Global analysis of infinite singularities of quadratic vector fields

Joan C. Artés∗  Jaume Llibre∗
Dana Schlomiuk†‡  Nicolae Vulpe§¶

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∗Departament de Matemàtiques, Universitat Autònoma de Barcelona
†Département de mathématiques et de statistiques, Université de Montréal
‡Work supported by NSERC.
§Institute of Mathematics and Computer Science, Academy of Science of Moldova
¶Partially supported by NSERC.
Abstract

In the topological classification of phase portraits no distinctions are made between a focus and a node and neither are they made between a strong and a weak focus or between foci of different orders. These distinctions are however important in the production of limit cycles close to the foci in perturbations of the systems. Weak saddles of various orders are also important, for example in perturbations of homoclinic loops. In the topological classification the distinction between the one direction and two directions nodes, which intervenes in understanding the behavior of solution curves around the singularities at infinity, is also missing.

In this work we introduce the notion of geometric equivalence relation which incorporates these important purely algebraic features. The geometric equivalence relation is finer than the topological one and also finer than the qualitative equivalence relation introduced in [21]. We also list all possibilities we have for singularities finite and infinite, taking into consideration these finer distinctions and we introduce notations for each one of them. Our long term goal is to use this finer equivalence relation to classify the quadratic family according to their geometrically different configurations of singularities, finite and infinite.

In this work we accomplish a first step of this larger project. We give a complete global classification, using the geometric equivalence relation, of the whole quadratic class according to the singularities at infinity of the systems. Our classification Theorem is stated in terms of polynomial invariants and hence it can be applied to any family of quadratic systems with respect to any particular normal form. The theorem we give also contains the bifurcation diagram, done in the 12-parameter space, of the geometric configurations of singularities at infinity, and this bifurcation set is algebraic in the parameter space. To determine the bifurcation diagram of configurations of singularities at infinity for any family of quadratic systems, given in any normal form, becomes thus a simple task using computer algebra calculations.
Résumé

Dans la classification topologique des portraits de phase on ne fait pas de distinction entre un foyer et un noyau ou entre un foyer fort et un foyer faible, ou bien entre les foyers de différents ordres. Ces distinctions sont pourtant importantes dans la production de cycles limites au voisinage des foyers, dans les perturbations des systèmes. Les cols faibles de différents ordres sont aussi importants, par exemple dans les perturbations des boucles homocliniques. La classification topologique ne fait pas de distinction entre les noeuds à une ou à deux directions, distinction qui intervient dans le comportement des solutions autour des singularités à l’infini.

Dans ce travail nous introduisons la notion d’égualité géométrique qui incorpore toutes ces distinctions algébriques importantes. La relation d’égalité géométrique est plus fine que celle topologique et elle est aussi plus fine que la relation d’égalité qualitative introduite dans [21]. Considérant la relation d’égalité géométrique, nous énumérons toutes les possibilités que nous avons pour les singularités, finies et infinies, pour toutes ces distinctions plus fines et nous introduisons des notations pour chaque type de ces singularités. Notre but à long terme est d’utiliser cette relation d’égalité plus fine afin de classifier la famille des systèmes quadratiques à l’aide de leurs configurations de singularités, finies et infinies.

Dans ce travail nous réalisons le premier pas de ce plus large projet. Nous donnons une classification complète globale, utilisant la relation d’égalité géométrique, de toutes la classe des systèmes quadratiques, suivant leurs configurations de singularités à l’infini. Notre théorème est énoncé en termes de polynômes invariants et donc il est applicable à toute famille de systèmes quadratiques présentée dans n’importe quelle forme normale particulière. Le théorème que nous donnons contient aussi le diagramme de bifurcation des configurations géométriques des singularités à l’infini, réalisée dans l’espace de 12 paramètres. L’ensemble de bifurcation est algébrique dans cet espace de paramètres. La construction du diagramme de bifurcation des configurations de singularités à l’infini pour une famille quelconque de systèmes quadratiques, donnée dans n’importe quelle forme normale, devient alors une simple tâche utilisant le calcul symbolique à l’ordinateur.
1 Introduction and statement of main results

We consider here differential systems of the form

\[
(S) \quad \frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y),
\]

where \( p, q \in \mathbb{R}[x, y] \), i.e. \( p, q \) are polynomials in \( x, y \) over \( \mathbb{R} \). We call degree of a system (1) the integer \( m = \max(\deg p, \deg q) \). In particular we call quadratic a differential system (1) with \( m = 2 \).

The study of the class of quadratic differential systems has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. The complete characterization of the phase portraits for real quadratic vector fields is not known and attempting to topologically classify these systems, which occur rather often in applications, is a very complex task. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time homotheties, the class ultimately depends on five parameters. This is still a large number of parameters and for the moment only subclasses depending on at most three parameters were studied globally. On the other hand we can restrict the study of this class by focusing on specific global features of the class. We may thus focus on the global study of singularities and their bifurcation diagram. The singularities are of two kinds: finite and infinite. The infinite singularities are obtained by compactifying the differential systems on the sphere or on the Poincaré disk (see [22]).

The global study of quadratic vector fields in the neighborhood of infinity was initiated by Nikolaev and Vulpe in [24] where they classified the singularities at infinity in terms of invariant polynomials. To simplify these invariants and the classification, Schliomiuk and Vulpe used geometrical concepts in [29], they introduced new geometrical concepts and refined the classification in [30]. To reduce the number of phase portraits in half, in both cases the topological equivalence relation was taken to mean the existence of a homeomorphism carrying orbits to orbits and preserving or reversing the orientation. In [4] the authors studied globally and for the whole quadratic class their finite singularities.

The goal of our present work is to go deeper into these classifications by using a finer equivalence relation. In the topological classification no distinction was made among the various types of foci or saddles, strong or weak of various orders. However these distinctions are very important in the study of perturbations of systems possessing such singularities. Indeed, the maximum number of limit cycles which can be produced close to the weak foci in perturbations depends on the orders of the foci.

The distinction among weak saddles is also important since for example when a loop is formed using two separatrices of one weak saddle, the maximum number of limit cycles that can be obtained close to the loop in perturbations is the order of weak saddle.

There are also three kinds of nodes as we can see in Figure 1 below where the local phase portraits around the singularities are given.

