RANDOM GRAPHS ARISING FROM L-FUNCTIONS AND MODULAR FORMS: TOPOLOGICAL PROPERTIES AND PHASE TRANSITIONS

PIERGIULIO TEMPESTA

Abstract. A construction is proposed, relating L-functions and modular forms with the theory of random graphs. A new class of graphs with a power-law type degree distribution, called zeta graphs is introduced: each model is realized by assigning a specific Dirichlet series, that governs the topology of the associated network. A large family of scale-free networks is obtained by introducing a product in the space of random graphs, which allows to generate new graphs from old ones. A particularly relevant class of zeta graphs, called modular graphs, is attached to modular forms of integer weight, and specifically to Eisenstein series. In this construction, Hecke-type operators are defined, mapping modular graphs into modular graphs.

A crucial feature of the models presented is the existence of a phase transition in terms of the creation of a giant cluster. A theorem is proved giving the analytic condition for the existence of a giant component. The critical point at which the phase transition can occur is defined by a functional equation for the L-function governing the model considered.

This transition can be interpreted in the context of percolation theory as a threshold for describing percolation phenomena.

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Contents

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1. Introduction

Random graphs, as models of complex and interacting systems, have been recognized in the last years as ubiquitous structures, since they play a distinguish role in mathematical physics, biology, information theory, social sciences, etc. [2], [9], [41].

In a random graph, a probability distribution for the degrees of the vertices is assigned, which controls the main topological and analytic properties. Since the pioneering work of Erdös and Rényi [25], and Solomonoff and Rapoport [46], this field has known recently a dramatic development [8], [11], [41]. In particular, *scale–free* models, i.e. models exhibiting a power–law degree distribution, are among the best known examples of complex networks [12], [13]. The first example of them was offered by the Price model [43]. Among the most important ones are those studied by Barabasi, Albert and collaborators in [5], [6] and by Aiello et al. in [1].

Recently, the connection between random graphs and random matrices has been extensively investigated [23], [24].

Starting with the works of Montgomery and Odlyzko in the '70s, relating the Gaussian unitary ensemble with the zeros of the Riemann zeta function $\zeta(s)$, the study of the connections between statistical mechanics and analytic number theory has been largely investigated (see [15], [16] and [35] for general reviews). Among the recent applications, in [30], it has been shown that the Riemann zeta function is the canonical partition function of a free bosonic gas. Also, the zeros of $\zeta(s)$ have been related to Landau levels [45] for a charged particle on a planar surface in an electric potential and uniform magnetic field.

Since the work of Ihara [28] in the 60's, zeta functions on finite connected graphs have been intensively investigated under different perspectives [27], [34], [47]–[49]. The zeta functions associated with graphs are defined by an Euler product and possess a functional equation and an analytic continuation to a meromorphic function. The definitions available essentially try to reproduce on a graph the usual properties of Selberg's zeta function and Artin L–functions.

The purpose of this paper is to build a conceptual bridge between number-theoretical objects as Dirichlet series and modular forms from one side and the theory of random graphs and complex networks from the other side. This is realized by large class of new models, defined on *undirected uncorrelated random graphs.*

Let $\mathbb{N}$ denote the set of positive integers. We shall specify a degree probability distribution $\{p_k\}_{k\in\mathbb{N}}$, where p_k is the probability that a uniformly randomly chosen vertex has degree k. The distribution a priori defines a set of possible graphs. Then (as we shall discuss below), a random graph is chosen uniformly at random in the class of all graphs with that degree distribution.

Definition 1. *A zeta graph (or network) is a unipartited undirected random whose associated degree probability distribution is of the form*

$$p_k = \begin{cases} 0 & for \quad k = 0, \\ a_k k^{-\alpha}/L(\alpha) & for \quad k \in \mathbb{N}, \end{cases} \tag{1}$$

where

$$L(\alpha)=\sum_{k=1}^{\infty}\frac{a_k}{k^{\alpha}} \tag{2}$$

is a Dirichlet series, $\{a_k\}_{k\in\mathbb{N}}$ *is a sequence of real nonnegative numbers, and* $\alpha>0$ *is a real parameter. We shall always assume that there exists a* $\sigma_c\in[-\infty,+\infty]$ *such that* $L(\alpha)$ *is absolutely convergent for* $\alpha>\sigma_c$ *and, for any* $\epsilon>0$*, uniformly convergent for* $\alpha\geq\sigma_c+\epsilon$*.*

Under our assumptions, $L(\alpha)$ also defines a real analytic function in the same domain (which by abuse of notation we will again denote by $L(\alpha)$). In general, the sequence $\{a_k\}_{k\in\mathbb{N}}$ will satisfy suitable growth conditions. The most basic example of our construction is the scale–free network connected to $\zeta(s)$, corresponding to the choice $a_k=1$ for all $k\in\mathbb{N}$.

In the subsequent discussion, we shall consider essentially the case of unbounded probability distributions, i.e. the case of Dirichlet series, possibly represented by distributions with an exponential cut-off. The case of bounded distributions, i.e. of Dirichlet polynomials will not be treated here.

In some sense, our work is complementary to the constructions of zeta functions on graphs in [38], [27], [47]–[49]. Indeed, a Dirichlet series is now assigned, and an infinite graph is obtained from it, whose topology reflects the properties of the selected series.

Notice that the zeta graphs are associated with generalized power–law distributions. Apart their prominence in network theory and complex systems, they are very common in science: indeed, are found to be useful in many context, from nuclear physics to biological taxa, in describing the transverse spectra of charged particles in colliders [17], in the distribution of city populations, of words in human languages, econometrics, etc. (see [18] for a review).

