} c+2h & -d \\ 1 & 0 \end{pmatrix}$. So the next remark becomes obvious.

Remark 11 *The family of systems* (37) *could not have a finite star node.*

For systems (37) we calculate

$$\begin{gathered} \mu_0 = \mu_1 = \kappa = \widetilde{K} = 0, \ \ \eta = 4g^2(dg + h^2), \ \ \mu_2 = -d(gx^2 + 2hxy - dy^2), \\ \theta_2 = dg + h^2, \ \ \mathrm{U} = d^2x^2(gx^2 + 2hxy - dy^2)^2, \ \ G_8 = 2dg(dg + h^2), \\ W_4 = 16d^2g^2(dg + h^2)^2(c^2 + 4d)(c^2 - 4d + 4ch + 4h^2) = 16d^2g^2(dg + h^2)^2\tau_1\tau_2, \\ \mathrm{D} = -192d^4(dg + h^2), \ \ \tilde{L} = 8g(gx^2 + 2hxy - dy^2), \\ \mathcal{F}_1 = 2d(cg + h + gh), \ \ \mathcal{T}_4 = 4cdg(c + 2h)(dg + h^2). \end{gathered} \tag{39}$$

Remark 12 *The condition* $W_4 \neq 0$ *implies* $G_8\mathrm{D}\theta_2\eta \neq 0$ *for systems* (37).

6.2.2.1 The case $W_4 < 0$. Then by the above remark $G_8\mathrm{D} \neq 0$ and due to $\widetilde{K} = 0$, according to [5] (see Table 1, lines 157,162) we have a saddle and either a focus or a center.

6.2.2.1.1 The subcase $\mathcal{T}_4 \neq 0$ Then by [?] we have a strong saddle and a strong focus. On the other hand as by Remark 12 we have $\eta\theta_2 \neq 0$, considering Lemma 6 we obtain the following three global configurations of singularities:

- s, f; N^d, $\binom{1}{1}$©, $\binom{1}{1}$©: *Example* $\Rightarrow$ $(c = 1,\ d = -2,\ g = 1,\ h = 1)$ (if $\eta < 0$);
- s, f; $\binom{\overline{1}}{1}SN$, $\binom{\overline{1}}{1}SN$, N^d: *Example* $\Rightarrow$ $(c = 1,\ d = -2,\ g = -1,\ h = 1)$ (if $\eta > 0$, $\mu_2\tilde{L} < 0$);
- s, f; $\binom{\overline{1}}{1}SN$, $\binom{\overline{1}}{1}NS$, N^d: *Example* $\Rightarrow$ $(c = 1,\ d = 3,\ g = -1/4,\ h = 1)$ (if $\eta > 0$ $\mu_2\tilde{L} > 0$).

6.2.2.1.2 The subcase $\mathcal{T}_4 = 0$ Then one of the finite singularities is weak and by Remark 4 without loss of generality we may assume $\rho_1 = c = 0$, i.e. this weak singularity is $M_1(0, 0)$. In this case for systems (37) we calculate:

$$\begin{gathered} \mathcal{T}_3 = 8dgh(dg + h^2), \ \ \mathcal{F} = gh(dg + h^2), \ \ \mathcal{F}_1 = 2dh(1 + g), \\ \mathcal{F}_2 = \mathcal{F}_3 = 0, \ \ W_4 = -256d^3g^2(d - h^2)(dg + h^2)^2. \end{gathered} \tag{40}$$

a) *The possibility* $\mathcal{T}_3\mathcal{F} < 0$. According to [?] the systems possess a weak focus, the order of which is determined by the invariant polynomial $\mathcal{F}_1$.

a.1) *The case* $\mathcal{F}_1 \neq 0$. Then we have a first order weak focus and considering Lemma 6 and the condition $\eta\theta_2 \neq 0$ and $\kappa = 0$ we obtain the following three global configurations of singularities:

- $s, f^{(1)}$; N^d, $\binom{1}{1}$©, $\binom{1}{1}$©: *Example* $\Rightarrow$ $(c = 0,\ d = -1,\ g = 2,\ h = 1)$ (if $\eta < 0$);
- $s, f^{(1)}$; $\binom{\overline{1}}{1}SN$, $\binom{\overline{1}}{1}SN$, N^d: *Example* $\Rightarrow$ $(c = 0,\ d = -1,\ g = -1,\ h = 1)$ (if $\eta > 0$, $\mu_2\tilde{L} < 0$);
- $s, f^{(1)}$; $\binom{\overline{1}}{1}SN$, $\binom{\overline{1}}{1}NS$, N^d: *Example* $\Rightarrow$ $(c = 0,\ d = -1,\ g = 1/2,\ h = 1)$ (if $\eta > 0$ $\mu_2\tilde{L} > 0$).

a.2) *The case* $\mathcal{F}_1 = 0$. Then by (40) due to $\mathcal{T}_3 \neq 0$ we get $g = -1$ and then we have

$$\begin{gathered} \mu_2\tilde{L} = 8d(x^2 - 2hxy + dy^2)^2, \ \ \mathcal{T}_3\mathcal{F} = 8dh^2(d - h^2)^2, \\ \eta = 4(h^2 - d), \ \ W_4 = 256d^3(h^2 - d)^3. \end{gathered}$$

So we arrive at the remark

Remark 13 *If* $\mathcal{T}_4 = 0$ *and* $\mathcal{T}_3 \neq 0$ *then the condition* $\mathcal{F}_1 = 0$ *implies* $\operatorname{sign}(\mathcal{T}_3\mathcal{F}) = \operatorname{sign}(\mu_2\tilde{L}) = \operatorname{sign}(\eta W_4)$.

Since by (40) we have $\mathcal{F}_2 = \mathcal{F}_3 = 0$, according to [?] systems (37) possess a center. By the above remark the conditions $\mathcal{T}_3\mathcal{F} < 0$ and $W_4 < 0$ imply $\mu_2\tilde{L} < 0$ and $\eta > 0$. So considering Lemma 6 we obtain the configuration: $s, c; \binom{1}{1}SN, \binom{1}{1}SN, N^d$: *Example* $\Rightarrow (c = 0,\ d = -2,\ g = -1,\ h = 1)$.

b) *The possibility* $\mathcal{T}_3\mathcal{F} > 0$. In this case by [?] the systems possess a weak saddle.

b.1) *The case* $\mathcal{F}_1 \neq 0$. Then we have a first order weak focus and considering Lemma 6 and the condition $\eta\theta_2 \neq 0$ and $\kappa = 0$ we obtain the following three global configurations of singularities:

- $s^{(1)}, f;\ N^d,\ \binom{1}{1}$ ©, $\binom{1}{1}$ ©: *Example* $\Rightarrow (c = 0,\ d = 2,\ g = -2,\ h = 1)$ (if $\eta < 0$);
- $s^{(1)}, f; \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}SN,\ N^d$: *Example* $\Rightarrow (c = 0,\ d = 2,\ g = 1,\ h = 1)$ (if $\eta > 0$, $\mu_2\tilde{L} < 0$);
- $s^{(1)}, f; \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}NS,\ N^d$: *Example* $\Rightarrow (c = 0,\ d = 2,\ g = -1/3,\ h = 1)$ (if $\eta > 0\ \mu_2\tilde{L} > 0$).

b.2) *The case* $\mathcal{F}_1 = 0$. Then the saddle is integrable. On the other hand by Remark 13 the condition $W_4 < 0$ implies $\eta < 0$ and this leads to the configuration

- $s, f;\ N^d,\ \binom{1}{1}$ ©, $\binom{1}{1}$ ©: *Example* $\Rightarrow (c = 0,\ d = 2,\ g = -1,\ h = 1)$.

c) *The possibility* $\mathcal{T}_3 = 0$. By (40) due to $W_4 \neq 0$ the condition $\mathcal{T}_3 = 0$ implies $h = 0$ and then we obtain:

$$\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{F}_1 = 0,\ \ \mathcal{T}_2 = 4d^2g^2,\ \ \mathcal{B} = -2d^2g^4,$$
$$W_4 = -256d^6g^4,\ \ \eta = 4dg^3,\ \ \mu_2\tilde{L} = -8dg(gx^2 - dy^2)^2.$$

Remark 14 *If* $W_4 \neq 0$ *then the condition* $\mathcal{T}_4 = \mathcal{T}_3 = 0$ *implies* $W_4 < 0$.

So the condition $W_4 \neq 0$ implies $\mathcal{B} < 0$ and $\mathcal{T}_2 > 0$. Therefore by [?] we have a center and an integrable saddle. As the condition $\eta > 0$ implies $\mu_2\tilde{L} < 0$, taking into consideration Lemma 6 we get following two configurations:

- $s, c;\ N^d,\ \binom{1}{1}$ ©, $\binom{1}{1}$ ©: *Example* $\Rightarrow (c = 0,\ d = 1,\ g = -1,\ h = -1)$ (if $\eta < 0$);
- $s, c; \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}SN,\ N^d$: *Example* $\Rightarrow (c = 0,\ d = 1,\ g = 1,\ h = -1)$ (if $\eta > 0$).