![Figure 1](image)

In the three phase portraits of Figure 1 the corresponding three singularities are stable nodes. These portraits are topologically equivalent but the solution curves do not arrive at the nodes in the same way. In the first case, any two distinct non-trivial phase curves arrive at the node with distinct slopes. Such a node is called a star node. In the second picture all non-trivial solution curves excepting two of them arrive at the node with the same slope but the two exception curves arrive at the node with a different slope. This is the generic node with two directions. In the third phase portrait all phase curves arrive at the node with the same slope.
Analogously, the infinite simple nodes can be distinguished according to similar criteria. Furthermore, since a generic node can have the two exceptional curves lying on the line at infinity or not we split the generic nodes in two types.

We recall that the first and the third types of nodes could produce foci in perturbations and the first type of nodes is also involved in the existence of invariant straight lines of differential systems. For example if a quadratic differential system has two finite star nodes then necessarily the system possesses invariant straight lines of total multiplicity 6.

A finer equivalence relation which takes into account such distinctions is thus needed and we define it in this paper.

The distinctions among the nilpotent and linearly zero singularities finite or infinite can also be refined. In this article we introduce the geometric equivalence relation for singularities, finite or infinite of planar polynomial vector fields. This equivalence relation is finer than the qualitative equivalence relation introduced by Jian and Llibre in [21] since it distinguishes among the foci of different orders and among the various types of nodes. This equivalence relation also induces a finer distinction among the more complicated degenerate singularities.

We point out that to distinguish among the foci (or saddles) of various orders we use the algebraic concept of Poincaré-Lyapunov constants. We call strong focus (or strong saddle) a focus with non–zero trace of the linearization matrix at this point. Such a focus (or saddle) will be considered to have the order zero. A focus (or saddle) with trace zero is called a weak focus (weak saddle). For the nodes in Figure 1 the distinction is also made by algebraic means: the linearization matrices at these nodes and their eigenvalues. For details on Poincaré-Lyapunov constants and weak foci we refer to [22].

As it will be seen in the next section, for degenerate singularities the finer distinctions will also be based on algebraic tools. In fact the whole bifurcation diagram of the global configurations of singularities, finite and infinite, in quadratic vector fields and more generally in polynomial vector fields can be obtained by using the algebraic tool of polynomial invariants.

In [13] Coppel wrote:

"Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients. However, attempts in this direction have met with very limited success..."

This proved to be impossible to realize. Indeed, Dumortier and Fiddelers [16] and Roussarie [27] exhibited examples of families of quadratic vector fields which have non-algebraic bifurcation sets. The following are legitimate questions:

How much of the behavior of quadratic (or more generally polynomial) vector fields can be described by algebraic means? How far can we go in the global theory of these vector fields by using mainly algebraic means?

For certain subclasses of quadratic vector fields the full description of the phase portraits as well as of the bifurcation diagrams can be obtained using algebraic tools. Examples of such classes are:

- the quadratic vector fields possessing a center [38], [28], [40], [25];
- the quadratic Hamiltonian vector fields [2], [5];
- the quadratic vector fields with invariant straight lines of total multiplicity at least four [31], [32];
- the planar quadratic differential systems possessing a line of singularities at infinity [33];
- the family of Lotka-Volterra systems [34], [35];
- the quadratic vector fields possessing an integrable saddle [6].

In the case of other subclasses of the quadratic class QS, such as the subclass of systems with a weak focus of order 3 or 2 (see [22], [3]) the bifurcation diagrams were obtained by using an interplay of algebraic, analytic and numerical methods. These subclasses were of dimensions 2 and 3 modulo the action of the affine group and time rescaling. No 4-dimensional subclass of QS were studied so far and such problems are very difficult due to the number of parameters as well as the increased complexities of these classes. On the other hand we propose to study the whole class QS but limiting the study to just the singularities.
of systems in this whole class. In this paper we do this but only for singularities at infinity and for this purpose we shall use the notion of configuration of singularities at infinity.

We distinguish two cases: 1) We have a finite number of infinite singular points; 2) the line at infinity is filled up with singularities.

In the first case we call configuration of singularities at infinity the set of all these singularities together with their local phase portraits endowed with an additional geometric structure described in Section 2 and using the notations described in Section 3.

In the second case, if the line at infinity is denoted by \( z = 0 \), in each one of the charts \( x \neq 0 \) and \( y \neq 0 \), the system is degenerate and we need to do a rescaling of an appropriate degree of the system, so that the degeneracy is removed. The resulting systems have on the line \( z = 0 \) only a finite number of singularities. So in the second case we call configuration of singularities at infinity the set of singularities of the “reduced” systems taken together with their phase portraits endowed with their additional structure as indicated for the case 1) and to which we add that all points on the line \( z = 0 \) are singularities.

The goal of this article is to classify the configurations of singularities at infinity of planar quadratic vector fields using the geometric equivalence relation which we define in the next section. A much larger goal is to complete the geometrical classification of all global configurations of singular points (finite and infinite) of quadratic differential systems. We obtain the following

**Main Theorem.** (A) The configurations of singularities at infinity of all quadratic vector fields are classified in Diagrams 1–4 according to the geometric equivalence relation. Necessary and sufficient conditions for each one of the 168 different equivalence classes can be assembled from these diagrams in terms of 27 invariant polynomials with respect to the action of the affine group and time rescaling, given in Section 5.

(B) The Diagrams 1–4 actually contain the bifurcation diagram in the 12-dimensional space of parameters of the global configurations of singularities at infinity of quadratic differential systems.

The invariants and comitants of differential equations used for proving our main results are obtained following the theory of algebraic invariants, developed by Sibirsky and his disciples (see for instance [37], [39], [26], [8], [12]).

### 2 Equivalence relations for singularities of planar polynomial vector fields

We first recall the topological equivalence relation as it is used in most of the literature. Two singularities \( p_1 \) and \( p_2 \) are topologically equivalent if there exist open neighborhoods \( N_1 \) and \( N_2 \) of these points and a homeomorphism \( \Psi : N_1 \to N_2 \) carrying orbits to orbits and preserving their orientations. To reduce the number of cases, by topological equivalence we shall mean here that the homeomorphism \( \Psi \) preserves or reverses the orientation. In this article we use this second notion, sometimes also used elsewhere in the literature (see [21], [3]).

In [21] Jiang and Llibre introduced another equivalence relation for singularities which is finer than the topological equivalence:

We say that \( p_1 \) and \( p_2 \) are qualitatively equivalent if i) they are topologically equivalent through a homeomorphism \( \Psi \); and ii) two orbits are tangent to the same straight line at \( p_1 \) if and only if the corresponding two orbits are tangent to the same straight line at \( p_2 \).