The main results of the paper can be summarized as follows.

1) In Theorem 1, we establish the existence of a *topological phase transition* for the zeta graphs, marking the formation of a giant cluster. This condition is expressed in terms of a functional equation for the L–function associated with the graphs. The phase transition has a natural interpretation, in the context of percolation theory, as a threshold marking the occurrence of either site or bond percolation phenomena.

2) We introduce algebraic structures in suitable spaces of graphs. Relevant instances of our approach are obtained from Dirichlet series constructed in terms of *multiplicative arithmetic functions*: the Euler and χ_f–random graphs will be introduced and discussed. In particular, we shall focus on the space of zeta graphs coming from multiplicative functions. When the coefficients a_k are completely multiplicative, Definition 1 generalizes the well–known case of scale–free models. As usual in this context, the scale–free invariance of models (1) essentially means that the ratio $p_{\alpha\times k}/p_k$ depends only on α but not on k. Under suitable hypotheses, the operation of pointwise multiplication of graphs enables to construct in a natural way *new scale–free graph models from known ones.* An important source of zeta graphs comes from the theory of *modular forms.* The coefficients $a(n)$ of the Fourier expansion of entire modular forms of weight k, and especially of *Hecke eigenforms*, can be used to define certain spaces of graphs, inheriting the rich algebraic structure of the spaces M_k.

We mention that a connection with nonextensive statistical mechanics [55] is also realized via a specific zeta model, that we shall call the Hurwitz random graph. This graph appears as a natural network structure related to many cut–off models; its generating function is expressed in terms of sums of q–exponentials.

The main motivation for introducing the zeta models, apart their mathematical interest, is therefore the richness of their properties, and their plausibility if compared with the phenomenological models available in the literature. This makes them flexible tools for applications (see the overall discussion in Sections 4, 6 and 7).

A relevant possible extension of the present theory concerns the class of graphs arising from the construction of the Dirichlet series recently proposed in [50], [51]. In this construction, formal groups play a special role [52]. Each series of them is associated with an entropic functional called *group entropy* [53].

Also, a generalization of our construction to the case of half–integer weight modular forms, as well as to directed and bipartite graphs is in order (some preliminary results are in [54].

The paper is organized as follows. In Section 2, the generating function formalism is sketched. In Section 3, the notion of zeta random graph is introduced, and the main theorem on topological phase transitions is proved. In Section 4, the important case of scale–free networks is discussed and several new models of zeta graphs are explicitly constructed and solved. In Section 5, algebraic structures in the space of random graphs are discussed. In Section 6, a connection between the theory of modular forms and Hecke algebras and the theory of random graphs is proposed. In Section 7, Hurwitz zeta graphs are discussed in connection with statistical mechanics of long range systems. In the final Section 8, devoted to applications, percolation phenomena and epidemic thresholds are studied.

2. The generating function formalism

2.1. **Random models and degree distributions.** The study of large unipartite undirected graphs with arbitrary degree distributions of the vertices has been performed by several authors [7], [10], [11], [56], [36], [40]. The degrees of all vertices are supposed to be independent identically distributed random integers drawn from a given distribution. As pointed out in [36], instead of generating directly such graphs (which would represent a major difficulty), it is standard to study random configurations on fixed degree sequences (or degree probability distributions). Given a set of vertices and degrees sampled from a given distribution $\{p_k\}_{k\in\mathbb{N}}$, the random configurations are generated according to a simple procedure of random connection of the vertices. The configuration model is defined as the ensemble of graphs so obtained [7], [11], [39].

Therefore, once assigned a degree probability distribution, a class of equally probable random graphs [40] is associated to the distribution. A specific representative of the class is chosen uniformly at random among all graphs generated from the same distribution. All properties are averaged over the ensemble of graphs generated in this way. This procedure defines a kind of "microcanonical ensemble" for random graphs.

When the limit of large graph size is considered, another equivalent procedure can be followed. Only one particular degree sequence is selected, to approximate as closely as possible the desired probability distribution. Then one averages uniformly over all graphs with that sequence. This second procedure defines a "canonical ensemble" for graphs (see [40] fore more details). All the random graph models considered in the rest of the paper are supposed to be defined in the sense of the construction defined above.

Remark 1. Throughout this paper, we will stay in the limit of *large graph size.*

2.2. **Generating functions for complex networks.** The mathematical framework of generating functions offers a convenient language for the description of complex networks [41]. The generating function of the degree distribution is defined to be the series

$$G_0(x) := \sum_{k=0}^{\infty} p_k x^k. \tag{3}$$

The excess degree distribution is the distribution of the edges that leave a vertex reached by following a randomly chosen edge in the graph, other than the edge we arrived along. The excess distribution is generated by

$$G_1(x) := \frac{\sum_k k p_k x^{k-1}}{\sum_k k p_k} = \frac{1}{< k >} G_0'(x), \tag{4}$$

where the average number of first neighbors, equal to the average degree of the graph, is

$$z_1 =< k >= \sum_k k p_k = G_0'(1).$$

Let $H_1(x)$ be the generating function for the distribution of the sizes of the components reached by selecting randomly an edge and following it to one of its ends. The giant component (if it exists) is excluded from $H_1(x)$. It is easy to see that this function should satisfy the self–consistency condition

$$H_1(x) = x G_1(H_1(x)).$$

If we chose randomly a vertex, the size of the component to which the chosen vertex belongs, or alternatively the total number of vertices reachable from the vertex is generated by a function $H_0(x)$, satisfying

$$H_0(x) = x G_0\left(H_1(x)\right). \tag{5}$$

The size of the mean component is given by

$$< s >= H_0'(1) = 1 + \frac{G_0'(1)}{1 - G_1'(1)}. \tag{6}$$

The threshold condition for the phase transition, giving rise to a giant cluster in the graph is therefore

$$G_1'(1) = 1, \tag{7}$$

i.e.

