INTEGRABLE MAPS FROM GALOIS DIFFERENTIAL ALGEBRAS, BOREL TRANSFORMS AND NUMBER SEQUENCES

PIERGIULIO TEMPESTA

Abstract. A new class of integrable maps, obtained as lattice versions of polynomial dynamical systems is introduced. These systems are obtained by means of a discretization procedure that preserves several analytic and algebraic properties of a given differential equation, in particular symmetries and integrability [40]. Our approach is based on the properties of a suitable Galois differential algebra, that we shall call a Rota algebra. A formulation of the procedure in terms of category theory is proposed. In order to make the lattice dynamics confined, a Borel regularization is also adopted. As a byproduct of the theory, a connection between number sequences and integrability is discussed.

Contents

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1. Introduction

The analysis of evolution equations on a discrete background is a very active research area, due to their ubiquitousness both in mathematics and in theoretical physics. The formulation of quantum physics on a lattice is motivated, for instance, by the need for regularizing divergencies in field theory [44], [21], [28] and in several scenarios of quantum gravity [3], [17], [42], [36], [13]. This would imply the discreteness of space–time geometry. Discrete versions of nonrelativistic quantum mechanics have also been proposed (see, for instance, [12], [45] and references therein).

Integrable discrete systems have been widely investigated as well. For several respects, they seem to be more fundamental objects than the continuous ones. For this reason, many efforts have been devoted to the construction of discrete systems possessing an algebro–geometric structure reminiscent of that of continuous models. In particular, a challenging issue is to *discretize nonlinear ordinary and partial differential equations* in such a way that symmetry and integrability properties be preserved (see the recent interesting monograph [38] for the treatment of the Hamiltonian point of view). Among the classical examples of discrete integrable systems are the Toda systems and the Ablowitz–Ladik hierarchies [1]–[2].

A rich literature exists on discrete differential geometry and related algebraic aspects [4], [5], [30], [18], [11], [14], [19], [26], [29], [31], [32]. Frobenius manifolds are also relevant in the discussion of generalized Toda systems [8], [9]. The problem of physically consistent discretizations has been considered in the context of field theories and Hamiltonian gravity, for instance in [13], [15], [36].

In the paper [40], a new approach, offering a possible solution to this problem has been proposed. Indeed, by means of this approach one can preserve both Lie symmetries and Lax pairs associated to a given nonlinear PDE. Also, it allows to constructs new hierarchies of integrable discrete equations, representing discrete versions of the Gelfand–Dikii hierarchies. In this context, *the notion of integrability is equivalent to the requirement that a large class of exact solutions can be analytically constructed*, starting from those of the continuous models we wish to discretize. In particular, analytic solutions of the continuous systems are converted into exact solutions of the corresponding systems on the lattice.

In the present work, by extending the results of [40], we focus on the discretization of continuous *dynamical systems*. Precisely, the problem we address is that of the discretization of polynomial vector fields of the form:

$$\frac{d}{dt}z = a_N z^N + a_{N-1}z^{N-1} + \ldots + a_1 z + a_0, \tag{1}$$

with $N \in \mathbb{N}$, $z = z(t) \in \mathbb{R}$, $a_0, \ldots, a_N \in \mathbb{R}$.

To this aim, a suitable Galois differential algebra is introduced. It is a pair $(\mathcal{F}_{\mathcal{L}}, Q)$, where $(\mathcal{F}_{\mathcal{L}}, +, *_{\mathcal{Q}})$ is an associative algebra and Q is a delta operator, that can be represented by continuous or discrete derivatives. The algebras $\mathcal{F}_{\mathcal{L}}$ are defined as spaces of formal power series over a lattice $\mathcal{L}$, endowed with an associative and commutative product "*", depending on the choice of Q. The main feature of this product is that the operators $\mathcal{Q}$ act on these function spaces as *derivations*, i.e. the *Leibnitz rule* is preserved under their action (see also [43], [6]). An analogous product also has appeared, in a different context, in the theory of linear operators acting on spaces of polynomials [16].

The theoretical scheme we propose has a natural formulation in the language of Category Theory. We introduce the category $\mathcal{G}al(F,Q)$ of Galois differential algebras and that of abstract dynamical systems $\mathcal{E}$. The correspondence among continuous and discrete equations is expressed in terms of morphisms of algebras. A *functor* between these two categories is defined. Essentially, it enables to maps a differential equations into difference equations by preserving the underlying differential structure: the symmetry and integrability properties of the continuous model are naturally inherited by the discrete one associated and *solutions are mapped into solutions*. This construction is made explicit in the case when Q is a forward difference operator (Theorem 11). Consequently, the algebraic structure underlying the dynamical models involved becomes transparent.

Other approaches existing in the literature deal with the discretization of differential equations on suitable nonlinear lattices, adapted to the symmetries of the problem (see, for instance, the review [26]). Instead, in this work we introduce a new family of integrable maps, defined on a regular lattice of equally spaced points. This kind of lattice is usually more convenient for concrete applications, and is especially suitable for the implementation of numerical integrating algorithms.