We observe that the above definition is not entirely explicit. The statement “two orbits are tangent to the same line at a point ” is intuitively understood. This notion is made explicit as follows:

Let us assume that we have an isolated singularity \( p \). Suppose that in a neighborhood \( U \) of \( p \) there is no other singularity. Consider an orbit \( \gamma \) in \( U \) defined by a solution \( M(t) = (x(t), y(t)) \) such that \( \lim_{t \to \pm \infty} M(t) = p \) as \( t \) tends to infinity (or to \( -\infty \)). For a fixed \( t \) consider the unit vector \( C(t) = (M(t) - p)/\|M(t) - p\| \). Let \( L \) be a line passing through \( p \). We shall say that the orbit \( \gamma \) is tangent to a line \( L \) at \( p \) if \( \lim_{t \to \pm \infty} C(t) \) exists as \( t \) tends to infinity (or to \(-\infty\)) and \( L \) contains this limit point on the unit circle centered at \( p \). In this case we may also say that the solution curve \( M(t) \) tends to \( p \) with a well defined slope. A characteristic orbit at a singular point \( p \) is the orbit of a solution curve \( M(t) \) which tends to \( p \) with a well defined slope. The line through \( p \) with this well defined slope is called a characteristic direction.
Diagram 1: Configurations of ISPs in the case $\eta > 0$. 
We say that two simple finite nodes, with the respective eigenvalues $\lambda_1, \lambda_2$ and $\sigma_1, \sigma_2$, of a planar polynomial vector field are tangent equivalent if and only if they satisfy one of the following three conditions: 

a) $(\lambda_1 - \lambda_2)(\sigma_1 - \sigma_2) \neq 0$; b) $\lambda_1 - \lambda_2 = 0 = \sigma_1 - \sigma_2$ and both linearization matrices at the two singularities are diagonal; c) $\lambda_1 - \lambda_2 = 0 = \sigma_1 - \sigma_2$ and the corresponding linearization matrices are not diagonal.

We say that two infinite simple nodes $p_1$ and $p_2$ are tangent equivalent if and only if their corresponding singularities on the sphere are tangent equivalent and in addition, in case they are generic nodes, we have $(|\lambda_1| - |\lambda_2|)(|\sigma_1| - |\sigma_2|) > 0$ where $\lambda_1$ and $\sigma_1$ are the eigenvalues of the eigenvectors tangent to the line at infinity.

Finite and infinite singular points may either be real or complex. In case we have a complex singular point we will specify this with the symbols $\otimes$ and $\bar{\otimes}$ for finite and infinite points respectively. We point out that the sum of the multiplicites of all singular points of a quadratic system (with a finite number of singular points) is always 7. (Here of course we refer to the compactification on the complex projective space $P_2(\mathbb{R})$ of the foliation with singularities associated to the complexification of the vector field.) The sum of the multiplicities of the infinite singular points is always at least 3, more precisely it is always 3 plus the sum of the multiplicites of the finite points which have gone to infinity.

We use here the following terminology for singularities:

- We call elemental a singular point with its both eigenvalues not zero;
- We call semi–elemental a singular point with exactly one of its eigenvalues equal to zero;
- We call nilpotent a singular point with both its eigenvalues zero but with its Jacobian matrix at that point not identically zero;
- We call intricate a singular point with its Jacobian matrix identically zero.

The intricate singularities are usually called in the literature linearly zero. We use here the term intricate to indicate the rather complicated behavior of phase curves around such a singularity.

Roughly speaking a singular point $p$ of an analytic differential system $\chi$ is a multiple singularity of multiplicity $m$ if $p$ produces $m$ singularities, as close to $p$ as we wish, in analytic perturbations $\chi_\epsilon$ of this system and $m$ is the maximal such number. In polynomial differential systems of fixed degree $n$ we have
several possibilities for obtaining multiple singularities. i) A finite singular point splits into several finite singularities in n-degree polynomial perturbations. ii) An infinite singular point splits into some finite and some infinite singularities in n-degree polynomial perturbations. iii) An infinite singularity splits only in infinite singular points of the systems in n-degree perturbations. To all these cases we can give a precise mathematical meaning using the notion of intersection multiplicity at a point $p$ of two algebraic curves.

To define the notion of geometric equivalence relation of singularities we need the preliminary definition of blow-up equivalence, necessary for nilpotent and intricate singularities. In case several blow-ups are needed before the desingularization is complete, in the succession of blow-ups such as we can see in Example 3.2 in Fig. 3.4. of [17] we shall consider the final picture. In the first step of the desingularization of a singularity, the singularity is replaced by a circle and the induced vector field in neighborhoods of the singularity by vector fields in neighborhoods of this circle in charts of a larger manifold. The singularity is decomposed into a number of simpler singularities located on the circle. If one or more of them are nilpotent or intricate singularities, the blow-up procedure needs to be repeated. After a succession of such blow-ups the initial singularity is finally decomposed in several elemental or semi-elemental singularities lying on a compact one dimensional set $D$ ([17]), subset of a more complicated two-dimensional manifold.

We say that two nilpotent or intricate singularities $p_1$ and $p_2$ of two polynomial vector fields are blow-up equivalent if and only if there exists neighborhoods $U_1$ and $U_2$ on the complete blow-up surface of the respective compact sets $D_1$ and $D_2$ and a homeomorphism $\Psi : U_1 \rightarrow U_2$ which is a topological equivalence of the two foliations restricted to $U_1, U_2$, sending $D_1$ to $D_2$. This could also be more compactly expressed by saying that the germs of the two foliations with singularities around $D_1$ and $D_2$ associated to the initial vector fields are topologically equivalent.
We say that two infinite singularities \( p_1 \) and \( p_2 \) of two polynomial vector fields of the same degrees are blow-up equivalent if and only if the homeomorphism \( \Psi \) sends points on the equator (or on the line at infinity) to points on the equator and in addition if their corresponding singularities on the sphere are blow-up equivalent.

Two foci (or saddles) are order equivalent if their corresponding orders coincide.
Diagram 3 (cont.): Configurations of ISPs in the case $\eta = 0$, $M \neq 0$. 
Diagram 3 (cont.): Configurations of ISPs in the case $\eta = 0$, $M \neq 0$.

Semi–elemental saddle–nodes are always topologically equivalent.

**Definition 1.** Two singularities $p_1$ and $p_2$ of two polynomial vector fields are geometrically equivalent if and only if they are topologically equivalent, they have the same multiplicity and one of the following conditions is satisfied:

- $p_1$ and $p_2$ are order equivalent foci (or saddles);
- $p_1$ and $p_2$ are tangent equivalent simple nodes;
- $p_1$ and $p_2$ are both centers;
- $p_1$ and $p_2$ are both semi–elemental singularities;
- $p_1$ and $p_2$ are blow–up equivalent nilpotent or intricate singularities.