$$\mathcal{Q}(\{p_k\}) = \sum_k k(k-2) p_k = 0. \tag{8}$$

Given a random graph with N vertices, in [36] it has been proved that a giant component exists almost surely, i.e. with probability tending to 1 for $N \to \infty$, when $\mathcal{Q}(\{p_k\}) > 0$, for degree probability distributions well behaved and such that the maximum degree is less than $N^{1/4-\epsilon}$, when condition (8) applies

3. A new class of graphs with power–law type degree distribution: the zeta random graphs

One of the main motivations for the definition of zeta random graphs (1) is that they naturally inherit from the underlying Dirichlet series many good analytic properties, that make them suitable for describing a large variety of models, as will be shown in the next sections.

The requirement that $L(\alpha)$ be a L–function in the Selberg class [29], [31] implies that the sequence $\{a_k\}_{k\in\mathbb{N}}$ should satisfy the *Ramanujan condition* $a(k) \ll k^{\mathcal{R}}$, for any real constant $\mathcal{R} > 0$. We also observe that the condition $p_0 = 0$ is required to take into account the fact that in scale–free networks there are no unconnected or "orphan" nodes.

A particular example contained in Definition 1 is the well–known case of a random graph with a power–law degree distribution introduced in [1]:

$$p_k = \begin{cases} 0 & for \quad k = 0 \\ k^{-\alpha}/\zeta(\alpha) & for \quad k \in \mathbb{N}, \end{cases} \tag{9}$$

where $\zeta(\alpha)$ is the Riemann zeta function. In this case, we get

$$G_0(x) = \frac{\mathrm{Li}_\alpha(x)}{\zeta(\alpha)}, \quad G_1(x) = \frac{\mathrm{Li}_{\alpha-1}(x)}{x\zeta(\alpha-1)}, \tag{10}$$

where $\mathrm{Li}_s(x)$ is the classical polylogarithm. The condition (7) provides

$$\zeta(\alpha-2) = 2\zeta(\alpha-1), \tag{11}$$

which implies that the critical value for α is

$$\alpha_c = 3.4788\ldots$$

A giant component exists below this value; above it there is no giant component.

In [39] it has been proven that, under the (very mild) assumption that p_k is sufficiently small for $k \gtrsim k_{max}$, then the maximum degree k_{max} for a random graph with N vertices is a solution of the relation $\frac{dp_k}{dk} \simeq -Np_k^2$. For large but finite zeta graphs coming from a truncated L–function we deduce

$$k_{max} \sim N^{1/(\alpha-\mathcal{R}-1)}. \tag{12}$$

Now, the following main theorem provides a functional equation for the critical value at which the phase transition occurs in a zeta graph, giving rise to the formation of a cluster.

Theorem 1. *A zeta graph configuration possesses a phase transition to a giant cluster, characterized by the critical values a_c for which the quantity*

$$Q(\alpha) := L(\alpha-2) - 2L(\alpha-1) \tag{13}$$

vanishes. The giant component exists whenever $Q(\alpha) > 0$.

Proof. We shall introduce *generalized polylogarithms*, defined by the following series

$$\mathrm{Li}_G(s,x) := \sum_{k=1}^{\infty} \frac{a_k x^k}{k^s}, \quad s \in \mathbb{C}. \tag{14}$$

Here $\{a_k\}_{k\in\mathbb{N}}$ is the sequence of the coefficients of the Dirichlet series given in eq. (2), Re s > 1 and $|x| \leq 1$.

The case of the standard polylogarithm is recovered by putting $a_k = 1$ for all k. Notice that by construction the series $\mathrm{Li}_G(s,x)$ are absolutely convergent in their domain.

For a zeta random graph, the generating function (3) is given by

$$G_0(x) = \frac{\sum_{k=1}^{\infty} \frac{a_k}{k^\alpha} x^k}{L(\alpha)} = \frac{\mathrm{Li}_G(\alpha,x)}{L(\alpha)}, \tag{15}$$

whereas the distribution of the outgoing edges is found to be

$$G_1(x) = \frac{\mathrm{Li}_G(\alpha-1,x)}{xL(\alpha-1)}. \tag{16}$$

Consequently, we obtain

$$G_1'(x) = \frac{\mathrm{Li}_G(\alpha-2,x) - \mathrm{Li}_G(\alpha-1,x)}{x^2 L(\alpha-1)}.$$

By imposing the condition (7) for a phase transition to a giant cluster, we get

$$L(\alpha-2) = 2L(\alpha-1),$$

which generalizes directly eq. (11). □

For values of α such that $Q(\alpha) > 0$, there exists a giant component, that occupies a fraction S of the graph; when $Q(\alpha) < 0$, there is no giant component. It is also possible to prove, by analogy with the treatment in [36], [41] that this fraction S is determined by the following relation

$$S = 1 - G_0(u),$$

where u is defined to be the smallest real non–negative solution of the relation

$$u = G_1(u).$$

The quantity u can be interpreted as the probability that a vertex not belonging to the giant component is reached from a randomly chosen edge. Therefore, by using (16), we are led to the result

$$u^2 = \frac{\mathrm{Li}_G(\alpha-1,u)}{L(\alpha-1)}. \tag{17}$$

Usually, this equation can not be solved analytically. However, it can be useful to provide accurate numerical estimations of u.