The theory, being defined on the associative algebra $(\mathcal{F}_{\mathcal{L}},+,*_{\mathcal{Q}})$, is intrinsically a *nonlocal* one: the value of a dynamical observable on the lattice depends on different points of the lattice. This is actually a very common feature observed in the theory of discrete integrable models. When the dependence is on an infinite number of points, the approach still holds, but in a formal sense. A regularization procedure should be adopted afterwards, in order to give a physical content to this case. This reminds very much what happens in nonlocal theories when dealing with operator product expansions: all infinite contributions of every operator should be taken into account.

However, in several cases, the value of an observable depends on a *finite* number of lattice points, and the discretization will be said *effective*. With this terminology, we mean that the solutions obtained possess the general form

$$z_n = \sum_{k=0}^{n} a_k f(k), \tag{2}$$

where $a_k \in \mathbb{R}$ or $\mathbb{C}$ and $f(k)$ is a function of the point $k \in \mathbb{N}$ on the lattice only. In this work, we will focus on the more interesting case of effective discretizations.

An interesting byproduct of our approach is a connection between *integrability* and *number sequences*. Suppose we have a recurrence relation involving several points on a lattice. This recurrence relation defines a discrete dynamical system in an auxiliary space of variables. Once we solve the dynamical system, we have obtained a solution (usually a particular one) of the original recurrence relation. A different connection between number theory and difference operators, related to the theory of formal groups, has been established in [39], [41].

The structure of the paper is the following. In Section II, an introduction to the algebraic techniques relevant in the discretization procedure is proposed. In Section III, the notions of Rota algebra and the categories $\mathcal{G}al(F,Q)$ and $\mathcal{E}$ are introduced and the main Theorem 11 proved. In Section IV, some explicit examples of dynamical systems obtained according to the previous theory are constructed and some number theoretical aspects related to the approach are presented. Some open problems are discussed in Section VI.

2. Integrability preserving discretizations of differential equations: a general framework

2.1. **Basic definitions.** In order to make the paper self–consistent, in this Section we review some basic notions concerning the algebraic theory of polynomial sequences and of finite difference operators, much in the spirit of the monographs [33]–[35].

Let us denote by $\{p_n(x)\}_{n\in\mathbb{N}}$, $n = 0, 1, 2, \ldots$, a sequence of polynomials in x of order n. Let $\mathcal{F}$ denote the algebra of formal power series in one variable x, endowed with the operations of sum and multiplication of series. An element of $\mathcal{F}$ is expressed by the formal power series

$$f(x) = \sum_{k=0}^{\infty} b_k x^k. \tag{3}$$

Let T be the shift operator, whose action on a function is given by $Tf(x) = f(x+\sigma)$. The operator T can also be represented in terms of a differential operator as $T = e^{\sigma D}$, where D denotes the standard derivative.

Definition 1. *A linear operator S is said to be shift–invariant if it commutes with T. A shift–invariant operator Q is called a delta operator if $Qx = const \neq 0$.*

Definition 2. *A polynomial sequence $p_n(x)$ is called a sequence of basic polynomials for a delta operator Q if it satisfies the following conditions:*

1) $p_0(x) = 1$;
2) $p_n(0) = 0$ *for all* $n > 0$;
3) $Qp_n(x) = np_{n-1}(x)$.

It is possible to prove that every delta operator Q has a unique sequence of associated basic polynomials [35]. Apart the standard derivative operator ∂_x, a general class of difference operators can be defined by [24]

$$\Delta_p = \frac{1}{\sigma}\sum_{k=l}^{m} \alpha_k T^k, \quad l, m \in \mathbb{Z}, \quad l < m, \quad m - l = p, \tag{4}$$

where σ is the constant lattice spacing and α_k are constants such that

$$\sum_{k=l}^{m} \alpha_k = 0, \quad \sum_{k=l}^{m} k\alpha_k = c. \tag{5}$$

and $\alpha_m \neq 0$, $\alpha_l \neq 0$. We choose $c = 1$, to reproduce the derivative ∂_x in the continuous limit, when the lattice spacing goes to zero. A difference operator of the form (4), which satisfies equations (5), is said to be a delta operator of order p, if it approximates the continuous derivative up to terms of order σ^p.

As eq. (4) involves $m - l + 1$ constants α_k, subject to just the two conditions (5), we can fix all constants α_k by choosing $m - l - 1$ further conditions.