In this work we discuss the behavior of quadratic vector fields globally around their singularities.

**Definition 2.** Let $\chi_1$ and $\chi_2$ be two polynomial vector fields each having a finite number of singularities. We say that $\chi_1$ and $\chi_2$ have geometric equivalent configurations of singularities if and only if we have a bijection $\vartheta$ carrying the singularities of $\chi_1$ to singularities of $\chi_2$ and for every singularity $p$ of $\chi_1$, $\vartheta(p)$ is geometric equivalent with $p$. 


Diagram 4: Configurations of ISPs in the case $\eta = 0, M = 0$. 
3 Singularities of polynomial differential systems and notations for them

In this work we encounter all the possibilities we have for the geometric features of the infinite singularities in the whole quadratic class as well as the way they assemble in systems of this class. Since we want to describe precisely these geometric features and in order to facilitate understanding, it is important to have a clear, compact and congenial notation which conveys easily the information. Of course this notation must be compatible with the one used to describe finite singularities, and thus we must start with the finite. The notation we use, even though it is used here to describe finite and infinite singular points of quadratic systems, can easily be extended to general polynomial systems.

We start by first describing the finite and infinite singularities, denoting the first ones with lower case letters and the second with capital letters. When describing in a sequence both finite and infinite singular points, we will always place first the finite ones and only later the infinite ones, separating them by a semi-column.

Elemental points: We use the letters ‘s’, ‘S’ for “saddles”; ‘n’, ‘N’ for “nodes”; ‘f’ for “foci” and ‘c’ for “centers”. In order to augment the level of precision we will distinguish the finite nodes as follows:

- ‘n’ for a node with two distinct eigenvalues;
- ‘nd’ (a one–direction node) for a node with two identical eigenvalues whose Jacobian matrix cannot be diagonal;
- ‘ns’ (a star–node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

Moreover, in the case of an elemental infinite node, we want to distinguish whether the eigenvalue associated to the eigenvector directed towards the affine plane is greater or lower than the eigenvalue associated to the eigenvector tangent to the line at infinity. This is relevant because this determines if all the orbits except one arrive at infinity tangent to the line at infinity or transversal to this line. We will denote them as ‘N∞’ and ‘Nf’ respectively.

Finite elemental foci and saddles are classified as strong or weak foci, respectively strong or weak saddles. When the trace of the Jacobian matrix evaluated at those singular points is not zero, we call them strong saddles and strong foci and we maintain the standard notations ‘s’ and ‘f.’ But when the trace is zero, except for centers and saddles of infinite order (i.e. saddles with all their Poincaré-Lyapounov constants equal to zero), it is known that the foci and saddles, in the quadratic case, may have up to 3 orders. We denote them by ‘s(i)’ and ‘f(i)’ where \( i = 1, 2, 3 \) is the order. In addition we have the centers which we denote by ‘c’ and saddles of infinite order (integrable saddles) which we denote by ‘s’.

Foci and centers cannot appear as isolated singular points at infinity and hence there is no need to introduce their order in this case. In case of saddles, we can have weak saddles at infinity but it is premature at this stage to describe them since the maximum order of weak singularities in cubic systems is not yet known.

All non–elemental singular points are multiple points, in the sense that there are perturbations which have at least two elemental singular points as close as we wish to the multiple point. For finite singular points we denote with a subindex their multiplicity as in ‘s(5)’ or in ‘c8(3)’ (the meanings of ‘−’ and ‘∗’ will be explained below). In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [30]. Thus we denote by ‘(a)’...’ the maximum number \( a \) (respectively \( b \)) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example ‘(1)SN’ means a saddle–node at infinity produced by the collision of one finite singularity with an infinite one; ‘(1)SN’ means a saddle produced by the collision of 3 infinite singularities.

Semi–elemental points: They can either be nodes, saddles or saddle–nodes, finite or infinite. We will denote the semi–elemental ones always with an overline, for example ‘sn’, ‘s’ and ‘π’ with the corresponding multiplicity. In the case of infinite points we will put ‘π’ on top of the parenthesis with multiplicities.

Moreover, in cases that will be explained later, an infinite saddle–node may be denoted by ‘SN’.

Semi–elemental nodes could never be ‘nd’ or ‘ns’ since the eigenvalues are always different. In case of an
infinite semi–elemental node, the type of collision determines whether the point is denoted by ‘NFL’ or by ‘NFL∞’ where $\overline{(2)}N$ is an ‘NFL’ and $\overline{(3)}N$ is an ‘NFL∞’.

Nilpotent points: They can either be saddles, nodes, saddle–nodes, elliptic–saddles, cusps, foci or centers. The first four of these could be at infinity. We denote the nilpotent singular points with a hat ‘$\hat{\ }$’ as in $\hat{es}_1$ for a finite nilpotent elliptic saddle of multiplicity 3 and $\hat{p}_2$ for a finite nilpotent cusp point of multiplicity 2. In the case of nilpotent infinite points, we will put the ‘$\hat{\ }$’ on top of the parenthesis with multiplicity, for example $\hat{p}_2(3)\ PEP – H$. The relative position of the sectors of an infinite nilpotent point with respect to the line at infinity can produce topologically different phase portraits. This forces us to use a notation for these points similar to the notation which we will use for the intricate points.

It is known that the neighborhood of any singular point of a polynomial vector field (except foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [17]). Then, a reasonable way to describe intricate and nilpotent points at infinity is to use a sequence formed by the types of their sectors. The description we give is the one which appears in the clockwise direction once the blow–down of the desingularization is done. Thus in quadratic systems, we have just seven possibilities for finite intricate singular points (see [4]) which are the following ones:

- a) $phphp_{(4)}$;
- b) $phph_{(4)}$;
- c) $hh_{(4)}$;
- d) $hhhhhh_{(4)}$;
- e) $pep_{(4)}$;
- f) $pe_{(4)}$;
- g) $ee_{(4)}$.

We use lower case because of the finite nature of the singularities and add the subindex (4) since they are all of multiplicity 4.

For infinite intricate and nilpotent singular points, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. In this way we will distinguish between $\hat{p}_2(3)\ PEP – PHP$ and $\hat{p}_2(3)\ PPH – PPH$.