Another important topological property of a graph is the *clustering coefficient*, defined as

$$C := \frac{\text{3x number of triangles in the network}}{\text{number of connected triples of vertices}}, \tag{18}$$

where a connected triple consists of a single vertex whose edges connect it to an unordered pair of others. The clustering coefficient satisfies $0 \leq C \leq 1$. A different definition has been proposed in [57]. One can show [41] that, in general

$$C = \frac{1}{n}\frac{\left[\langle k^2\rangle - \langle k\rangle\right]^2}{\langle k\rangle^3}, \tag{19}$$

where n is the number of vertices of the graph. This coefficient, in the limit of large graphs, is not necessary negligible. In the case of the zeta random graphs, the clustering coefficient is a function of α. Therefore, the following general result holds

$$C = \frac{1}{n} \frac{\left[L(\alpha-2) - L(\alpha-1)\right]^2 L(\alpha)^2}{L(\alpha-1)^3}. \tag{20}$$

4. Scale–free zeta graphs from arithmetic multiplicative functions

To make transparent the connection between complex networks and arithmetic number theory, in this Section we construct several new models of zeta graphs, as a direct application of the previous approach. An interesting subclass of random graphs can be obtained by advocating the theory of multiplicative functions [4], [19].

An arithmetic multiplicative function is an application $f : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$), not identically zero, such that

$$f(mn) = f(m)f(n) \quad \text{whenever} \quad (m, n) = 1.$$

The arithmetic function f will be said to be completely multiplicative if

$$f(mn) = f(m)f(n) \quad \text{for all} \quad m, n \in \mathbb{N}.$$

Multiplicative functions possess many interesting arithmetical properties. A necessary condition for f to be multiplicative is that $f(1) = 1$. Also, hereafter we shall always assume that there exists a constant $\sigma_a \in \mathbb{R}$ such that the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad s \in \mathbb{C} \tag{21}$$

converges absolutely for Re s $> \sigma_a$. Then, we have the Euler product representation

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}}\right) \tag{22}$$

if f is multiplicative, and

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \frac{1}{1 - f(p)p^{-s}}, \quad \text{if} \quad \text{Re s} > \sigma_a, \tag{23}$$

if f is completely multiplicative.

Definition 2. *A zeta random graph is said to be (completely) multiplicative if in the associated L–series* (2) $a_k = |f(k)|$*, where f is a (completely) multiplicative function, such that $\sum f(n)n^{-s}$ is absolutely convergent for Re $s > \sigma_a$. We shall assume in* (2) *that $\alpha > \sigma_a$.*

From the previous definition follows directly the following interesting property.

Proposition 1. *Completely multiplicative zeta graphs are scale–free.*

In the following, we will present several new models of random graphs. Two of them arise from well–known examples of multiplicative functions, the Euler totient function and the divisor functions, and two come from completely multiplicative functions, the χ–functions and the Liouville one. Their most relevant topological properties will be discussed in detail.

4.1. **Euler random graph.** The Euler totient function $\phi(n)$ is defined to be the number of integers relatively prime to n not exceeding n. We have

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} = \prod_p \frac{1-p^{-s}}{1-p^{1-s}}, \quad \text{if} \quad \text{Re s} > 2. \tag{24}$$

Definition 3. *An Euler random graph is a random graph associated with a degree probability distribution of the form*

$$\begin{cases} p_0 = 0 \\ p_k = \frac{\phi(k)\zeta(k)}{k^\alpha \zeta(k-1)}, \quad k \in \mathbb{N}, \quad \alpha > 1. \end{cases}$$

The functional equation (13) for the phase transition provides

$$\zeta(\alpha-1)\zeta(\alpha-3) = 2\zeta(\alpha-2)^2. \tag{25}$$

Interestingly enough, the model possesses two critical values:

$$\alpha_c^{(1)} = 1.801, \qquad \alpha_c^{(2)} = 4.354. \tag{26}$$

We observe that $Q(\alpha) > 0$ for $\alpha_c^{(1)} < \alpha < 2$, and $4 < \alpha < \alpha_c^{(2)}$.

4.2. **The χ_f–random graphs.** Another source of interesting random graphs comes from the theory of Dirichlet characters.

Given $n \in \mathbb{Z}$, a Dirichlet character is a function $\chi(n)$ which satisfies the following assumptions:

a) *periodicity* : $\chi(n+f) = \chi(n)$;

b) *complete multiplicativity* : $\chi(mn) = \chi(m)\chi(n)$ for all n;

c) *degeneracy* : $\chi(n) = 0$, if $\gcd(n, f) > 1$.

The integer f is called the *conductor* of χ. The *principal character* χ_f with conductor f is defined to be

$$\chi_f(n) := \begin{cases} 1 & \text{if } n \text{ is prime to } f; \\ 0 & \text{otherwise.} \end{cases}$$

One of the most important L–series is the Dirichlet L–function

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}, \quad \text{for Re s} > 1.$$

If $\chi = \chi_f$, we observe that $\chi_f(p) = 0$ if $p|f$ and $\chi_f(p) = 1$ if $p \nmid f$. Consequently,

$$L(s,\chi_f) = \prod_{p \nmid f} \frac{1}{1-p^{-s}} = \zeta(s) \prod_{p|f} \left(1-p^{-s}\right), \quad \text{for Re s} > 1, \tag{27}$$

i.e. the Dirichlet L–function $L(s, \chi_f)$ factorizes into the product of $\zeta(s)$ for a finite number of terms.

We can introduce now a new model of a random graph related to a Dirichlet character.