2.2. **Associative algebras and nonlocal functional products.** Let $\mathcal{Q}$ be a delta operator, and $\{p_n(x)\}_{n\in\mathbb{N}}$ be the associated basic sequence. Let $\mathcal{L}$ be a lattice of equally spaced points on the line, indexed by an integer variable x. Denote by $\mathcal{F}_{\mathcal{L}}$ the vector space of the formal power series on $\mathcal{L}$. Also, denote by $\mathcal{P}$ the space

of polynomials in one variable $x \in \mathcal{L}$. Since the basic polynomials $p_n(x)$ for every $\mathcal{Q}$ provide a basis of $\mathcal{F}_{\mathcal{L}}$, f can be expanded into a series of the form

$$f = \sum_{n=0}^{\infty} a_n p_n(x). \tag{6}$$

We can endow the space $\mathcal{F}_{\mathcal{L}}$ with the structure of an algebra, by introducing a product of functions adapted to the specific discretization required. Precisely, in [40], the following result has been proved.

Theorem 3. *For any delta operator $\mathcal{Q}$, whose associated sequence is $\{p_n(x)\}_{n\in\mathbb{N}}$, the product*

$$* : \mathcal{P} \times \mathcal{P} \longrightarrow \mathbb{R},$$

defined as

$$p_n(x) * p_m(x) := p_{n+m}(x)$$

and extended by linearity on the lattice, is associative, commutative and satisfies the Leibnitz rule:

$$\mathcal{Q}\left(f(x) * g(x)\right) = \mathcal{Q}f(x) * g(x) + f(x) * \mathcal{Q}g(x) \tag{7}$$

where $f(x), g(x)$ are functions on the lattice.

As an immediate consequence of the previous theorem, we have the following result.

Proposition 4. *For any choice of $\mathcal{Q}$, the space $(\mathcal{F}_{\mathcal{L}}, +, *_{\mathcal{Q}})$ is an associative algebra.*

A general element of $\mathfrak{F}$ will be a series of the form $\sum a_n p_n(x)$.

One of the main feature of the $*$ product is the fact that it is *nonlocal*. Indeed, the product $(f * g)$ does depend on the values of f and g at distinct points of the lattice, with the exception of the standard point–wise product of functions, that corresponds to the choice $\mathcal{Q} = \partial_x$. The simplest version of this product, i.e. that associated with the forward difference operator Δ, has been proposed in [43].

Observe that the operators $\mathcal{Q}$ act as *derivations* on the associated function algebra $(\mathcal{F}_{\mathcal{L}}, +, *_{\mathcal{Q}})$. However, if the lattice consists of a finite number of points, in general these derivations are not defined on the whole $\mathcal{F}_{\mathcal{L}}$, since there could be several points where their action is meaningless. For instance, the shift operator T is not defined on the extreme point N of the interval $[0, N]$.

In the sequel, we will suppose that the lattice is either infinite, or that could contain sufficiently many points for the delta operators (4)–(5) to be well defined at least on a close subinterval $I \subset \mathbb{N}$. In general, the algebras $(\mathcal{F}_{\mathcal{L}}, +, *_{\mathcal{Q}})$ are infinite–dimensional. Nevertheless, the expansion (6) can also truncate in some cases, that are of special interest for the applications.

3. Category theory, differential equations and their discretization

The discretization approach we wish to propose can be formulated in a natural way in the context of Category Theory. To this aim, we need to propose several new definitions.

3.1. **Rota algebras.** We introduce first the notion of Rota algebra, as a natural Galois differential algebra of functions over which the discretization procedure is carried out. To motivate this definition, crucial for the subsequent discussion, we shall recall first that of Rota correspondence, that has been discussed extensively in the literature [35], [24]–[25], [10] [22].

Definition 5. *We shall call the correspondence expressed by the following diagram*

$$\begin{array}{ccc} \partial & \xrightarrow{f} & Q \\ i\downarrow & & i\downarrow \\ \{x^n\}_n & \xrightarrow{g} & \{p_n(x)\}_n \end{array} \tag{8}$$

the Rota (or umbral) correspondence. Here i is the isomorphism between a delta operator Q and its associated basic sequence $\{p_n(x)\}_{n\in\mathbb{N}}$, the application f transforms delta operators into delta operators and g basic sequences into basic sequences.

The Rota correspondence allows to map a linear differential equation (defined in the usual algebra of $\mathcal{C}^\infty$ functions, endowed with the point–wise multiplication of functions), into a linear difference equation in such a way that several algebraic properties are preserved. The application of this correspondence to the Schrödinger equation, for instance, allows to discretize it in such a way that integrals of motion are conserved [25].

Nevertheless, in order to treat the more general case of *nonlinear equations*, one needs also to define their discrete versions on suitable associative algebras: the Rota algebras. Therefore, the Rota correspondence will be generalized in the framework of category theory. As will be shown in the subsequent considerations, an important class of exact solutions in this way will be conserved.