6.2.2.2 The case $W_4 > 0$. Since $\widetilde{K} = 0$ and $G_8\mathbf{D} \neq 0$ (see Remark 12), according to [5] (see Table 1, line 151) systems (37) possess a saddle and a node.

6.2.2.2.1 The subcase $\mathcal{T}_4 \neq 0$ Then by [?] the saddle is strong and as $W_4 \neq 0$ the node is generic and $\theta_2 \neq 0$. Therefore considering Lemma 6 we obtain the following three global configurations of singularities:

- $s, n;\ N^d,\ \binom{1}{1}$ ©, $\binom{1}{1}$ ©: *Example* $\Rightarrow (c = 1,\ d = 1/2,\ g = -3,\ h = 1)$ (if $\eta < 0$);
- $s, n; \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}SN,\ N^d$: *Example* $\Rightarrow (c = 1,\ d = 1/2,\ g = 1,\ h = 1)$ (if $\eta > 0$, $\mu_2\tilde{L} < 0$);
- $s, n; \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}NS,\ N^d$: *Example* $\Rightarrow (c = 1,\ d = 1/2,\ g = -1,\ h = 1)$ (if $\eta > 0$, $\mu_2\tilde{L} > 0$).

6.2.2.2.2 The subcase $\mathcal{T}_4 = 0$ Then the saddle is weak and by Remark 4 without loss of generality we may assume that this saddle is located at $M_1(0,0)$, i.e. $c = 0$ and we consider the relations (40).

We observe that by Remark 14 the condition $W_4 > 0$ implies $\mathcal{T}_3 \neq 0$.

a) *The possibility* $\mathcal{F}_1 \neq 0$. In this case by [?] the weak saddle is of order one and considering Lemma 6 we arrive at the following three global configurations of singularities:

- $s^{(1)}, n;\ N^d,\ \binom{1}{1}$ ©, $\binom{1}{1}$ ©: *Example* $\Rightarrow (c = 0,\ d = 1/2,\ g = -3,\ h = 1)$.
- $s^{(1)}, n; \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}SN,\ N^d$: *Example* $\Rightarrow (c = 0,\ d = 1/2,\ g = 1,\ h = 1)$ (if $\mu_2\tilde{L} < 0$);
- $s^{(1)}, n; \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}NS,\ N^d$: *Example* $\Rightarrow (c = 0,\ d = 1/2,\ g = -1/2,\ h = 1)$ (if $\mu_2\tilde{L} > 0$).

b) *The possibility* $\mathcal{F}_1 = 0$. As $\mathcal{F}_2 = \mathcal{F}_3 = 0$ according to [?] systems (37) possess an integrable saddle. On the other hand by Remark 13 the condition $W_4 > 0$ implies $\eta > 0$ and this leads to the configuration

- $s, n; \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}NS,\ N^d$: *Example* $\Rightarrow (c = 0,\ d = 1/2,\ g = -1,\ h = 1)$.

6.2.2.3 The case $W_4 = 0$. Taking into account (39) we consider two possibilities: $\eta \neq 0$ and $\eta = 0$.

6.2.2.3.1 The subcase $\eta \neq 0$. Then due to $\mu_2 \neq 0$ the condition $W_4 = 0$ implies $\tau_1\tau_2 = (c^2 + 4d)(c^2 - 4d + 4ch + 4h^2) = 0$, i.e. one of singularities is a node with coinciding eigenvalues. Moreover by Remark 11 we have a node with one direction. According to Remark 4 we may assume that the singularity $M_1(0,0)$ is such a node and this implies $\tau_1 = c^2 + 4d = 0$, i.e. $d = -c^2/4/ne0$. So we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, yt/c)$ and this leads to the family of systems

$$\dot{x} = x - y/4 + gx^2 + 2hxy + y^2/4, \quad \dot{y} = x. \tag{41}$$

for which we calculate

$$\boldsymbol{\mu_2\tilde{L} = 2g(4gx^2 + 8hxy + y^2)^2/8,}$$
$$\boldsymbol{\mathcal{T}_4 = g(1+2h)(g-4h^2)/4,\ \ \theta_2 = (4h^2-g)/4,\ \ \eta = g^2(4h^2-g).}$$

1) *The possibility* $\boldsymbol{\mathcal{T}_4 \neq 0}$. Then by [?] we have a strong saddle. We observe that the condition $\boldsymbol{\eta \neq 0}$ implies $\boldsymbol{\theta_2 \neq 0}$ and considering Lemma 6 we obtain the following three global configurations of singularities:

- $\boldsymbol{s, n^d;\ N^d,\ (^1_1)\,©, (^1_1)\,©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=-1/4,\ g=1,\ h=0)}$ (if $\boldsymbol{\eta<0}$);
- $\boldsymbol{s, n^d;\ (^{\overline{1}}_1)SN, (^{\overline{1}}_1)SN,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=-1/4,\ g=-1,\ h=0)}$ (if $\boldsymbol{\eta>0,\ \mu_2\tilde{L}<0}$);
- $\boldsymbol{s, n^d;\ (^{\overline{1}}_1)SN, (^{\overline{1}}_1)NS,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=-1/4,\ g=1,\ h=1)}$ (if $\boldsymbol{\eta>0,\ \mu_2\tilde{L}>0}$).

2) *The possibility* $\boldsymbol{\mathcal{T}_4 = 0}$ Then the saddle $\boldsymbol{M_2}$ is weak. As $\boldsymbol{\eta \neq 0}$ we get $\boldsymbol{h=-1/2}$ and then

$$\boldsymbol{\eta = g^2(1-g),\ \ \mathcal{F}_1 = (1-g)/4,\ \ \theta_2 = (1-g)/4.}$$

So due to $\boldsymbol{\eta \neq 0}$ we have $\boldsymbol{\theta_2\mathcal{F}_1 \neq 0}$, i.e. the weak saddle is of order one. Therefore we get the following three configurations of singularities:

- $\boldsymbol{s^{(1)}, n^d;\ N^d,\ (^1_1)\,©, (^1_1)\,©}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=-1/4,\ g=2,\ h=-1/2)}$ (if $\boldsymbol{\eta<0}$);
- $\boldsymbol{s^{(1)}, n^d;\ (^{\overline{1}}_1)SN, (^{\overline{1}}_1)SN,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=-1/4,\ g=-1,\ h=-1/2)}$ (if $\boldsymbol{\eta>0, \mu_2\tilde{L}<0}$);
- $\boldsymbol{s^{(1)}, n^d;\ (^{\overline{1}}_1)SN, (^{\overline{1}}_1)NS,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=-1/4,\ g=1/2,\ h=-1/2)}$ (if $\boldsymbol{\eta>0,\ \mu_2\tilde{L}>0}$).

6.2.2.3.2 The subcase $\boldsymbol{\eta = 0}$. Considering (39) we have $\boldsymbol{g(dg+h^2)=0}$ and we consider two possibilities: $\boldsymbol{\tilde{L}\neq 0}$ (then $\boldsymbol{g\neq 0}$) and $\boldsymbol{\tilde{L}=0}$ (then $\boldsymbol{g=0}$).

1) *The possibility* $\boldsymbol{\tilde{L}\neq 0}$. In this case $\boldsymbol{g\neq 0}$ and as $\boldsymbol{d\neq 0}$ we obtain $\boldsymbol{g=-h^2/d\neq 0}$. Then we may assume $\boldsymbol{h=1}$ due to the rescaling $\boldsymbol{(x,y,t)\mapsto(hx, yt/h)}$ (and by introducing new parameters $\boldsymbol{\tilde{c}=c/h}$, $\boldsymbol{\tilde{d}=d/h^2}$). So we arrive at the family of systems (we keep the old parameters):

$$\boldsymbol{\dot{x} = cx + dy - x^2/d + 2xy - dy^2,\quad \dot{y} = x.} \tag{42}$$

for which we calculate

$$\begin{gathered}
\boldsymbol{\kappa=\eta=\widetilde{K}=\mathrm{D}=0,\ \ \kappa_1=-32(1+c-d)/d,\ \ \theta_5=96(1+c-d)(x-dy)^3)/d^2,}\\
\boldsymbol{\widetilde{M}=-8(x-dy)^2/d^2=-\tilde{L},\ \ \mu_2=(x-dy)^2,\ \ G_3=-2(1+c-d),}\\
\boldsymbol{W_8=2^{12}3^3(1+c-d)^2(c^2+4d)\big[(c+2)^2-4d\big]\ =\ 2^{12}3^3(1+c-d)^2\tau_1\tau_2,}\\
\boldsymbol{\mathcal{T}_i=0,\ i=1,2,3,4,\ \ \sigma=c-2x/d+2y,\ \ \mathcal{F}_1=-2(1+c-d),\ \ \mathcal{H}=0,}\\
\boldsymbol{\mathcal{B}_1=-2c(2+c)(1+c-d)=-2(1+c-d)\rho_1\rho_2,\ \ \mathcal{B}_2=4(1+c)(1+c-d)^2/d.}
\end{gathered} \tag{43}$$

Remark 15 *The condition* $\boldsymbol{W_8\neq 0}$ *implies* $\boldsymbol{G_3\kappa_1\theta_5\mathcal{F}_1\neq 0}$ *for systems* (42). *Moreover in this case the condition* $\boldsymbol{\mathcal{B}_1=0}$ *is equivalent to* $\boldsymbol{\rho_1\rho_2=0}$.