Definition 4. *A random graph associated to a probability distribution given by*

$$\begin{cases} p_0 = 0 \\ p_k = \frac{\chi_f}{k^\alpha L(\alpha,\chi_f)}, \quad k \in \mathbb{N}, \quad \alpha > 1, \end{cases}$$

will be called a χ_f–random graph of conductor f.

As a consequence of (27), the functional equation acquires a simpler expression:

$$\zeta(\alpha-2)\prod_{p|f}\left(1-p^{2-\alpha}\right)=2\zeta(\alpha-1)\prod_{p|f}\left(1-p^{1-\alpha}\right), \tag{28}$$

where f is the conductor of χ_f. It is evident that this model directly generalizes that of distribution (9), that is recovered for $f=1$.

Remark 2. A comparison with the existing literature shows the plausibility of the models proposed here. Indeed, most of the scaling exponents for the degree distributions found in the phenomenological studies devoted to social networks lie in the range $2<\alpha<3.4$ (see e.g. [13]), with some exceptions (for instance, the case of e–mail networks shows an exponent $\alpha=1.81$; that of co–authorship in high energy physics an exponent $\alpha=1.2$ [41]).

From the previous analysis it is clear that (infinitely) many other examples of exactly solvable analytic models of random graphs can be constructed by using the theory of arithmetic multiplicative functions.

5. Products of graphs

The beauty of the theory of multiplicative functions relies also on the possibility of defining analytical operations with them.

Let us denote by $\mathcal{G}_{\mathcal{M}}$ the set of multiplicative zeta random graphs, and by $\mathcal{G}_{\mathcal{CM}}$ that of the completely multiplicative random graphs, according to Definition (2). As usual, a graph is intended to be randomly selected from the set of all graphs associated with an assigned probability distribution.

A product can be defined in the following way. Let f and g be two (completely) multiplicative functions, such that their associated Dirichet series (21) are absolutely convergent in the same half plane. We can construct two associated random graphs $G_1(f)$ and $G_2(g)$, according to Definition 2. Let

$$h=(f\cdot g)(n)=f(n)g(n). \tag{29}$$

Definition 5. *A random graph configuration $G_{12}(h)$ will be said to be the pointwise product of two (completely) multiplicative random graphs $G_1(f)$ and $G_2(g)$ if eq.* (29) *holds.*

Particularly interesting is the case of the set $\mathcal{G}_{\mathcal{CM}}(\cdot)$: *scale–free networks are transformed into scale–free networks.*
A different algebraic structure can be introduced by means of the Dirichlet multiplication of arithmetic functions.

Definition 6. *A random graph configuration $G_{12}(h)$ will be said to be the Dirichlet product of two multiplicative random graphs $G_1(f)$ and $G_2(g)$ if*

$$h=(f*g)(mn)=\sum_{x|mn}f(x)g\left(\frac{mn}{x}\right).$$

The previous definition is motivated by the fact that the product of two multiplicative functions is also multiplicative (see, for instance, [4]).

However, the Dirichlet product of two completely multiplicative functions need not be completely multiplicative. Nevertheless, the Dirichlet inverse of a completely multiplicative function still is completely multiplicative. In addition, this inverse is especially simple to determine in terms of the Möbius function. It is defined as follows. Given $n \in \mathbb{N}$, we write it in the form $n = p_1^{a_1} \cdots p_k^{a_k}$. Then

$$\mu(n) = 1 \qquad\qquad \text{if} \quad n = 1;$$

$$\text{for} \quad n > 1, \quad \mu(n) = \begin{cases} (-1)^k & \text{if} \quad a_1 = a_2 = \cdots = a_k = 1 \\ \mu(n) = 0 & \text{otherwise.} \end{cases}$$

In other words, $\mu(n) = 0$ if and only if n has a square factor > 1. It is related to the Euler function $\phi(n)$ by the formula $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}, n \geq 1$.

The construction proposed here, apart its intrinsic mathematical interest, allows us to construct infinitely many more multiplicative graph configurations by “composing” known ones, in the sense specified above.

6. Modular forms and random graphs

6.1. **Definitions.** We wish to present a natural extension of the construction of random graphs presented in the previous sections, based on the theory of modular forms [29], [58], [44], [32]. Let us denote by $H = \{\tau : \mathrm{Im}\tau > 0\}$ the upper half–plane of $\mathbb{C}$, and by $\Gamma_1 = PSL_2(\mathbb{Z})$ the full modular group.

We recall that a function f is said to be an entire modular form of weight $\kappa \in \mathbb{Z}$ if it satisfies the following requirements.

(1) The function f is analytic in H;

(2) $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^\kappa f(\tau), \qquad \text{where} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1.$

(3) The Fourier expansion of f has the form

$$f(\tau) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n \tau}. \tag{30}$$

If $a(0) = 0$, the function f is called a *cusp form* (or Spitzenform), whereas if $a(1) = 1$, f is said to be a normalized form. As usual, we will denote by M_κ the linear space over $\mathbb{C}$ of all entire forms of weight κ, and by $M_{\kappa,0}$ the subspace of all cusp forms in M_κ.

The following estimations for the coefficients $a(n)$ of modular forms of weight κ hold. If $f \in M_\kappa$ and is not a cusp form, then $a(n) = O(n^{\kappa-1})$. If $f \in M_{\kappa,0}$, then Deligne established, as a consequence of his proof of the Weil conjecture, that $a(n) = O(n^{\frac{\kappa-1}{2}+\epsilon})$ [21]. By analogy with the classical Hecke theory, we construct with the Fourier coefficients of f the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}, \tag{31}$$

where $c(n) =| a(n) |$. We shall call $\phi(s)$ *the absolute Dirichlet series* associated with the modular form f. This series is indeed absolutely and uniformly convergent for $\mathrm{Re}s > k_0$, where $k_0 = k$ if $f \in M_k$ and $k_0 = \frac{k+1}{2}$ if $f \in M_{k,0}$. It will play a major role in the subsequent construction.