Definition 6. *A Rota algebra is a Galois differential algebra $(\mathfrak{F}, Q)$, where $\mathfrak{F}(+,\cdot,*_{\mathcal{Q}})$ is an associative algebra of formal power series, endowed with the usual sum of series and multiplication by a scalar, Q is a delta operator and $*$ is a composition law such that Q acts as a derivation on $\mathfrak{F}$:*

$$i)\quad \mathcal{Q}(a+b) = \mathcal{Q}(a) + \mathcal{Q}(b), \tag{9}$$

$$ii)\quad \mathcal{Q}(a*b) = \mathcal{Q}(a)*b + a*\mathcal{Q}(b). \tag{10}$$

Proposition 7. *Given a delta operator Q, there exists a unique Rota algebra $(\mathfrak{F}, Q)$ associated with it.*

Proof. It follows from the uniqueness both of the "*" product associated with Q and of the sequence of polynomials $\{p_n(x)\}_{n\in\mathbb{N}}$, that ensures that, once Q is assigned, the algebra $\mathfrak{F}(+,\cdot,*_{\mathcal{Q}})$ is uniquely determined. □

3.2. **The Galois category.** We introduce a subcategory of the well known category of associative algebras [27]: the *Galois category.*

Definition 8. *The Galois category, denoted by $\mathcal{G}al(\mathfrak{F}, Q)$ is the collection of all Rota algebras $(\mathfrak{F}, Q)$, with morphisms defined by*

$$\mu_{Q,Q'} : \mathcal{G}al(\mathfrak{F}, Q) \longrightarrow \mathcal{G}al(\mathfrak{F}, Q)$$

$$\mathfrak{F}(+,\cdot,*_{\mathcal{Q}}) \longrightarrow \mathfrak{F}'(+,\cdot,*_{\mathcal{Q}'}) \tag{11}$$

which are closed under composition.

The action of the morphism $\mu_{Q,Q'}$ on formal power series (or a $\mathcal{C}^\infty$ functions) is defined by

$$\sum_n a_n p_n(x) \longrightarrow \sum_m a_m q_m(x), \tag{12}$$

where $\{p_n(x)\}_{n\in\mathbb{N}}$ and $\{q_m(x)\}_{m\in\mathbb{N}}$ are the basic sequences associated with Q and Q' respectively. The property of closure under composition is trivial.

We also introduce the new category of the *abstract dynamical systems.*

Definition 9. *The category of abstract dynamical systems, denoted by $\mathcal{E}$, is the collection of equations of the form*

$$eq(Q, z, *) := Qz - a_N z^{*N} - a_{N-1} z^{*N-1} - \ldots + a_1 z - a_0 = 0, \tag{13}$$

where $a_0, \ldots, a_N$ are indeterminates over $\mathbb{R}$ and N is a fixed integer. The set of correspondences

$$u_{Q,Q'} : \mathcal{E} \longrightarrow \mathcal{E}, \tag{14}$$

$$eq(Q, z, *) \longrightarrow eq(Q', z, *'), \tag{15}$$

defines the class of morphisms of the category.

The closure of these morphisms under composition is easily verified.

As a consequence of the previous construction, we can define a functor relating the categories of Galois differential algebras and abstract dynamical systems.

Theorem 10. *The application*

$$F : \mathcal{G}al(\mathcal{F}, Q) \longrightarrow \mathcal{E},$$

$$\mathfrak{F}(+, \cdot, *_{\mathcal{Q}}) \longrightarrow eq(Q, z, *), \tag{16}$$

$$\mu_{Q,Q'} \longrightarrow u_{Q,Q'}, \tag{17}$$

is a functor between categories.

Proof. A direct verification shows that F preserves the composition of morphisms:

$$F(\mu_{Q'',Q'} \circ \mu_{Q',Q}) = F(\mu_{Q'',Q'}) \circ F(\mu_{Q',Q})$$

and identity morphisms:

$$F(\mu_{Q,Q}(\mathcal{A})) = \mu_{Q,Q}(F(\mathcal{A})),$$

where $\mathcal{A} \in \mathcal{G}al(\mathfrak{F}, Q)$. □

This functor encodes all the main features of the discretization procedure we propose. It generalizes considerably the Rota correspondence (8).

3.3. **Main theorem.** The following Theorem 11 represents the main result of this Section. As a particular instance of the action of the functor F, it guarantees the discretization of a differential equation into a difference equation belonging to the same category $\mathcal{E}$, with the property that their general solutions share the same algebraic structure.

Theorem 11. *Given a dynamical system of the form*

$$\frac{d}{dt}z = a_N z^N + a_{N-1}z^{N-1} + \ldots + a_1 z + a_0, \tag{18}$$

where $z : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function, $a_0, \ldots, a_N \in \mathbb{R}$, let

$$\mathcal{Q}z = a_N z^{*N} + a_{N-1}z^{*(N-1)} + \ldots + a_1 z + a_0 \tag{19}$$

be the abstract operator equation associated with it, defined in the Rota algebra $(\mathfrak{F}, Q)$ where $z \in \mathfrak{F}(+, \cdot, *_{\mathcal{Q}})$, and $z^{*N} := \underbrace{z * \ldots * z}_{N-times}$. Let

$$z = \sum_{k=0}^{\infty} b_k t^k \tag{20}$$

be a solution of (18) *in the ring of formal power series in t. Then the equation*