Since $\boldsymbol{\widetilde{K}=\mathrm{D}=0}$ according to [5] the types of the finite singularities are governed by the invariant polynomials $\boldsymbol{W_8, W_9}$, $\boldsymbol{G_3}$ and $\boldsymbol{D_2}$.

$\boldsymbol{a}$) *The case* $\boldsymbol{W_8<0}$. Then by the above remark we have $\boldsymbol{G_3\neq 0}$ and by [5] (see Table 1, line 159) systems (42) possess a saddle and a focus.

$\boldsymbol{a}$.1) *The subcase* $\boldsymbol{\mathcal{B}_1\neq 0}$. Then $\boldsymbol{\rho_1\rho_2\neq 0}$ and we do not have weak singularities. On the other hand as $\boldsymbol{\widetilde{M}\tilde{L}\kappa_1\theta_5\neq 0}$, considering Lemma 6, we obtain the following global configuration of singularities:

- $\boldsymbol{s, f;\ (\widehat{^2_2})\,\widehat{P}_{\curlywedge}\widehat{P}H_{\curlywedge}-H,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=-1,\ g=1,\ h=1)}$.

$\boldsymbol{a}$.2) *The subcase* $\boldsymbol{\mathcal{B}_1=0}$. Then one of the finite singularities is weak and by Remark 4 without loss of generality we may assume such a point to be $\boldsymbol{M_1(0,0)}$, i.e. for systems (42) the condition $\boldsymbol{\rho_1=c=0}$ holds. So we have:

$$\boldsymbol{\mathcal{B}_1=0,\ \ \mathcal{B}_2=4(1-d)^2/d,\ \ W_8=2^{16}3^3d(1-d)^3.} \tag{44}$$

Therefore the condition $\boldsymbol{W_8<0}$ implies $\boldsymbol{\mathcal{B}_2\neq 0}$ and by [?] the type of weak singularity (which is of order one due to $\boldsymbol{\mathcal{F}_1\neq 0}$) depends of the sign of $\boldsymbol{\mathcal{B}_2}$. Thus considering Lemma 6 we get the following two global configurations of singularities:

- $\boldsymbol{s, f^{(1)};\ (\widehat{^2_2})\,\widehat{P}_{\curlywedge}\widehat{P}H_{\curlywedge}-H,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=0,\ d=-1,\ g=1,\ h=1)}$ (if $\boldsymbol{\mathcal{B}_2<0}$);
- $\boldsymbol{s^{(1)}, f;\ (\widehat{^2_2})\,\widehat{P}_{\curlywedge}\widehat{P}H_{\curlywedge}-H,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=0,\ d=2,\ g=-1/2,\ h=1)}$ (if $\boldsymbol{\mathcal{B}_2>0}$).

b) *The case* $\boldsymbol{W_8 > 0}$. Then by [5] (see Table 1, lines 153) systems (42) possess a saddle and a node which is generic (due to $\boldsymbol{W_8 \neq 0}$).

b.1) *The subcase* $\boldsymbol{\mathcal{B}_1 \neq 0}$. Then the saddle is strong and considering Remark 15 and Lemma 6 we arrive at the configuration $\bullet$ $\boldsymbol{s, n;}$ $\boldsymbol{\widehat{\binom{2}{2}}\,\widehat{P}_{\curlywedge}\,\widehat{P}\,H_{\curlywedge}-H,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=1,\ g=-1,\ h=1)}$.

b.2) *The subcase* $\boldsymbol{\mathcal{B}_1 = 0}$. It was mentioned earlier that in this case we may assume $\boldsymbol{c = 0}$, i.e. the weak saddle of the first order (as $\boldsymbol{\mathcal{F}_1 \neq 0}$) is located at the origin of coordinates. Considering Remark 15 and Lemma 6 we obtain the configuration $\bullet$ $\boldsymbol{s^{(1)}, n;}$ $\boldsymbol{\widehat{\binom{2}{2}}\,\widehat{P}_{\curlywedge}\,\widehat{P}\,H_{\curlywedge}-H,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=0,\ d=1/2,\ g=-2,\ h=1)}$.

c) *The case* $\boldsymbol{W_8 = 0}$. Then $\boldsymbol{(1+c-d)\tau_1\tau_2 = 0}$ and considering (43) we have to distinguish two subcases given by the invariant polynomial $\boldsymbol{\kappa_1}$.

c.1) *The subcase* $\boldsymbol{\kappa_1 \neq 0}$. Then $\boldsymbol{(1+c-d) \neq 0}$ and the condition $\boldsymbol{W_8 = 0}$ gives $\boldsymbol{\tau_1\tau_2 = 0}$. Therefore we have a node with coinciding eigenvalues and by Remark 4 we may assume that this node is located at the the origin of coordinates, i.e. the condition $\boldsymbol{\tau_1 = c^2 + 4d = 0}$ holds. We note that by Remark 11 this node could not be a star node. So we have $\boldsymbol{d = -c^2/4 \neq 0}$ and we calculate

$$\boldsymbol{\mathcal{B}_1 = -c(2+c)^3/2 =,\ \kappa_1 = 32(2+c)^2/c^2,\ \ \theta_5 = -6(2+c)^2(4x+c^2y)^3/c^4.}$$

Since $\boldsymbol{\kappa_1 \neq 0}$ (in this case we also have $\boldsymbol{c \neq 0}$) by [**?**], the saddle is strong. On the other other hand the condition $\boldsymbol{\kappa_1 \neq 0}$ implies $\boldsymbol{\theta_5 \neq 0}$ and considering Lemma 6 we arrive at the configuration

$\bullet$ $\boldsymbol{s, n^d;}$ $\boldsymbol{\widehat{\binom{2}{2}}\,\widehat{P}_{\curlywedge}\,\widehat{P}\,H_{\curlywedge}-H,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=-1/4,\ g=4,\ h=1)}$.

c.2) *The subcase* $\boldsymbol{\kappa_1 = 0}$ Then by (43) we have $\boldsymbol{c = d-1}$ and for systems (42) we calculate

$$\begin{gathered}
\boldsymbol{\mu_0 = \mu_1 = \widetilde{K} = \eta = \mathrm{D} = \kappa = \kappa_1 = 0,\ \ \widetilde{M} = -8(x-dy)^2/d^2 = \mathcal{G},}\\
\boldsymbol{\mu_2 = (x-dy)^2,\ \ K_2 = 96(x-dy)^2/d^2,\ \ \theta_6 = 8(x-dy)^4/d^2,}\\
\boldsymbol{G_3 = 0,\ \ D_2 = 2(x-dy)/d,\ \ \mathrm{U} = x^2(x-dy)^4,}\\
\boldsymbol{\mathcal{T}_i = 0,\ i = 1,2,3,4,\ \ \sigma = d-1-2x/d+2y,\ \ \mathcal{F}_1 = \mathcal{H} = 0,}\\
\boldsymbol{\mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = 0,\ \ \mathcal{B}_3 = 72(d-1)(1+d)(x-dy)/d^2.}
\end{gathered} \tag{45}$$

Since $\boldsymbol{\mathrm{D} = \widetilde{K} = G_3 = 0}$ and $\boldsymbol{D_2 \neq 0}$, according to [5] (see Table 1, line 154) we have a saddle and a node. Considering (45), by [**?**] (see the Main Theorem, the statement $\boldsymbol{(e_3)[\delta]}$) the saddle will be weak (more precisely it will be an integrable one) if and only if $\boldsymbol{\mathcal{B}_3 = 0}$. Moreover for the singular points $\boldsymbol{M_1(0,0)}$ and $\boldsymbol{M_2(0,1)}$ we have, respectively:

$$\boldsymbol{\rho_1 = d-1,\ \ \tau_1 = (d+1)^2,\quad \rho_2 = d+1,\ \ \tau_1 = (d-1)^2.}$$

Therefore we observe that if one of the singular points is a weak saddle, the second one becomes a node with coinciding eigenvalues, which by Remark 11 is a node $\boldsymbol{n^d}$.

Since $\boldsymbol{K_2 > 0}$ and $\boldsymbol{\theta_6 \neq 0}$, considering Lemma 6, we obtain the following two global configuration of singularities:

$\bullet$ $\boldsymbol{s, n;}$ $\boldsymbol{\binom{2}{2}\,\widehat{P}\,\widehat{P}\,H - \widehat{P}\,\widehat{P}\,H,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=2,\ g=-1/2,\ h=1)}$ (if $\boldsymbol{\mathcal{B}_3 \neq 0}$);

$\bullet$ $\boldsymbol{\$, n^d;}$ $\boldsymbol{\binom{2}{2}\,\widehat{P}\,\widehat{P}\,H - \widehat{P}\,\widehat{P}\,H,\ N^d}$: *Example* $\Rightarrow$ $\boldsymbol{(c=0,\ d=1,\ g=-1,\ h=1)}$ (if $\boldsymbol{\mathcal{B}_3 = 0}$).