We can introduce a class of random graphs related to modular forms.

Definition 7. *Given a modular form f of weight κ, with Fourier expansion* (30) *and absolute Dirichlet series* (31), *a modular random graph is a zeta graph associated* *with a degree probability distribution of the form*

$$\begin{cases} p_0 = 0 \\ p_m = c(m)/n^{\alpha}\varphi(\alpha) \quad m \in \mathbb{N}, \quad \alpha > k_0. \end{cases} \tag{32}$$

Here $k_0 = k$ *if* $f \in M_k$ *and* $k_0 = \frac{k+1}{2}$ *if* $f \in M_{k,0}$.

We shall denote by $\mathcal{G}_{\Gamma_1,\kappa}$ the set of all of modular graphs of weight κ.

6.2. **Algebraic structure: Hecke theory on graphs.** An important class of operators in the theory of modular forms is represented by Hecke operators. We recall that the *m–th Hecke operator of weight k* is defined by the formula

$$T_\kappa(m)f = m^{\kappa-1} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \backslash \mathcal{M}_m} (c\tau + d)^{-\kappa} f\left(\frac{a\tau + b}{c\tau + d}\right), \tag{33}$$

where $\mathcal{M}_n$ denotes the set of 2×2 integral matrices of determinant m and $\Gamma_1 \backslash \mathcal{M}_m$ the set of orbits of $\mathcal{M}_m$ under left multiplication by elements of Γ_1. Hereafter we will often write for short T_m instead of $T_\kappa(m)$. An important property is the multiplication rule

$$T_n T_m = \sum_{d|n,m} d^{\kappa-1} T_{nm/d^2}. \tag{34}$$

In our construction, Hecke theory plays a twofold role.
i) Since

$$f \in M_\kappa \implies T_n f \in M_\kappa, \tag{35}$$

we deduce that the action of the Hecke operators T_κ induces homomorphically a class of operators H_κ acting on $\mathcal{G}_{\Gamma_1,\kappa}$. Precisely, let us denote by $\psi : M_\kappa \to \mathcal{G}_{\Gamma_1,\kappa}$ the operator that associates with a modular form f a random graph configuration G_f, coming from the distribution $\{p_m\}_{m\in\mathbb{N}}$ given by eq. (32), obtained by means of the (moduli of the) coefficients of the Fourier expansion of f.

Definition 8. *Let* $f \in M_\kappa$. *We shall call graph Hecke–type operators the operators* $H_\kappa : M_\kappa \times \mathcal{G}_{\Gamma_1,\kappa} \to M_\kappa \times \mathcal{G}_{\Gamma_1,\kappa}$ *defined by*

$$(f, G_f) \to (g, G_g) \tag{36}$$

where $g = T_n f$.

Explicitly,

$$H_\kappa(f, G_f) = (T_n f, \psi\left(T_\kappa(f)\right)). \tag{37}$$

Let $h_k : \mathcal{G}_{\Gamma_1,\kappa} \to \mathcal{G}_{\Gamma_1,\kappa}$ denote the operator induced by H_k on $\mathcal{G}_{\Gamma_1,\kappa}$. The following commutative diagram holds

$$\begin{array}{ccc} f & \xrightarrow{T_\kappa} & g \\ \psi \downarrow & & \psi \downarrow \\ G_f & \xrightarrow{h_\kappa} & G_g \end{array} \tag{38}$$

The reason for introducing the graph Hecke–type operators lies in the following observation.

Proposition 2. *The space $\mathcal{G}_{\Gamma_1,\kappa}$ is stable under the action of the operators h_κ.*

Remark 3. We can endow the set $\mathcal{G}_{\Gamma_1,\kappa}$ with the structure of a vector space over $\mathbb{C}$ by making use of that of M_κ. The composition law $+$ of two random graph configurations associated with f, $g \in M_\kappa$ is defined by the rule $G_f + G_g := G_{f+g}$, and similarly for the multiplication by a complex scalar.

ii) We are especially interested in Hecke eigenforms, i.e. common eigenfuntions of all the Hecke operators of M_κ. A normalized modular form satisfying the property

$$T_n f = \lambda_n f \tag{39}$$

for some complex numbers λ_n is called a normalized Hecke eigenform, or Hecke form for short. In particular, Hecke eigenforms represent a basis of M_κ for every κ. The relevance of Hecke forms in the context of our construction is due to the fact that the Fourier coefficients $a(n)$ are now *real* and *multiplicative*, as a consequence of eq. (34) whenever $(m, n) = 1$. Therefore, by repeating the construction of Definition 7, we introduce the space $\mathcal{G}_{H,\kappa}$ of *Hecke modular graphs of weight* κ, with $\mathcal{G}_{H,\kappa} \subset \mathcal{G}_{\mathcal{M}}$.

Let us discuss the algebraic properties of the classes of modular graphs defined above. We shall introduce the graded ring $\mathcal{G}_{\Gamma_1,\infty} = \oplus_{\kappa=1}^{\infty} \mathcal{G}_{\Gamma_1,\kappa}$. We can endow the space $\mathcal{G}_{\Gamma_1,\infty}$ both with the pointwise and the Dirichlet products. Both products map modular graphs into modular graphs, although of different weights.