$$\begin{aligned} z_{n+1} - z_n \quad &= a_N \sum_{k_1,\ldots,k_N=0}^{n} \frac{(-1)^{k_1+\ldots k_N+n}}{k_1!\ldots k_N!} z_{k_1} z_{k_2} \cdots z_{k_N} \frac{n!}{(n-k_1-k_2-\ldots-k_N)!} + \\ a_{N-1} \quad & \sum_{k_1,\ldots,k_{N-1}=0}^{n} \frac{(-1)^{k_1+\ldots k_{N-1}+n}}{k_1!\ldots k_{N-1}!} z_{k_1} z_{k_2} \cdots z_{k_{N-1}} \frac{n!}{(n-k_1-k_2-\ldots-k_{N-1})!} + \ldots + \\ a_2 \quad & \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} \frac{(-1)^{k_1+k_2+n}}{k_1!k_2!} z_{k_1} z_{k_2} \frac{n!}{(n-k_1-k_2)!} + a_1 z_n + a_0, \end{aligned}$$

that represents eq. (19) *for $Q = \Delta^+$ on a regular lattice of points, indexed by the variable $n \in \mathbb{N}$, admits as a solution the series*

$$z_n = \sum_{k=0}^{n} b_k \frac{n!}{(n-k)!}. \tag{21}$$

Proof. The proof entails several steps. First, we prove that the equation (19) converts into the discrete map (21) for $Q = \Delta^+$. To this aim, we assume that $z \in \mathcal{C}^\infty$ is defined on an equally spaced regular lattice of points $\mathcal{L}$, and that $t = n\sigma$, with $n \in \mathbb{N}$ and $\sigma \in \mathbb{R}^+ \cup \{0\}$. The continuum limit is restored when $\sigma \to 0$. Unless specifically stated, in what follows we put $\sigma = 1$. Also, we introduce an auxiliary space of variables, by means of a finite transformation on the lattice.

Definition 12. *Let $p_n(t)$ be a basic sequence for a given delta operator Q. We call the transformation*

$$z(t) = \sum_{n=0}^{\infty} \widehat{z}_n p_n(t) \tag{22}$$

a discrete interpolating transformation with coefficients $\widehat{z}_n \in \mathbb{R}$.

In the specific case when $p_n(t)$ are the lower factorial polynomials, the discrete transform (22) is *finite*. Indeed, we have

$$p_n(k) = \begin{cases} 0 & if \quad k < n \\ \frac{k!}{(k-n)!} & if \quad k \geq n. \end{cases} \tag{23}$$

It is straightforward to prove that

$$z_n = \sum_{l=0}^{n} \frac{n!}{(n-l)!} \widehat{z}_l \tag{24}$$

and, for the *inverse interpolating transform*,

$$\widehat{z}_n = \sum_{l=0}^{n} (-1)^{n-l} \frac{1}{l!(n-l)!} z_l. \tag{25}$$

To derive the r.h.s. of eq. (21), we start computing explicitly the product z^{*2}. We get

$$(z * z)_n = \sum_{l_1,l_2=0}^{\infty} \widehat{z}_{l_1} \widehat{z}_{l_2} P_{l_1+l_2}(n) =$$

$$\sum_{l_1=0}^{n} \sum_{l_2=0}^{n} \sum_{k_1=0}^{l_1} \sum_{k_2=0}^{l_2} (-1)^{l_1-k_1+l_2-k_2} [\frac{z_{k_1} z_{k_2}}{k_1!(l_1-k_1)!k_2!(l_2-k_2)!}$$

$$\cdot \frac{n!}{(n-l_1-l_2)!}] = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \frac{(-1)^{k_1+k_2}}{k_1!k_2!} z_{k_1} z_{k_2} \mathrm{K}_{n,k_1,k_2},$$

where we have introduced the kernel

$$\mathrm{K}_{n,k_1,k_2} = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} (-1)^{l_1+l_2} \frac{1}{(l_1-k_1)!} \cdot \frac{1}{(l_2-k_2)!} \frac{n!}{(n-l_1-l_2)!}. \tag{26}$$

This expression, after some algebraic manipulations, reduces to

$$\mathrm{K}_{n,k_1,k_2} = \sum_{l=k_1+k_2}^{n} \frac{(-1)^l n!}{(n-l)!} \frac{2^{l-k_1-k_2}}{(l-k_1-k_2)!}. \tag{27}$$

Putting $s = l - k_1 - k_2$, and summing over s, we arrive at the final expression for the kernel (26):

$$\mathrm{K}_{n,k_1,k_2} = \frac{(-1)^n n!}{(n-k_1-k_2)!}, \qquad n > k_1 + k_2. \tag{28}$$

By generalizing the previous reasoning, we prove the first statement of Theorem 11, i.e. that formula (21) is the discrete analog of eq. (18) on the regular lattice $t = n$.