2) *The possibility* $\boldsymbol{\tilde{L} = 0}$. Then $\boldsymbol{g = 0}$ and we obtain the family of systems

$$\boldsymbol{\dot{x} = cx + dy + 2hxy - dy^2,\quad \dot{y} = x.} \tag{46}$$

for which we calculate

$$\begin{gathered}
\boldsymbol{\mu_0 = \mu_1 = \widetilde{K} = \kappa = \eta = G_8 = 0,\ \ \mu_2 = dy(-2hx+dy),\ \ \mathrm{D} = -192d^4h^2,}\\
\boldsymbol{\mathrm{U} = d^2x^2y^2(2hx-dy)^2,\ \ W_7 = 12d^2h^6(c^2+4d)\big[(c+2h)^2-4d\big] = 12d^2h^6\tau_1\tau_2,}\\
\boldsymbol{\mathcal{T}_i = 0,\ i = 1,2,3,4,\ \ \sigma = c+2hy,\ \ \mathcal{F}_1 = 2dh,\ \ \mathcal{H} = 0,}\\
\boldsymbol{\mathcal{B}_1 = 2cdh(c+2h) = 2dh\rho_1\rho_2,\ \ \mathcal{B}_2 = 4dh^3(c+h),\ \ \widetilde{M} = -32h^2y^2.}
\end{gathered} \tag{47}$$

For the singular points $\boldsymbol{M_1(0,0)}$ and $\boldsymbol{M_2(0,1)}$ of the above systems we have $\boldsymbol{\Delta_2 = d = -\Delta_1}$ and hence these systems possess a saddle and an anti-saddle.

a) *The case* $\boldsymbol{W_7 < 0}$. Then we have a saddle and a focus or a center and considering (47) we observe that the condition $\boldsymbol{\mathcal{B}_1 = 0}$ is equivalent to $\boldsymbol{\rho_1\rho_2 = 0}$.

a.1) *The subcase* $\boldsymbol{\mathcal{B}_1 \neq 0}$. Then both singularities are strong and considering Lemma 6 we arrive at the configuration $\bullet$ $\boldsymbol{s, f;}$ $\boldsymbol{\widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge}\,E\widehat{P}_{\curlywedge}-H, \overline{\binom{1}{1}}SN}$: *Example* $\Rightarrow$ $\boldsymbol{(c=1,\ d=-1,\ g=0,\ h=1)}$

$\boldsymbol{a}$.2) *The subcase* $\boldsymbol{\mathcal{B}_1 = 0}$. Then one of the finite singularities is weak and by Remark 4 without loss of generality we may assume such a point is $\boldsymbol{M_1(0,0)}$, i.e. for systems (46) the condition $\boldsymbol{\rho_1 = c = 0}$ holds. In this case we calculate:

$$\boldsymbol{\mathcal{B}_1 = 0,\ \ \mathcal{B}_2 = 4dh^4,\ \ \mathcal{F}_1 = 2dh,\ \ W_7 = -192d^3h^6(d - h^2).} \tag{48}$$

Therefore the condition $\boldsymbol{W_7 \neq 0}$ implies $\boldsymbol{\mathcal{F}_1\mathcal{B}_2 \neq 0}$ and by [**?**] the type of the weak singularity (which is of order one due to $\boldsymbol{\mathcal{F}_1 \neq 0}$) depends on the sign of $\boldsymbol{\mathcal{B}_2}$. Thus considering considering Lemma 6 we get the following two global configurations of singularities

- $\boldsymbol{s, f^{(1)};\ \widehat{\binom{1}{2}}\,\widehat{P}_\lambda E\widehat{P}_\lambda - H, \overline{\binom{1}{1}}SN}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 0,\ d = -1,\ g = 0,\ h = 1)}$ (if $\boldsymbol{\mathcal{B}_2 < 0}$);
- $\boldsymbol{s^{(1)}, f;\ \widehat{\binom{1}{2}}\,\widehat{P}_\lambda E\widehat{P}_\lambda - H, \overline{\binom{1}{1}}SN}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 0,\ d = 2,\ g = 0,\ h = 1)}$ (if $\boldsymbol{\mathcal{B}_2 > 0}$).

$\boldsymbol{b}$) *The case* $\boldsymbol{W_7 > 0}$. Then by [5] (see Table 1, lines 152) systems (46) possess a saddle and a node which is generic (due to $\boldsymbol{W_7 \neq 0}$).

So taking into account the fact that the saddle is weak if and only if $\boldsymbol{\mathcal{B}_1 = 0}$ we arrive at the following two global configurations of singularities

- $\boldsymbol{s, n;\ \widehat{\binom{1}{2}}\,\widehat{P}_\lambda E\widehat{P}_\lambda - H, \overline{\binom{1}{1}}SN}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1,\ d = 1/2,\ g = 0,\ h = 1)}$ (if $\boldsymbol{\mathcal{B}_1 \neq 0}$);
- $\boldsymbol{s^{(1)}, n;\ \widehat{\binom{1}{2}}\,\widehat{P}_\lambda E\widehat{P}_\lambda - H, \overline{\binom{1}{1}}SN}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 0,\ d = 1/2,\ g = 0,\ h = 1)}$ (if $\boldsymbol{\mathcal{B}_1 = 0}$).

$\boldsymbol{c}$) *The case* $\boldsymbol{W_7 = 0}$. Since $\boldsymbol{\mu_2 \neq 0}$ by (47) we have $\boldsymbol{h^6\tau_1\tau_2 = 0}$ and we consider two subcases: $\boldsymbol{\widetilde{M} \neq 0}$ and $\boldsymbol{\widetilde{M} = 0}$.

$\boldsymbol{c}$.1) *The subcase* $\boldsymbol{\widetilde{M} \neq 0}$. Then $\boldsymbol{h \neq 0}$ and we have a node with coinciding eigenvalues. By Remark 4 we may assume that this node is located at the the origin of coordinates, i.e. we have $\boldsymbol{\tau_1 = c^2 + 4d = 0}$. So we have a node $\boldsymbol{n^d}$ (see Remark 11) and setting $\boldsymbol{d = -c^2/4 \neq 0}$ we may assume $\boldsymbol{c = 1}$ due to the rescaling$\boldsymbol{(x, y, t) \mapsto (cx, yt/c)}$. This leads to the family of systems

$$\boldsymbol{\dot{x} = x - y/4 + 2hxy + y^2/4,\quad \dot{y} = x} \tag{49}$$

and we calculate

$$\boldsymbol{\sigma = 1 + 2hy,\ \ \mathcal{B}_1 = -h(1 + 2h)/2,\ \ \mathcal{B}_2 = -h^3(1 + h),\ \ \mathcal{F}_1 = -h/2,\ \ \widetilde{M} = -32h^2y^2.}$$

Since $\boldsymbol{\widetilde{M} \neq 0}$ we could have a weak saddle if and only if $\boldsymbol{h = -1/2}$ and the weak saddle is of order one due to $\boldsymbol{\mathcal{F}_1 \neq 0}$ (see [**?**], Main Theorem, the statement $\boldsymbol{(e_1)}$). So considering Lemma 6 we get the following two global configurations of singularities

- $\boldsymbol{s, n^d;\ \widehat{\binom{1}{2}}\,\widehat{P}_\lambda E\widehat{P}_\lambda - H, \overline{\binom{1}{1}}SN}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1,\ d = -1/4,\ g = 0,\ h = 1)}$ (if $\boldsymbol{\mathcal{B}_1 \neq 0}$);
- $\boldsymbol{s^{(1)}, n^d;\ \widehat{\binom{1}{2}}\,\widehat{P}_\lambda E\widehat{P}_\lambda - H, \overline{\binom{1}{1}}SN}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1,\ d = -1/4,\ g = 0,\ h = -1)}$ (if $\boldsymbol{\mathcal{B}_1 = 0}$).

$\boldsymbol{c}$.2) *The subcase* $\boldsymbol{\widetilde{M} = 0}$. Then $\boldsymbol{h = 0}$ and we get the family of systems

$$\boldsymbol{\dot{x} = cx + dy - dy^2,\quad \dot{y} = x} \tag{50}$$

and we calculate

$$\begin{gathered}\boldsymbol{\widetilde{M} = 0,\ \ C_2 = -dy^3,\ \ \mu_2 = d^2y^2,\ \ W_9 = 12(c^2 + 4d)(c^2 - 4d) = 12\tau_1\tau_2,}\\ \boldsymbol{\mathcal{T}_i = 0,\ i = 1, 2, 3, 4,\ \ \sigma = c = \rho_1 = \rho_2,\ \ \Delta 2 = d = -\Delta_1.}\end{gathered} \tag{51}$$

So the above systems possess a saddle and an anti-saddle and clearly the type of the anti-saddle is governed by the invariant polynomial $\boldsymbol{W_9}$.