When describing a single finite nilpotent or intricate singular point, one can always apply an affine change of coordinates to the system, so it does not really matter which sector starts the sequence, or the direction (clockwise or counter–clockwise) we choose. If it is an infinite nilpotent or intricate singular point, then we will always start with a sector bordering the infinity (to avoid using two dashes). However, when one needs to describe several singular points at infinity together, then the relative positions of one with respect with those other points, may be relevant. This will happen when trying to describe at the same time at least two non–elemental points together. In this paper this situation only occurs once for systems having two semi–elemental saddle–nodes at infinity and a third singular point which is elemental. In this case we may need to write $NS$ instead of $SN$ to denote the different relative positions of the parabolic (nodal) sector and the hyperbolic (saddle) one with respect of the other infinite singular points. More concretely, $Figure 3$ from [30] (which corresponds to Config. 3 in Figure 4) must be described as $\overline{(1)}SN, \overline{(1)}SN, N$ since the elemental node lies always between the hyperbolic sectors of one saddle–node and the parabolic ones of the other. However, $Figure 4$ from [30] (which corresponds to Config. 4 in Figure 4) must be described as $\overline{(1)}SN, \overline{(1)}NS, N$ since the hyperbolic sectors of each saddle–node lie between the elemental node and the parabolic sectors of the other saddle–node. These two configurations have exactly the same description of singular points but their relative position produces topologically (and geometrically) different portraits.

For the description of the topological phase portraits around the isolated singular points the information described above is sufficient. However we are interested in additional geometrical features such as the number of directions which figure in the final global picture of the desingularization. In order to add this information we need to introduce more notation. If the two limiting orbits of a sector arrive at the singular
point with the same slope and direction, then the sector will be denoted by $H, E$ or $R$. The index in this notation refers to the cusp–like form of limiting trajectories of the sectors. Moreover, in the case of parabolic sectors we want to make precise whether the orbits arrive tangent to one or the other limiting trajectories. We distinguish the two cases by $\vartriangleleft P$ if they arrive tangent to the trajectory limiting the previous sector in clock–wise sense or $\vartriangleright P$ if they arrive tangent to the trajectory limiting the next sector. Clearly, a parabolic sector denoted by $P^*$ would correspond to a sector in which orbits arrive with all possible slopes in between the limiting trajectories. In the case of a cusp–like parabolic sector, all orbits must arrive with only one slope, but the distinction between $\vartriangleleft P$ and $\vartriangleright P$ is still valid if we consider the limits of the second derivatives of integral curves within the sector and we compare them with those of the limiting trajectories. Thus, complicated intricate singular points like the two we see in 2 may be described as $(\frac{1}{2}) \vartriangleleft P E \vartriangleright H H H$ (case (a)) and $(\frac{3}{4}) E \vartriangleleft P H H H \vartriangleright P E$ (case (b)), respectively.

Figure 2:

The lack of finite singular points will be encapsulated in the notation $\emptyset$. In the cases we need to point out the lack of an infinite singular point, we will use the symbol $\emptyset$.

Finally there is also the possibility that we have an infinite number of finite or infinite singular points. In the first case, this means that the polynomials defining the differential system are not coprime. Their common factor may produce a line or conic with real coefficients filled up with singular points.

**Line at infinity filled up with singularities:** It is known that any such system has in a sufficiently small neighborhood of infinity one of 6 topological distinct phase portraits (see [33]). The way to determine these portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. In case a singular point still remains on the line at infinity we study such a point. In [33] the tangential behavior of the solution curves was not considered in the case of a node. If after the removal of the degeneracy in the local charts at infinity a node remains, this could either be of the type $N^d$, $N$ and $N^*$ (this last case does not occur in quadratic systems as we will see in this paper).

Since no eigenvector of such a node $N$ (for quadratic systems) will have the direction of the line at infinity we do not need to distinguish $N^f$ and $N^\infty$. Other singular points at infinity of quadratic systems, after removal of the degeneracy can be a saddle, a center, a semi–elemental saddle–node or a nilpotent elliptic saddle. We also have the possibility of no singularities after the removal of the degeneracy. To convey the way these singularities were obtained we use the notation $[\infty; \emptyset], [\infty; N], [\infty; N^d], [\infty; S], [\infty; C], [\infty; (\frac{1}{2}) S N]$ or $[\infty; (\frac{3}{2}) E S]$.

**Degenerate systems:** We will denote with the symbol $\ominus$ the case when the polynomials defining the system have a common factor. The symbol stands for the most generic of these cases which corresponds to a real line filled up of singular points. The degeneracy can be also be produced by a common quadratic factor which defines a conic. It is well known that by an affine transformation any conic over $\mathbb{R}$ can be brought to one of the following forms: $x^2 + y^2 - 1 = 0$ (real ellipse), $x^2 + y^2 + 1 = 0$ (complex ellipse), $x^2 - y^2 = 1$ (hyperbola), $y - x^2 = 0$ (parabolla), $x^2 - y^2 = 0$ (pair of intersecting real lines), $x^2 + y^2 = 0$ (pair of intersecting complex lines), $x^2 - 1 = 0$ (pair of parallel real lines), $x^2 + 1 = 0$ (pair of parallel complex lines), $x^2 = 0$ (double line).

We will indicate each case by the following symbols:

- $\ominus[\emptyset]$ for a real straight line;
• \(\Theta[\sigma]\) for a real ellipse;
• \(\Theta[\bigcirc]\) for a complex ellipse;
• \(\Theta[\chi]\) for an hyperbola;
• \(\Theta[\cup]\) for a parabola;
• \(\Theta[\times]\) for two real straight lines intersecting at a finite point;
• \(\Theta[\cdot]\) for two complex straight lines which intersect at a real finite point.
• \(\Theta[\|]\) for two real parallel lines;
• \(\Theta[\|\cdot]\) for two complex parallel lines;
• \(\Theta[\times]\) for a double real straight line;

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If the reduced system has no finite singularity on this curve, we will use the symbol \(\emptyset\) to describe this situation. If some singular points remain we will use the corresponding notation of their types. As an example we complete the notation above as follows:

• \((\Theta[\|];\emptyset)\) denotes the presence of a real straight line filled up with singular points such that the reduced system has no singularity on this line;
• \((\Theta[\|];f)\) denotes the presence of the same straight line such that the reduced system has a strong focus on this line;
• \((\Theta[\|\cdot];\emptyset)\) denotes the presence of a parabola filled up with singularities such that no singular point of the reduced system is situated on this parabola.

**Degenerate systems with non–isolated singular points at infinity, which are isolated on the line at infinity:** The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points. We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non–isolated singularity.