The second part of the Theorem, concerning the existence of the solution (21) for eq. (21), can be proved as follows. By means of the previous construction, we have shown that eq. (21) is nothing but the image of (18) under the action of the morphism u_{∂,Δ^+} acting in $\mathcal{E}$. The morphism μ_{∂,Δ^+} gives the correspondence

$$\sum_k b_k t^k \longrightarrow \sum_k b_k p_n(k) \tag{29}$$

By means of the action of the functor F, any formal series solution z of eq. (19) corresponds to a solution z of eq. (1) and z_n of eq. (21). In addition, on the lattice $\mathcal{L}$, the sum $\sum_k a_k p_n(k)$ truncates and transforms into the finite sum (21).

□

Remark 13. *The scheme offered by Theorem 10 can be used to generalize Theorem 11 to infinitely many other integrable maps, each of them defined in a specific Rota algebra* $(\mathcal{F}, Q)$.

4. Applications: recurrences, number sequences and Borel regularization

In this Section, a connection between *integrability* and *number theory* will be illustrated. It reposes on the fact that a recurrence relation, defined on a suitable space of variables, will be associated to each of the dynamical systems on the lattice previously introduced. The solutions of these dynamical models are strictly related to those of the recurrences, that are expressed in terms of specific number sequences. For completeness, in the following we will study in detail some particular cases of this construction.

4.1. **A family of new quadratic dynamical systems.** Let $z = z(t) : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^\infty$ function. Consider the dynamical system

$$\frac{d}{dt} z = z^2. \tag{30}$$

By applying Theorem 11, we discretize this model in such a way that the solutions of its discrete versions conserve the same structure of the solutions of (30).
We associate with the system (30) an abstract operator equation

$$\mathcal{Q} z = z^{*2}. \tag{31}$$

Here $z \in \mathfrak{F}(+, \cdot, *_{\mathcal{Q}})$. In order to get a discrete map, we represent the equation (31) on a specific function algebra. As customarily, we denote by z_n the value of the function z at the point n. Eq. (31) becomes on the lattice

$$z_{n+1} - z_n = z_n^{*2} = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \frac{(-1)^{k_1+k_2+n}}{k_1! k_2!} z_{k_1} z_{k_2} \frac{n!}{(n-k_1-k_2)!}, \tag{32}$$

which is a particular instance of eq. (21).

This system possesses an interesting *alter ego* in the auxiliary space $\mathcal{A}$ of the variables $\widehat{z}$, defining a *recurrence relation*. Indeed, observe that

$$\Delta z_n = \sum_{k=0}^{\infty} \widehat{z}_l l p_{l-1}(n) = \sum_{l=0}^{n} \frac{n!}{(n-l)!} (l+1) \widehat{z}_{l+1} \tag{33}$$

and

$$z_n^{*2} = \sum_{l_1, l_2 = 0}^{\infty} \widehat{z}_{l_1} \widehat{z}_{l_2} P_{l_1+l_2} = \sum_{l=0}^{n} \frac{n!}{(n-l)!} \sum_{l_1=0}^{l} \widehat{z}_{l_1} \widehat{z}_{l-l_1}. \tag{34}$$

We conclude that the following recurrence holds.

$$(l+1) \widehat{z}_{l+1} = \sum_{l'=0}^{l} \widehat{z}_{l'} \widehat{z}_{l-l'} \tag{35}$$

This recurrence can be solved by using the ansatz

$$\widehat{z}_l = z_0^{l+1} \tag{36}$$

We deduce the solution

$$z_n = \sum_{l=0}^{n} \frac{n!}{(n-l)!} z_0^{l+1}, \tag{37}$$

which can be also expressed in the compact form in terms of a hypergeometric function:

$$z_n = z_0 \cdot_2 F_0(1, n; z_0). \tag{38}$$

Therefore, the discretization procedure we adopted is *effective*. Also, it preserves integrability. Indeed, if we write the general solution of (30) in the functional space $\mathfrak{F}(+, \cdot, *_\Delta)$, we get the corresponding solution of the discrete model (32).

To prove this, observe that the general solution of eq. (30) is provided by

$$z(t) = \frac{z_0}{1 - z_0 t}. \tag{39}$$

We get

$$z(t) = z_0 + z_0^2 t + z_0^3 t^2 + \dots \tag{40}$$

and by virtue of the correspondence $t^n \longrightarrow p_n(t)$ defined by the diagram (8), taking into account (23), we obtain immediately eq. (37).

4.2. **Borel regularization.** The solution (37) of the discrete dynamical system (32) diverges as $n \to \infty$. Instead, the solution (39) of the continuous model (30) has a stable point at $z(\infty) = 0$. In order to cure this pathology, and to construct a system possessing localized dynamics, we introduce a *finite Borel regularization* procedure.