A relevant example of application of the previous theory is provided by the Eisenstein series of weight κ [44]. They possesses the Fourier expansion

$$G_\kappa = -\frac{B_\kappa}{2\kappa} + \sum_{m=1}^{\infty} \sigma_{\kappa-1}(m) e^{2\pi i \tau}. \tag{40}$$

Here B_κ denotes the κ–Bernoulli number; also, given $n \in \mathbb{N}$, the divisor functions $\sigma_\gamma(n)$, where $\gamma \in \mathbb{R}$ or $\mathbb{C}$, are defined to be the sums of the γ powers of the divisors of n:

$$\sigma_\gamma(n) := \sum_{d|n} d^\gamma. \tag{41}$$

The series (40) is a Hecke form for all $\kappa \geq 4$. Let us consider the Dirichlet series (31) associated with an Eisenstein series. One can prove the summation property [58]

$$\sum_{n=1}^{\infty} \frac{\sigma_\gamma(n)}{n^s} = \zeta(s)\zeta(s-\gamma). \tag{42}$$

that motivates the following definition.

Definition 9. *An Eisenstein random graph of weight k is a random graph associated with a probability distribution, for a fixed $\kappa \in \mathbb{N}$, given by*

$$\begin{cases} p_0 = 0 \\ p_m = \frac{\sigma_{\kappa+1}(n)}{m^s \zeta(\alpha)\zeta(\alpha-\kappa-1)} \quad m \in \mathbb{N}, \quad \alpha > 1. \end{cases}$$

The functional equation for the transition towards a giant component is readily obtained:

$$\zeta(\alpha-2)\zeta(\alpha-\kappa-3) - 2\zeta(\alpha-1)\zeta(\alpha-\kappa-2) = 0. \tag{43}$$

This equation can be easily studied numerically as a function of the modular weight κ. For $\kappa \geq 4$, the model possesses the threshold value $\alpha_c = \kappa + 4.482$. A giant cluster is always formed for $4 + \kappa < \alpha < \alpha_c$.

7. Some relevant models: exponentially bounded scale–free graphs and the Hurwitz random graph

The class of random graphs proposed in Definition (1) contains as particular cases several models presenting a cut–off, leading to (asymptotically) bounded scale–free networks (BSF). A most studied model is that presenting an exponential cut–off, that seems to be common in many real–world networks [3], [14]. In this case, the probability degree distribution is of the form $p_k = Ck^{-\alpha}e^{-k/\tau}$, where $C \in \mathbb{R}$ is a normalization constant and $\tau \in \mathbb{R}$ is the cut–off tuning parameter. Also, generalizations have been studied, in which the degree distribution is of the form $p_k \sim k^{-\alpha} f(k\tau^{-1/\alpha-1})$, for suitable choices of $f(x)$ [20]. All of them fall into the class of zeta random graphs, with the choice $a_k = g(k/\tau)$, where $g(x)$ depends on the model considered, and generally speaking is a function that decreases very rapidly for $x > 1$.

In this section, we propose the following model of a random graph related to the classical Hurwitz zeta function

$$\zeta_H(s, k_0) = \sum_{k=1}^{\infty} \frac{1}{(k+k_0)^s}, \quad s \in \mathbb{C}, \quad \operatorname{Re} s > 1.$$

Definition 10. *A Hurwitz random graph is a random graph associated with a probability distribution of the form*

$$\begin{cases} p_0 = 0, \\ p_k = \frac{(k_0+k)^{-\alpha}}{\zeta_H(\alpha)}, \quad k \in \mathbb{N}, \quad \alpha > 1. \end{cases}$$

This graph has an interesting relation with the theoretical framework known as nonextensive statistical mechanics [55]. Indeed, this distribution, once written in terms of the q–exponential function $e_q(-k/\tau)$, where $e_q(x) := [1 + (1-q)x]^{\frac{1}{1-q}}$, and $\alpha = \frac{1}{q-1}$, $k_0 = \frac{\tau}{q-1}$, is nothing but the optimizing distribution for the Tsallis entropy

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{1-q}, \quad i = 1, \dots, W,$$

arising in the description of the stationary state associated with the canonical ensemble in the nonextensive scenario. This model can be interpreted as a cut–off model with a q–exponential as the cut–off function. It should be noticed that it does not come from multiplicative functions, as the previous ones.

In order to prove the existence of a phase transition, one can use an argument similar to that of Theorem 1, where the role of the generalized polylogarithm (14) is replaced by the *Lerch transcendent*

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}.$$

Then the phase transition is described by the critical equation

$$\zeta_H(\alpha - 2, k_0) = 2\zeta_H(\alpha - 1, k_0),$$

which depends on the entropic index q only (for simplicity, we put $\tau = 1$). An interesting consequence of this simple result is the following. Given a physical system in a regime governed by the Tsallis thermostatistics, with a sufficiently large number W of particles, once the probability distribution $\{p_n\}_{n\in\mathbb{N}}$ is assigned, one can construct a Hurwitz random graph, whose critical properties are determined by the entropic index q. This index is *determined by the system*, i.e., fixed by the extensivity requirement of its associated S_q entropy.

Appendix A. Applications: giant clusters, percolation on zeta graphs and epidemic thresholds

Percolation processes on random graphs has been widely investigated [2], [20], [41]. They provide a simple representation of realistic situations in which connection patterns show to be robust or sensitive to removal of nodes [14]. Also, percolation models are suitable for the description of the spread of diseases in communities [26]. To this aim, particularly relevant is the determination of possible *epidemic thresholds* [22]. Epidemic spreading has also been investigated for the specific case of scale–free networks [42].