In order to describe correctly the singularities at infinity, we must mention also this kind of phenomena and describe what happens to such points at infinity after the removal of the common factor. To show the existence of the common factor we will use the same symbol as before \(\Theta\), and for the type of degeneracy we use the symbols introduced above. We will use the symbol \(\emptyset\) to denote the non–existence of real infinite singular points after the removal of the degeneracy. We will use the corresponding capital letters to describe the singularities which remain there. Let us take note that a simple straight line, two parallel lines (real or complex), one double line or one parabola defined by the common factor (all taken over the reals) imply the existence of one real non–isolated singular point at infinity in the original degenerate system. However a hyperbola and two real straight lines intersecting at a finite point imply the presence of two real non–isolated singular points at infinity in the original degenerate system. Finally, a complex ellipse and two complex straight lines which intersect at a real finite point imply the presence of two complex non–isolated singular points at infinity in the original degenerate system. Thus, in the reduced system these points may disappear as singularities and in case they remain, they must be described. In the above mentioned first five cases we will give the description of the corresponding infinite point. In the next five cases we will give the description of the corresponding two singular points. As agreed, we will use capital letters to denote them since they are on the line at infinity. Thus, as an example, we have:

• \(N^f, S, (\Theta[\|];\emptyset)\) means that the system has a node at infinity such that an infinite number of orbits arrive tangent to the eigenvector in the affine part, a saddle, and one non–isolated singular point which is part of a real finite straight line filled up with singularities, and that the reduced linear system has no infinite singular point in that position;
• \( S, (\ominus [\cdot]; N^*) \) means that the system has a saddle at infinity, and one non–isolated singular point which is part of a real finite straight line filled up with singularities, and that the reduced linear system has a star node in that position;

• \( S, (\ominus [\cdot]; 0, 0) \) means that the system has a saddle at infinity, and two non–isolated singular points which are part of a hyperbola filled up with singularities, and that the reduced constant system has no singularity in those positions;

• \( (\ominus [\times]; N^*, \emptyset) \) means that the system has two non–isolated singular points at infinity which are part of two real intersecting straight lines filled up with singularities, and that the reduced constant system has a star node in one of those positions and no singularity in the other;

• \( S, (\ominus [\cdot]; 0, 0) \) means that the system has a saddle at infinity, and two non–isolated (complex) singular points which are located on the complexification of a real conic which has no real point at infinity, and the reduced constant system has no singularity in those positions.

When there is a non–isolated infinite singular point such that the reduced system has a singularity at that position, it may happen that one or several characteristic directions at this point, directed towards the affine plane, could coincide with a tangent line to the curve of singularities at this point. This situation could produce many different geometrical (or even topological) combinations but in the quadratic case we only have a few of them for which we introduce a coherent notation. This notation can be further developed for higher degree systems. In quadratic systems we need only to distinguish among some situations in which a characteristic direction of the infinite singular point, after the removal of the degeneracy, may coincide or not with a tangent line to the curve of singularities at this point. We show two cases that need to be distinguished in Figure 3 (case \((a)\) and \((b)\)). Here we will use a numerical subscript which denotes the cardinal number \( \parallel \) of the union of the characteristic directions, together with the tangent lines to the curve of singularities at this point, all of them considered in a neighborhood of the point at infinity on the Poincaré sphere. The singularities at infinity of the examples of Figure 3 would then be denoted by \( S, (\ominus [\cdot]; N^*_3) \) (case \((a)\)) and \( S, (\ominus [\cdot]; N^*_2) \) (case \((b)\)).

![Figure 3: Degenerate systems with the line at infinity filled up with singularities](image)

**Degenerate systems with the line at infinity filled up with singularities:** For a quadratic system this implies that the polynomials must have a common linear factor and there are only two possible phase portraits, which can be seen in Figure 3 (the portraits \((c)\) and \((d)\)). In order to be consistent with our notation and considering generalization to higher degree systems, we describe the two cases in a coherent way with what we have done up to now.

The case \((c)\) is denoted by \([\infty; (\Theta [\cdot]; 0_3)]\) which means:

• the line at infinity is filled up with singular points;

• the reduced quadratic system on the infinite local charts has a non–isolated singular point on the line at infinity due to the affine line of degeneracy;

• once the original system is reduced to a linear one by removing the common factor, the infinity continues to be filled up with singular points;
• once the system on a local chart at infinity around the singularity which is common to both lines filled up with singular points, is reduced by completely removing the degeneracy, there is no singular point on that intersection;

• the cardinal number $||$ is 3. That is beyond the line of singularities and the line at infinity, we have another characteristic direction which is affine.

The second case is denoted by $[\infty; (\emptyset ||; \emptyset_{2})]$, which means exactly the same items as above with the exception that cardinal number $||$ is 2. That is, beyond the line of singularities and the line at infinity, we have no other characteristic direction.

4 Assembling multiplicities for global configurations of singularities at infinity using divisors

The singular points at infinity belong to compactifications of planar polynomial differential systems, defined on the affine plane. We begin this section by briefly recalling these compactifications.

4.1 Compactifications associated to planar polynomial differential systems

4.1.1 Compactification on the sphere and on the Poincaré disc

Planar polynomial differential systems (1) can be compactified on the sphere. For this we consider the affine plane of coordinates $(x, y)$ as being the plane $Z = 1$ in $\mathbb{R}^3$ with the origin located at $(0, 0, 1)$, the $x$-axis parallel with the $X$-axis in $\mathbb{R}^3$, and the $y$-axis parallel to the $Y$-axis. We use central projection to project this plane on the sphere as follows: for each point $(x, y, 1)$ we consider the line joining the origin with $(x, y, 1)$. This line intersects the sphere in two points $P_1 = (X, Y, Z)$ and $P_2 = (-X, -Y, -Z)$ where $(X, Y, Z) = (1/\sqrt{x^2 + y^2 + 1})(x, y, 1)$. The applications $(x, y) \mapsto P_1$ and $(x, y) \mapsto P_2$ are bianalytic and associate to a vector field on the plane $(x, y)$ an analytic vector field $\Psi$ on the upper hemisphere and also an analytic vector field $\Psi'$ on the lower hemisphere. A theorem stated by Poincaré and proved in [18] says that there exists an analytic vector field $\Theta$ on the whole sphere which simultaneously extends the vector fields on the two hemispheres. By the Poincaré compactification on the sphere of a planar polynomial vector field we mean the restriction $\Psi$ of the vector field $\Theta$ to the union of the upper hemisphere with the equator. For more details we refer to [22]. The vertical projection of $\Psi$ on the plane $Z = 0$ gives rise to an analytic vector field $\Phi$ on the unit disk of this plane. By the compactification on the Poincaré disc of a planar polynomial vector field we understand the vector field $\Phi$. By singular point at infinity of a planar polynomial vector field we mean a singular point of the vector field $\Psi$ which is located on the equator of the sphere, respectively a singular point of the vector field $\Phi$ located on the circumference of the Poincaré disc.