Definition 14. *Given a series $S = \sum_{k=0}^{\infty} b_k z^k$, we call finite Borel regularization of S the following transformation of the sequence of partial sums $\{S_n\}_{n\in\mathbb{N}}$ of S, i.e.*

$$\mathcal{B}(S_n) = \sum_{k=0}^{n} b_k \frac{z^k}{k!}. \tag{41}$$

Let

$$w_n = \mathcal{B}(z_n) = \sum_{l=0}^{n} \frac{1}{(n-l)!} z_0^{l+1}. \tag{42}$$

Consequently, the Borel regularized dynamical system associated with (32) is

$$w_{n+1} - w_n = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} (-1)^{k_1+k_2+n} w_{k_1} w_{k_2} \frac{1}{(n-k_1-k_2)!}. \tag{43}$$

This new dynamical system is still in correspondence with the original model (30). Indeed, its solution is nothing but the Borel transform of the series obtained by applying to (40) the isomorphism $t^n \to p_n(t)$.
Consider now the differential equation

$$z'(t) = a_2 z(t)^2 + a_1 z(t) + a_0, \qquad a_0, a_1, a_2 \in \mathbb{R}. \tag{44}$$

It admits the general solution

$$z(t) = \frac{-a_1 + \sqrt{\Gamma}\tan\left(\frac{1}{2}\sqrt{\Gamma}(t+c_1)\right)}{2a_2}, \tag{45}$$

where $\Gamma = 4a_2a_0 - a_1^2$, and $c_1 \in \mathbb{R}$ is fixed by the initial condition. Following the procedure described above, we obtain the discrete dynamical system

$$z_{n+1} - z_n = a_2 \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} \frac{(-1)^{k_1+k_2+n}}{k_1!k_2!} z_{k_1}z_{k_2}\frac{n!}{(n-k_1-k_2)!} + a_1 z_n + a_0, \tag{46}$$

whose solution is

$$z_n = \sum_{k=0}^{n} \frac{\sqrt{\Gamma}}{2a_2} b_k \frac{n!}{(n-k)!}, \tag{47}$$

where $\{b_k\}_{k\in\mathbb{N}}$ is a number sequence, whose generating function is

$$b_k = \left[\frac{1}{k!}\frac{d^n}{dx^n}\tan\left(\frac{1}{2}\sqrt{\Gamma}(x+c_1)\right)\right]_{|x=0}. \tag{48}$$

Its Borel regularized version defines the map

$$w_{n+1} - w_n = a_2 \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} (-1)^{k_1+k_2+n} w_{k_1}w_{k_2}\frac{1}{(n-k_1-k_2)!} + a_1 w_n + a_0, \tag{49}$$

whose solution is given by the Borel transform of z_n:

$$w_n = \sum_{k=0}^{n} \frac{\sqrt{\Gamma}}{2a_2}\frac{b_k}{(n-k)!}. \tag{50}$$

Remark 15. *Apart the Borel regularization adopted in this work, other regularization procedures are possible (for instance, the Mittag–Leffler one). The other procedures a priori could lead to discrete dynamical systems different than* (43) *or* (49).

4.3. **A more general family of dynamical systems and associated recurrences.** Let us consider the dynamical system

$$\frac{d}{dt}z = a_N z^N, \qquad N \in \mathbb{N} \qquad a_N \in \mathbb{R}. \tag{51}$$

where $z : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function. The third system of the hierarchy is the dynamical system

$$\Delta z = a_3 z^{*3}. \tag{52}$$

Its realization in the space $\mathfrak{F}(+,\cdot,*_{\Delta^+})$ can be performed in a similar way. Therefore, eq. (52) becomes on the lattice

$$z_{n+1} - z_n = a_3 \sum_{k_1,k_2,k_3=0}^{n} \frac{(-1)^{k_1+k_2+k_3+n}}{k_1!k_2!k_3!} z_{k_1}z_{k_2}z_{k_3}\frac{n!}{(n-k_1-k_2-k_3)!}. \tag{53}$$

This system, as before, can be written in terms of the transformed variables (25). It has the form of a recurrence relation

$$(l+1)\widehat{z}_{l+1} = a_3 \sum_{\substack{l_1,l_2=0\\ l_1+l_2\le l}} \widehat{z}_{l_1}\widehat{z}_{l_2}\widehat{z}_{l-l_1-l_2}. \tag{54}$$

The solution of system (1), for $N = 3$, is provided (up to a sign) by

$$z(t) = \frac{1}{\sqrt{2}\sqrt{-a_3 t + c_0}}, \tag{55}$$

where c_0 is an arbitrary constant. Since our discretization preserves integrability, we get the following series expression for the solution of the system

$$z_n = \sum_{k=0}^{n} \frac{b_k}{\sqrt{2}} \frac{n!}{(n-k)!} \frac{a^k}{c_0^{(2k+1)/2}}, \qquad n = 0, 1, 2, \ldots \tag{56}$$

where the first terms of the sequence $\{b_k\}_{k\in\mathbb{N}}$ are

$$b_0 = 1, b_1 = \frac{1}{2}, b_2 = \frac{3}{8}, b_3 = \frac{5}{16},$$

$$b_4 = \frac{35}{128}, \quad b_5 = \frac{63}{256}, \quad etc. \tag{57}$$

The generating function of this sequence is given by

$$b_k = \frac{1}{k!} \left[\frac{d^k}{dx^k} \frac{1}{\sqrt{1-x}} \right] |_{x=0} \, . \tag{58}$$

The procedure can be easily generalized to any $N \in \mathbb{N}$: to each value of N, it corresponds naturally a specific sequence of rational numbers.