$\boldsymbol{\alpha}$) *The possibility* $\boldsymbol{W_9 < 0}$. In this case we have a saddle and a focus or a center and considering (51) we observe that we could have a weak singularity if and only if we have $\boldsymbol{\sigma = 0}$. However in this case we get Hamiltonian systems possessing a center and an integrable saddle. As $\boldsymbol{C_2 \neq 0}$ (due to $\boldsymbol{\mu_2 \neq 0}$) considering Lemma 6 we obtain the following two global configurations of singularities

- $\boldsymbol{s, f;\ \widehat{\binom{2}{3}}\,\widehat{P}_\lambda\widehat{P} - \widehat{P}_\lambda\widehat{P}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 1,\ d = 1,\ g = 0,\ h = 0)}$ (if $\boldsymbol{\sigma \neq 0}$);
- $\boldsymbol{\$, c;\ \widehat{\binom{2}{3}}\,\widehat{P}_\lambda\widehat{P} - \widehat{P}_\lambda\widehat{P}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 0,\ d = 1,\ g = 0,\ h = 0)}$ (if $\boldsymbol{\sigma = 0}$).

$\boldsymbol{\beta}$) *The possibility* $\boldsymbol{W_9 \geq 0}$ Then we have a saddle and a node and in this case we have $\boldsymbol{\sigma \neq 0}$, otherwise if $\boldsymbol{c = 0}$ we get $\boldsymbol{W_9 = -192d^2 < 0}$. So the saddle is strong and the node is generic if $\boldsymbol{W_9 > 0}$ and it is a node with one direction (see Remark 11) if $\boldsymbol{W_9 = 0}$. Therefore considering Lemma 6 we obtain the following two global configurations of singularities

- $\boldsymbol{s, n;\ \widehat{\binom{2}{3}}\,\widehat{P}_\lambda\widehat{P} - \widehat{P}_\lambda\widehat{P}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 3,\ d = 1,\ g = 0,\ h = 0)}$ (if $\boldsymbol{W_9 > 0}$);
- $\boldsymbol{s, n^d;\ \widehat{\binom{2}{3}}\,\widehat{P}_\lambda\widehat{P} - \widehat{P}_\lambda\widehat{P}}$: *Example* $\Rightarrow$ $\boldsymbol{(c = 2,\ d = 1,\ g = 0,\ h = 0)}$ (if $\boldsymbol{W_9 = 0}$).

As all the possibilities are investigated we have ended the examination of systems (37).

B. Systems (38)

We consider the systems (38), i.e. the systems of the form

$$\dot{x} = cx + dy - cx^2 + 2hxy + ky^2, \quad \dot{y} = y \tag{52}$$

which possess the finite the singularities $M_1(0,0)$ and $M_2(0,1)$. We observe that the Jacobian matrices corresponding to these singular points are respectively $\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -c & d+2h \\ 0 & 1 \end{pmatrix}$. We observe that

$$\rho_1 = c+1, \ \ \Delta_1 = c, \ \ \tau_1 = (c-1)^2; \quad \rho_2 = 1-c, \ \ \Delta_2 = -c, \ \ \tau_2 = (c+1)^2. \tag{53}$$

So obviously we have the next remark.

Remark 16 *The family of systems* (52) *have a finite node and a finite saddle. The node has coinciding eigenvalues if and only if* $(c-1)(c+1) = 0$ *and in this case the systems simultaneously have a weak saddle. Moreover these systems have a star node if and only if either* $c-1 = d = 0$ *or* $c+1 = d+h = 0$.

For systems (52) we calculate

$$\begin{gathered} \mu_0 = \mu_1 = \kappa = \widetilde{K} = 0, \ \ \eta = 4c^2(h^2+ck), \ \ \mu_2 = c(cx^2 - 2hxy - ky^2), \\ \theta_2 = 0, \ \ \mathrm{U} = c^2y^2(cx^2 - 2hxy - ky^2)^2, \ \ G_8 = -2c^2(h^2+ck), \\ W_4 = 16c^4(c-1)^2(c+1)^2(h^2+ck)^2, \\ \mathrm{D} = -192c^4(h^2+ck), \ \ \tilde{L} = 8c(cx^2 - 2hxy - ky^2), \\ \mathcal{F}_1 = -2c(cd - h + ch), \ \ \mathcal{T}_4 = 4c^2(c+1)(c-1)(h^2+ck). \end{gathered} \tag{54}$$

We observe that for systems (52) the following conditions hold

$$W_4 \geq 0, \ \ \mu_2\tilde{L} > 0, \theta_2 = 0. \tag{55}$$

6.2.2.4 The case $W_4 \neq 0$. Then by (55) we have $W_4 > 0$ and we have $\eta\mathcal{T}_4 \neq 0$. Hence obviously the node is generic and the saddle is strong.

Considering (55) and Lemma 6 we obtain the following two global configurations of singularities:

- $s, n;\ N^*,\ \binom{1}{1}$ ©, $\binom{1}{1}$ ©: *Example* $\Rightarrow$ $(c=2,\ d=0,\ h=1,\ k=-1)$ (if $\eta < 0$).
- $s, n;\ \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}NS,\ N^*$: *Example* $\Rightarrow$ $(c=2,\ d=0,\ h=1,\ k=1)$ (if $\eta > 0$).

6.2.2.5 The case $W_4 = 0$. Taking into account (54), we consider two possibilities: $\eta \neq 0$ and $\eta = 0$.

6.2.2.5.1 The subcase $\eta \neq 0$. Then the condition $W_4 = 0$ implies $\tau_1\tau_2 = (c-1)^2(c+1)^2 = 0$, i.e. one of singularities is a node with coinciding eigenvalues. According to Remark 4 we may assume that the singularity $M_1(0,0)$ is such a node and this implies $\tau_1 = (c-1)^2 = 0$, i.e. $c = 1$. Then we calculate

$$\mathcal{T}_4 = \mathcal{F}_2 = \mathcal{F}_3 = 0, \ \ \mathcal{T}_3\mathcal{F} = 8(h^2+k)^2, \ \ \mathcal{F}_1 = -2d, \ \ \eta = 4(h^2+k).$$

We observe that the condition $\eta \neq 0$ implies $\mathcal{T}_3\mathcal{F} > 0$ and by [?] we have a weak saddle of order one if $\mathcal{F}_1 \neq 0$ and an integrable saddle if $\mathcal{F}_1 = 0$. On the other hand by Remark 16 we have a node n^d if $\mathcal{F}_1 \neq 0$ and a star node if $\mathcal{F}_1 = 0$.

Thus considering (55) and Lemma 6 we obtain the following two global configurations of singularities:

- $s^{(1)}, n^d;\ N^*,\ \binom{1}{1}$ ©, $\binom{1}{1}$ ©: *Example* $\Rightarrow$ $(c=1,\ d=1,\ h=1,\ k=-2)$ (if $\eta < 0$, $\mathcal{F}_1 \neq 0$);
- $\$, n^*;\ N^*,\ \binom{1}{1}$ ©, $\binom{1}{1}$ ©: *Example* $\Rightarrow$ $(c=1,\ d=0,\ h=1,\ k=-2)$ (if $\eta < 0$, $\mathcal{F}_1 = 0$);
- $s^{(1)}, n^d;\ \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}NS,\ N^*$: *Example* $\Rightarrow$ $(c=1,\ d=1,\ h=1,\ k=0)$ (if $\eta > 0$, $\mathcal{F}_1 \neq 0$);
- $\$, n^*;\ \overline{\binom{1}{1}}SN, \overline{\binom{1}{1}}NS,\ N^*$: *Example* $\Rightarrow$ $(c=1,\ d=0,\ h=1,\ k=-2)$ (if $\eta > 0$, $\mathcal{F}_1 = 0$).

6.2.2.5.2 The subcase $\eta = 0$. Considering (54) due to $\mu_2 \neq 0$ (i.e. $c \neq 0$) we obtain $k = -h^2/c \neq 0$. Then we may assume $h = 1$ due to the rescaling $(x, y, t) \mapsto (x, y/ht)$. So we arrive at the family of systems

$$\dot{x} = cx + dy - cx^2 + 2xy - y^2/c, \quad \dot{y} = y. \tag{56}$$

for which we calculate

$$\kappa = \eta = \kappa = \widetilde{K} = \mathbf{D} = \theta_5 = 0,\ \ \kappa_1 = -32c(c-1+cd) = 16G_3,$$
$$\widetilde{M} = -8(cx-y)^2 = -\tilde{L},\ \ \mu_2 = (cx-y)^2,\ \ \sigma = 1+c-2cx+2y,$$
$$W_8 = 2^{12}3^3c^2(c-1)^2(1+c)^2(c-1+cd)^2,\ \ \mathcal{H} = 0,$$
$$\mathcal{T}_i = 0,\ i = 1,2,3,4,\ \ \mathcal{F}_1 = -2(c-1+cd),\ \ \mathcal{B}_2 = 4c^2(c-1+cd)^2,$$
$$\mathcal{B}_1 = 2c(c-1+cd)(c+1)(c-1) = 2c(c-1+cd)\rho_1\rho_2. \tag{57}$$

We note that by Remark 16 the above systems possess a node and a saddle.

a) *The case* $W_8 \neq 0$. Then by (57) we have $W_8 > 0$ and this implies $\kappa_1\mathcal{B}_1 \neq 0$. Hence obviously the node is generic and the saddle is strong. Since $\kappa_1\widetilde{M}\tilde{L} \neq 0$ and $\theta_5 = 0$ considering Lemma 6 we obtain the following configuration $\bullet\ s, n;\ \binom{2}{2}\ \widehat{P}_{\curlywedge}\widehat{P}H_{\curlywedge} - H,\ N^*$: *Example* $\Rightarrow (c=2,\ d=0,\ h=1,\ k=-1/2)$.