4.1.2 Compactification on the projective plane

To a polynomial system (1) we can associate a differential equation $\omega_1 = q(x, y)dx - p(x, y)dy = 0$. Assuming the differential system (1) is with real coefficients, we may associate to it foliations with singularities on the real, respectively complex, projective plane as indicated below. The equation $\omega_1 = 0$ defines a foliations with singularities on the real or complex plane depending if we consider the equation as being defined over the real or complex affine plane. It is known that we can compactify these foliations with singularities on the real respectively complex projective plane. In the study of real planar polynomial vector fields, their associated complex vector fields and their singularities play an important role. In particular such a vector field could have complex, non-real singularities, by this meaning singularities of the associated complex vector field. We briefly recall below how these foliations with singularities are defined.

The application $\Upsilon : K^2 \rightarrow P_2(K)$ defined by $(x, y) \mapsto [x : y : 1]$ is an injection of the plane $K^2$ over the field $K$ into the projective plane $P_2(K)$ whose image is the set of $[X, Y, Z]$ with $Z \neq 0$. If $K$ is $\mathbb{R}$ or $\mathbb{C}$ this application is an analytic injection. If $Z \neq 0$ then $(\Upsilon)^{-1}([X : Y : Z]) = (x, y)$ where $(x, y) = (X/Z, Y/Z)$. We obtain a map $i : K^3 - \{Z = 0\} \rightarrow K^2$ defined by $[X : Y : Z] \mapsto (X/Z, Y/Z)$. 

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Considering that $dx = d(X/Z) = (ZdX - XdZ)/Z^2$ and $dy = (ZdY - YdZ)/Z^2$, the pull-back of the form $\omega_1$ via the map $i$ yields the form $i^* (\omega_1) = q(X/Z, Y/Z)(ZdX - XdZ)/Z^2 - p(X/Z, Y/Z)(ZdY - YdZ)/Z^2$ which has poles on $Z = 0$. Then the form $\omega = Z^{m+2}i^* (\omega_1)$ on $K^3 - \{Z = 0\}$, $K$ being $\mathbb{R}$ or $\mathbb{C}$ and $m$ being the degree of systems (1) yields the equation $\omega = 0$:

$$A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ = 0$$

on $K^3 - \{Z = 0\}$ where $A, B, C$ are homogeneous polynomials over $K$ with $A(X, Y, Z) = ZQ(X, Y, Z)$, $Q(X, Y, Z) = Z^m q(X/Z, Y/Z)$, $B(X, Y, Z) = ZP(X, Y, Z)$, $P(X, Y, Z) = Z^m p(X/Z, Y/Z)$ and $C(X, Y, Z) = YP(X, Y, Z) - XQ(X, Y, Z)$.

The equation $AdX + BdY + CdZ = 0$ defines a foliation $F$ with singularities on the projective plane over $K$ with $K$ either $\mathbb{R}$ or $\mathbb{C}$. The points at infinity of the foliation defined by $\omega_1 = 0$ on the affine plane are the points $[X : Y : 0]$ and the line $Z = 0$ is called the line at infinity of the foliation with singularities generated by $\omega_1 = 0$.

The singular points of the foliation $F$ are the solutions of the three equations $A = 0, B = 0, C = 0$. In view of the definitions of $A, B, C$ it is clear that the singular points at infinity are the points of intersection of $Z = 0$ with $C = 0$.

### 4.2 Assembling data on infinite singularities in divisors of the line at infinity

In the previous sections we have seen that there are two types of multiplicities for a singular point $p$ at infinity: one expresses the maximum number $m$ of infinite singularities which can split from $p$, in small perturbations of the system and the other expresses the maximum number $m'$ of finite singularities which can split from $p$, in small perturbations of the system. In Section 2 we mentioned that we shall use a column $(m, m')^t$ to indicate this situation.

We are interested in the global picture which includes all singularities at infinity. Therefore we need to assemble the data for individual singularities in a convenient, precise way. To do this we use for this situation the notion of cycle on an algebraic variety as indicated in [25] and which was used in [22] as well as in [30].

We briefly recall here the definition of this notion. Let $V$ be an irreducible algebraic variety over a field $K$. A cycle of dimension $r$ or $r$-cycle on $V$ is a formal sum $\sum_W n_W W$, where $W$ is a subvariety of $V$ of dimension $r$ which is not contained in the singular locus of $V$, $n_W \in \mathbb{Z}$, and only a finite number of the coefficients $n_W$ are non-zero. The degree $\deg(J)$ of a cycle $J$ is defined by $\sum_W n_W$. An $(n - 1)$-cycle is called a divisor on $V$. These notions were used for classification purposes of planar quadratic differential systems in [25, 22, 30].

To a system (1) we can associate two divisors on the line at infinity $Z = 0$ of the complex projective plane: $D_S(P, Q; Z) = \sum_w I_w(P, Q)w$ and $D_S(C, Z) = \sum_w I_w(C, Z)w$ where $w \in \{Z = 0\}$ and where by $I_w(F, G)$ we mean the intersection multiplicity at $w$ of the curves $F(X, Y, Z) = 0$ and $G(X, Y, Z) = 0$, with $F$ and $G$ homogeneous polynomials in $X, Y, Z$ over $\mathbb{C}$. For more details see [22].