In order to describe a percolation process, each vertex of a graph is declared to be either "occupied" or "unoccupied" (site percolation). Sometimes, is preferable to define the edge occupation of a graph (bond percolation).

If Ω is the overall fraction of occupied sites ($0 \leq \Omega \leq 1$), it has been proved in [14] that the mean cluster size is given by

$$\langle s\rangle = \Omega\left[1 + \frac{\Omega G_0'(1)}{1-\Omega G_1'(1)}\right].$$

The previous condition yields for the critical occupation probability $\Omega_c = \frac{1}{G_1'(1)}$. Consequently, we deduce for the zeta graphs the analytic threshold condition

$$\Omega_c = \frac{L(\alpha-1)}{L(\alpha-2)-L(\alpha-1)}. \tag{44}$$

This results holds both for site and bond percolation. It shows that the percolation transition corresponds to the formation of a giant cluster.

Let us determine analytically and numerically the percolation thresholds for the zeta models previously discussed.

a1) For the *Euler random graph*, we obtain

$$\Omega_c = \frac{\zeta(\alpha-2)^2}{\zeta(\alpha-3)\zeta(\alpha-1)-\zeta(\alpha-2)^2}. \tag{45}$$

A numerical analysis shows that Ω_c is negative (therefore unphysical) for $\alpha < 1.6$ and $2 < \alpha < 4$. Also, $\Omega_c > 1$ for the range of values $1.6 < \alpha < 1.8$, and for $\alpha > 4.3504..$ which again should be considered of no physical interpretation. Instead, the model percolates for the range of values $1.8 < \alpha < 2$ and $4 < \alpha < 4.3504....$ This is in agreement with the analysis concerning the existence of a giant cluster for the model.

b1) For the *χ–random graph* with conductor $f = 10$, we get the critical equation

$$\frac{\zeta(\alpha-2)}{\zeta(\alpha-1)} = \frac{2\left(1-2^{1-\alpha}\right)\left(1-5^{1-\alpha}\right)}{\left(1-2^{2-\alpha}\right)\left(1-5^{2-\alpha}\right)}. \tag{46}$$

We observe that for $1 < \alpha < \alpha_{c_1} = 1.34$, there always exists a giant component (in the limit of large graph size). This component is not present for values $\alpha_{c_1} < \alpha < 3$. Above $\alpha_{c_2} = 3$, there exists again a giant component. Another critical value is $\alpha_{c_3} = 3.26106$, above which there is no giant cluster. Therefore, the giant cluster exists for $\alpha_{c_2} < \alpha < \alpha_{c_3}$.

One of the most important epidemic real–worlds networks is that studied by Liljeros et al [33]. This model is numerically found to be governed by a power–law distribution, with exponent $\alpha \simeq 3.2$. We observe that this empiric behaviour is very closely reproduced by the topology of the χ–random graphs. Indeed, it percolates for $3 < \alpha < \alpha_{c_3} = 3.26$, in excellent agreement with the experimental data.

Let us discuss the case of epidemic models defined on zeta graphs. Fairly most important are the SIR and SIS models [37], [41]. A vertex is said to be susceptible if can be infected. Let λ be the probability that a susceptible vertex becomes infected when at least one of its nearest neighbors is infected. The SIR model assumes that each individual can be in one of three possible states, i.e. susceptible, infected or removed; the SIS model only considers two states: susceptible and infected. In the SIR model, infected vertices become recovered with unit rate, whereas in the SIS model, infected vertices become susceptible with unit rate. The parameter λ is the only control parameter for these models. By analyzing the dynamical equations defining the models, one can establish the existence of an epidemic threshold for both cases. In the general setting of zeta random graphs, for the SIR model the epidemic threshold λ_t^{SIR} coincides with the percolation threshold Ω_c, given by (44). Instead, we get for the SIS model the critical equation

$$\lambda_t^{SIS} = \frac{L(\alpha-1)}{L(\alpha-2)}. \tag{47}$$

For values $\lambda > \lambda_t^{SIS}$ the infection spreads and becomes endemic [42]. Instead, for $\lambda < \lambda_c$, the infection tends to disappear exponentially fast.

Let us estimate the epidemic thresholds for the zeta models previously discussed.

a2) For the *Euler random graph*, we obtain

$$\lambda_t^E = \frac{\zeta(\alpha-2)^2}{\zeta(\alpha-1)\zeta(\alpha-3)}. \tag{48}$$

b2) For the χ*–random graph*, with $f = 10$, we have

$$\lambda_t^\chi = \frac{\zeta(\alpha-1)}{\zeta(\alpha-2)} \frac{\left(1-2^{1-\alpha}\right)\left(1-5^{1-\alpha}\right)}{\left(1-2^{2-\alpha}\right)\left(1-5^{2-\alpha}\right)}. \tag{49}$$

A very similar analysis can be performed for the other models proposed above (and in particular for modular graphs), and is left to the reader.

Remark 4. For zeta graphs whose degree probability distributions are such that the second moment diverges, we obtain that $\lambda_t^{SIS} = 0$. This generalizes the well–known case of the purely power–law distribution when $2 < \alpha \leq 3$ [1]. Consequently, for any value of λ positive the infection can pervade the network associated with the zeta graph considered, with a finite nonzero density of infected individuals.

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Departamento de Física Teórica II (Métodos Matemáticos de la física), Facultad de Físicas, Universidad Complutense de Madrid, 28040 – Madrid, Spain and Instituto de Ciencias Matemáticas, C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain

E-mail address: p.tempesta@fis.ucm.es, piergiulio.tempesta@icmat.es