b) *The case* $W_8 = 0$. Then $(c-1)(1+c)(c-1+cd) = 0$ (as $c \neq 0$ and considering (57) we have to distinguish two subcases: $\kappa_1 \neq 0$ and $\kappa_1 = 0$.

b.1) *The subcase* $\kappa_1 \neq 0$. Then $(c-1+cd) \neq 0$ and hence we get $(c-1)(1+c) = 0$. So one of singularities is a node with coinciding eigenvalues and by Remark 4 we may assume that the singularity $M_1(0,0)$ is such a node, i.e. $c = 1$. Then we calculate

$$\mathcal{H} = \mathcal{B}_1 = 0,\ \ \mathcal{F}_1 = -2d,\ \ \mathcal{B}_2 = 4d^2,\ \ \kappa_1 = -32d.$$

Therefore the condition $\kappa_1 \neq 0$ implies $\mathcal{F}_1 \neq 0$ and $\mathcal{B}_2 > 0$. By [?] (see the Main Theorem, the statement (e_1)) and by Remark 16, systems (56) possess a weak saddle of order one and a node n^d.

Thus considering the condition $\theta_5 = 0$ by Lemma 6 we arrive at the following global configuration of singularities: $\bullet\ s^{(1)}, n^d;\ \binom{2}{2}\ \widehat{P}_{\curlywedge}\widehat{P}H_{\curlywedge} - H,\ N^*$: *Example* $\Rightarrow (c=1,\ d=1,\ h=1,\ k=-1)$.

b.2) *The subcase* $\kappa_1 = 0$. Then due to $\mu_2 \neq 0$ (i.e. $c \neq 0$) by (57) we have $d = (1-c)/c$. So for systems (56) we calculate

$$\widetilde{K} = \kappa_1 = \theta_6 = 0,\ \ \widetilde{M} = -8(cx-y)^2 = \mathcal{G},\ \ K_2 = 96c^2(cx-y)^2,$$
$$\mu_2 = (cx-y)^2,\ \ \sigma = 1+c-2cx+2y,\ \ \mathcal{F}_1 = \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = 0,$$
$$\mathcal{B}_3 = -72(-1+c)(1+c)(cx-y)^2 = -\rho_1\rho_2(cx-y)^2. \tag{58}$$

Considering (58) by [?] (see the Main Theorem, the statement $(e_3)[\delta]$) the saddle will be weak (more precisely will be an integrable one) if and only if $\mathcal{B}_3 = 0$. Moreover by Remark 16 besides the integrable saddle we have a star node. Since $K_2 > 0$ and $\theta_6 = 0$ considering Lemma 6 we obtain the following two global configuration of singularities:

- $s, n;\ \binom{2}{2}\ \widehat{P}\widehat{P}H - \widehat{P}\widehat{P}H,\ N^*$: *Example* $\Rightarrow (c=2,\ d=-1/2,\ h=1,\ k=-1/2)$ (if $\mathcal{B}_3 \neq 0$);
- $s, n^*;\ \binom{2}{2}\ \widehat{P}\widehat{P}H - \widehat{P}\widehat{P}H,\ N^*$: *Example* $\Rightarrow (c=1,\ d=0,\ h=1,\ k=-1)$ (if $\mathcal{B}_3 = 0$).

Since all the possibilities are investigated the examination of systems (52) is complete.

6.3 The family of quadratic differential systems with only one finite singularity which is of multiplicity two

Assuming that quadratic systems (2) possess a double singular point, according to [?] we have to consider two cases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

6.3.1 Systems with $\widetilde{K} \neq 0$

In this case, following [?], we consider the following family of systems

$$\dot{x} = dy + gx^2 + 2dxy, \quad \dot{y} = fy + lx^2 + 2fxy \tag{59}$$

possessing the double singular point $M_{1,2}(0,0)$. For these systems calculations yield

$$\mu_0 = \mu_1 = 0,\ \ \mu_2 = (dl-fg)^2x^2,\ \ \kappa = 128d^2(dl-fg),\ \ \mathcal{T}_4 = 4d^2f^2(dl-fg)^2. \tag{60}$$

Remark 17 *We observe that the family of systems* (59) *depends on four parameters. However due to a rescaling we can reduce the number of the parameters to two. More precisely since by the condition* $\mu_2 \neq 0$ *we have* $d^2 + f^2 \neq 0$, *then we may assume* $d, f \in \{(1,1),(1,0),(0,1)\}$ *due to the rescaling:* (i) $(x,y,t) \mapsto (x, fy/d, t/f)$ *if* $df \neq 0$; (ii) $(x,y,t) \mapsto (x, y/d, t)$ *if* $f = 0$ *and* (iii) $(x,y,t) \mapsto (x,y,t/f)$ *if* $d = 0$.

Considering (60) and $\mu_2 \neq 0$, the condition $d = 0$ is equivalent to $\kappa = 0$ and in the case $\kappa \neq 0$ the condition $f = 0$ is equivalent to $\mathcal{T}_4 = 0$.

6.3.1.1 The case $\kappa \neq 0$

6.3.1.1.1 The subcase $\mathcal{T}_4 \neq 0$ Then we have $df \neq 0$ and considering Remark 17 we may assume $d = f = 1$. So we obtain the 2-parameter family of systems:

$$\dot{x} = y + gx^2 + 2xy, \quad \dot{y} = y + lx^2 + 2xy$$

for which calculations yield

$$\begin{gathered}\mu_0 = \mu_1 = 0, \ \ \mu_2 = (g-l)^2x^2, \ \ \widetilde{K} = 4(g-l)x^2, \ \ G_8 = 2(g-l)^2,\\ \mathcal{T}_4 = 4(g-l)^2 = G_1, \ \ \kappa = 128(l-g), \ \ \eta = 4\big[(g+2)^2 + 8(l-g)\big],\\ \widetilde{M} = -8\big[(g-2)^2 + 6l\big]x^2 + 16(2-g)xy - 32y^2.\end{gathered}$$

Remark 18 *We observe that* $\widetilde{M} \neq 0$ *and* $\mu_2 > 0$. *Moreover the condition* $\kappa > 0$ *implies* $\eta > 0$.

As $G_8G_1 \neq 0$ according to [5] (see Table 1, line 171) the double finite singular point is a saddle-node.

1) *The possibility* $\kappa < 0$. As $\widetilde{M} \neq 0$ and $\mu_2 > 0$, by Lemma 6 we get the following three global configurations of singularities

- $\overline{sn}_{(2)}; \binom{\overline{2}}{1}N, ©, ©$: *Example* $\Rightarrow (g = 0,\ l = -1)$ (if $\eta < 0$);
- $\overline{sn}_{(2)}; \binom{\overline{2}}{1}N, S, N^\infty$: *Example* $\Rightarrow (g = 0,\ l = -1/4)$ (if $\eta > 0$);
- $\overline{sn}_{(2)}; \binom{0}{2}SN, \binom{\overline{2}}{1}N$: *Example* $\Rightarrow (g = 0,\ l = -1/2)$ (if $\eta = 0$).

2) *The possibility* $\kappa > 0$. By Remark 18 we have $\eta > 0$ and according to Lemma 6 we arrive at the configuration
- $\overline{sn}_{(2)}; \binom{\overline{2}}{1}S, N^f, N^f$: *Example* $\Rightarrow (g = 0,\ l = 1)$.

6.3.1.1.2 The subcase $\mathcal{T}_4 = 0$ Then we have $d \neq 0$, $f = 0$ and considering Remark 17 we may assume $d = 1$.

So we obtain the 2-parameter family of systems:

$$\dot{x} = y + gx^2 + 2xy, \quad \dot{y} \ = lx^2 \tag{61}$$

for which calculations yield

$$\begin{gathered}\mu_0 = \mu_1 = 0, \ \ \mu_2 = l^2x^2, \ \ \widetilde{K} = -4lx^2, \ \ G_8 = 2l^2, \ \ G_1 = 0,\\ \kappa = 128l, \ \ \eta = 4(8l + g^2), \ \ \widetilde{M} = -8(6cl + g^2)x^2 - 16gxy - 32y^2.\end{gathered} \tag{62}$$

As $G_8 \neq 0$ and $G_1 = 0$ according to [5] (see Table 1, line 175) the double finite singular point is a cusp.

1) *The possibility* $\kappa < 0$. As $\widetilde{M} \neq 0$ and $\mu_2 > 0$, by Lemma 6 we get the following three global configurations of singularities

- $\widehat{cp}_{(2)}; \binom{\overline{2}}{1}N, ©, ©$: *Example* $\Rightarrow (g = 2,\ l = -1)$ (if $\eta < 0$);
- $\widehat{cp}_{(2)}; \binom{\overline{2}}{1}N, S, N^\infty$: *Example* $\Rightarrow (g = 2,\ l = -1/4)$ (if $\eta > 0$);
- $\widehat{cp}_{(2)}; \binom{0}{2}SN, \binom{\overline{2}}{1}N$: *Example* $\Rightarrow (g = 2,\ l = -1/2)$ (if $\eta = 0$).