The Borel regularization procedure, described above in the simpler case of quadratic dynamical systems, can be extended in a completely analogous way to the case (53) or to the maps arising from more general polynomial vector fields. The number sequence $\{b_k\}_{k\in\mathbb{N}}$ appearing in the explicit solution of the maps and of their Borel–transformed analogs stays the same.

Consider now a general recurrence relation of the form

$$(l+1)\widehat{z}_{l+1} = a \sum_{\substack{l_1, l_2, \ldots, l_N = 0 \\ l_1 + \ldots + l_N \leq l}} \widehat{z}_{l_1} \widehat{z}_{l_2} \cdots \widehat{z}_{l_N} \widehat{z}_{l - l_1 - l_2 - \cdots l - N} \cdot \tag{59}$$

We wish to find an exact solution of it. To this aim, for every N, we interpret the recurrence (59) as the difference equation defining an abstract dynamical system, in the auxiliary space $\mathcal{A}$. Then, a particular solution of the recurrence can be obtained by associating to it the discrete map

$$z_{n+1} - z_n = a \sum_{k_1, \ldots, k_N = 0}^{n} \frac{(-1)^{k_1 + \cdots + k_N + n}}{k_1! \cdots ! k_N!} z_{k_1} \cdots \cdot z_{k_N} \frac{n!}{(n - k_1 - k_2 - \cdots k_N)!} \cdot \tag{60}$$

obtained by means of the inverse transform (25). Then, we come back to the continuous system (51), of whom we consider the general solution

$$z(t) = \left[(1-N)(at + c_0) \right]^{1/(1-N)} . \tag{61}$$

Once we expand this solution and discretize it on the lattice $\mathcal{L}$, exactly in the same way as before, we get a particular solution of the recurrence (59), which depends

on the choice of the initial condition of (51), in terms of a sequence of rational numbers.

Precisely, we get

$$z_n = (1-N)^{\frac{1}{1-N}} c_0^{\frac{1}{1-N}} \cdot \sum_{k=0}^{n} \left[\frac{a}{c_0}\right]^k \frac{1}{(N-1)^k} \prod_{j=0}^{k-2} [(j+1)N - j] \binom{n}{k}, \tag{62}$$

and from (25) we deduce a solution of the recurrence (59).

5. Some open problems

The examples proposed reveal the potential of the theory constructed in this work. Its categorical formulation makes it possible to define a whole hierarchy of dynamical systems, associated with the differential equation (1), of which (21) is an example, related with the operator $Q = \Delta^+ = T - 1$. The case of a discretization involving $\Delta^- = 1 - T^{-1}$ is very similar, and left to the reader. The basic polynomials associated to Δ^- are defined by $p_n(x) = x(x+1)(x+2)...(x+(n-1))$.

Higher order operators $\mathcal{Q}$, as in (4), starting from the case of $\Delta^s = (T - T^{-1})/2$, a priori would lead to non effective discretizations, if we stay on a regular lattice of points, isomorphous to $\mathbb{N}$ (or to a finite subset $S \in \mathbb{N}$).

The general prescription to make effective the procedure is that the point of the lattice $\mathcal{L}$ we work over should correspond to the set of zeroes of the basic polynomials $\{p_n(x)\}_{n\in\mathbb{N}}$ associated to $\mathcal{Q}$. Therefore, for higher order delta operators, nonlinear lattices should be used. A procedure to compute the basic polynomials has been proposed in [10], [24]. The problem of the determination of their zeroes in a closed form is essentially open [23].

If we use discrete derivatives of the form (4)–(5), but we stay on regular lattices, we get discretizations involving a countable number of points. However, the underlying continuous model is represented by eq. (1), so a priori its solutions may be used to produce exact solutions of infinite order difference equations.

It would be interesting to generalize the procedure proposed in this work to the case of recurrences possessing other functional forms, especially to the case of infinite order recurrences. They can be constructed by using alternative discrete derivatives.

The approach developed here can be easily adapted to the study of physically relevant models, especially nonlinear lattice field theories, as for instance the Liouville theory. A priori, a higher–dimensional theory can be developed on the same footing. This could also be related to the choice of different geometries on the lattice [4], [18], [11], [37]. Work is in progress on these aspects.

Another interesting open problem is to extend the proposed technique to the discretization of *isochronous dynamical systems*, in the spirit of the general framework proposed in [7].

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Departamento de Física Teórica II (Métodos Matemáticos de la física), Facultad de Físicas, Universidad Complutense de Madrid, 28040 – Madrid, Spain