2) *The possibility* $\kappa > 0$. We observe that the condition $\kappa > 0$ implies $l > 0$ and then $\eta = 4(8l + g^2) > 0$. So considering Lemma 6 we arrive at the global configuration of singularities
- $\widehat{cp}_{(2)}; \binom{\overline{2}}{1}S, N^f, N^f$: *Example* $\Rightarrow (g = 1,\ l = 1)$.

6.3.1.2 The case $\kappa = 0$.

Then for systems (59) we have $d = 0$ and by Remark 17 we may assume $f = 1$. So we get the family of systems

$$\dot{x} = gx^2, \quad \dot{y} = y + lx^2 + 2xy, \tag{63}$$

for which calculations yield

$$\begin{gathered}\mu_0 = \mu_1 = 0, \ \ \mu_2 = g^2x^2, \ \ \widetilde{K} = 4gx^2, \ \ \eta = \kappa = G_8 = 0, \ \ K_2 = 0,\\ \tilde{L} = 8g(g-2)x^2, \ \ \widetilde{M} = -8(g-2)^2x^2, \ \ C_2 = -lx^3 + (g-2)x^2y.\end{gathered} \tag{64}$$

As $G_8 = 0$ and $\widetilde{K} \neq 0$ according to [5] (see Table 1, line 172) the double finite singular point is a saddle-node.

6.3.1.2.1 The subcase $\widetilde{K} < 0$ Then $g < 0$ and this implies $\widetilde{M} \neq 0$. As $\mu_2 > 0$ and $K_2 = 0$ considering Lemma 6 we obtain the configuration • $\overline{sn}_{(2)}$; $\binom{2}{2}$ $\widehat{P}H - \widehat{P}H$, N^f : *Example* $\Rightarrow$ $(g = -1,\ l = 0)$.

6.3.1.2.2 The subcase $\widetilde{K} > 0$. In view of Lemma 6 we consider two possibilities: $\tilde{L} \neq 0$ and $\tilde{L} = 0$.

1) *The possibility* $\tilde{L} \neq 0$. Then $g - 2 \neq 0$ and we have $\widetilde{M} \neq 0$. So taking into account the conditions $\mu_2 > 0$ and $K_2 = 0$, by Lemma 6 we arrive at the following two global configurations of singularities

• $\overline{sn}_{(2)}$; $\binom{2}{2}$ $\widehat{P}E - \widehat{P}E$, S : *Example* $\Rightarrow$ $(g = 1,\ l = 0)$ (if $\tilde{L} < 0$);

• $\overline{sn}_{(2)}$; $\binom{2}{2}$ $\widehat{P}H - \widehat{P}H$, N^∞ : *Example* $\Rightarrow$ $(g = 3,\ l = 0)$ (if $\tilde{L} > 0$).

2) *The possibility* $\tilde{L} = 0$. In this case $g = 2$ and this implies $\widetilde{M} = 0$. As $\mu_2 > 0$ and $K_2 = 0$, by Lemma 6 we arrive at the following two global configurations of singularities

• $\overline{sn}_{(2)}$; $\binom{2}{3}$ $HE - \widehat{P}\,\widehat{P}$: *Example* $\Rightarrow$ $(g = 2,\ l = 1)$ (if $C_2 \neq 0$);

• $\overline{sn}_{(2)}$; $[\infty;\ N^d]$: *Example* $\Rightarrow$ $(g = 2,\ l = 0)$ (if $C_2 = 0$).

6.3.2 Systems with $\widetilde{K} = 0$.

In this case according to [?] we consider the following family of systems

$$\dot{x} = cx + dy, \quad \dot{y} = lx^2 + 2mxy + ny^2, \quad 0 \neq c^2 n - 2cdm + d^2 l \equiv \mathcal{Z}, \tag{65}$$

for which we calculate

$$\begin{gathered} \mu_0 = \mu_1 = \kappa = 0,\ \ \mu_2 = \mathcal{Z}(lx^2 + 2mxy + ny^2),\ \ \widetilde{K} = 0, \\ \tilde{L} = 8n(lx^2 + 2mxy + ny^2),\ \ \eta = 4n^2(m^2 - ln), \\ \theta_2 = -d(m^2 - ln),\ \ \widetilde{M} = -8(4m^2 - 3ln)x^2 - 16mnxy - 8n^2y^2, \\ G_8 = 2n(ln - m^2)\mathcal{Z},\ \ G_1 = 4c^2n(ln - m^2)\mathcal{Z} = \mathcal{T}_4, \end{gathered} \tag{66}$$

6.3.2.1 The case $\tilde{L} \neq 0$. We consider two subcases: $\eta \neq 0$ and $\eta = 0$.

6.3.2.1.1 The subcase $\eta \neq 0$. In this case $G_8 \neq 0$ and we examine two possibilities: $\mathcal{T}_4 \neq 0$ and $\mathcal{T}_4 = 0$.

1) *The possibility* $\mathcal{T}_4 \neq 0$. Then we have $cn \neq 0$ and due to the rescaling $(x, y, t) \mapsto (x, cy/n, t/c)$ we may assume $c = n = 1$. So we get the family of systems

$$\dot{x} = x + dy, \quad \dot{y} = lx^2 + 2mxy + y^2, \quad 0 \neq 1 - 2dm + d^2 l \equiv \mathcal{Z}', \tag{67}$$

where we may consider $d \in \{0, 1\}$ due to the rescaling $x \to dx$ if $d \neq 0$. For these systems we have

$$\begin{gathered} \mathcal{T}_4 = 4(l - m^2)\mathcal{Z}' = G_1 = 2G_8,\ \ \eta = 4(m^2 - l),\ \ \theta_2 = d(l - m^2), \\ \mu_2 = \mathcal{Z}'(lx^2 + 2mxy + y^2),\ \ \tilde{L} = 8(lx^2 + 2mxy + y^2). \end{gathered}$$

As $G_8 G_1 \neq 0$ according to [5] (see Table 1, line 171) the finite singular point is a saddle-node.

a) *The case* $\eta < 0$. Considering Lemma 6 we arrive at the following two global configurations of singularities:

• $\overline{sn}_{(2)}$; N^d, $\binom{1}{1}$ ©, $\binom{1}{1}$ © : *Example* $\Rightarrow$ $(d = 1,\ l = 1,\ m = 0)$ (if $\theta_2 \neq 0$);

• $\overline{sn}_{(2)}$; N^*, $\binom{1}{1}$ ©, $\binom{1}{1}$ © : *Example* $\Rightarrow$ $(d = 0,\ l = 1,\ m = 0)$ (if $\theta_2 = 0$).

b) *The case* $\eta > 0$. We observe, that $\operatorname{sign}(\mu_2 \tilde{L}) = \operatorname{sign}(\mathcal{Z}')$. Moreover if $d = 0$ we obtain $\mathcal{Z}' = 1 > 0$. So considering Lemma 6 we arrive at the following three global configurations of singularities

• $\overline{sn}_{(2)}$; $\overline{\binom{1}{1}}SN$, $\binom{1}{1}SN$, N^d : *Example* $\Rightarrow$ $(d = 1,\ l = 0,\ m = 1)$ (if $\mu_2 \tilde{L} < 0$);

• $\overline{sn}_{(2)}$; $\overline{\binom{1}{1}}SN$, $\overline{\binom{1}{1}}NS$, N^d : *Example* $\Rightarrow$ $(d = 1,\ l = 0,\ m = -1)$ (if $\mu_2 \tilde{L} > 0$, $\theta_2 \neq 0$);

• $\overline{sn}_{(2)}$; $\overline{\binom{1}{1}}SN$, $\overline{\binom{1}{1}}NS$, N^* : *Example* $\Rightarrow$ $(d = 0,\ l = 0,\ m = 1)$ (if $\mu_2 \tilde{L} > 0$, $\theta_2 = 0$).

2) *The possibility* $\mathcal{T}_4 = 0$. By (66) due to the condition $\eta \neq 0$ we obtain $c = 0$ and then $dln \neq 0$. So via the rescaling $(x, y, t) \mapsto (dx/n, y/n, t)$ we may assume $n = d = 1$ and we arrive at the family of systems

$$\dot{x} = y, \quad \dot{y} = lx^2 + 2mxy + y^2. \tag{68}$$

We observe that we may assume $m \in \{0, 1\}$ due to the rescaling $(x, y, t) \mapsto (x, my, t/m)$ if $m \neq 0$. For these systems we calculate:

$$\begin{gathered} G_1 = 0,\ \ G_8 = 2l(l - m^2),\ \ \eta = 4(m^2 - l),\ \ \theta_2 = l - m^2, \\ \mu_2 = l(lx^2 + 2mxy + y^2),\ \ \tilde{L} = 8(lx^2 + 2mxy + y^2). \end{gathered}$$

So the condition $\eta\mu_2 \neq 0$ implies $G_8 \neq 0$ and due to $G_1 = 0$ by [5] (see Table 1, line 175) the double finite singular point is a cusp.

On the other hand we have $\theta_2 \neq 0$ and sign $(\mu_2\tilde{L}) =$ sign (l). So considering Lemma 6 we arrive at the following three global configurations of singularities

- $\widehat{cp}_{(2)}$; N^d, $\binom{1}{1}$ ©, $\binom{1}{1}$ ©: *Example* $\Rightarrow$ $(l = 1,\ m = 0)$ (if $\eta < 0$);
- $\widehat{cp}_{(2)}$; $\overline{\binom{1}{1}}SN$, $\overline{\binom{1}{1}}SN$, N^d : *Example* $\Rightarrow$ $(l = -1,\ m = 1)$ (if $\eta > 0$, $\mu_2\tilde{L} < 0$);
- $\widehat{cp}_{(2)}$; $\overline{\binom{1}{1}}SN$, $\overline{\binom{1}{1}}NS$, N^d : *Example* $\Rightarrow$ $(l = 1,\ m = 2)$ (if $\eta > 0$, $\mu_2\tilde{L} > 0$).

6.3.2.1.2 The subcase $\eta = 0$ As $n \neq 0$ (due to $\tilde{L} \neq 0$) we may assume $n = 1$ due to a rescaling. So considering (66) the condition $\eta = 0$ gives $m^2 - l = 0$ and we obtain $l = m^2$. Then for systems (65) we have

$$\mu_2 = (c - dm)^2(mx + y)^2, \quad \kappa_1 = 32m(c - dm)$$

and as $\mu_2 \neq 0$ the condition $\kappa_1 = 0$ is equivalent to $m = 0$.

1) *The possibility* $\kappa_1 \neq 0$. Then $m \neq 0$ and we may assume $m = 1$ due to the rescaling $x \to x/m$. Therefore we arrive at the family of systems

$$\dot{x} = cx + dy, \quad \dot{y} = (x + y)^2, \tag{69}$$

for which we calculate

$$\begin{gathered}\eta = \kappa = G_8 = 0, \quad \mu_2 = (c - d)^2(x + y)^2, \quad \widetilde{K} = \widetilde{N} = 0,\\ \tilde{L} = 8(x + y)^2, \quad \widetilde{M} = -8(x + y)^2, \quad \theta_5 = 96(c - d)d(x + y)^3,\\ \kappa_1 = 32(c - d), \quad F_3 = 24c(c - d)(x + y), \quad \mathcal{B}_1 = 2c^2(c - d)^2.\end{gathered} \tag{70}$$

a) *The case* $\mathcal{B}_1 \neq 0$. Then $c \neq 0$ and we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. We observe that in this case $F_3 \neq 0$ and as $G_8 = \widetilde{K} = \widetilde{N} = 0$, according to [5] (see Table 1, line 174), the finite singular point is a saddle-node.

On the other hand we have $\eta = \kappa = \widetilde{K} = 0$ and $\widetilde{M}\tilde{L}\kappa_1 \neq 0$. Therefore considering Lemma 6 we obtain the following two global configurations of singularities

- $\overline{sn}_{(2)}$; $\widehat{\binom{2}{2}}\,\widehat{P}_{\curlywedge}\widehat{P}H_{\curlywedge} - H$, N^d : *Example* $\Rightarrow$ $(c = 1,\ d = 2)$ (if $\theta_5 \neq 0$);
- $\overline{sn}_{(2)}$; $\widehat{\binom{2}{2}}\,\widehat{P}_{\curlywedge}\widehat{P}H_{\curlywedge} - H$, N^* : *Example* $\Rightarrow$ $(c = 1,\ d = 0)$ (if $\theta_5 = 0$).

b) *The case* $\mathcal{B}_1 = 0$. Then $c = 0$ and this implies $F_3 = 0$. So by [5] (see Table 1, line 177) the double finite singular point is a cusp. As in this case $\theta_5 = -96d^2(x + y)^3 \neq 0$ (due to $\mu_2 = d^2(x + y)^2 \neq 0$), considering (70) and Lemma 6 we get the configuration • $\widehat{cp}_{(2)}$; $\widehat{\binom{2}{2}}\,\widehat{P}_{\curlywedge}\widehat{P}H_{\curlywedge} - H$, N^d: *Example* $\Rightarrow$ $(c = 0,\ d = 1)$.

1) *The possibility* $\kappa_1 = 0$. In this case we have $m = 0$ and this leads to the family of systems

$$\dot{x} = cx + dy, \quad \dot{y} = y^2, \tag{71}$$

for which we calculate $\mu_2 = c^2y^2$, $F_3 = 24c^2y$, $K_2 = 0$, $\theta_6 = -8dy^4$ and therefore the condition $\mu_2 \neq 0$ implies $F_3 \neq 0$. So the double finite singularity is a saddle-node and considering Lemma 6 we obtain the following two global configurations of singularities

- $\overline{sn}_{(2)}$; $\binom{2}{2}\,\widehat{P}H - \widehat{P}H$, N^d : *Example* $\Rightarrow$ $(c = 1, d = 1)$ (if $\theta_6 \neq 0$);
- $\overline{sn}_{(2)}$; $\binom{2}{2}\,\widehat{P}H - \widehat{P}H$, N^* : *Example* $\Rightarrow$ $(c = 1, d = 0)$ (if $\theta_6 = 0$).

6.3.2.2 The case $\tilde{L} = 0$ Then for systems (65) we have $n = 0$ and then $d \neq 0$ (otherwise we get degenerate systems). So we may assume $d = 1$ (due to a rescaling) and we obtain the family of systems

$$\dot{x} = cx + y, \quad , \quad \dot{y} = lx^2 + 2mxy, \tag{72}$$

for which we calculate

$$\begin{gathered}\eta = \kappa = G_8 = \widetilde{K} = \tilde{L} = 0, \quad \mu_2 = (l - 2cm)x(lx + 2my), \quad \widetilde{N} = -4m^2x^2,\\ \widetilde{M} = -32m^2x^2, \quad G_{10} = c^2m^3(l - 2cm), \quad \mathcal{B}_1 = 2c^2m(l - 2cm)\end{gathered} \tag{73}$$

6.3.2.2.1 The subcase $\widetilde{M} \neq 0$ Then $m \neq 0$ and we may assume $m = 1$ due to the rescaling $(x, y, t) \mapsto (x/m, y/m, t)$. Moreover for systems above with $m = 1$ we may consider $c \in \{0, 1\}$ due to the rescaling $(x, y, t) \mapsto (cx, c^2y, t/c)$ if $c \neq 0$.

Therefore we have $\widetilde{N} \neq 0$ and by [5] (see Table 1, lines 173,176) the finite singular point is a saddle–node if $G_{10} \neq 0$ and it is a cusp if $G_{10} = 0$. We observe that the condition $G_{10} = 0$ is equivalent to $\mathcal{B}_1 = 0$. So as $\widetilde{M} \neq 0$ and $\tilde{L} = 0$, considering Lemma 6 we obtain the following two global configurations of singularities

- $\widehat{\overline{sn}_{(2)}}$; $\binom{\widehat{1}}{2}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H, \binom{\overline{1}}{1} SN$: *Example* $\Rightarrow$ $(c = 1,\ m = 1,\ l = 1)$ (if $\mathcal{B}_1 \neq 0$);
- $\widehat{cp}_{(2)}$; $\binom{\widehat{1}}{2}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H, \binom{\overline{1}}{1} SN$: *Example* $\Rightarrow$ $(c = 0,\ m = 1,\ l = 0)$ (if $\mathcal{B}_1 = 0$).

6.3.2.2.2 The subcase $\widetilde{M} = 0$ Then $m = 0$ and $l \neq 0$ (otherwise we get degenerate systems). Hence we may assume $l = 1$ due to the rescaling $(x, y, t) \mapsto (x/l, y/l, t)$. Then for systems (72) with $m = 0$ and $l = 1$ considering (73) we obtain

$$\widetilde{M} = \widetilde{K} = \widetilde{N} = G_8 = 0,\ \ C_2 = -x^3,\ \ F_3 = -24cx,\ \ \mathcal{B}_4 = 6cx^2(cx + y).$$

So by [5] (see Table 1, lines 174,177) the finite singular point is a saddle–node if $F_3 \neq 0$ and it is a cusp if $F_3 = 0$. We observe that the condition $F_3 = 0$ is equivalent to $\mathcal{B}_4 = 0$. Considering the conditions above according to Lemma 6 we obtain the following two global configurations of singularities

- $\widehat{\overline{sn}_{(2)}}$; $\binom{\widehat{2}}{3}\ \widehat{P}_{\lambda} \widehat{P} - \widehat{P}_{\lambda} \widehat{P}$: *Example* $\Rightarrow$ $(c = 1,\ m = 0,\ l = 1)$ (if $\mathcal{B}_4 \neq 0$);
- $\widehat{cp}_{(2)}$; $\binom{\widehat{2}}{3}\ \widehat{P}_{\lambda} \widehat{P} - \widehat{P}_{\lambda} \widehat{P}$: *Example* $\Rightarrow$ $(c = 0,\ m = 0,\ l = 1)$ (if $\mathcal{B}_4 = 0$).

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