Following [30] we assemble the above two divisors on the line at infinity into just one but with values in the ring $\mathbb{Z}^2$:

$$D_S = \sum_{\omega \in \{Z = 0\}} \begin{pmatrix} I_w(P, Q) \\ I_w(C, Z) \end{pmatrix} w$$

of the complex projective plane. This divisor signals to us the total number of singularities at infinity of a system 1 as well as the two kinds of multiplicities which each singularity has. The meaning of these two kinds of multiplicities are described in the definition of the two divisors $D_S(P, Q; Z)$ and $D_S(C, Z)$ on the line at infinity.
5 Invariant polynomials and preliminary results

Consider real quadratic systems of the form:

\[
\begin{align*}
\frac{dx}{dt} &= p_0 + p_1(x, y) + p_2(x, y) \equiv P(x, y), \\
\frac{dy}{dt} &= q_0 + q_1(x, y) + q_2(x, y) \equiv Q(x, y)
\end{align*}
\]

with homogeneous polynomials \( p_i \) and \( q_i \) (\( i = 0, 1, 2 \)) of degree \( i \) in \( x, y \):

\[
\begin{align*}
p_0 &= a_{00}, & p_1(x, y) &= a_{10}x + a_{01}y, & p_2(x, y) &= a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\
q_0 &= b_{00}, & q_1(x, y) &= b_{10}x + b_{01}y, & q_2(x, y) &= b_{20}x^2 + 2b_{11}xy + b_{02}y^2.
\end{align*}
\]

Let \( \bar{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}) \) be the 12-tuple of the coefficients of systems (2) and denote \( \mathbb{R}[\bar{a}, x, y] = \mathbb{R}[a_{00}, \ldots, b_{02}, x, y] \).

5.1 Affine invariant polynomials associated to infinite singularities

It is known that on the set \( \mathbb{Q}_S \) of all quadratic differential systems (2) acts the group \( \text{Aff}(2, \mathbb{R}) \) of the affine transformation on the plane (cf. [30]). For every subgroup \( G \subseteq \text{Aff}(2, \mathbb{R}) \) we have an induced action of \( G \) on \( \mathbb{Q}_S \). We can identify the set \( \mathbb{Q}_S \) of systems (2) with a subset of \( \mathbb{R}^{12} \) via the map \( \mathbb{Q}_S \to \mathbb{R}^{12} \) which associates to each system (2) the 12–tuple \((a_{00}, \ldots, b_{02})\) of its coefficients.

For the definitions of a \( \text{GL} \)–comitant and invariant as well as for the definitions of a \( T \)–comitant and a \( \text{CT} \)–comitant we refer the reader to the paper [30] (see also [37]). Here we shall only construct the necessary \( T \)–comitants and \( \text{CT} \)–comitants associated to configurations of infinite singularities (including multiplicities) of quadratic systems (2).

Consider the polynomial \( \Phi_{\alpha, \beta} = \alpha P^* + \beta Q^* \in \mathbb{R}[\bar{a}, X, Y, Z, \alpha, \beta] \) where \( P^* = Z^2P(X/Z, Y/Z) \), \( Q^* = Z^2Q(X/Z, Y/Z) \), \( P, Q \in \mathbb{R}[\bar{a}, x, y] \) and \( \max(\deg_{(x,y)}P, \deg_{(x,y)}Q) = 2 \). Then

\[
\Phi_{\alpha, \beta} = s_{11}(\bar{a}, \alpha, \beta)X^2 + 2s_{12}(\bar{a}, \alpha, \beta)XY + s_{22}(\bar{a}, \alpha, \beta)Y^2 + 2s_{13}(\bar{a}, \alpha, \beta)XZ + 2s_{23}(\bar{a}, \alpha, \beta)YZ + + s_{33}(\bar{a}, \alpha, \beta)Z^2
\]

and we denote \( \bar{D}(\bar{a}, x, y) = 4 \det ||s_{ij}(\bar{a}, y, -x)||_{i,j \in \{1,2,3\}} \), \( \bar{H}(\bar{a}, x, y) = 4 \det ||s_{ij}(\bar{a}, y, -x)||_{i,j \in \{1,2\}} \).

We consider the polynomials

\[
\begin{align*}
C_i(a, x, y) &= yp_i(a, x, y) - xq_i(a, x, y) \in \mathbb{R}[a, x, y], & i &= 0, 1, 2, \\
D_i(a, x, y) &= \frac{\partial}{\partial x}p_i(a, x, y) + \frac{\partial}{\partial y}q_i(a, x, y) \in \mathbb{R}[a, x, y], & i &= 1, 2. 
\end{align*}
\]

Using the so–called *transvectant of order \( k \)* (see [19], [23]) of two polynomials \( f, g \in \mathbb{R}[a, x, y] \)

\[
(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}},
\]

we construct the following \( \text{GL} \)–comitants of the second degree with the coefficients of the initial system

\[
\begin{align*}
T_1 &= (C_0, C_1)^{(1)}, & T_2 &= (C_0, C_2)^{(1)}, & T_3 &= (C_0, D_2)^{(1)}, \\
T_4 &= (C_1, C_1)^{(2)}, & T_5 &= (C_1, C_2)^{(1)}, & T_6 &= (C_1, C_2)^{(2)}, \\
T_7 &= (C_1, D_2)^{(1)}, & T_8 &= (C_2, C_2)^{(2)}, & T_9 &= (C_2, D_2)^{(1)}.
\end{align*}
\]

Next using these \( \text{GL} \)–comitants as well as the polynomials (3) we shall construct the following needed
invariant polynomials (see also [30])

\[ \tilde{M}(\tilde{a}, x, y) = (C_2, C_2)^{(2)} \equiv 2 \text{Hess} \ (C_2(\tilde{a}, x, y)); \]
\[ \eta(\tilde{a}) = (\tilde{M}, \tilde{M})^{(2)}/384 \equiv \text{Discrim} \ (C_2(\tilde{a}, x, y)); \]
\[ \tilde{K}(\tilde{a}, x, y) = \text{Jacob} \ (p_2(x, y), q_2(x, y)); \]
\[ K_1(\tilde{a}, x, y) = p_1(x, y)q_2(x, y) - p_2(x, y)q_1(x, y); \]
\[ K_2(\tilde{a}, x, y) = 4(T_2, \tilde{M} - 2\tilde{K})^{(1)} + 3D_1(C_1, \tilde{M} - 2\tilde{K})^{(1)} - (\tilde{M} - 2\tilde{K})(16T_3 - 3T_4/2 + 3D_1^2); \]
\[ K_3(\tilde{a}, x, y) = C_2^2(4T_3 + 3T_4) + C_2(3C_0\tilde{K} - 2C_1T_7) + 2K_1(3K_1 - C_1D_2); \]
\[ \tilde{L}(\tilde{a}, x, y) = 4\tilde{K} + 8\tilde{H} - \tilde{M}; \]
\[ L_1(\tilde{a}, x, y) = (C_2, \tilde{D})^{(2)}; \]
\[ L_2(\tilde{a}, x, y) = (C_2, \tilde{D})^{(1)}; \]
\[ L_3(\tilde{a}, x, y) = C_1^2 - 4C_0C_2; \]
\[ \tilde{R}(\tilde{a}, x, y) = \tilde{L} + 8\tilde{K}; \]
\[ \kappa(\tilde{a}) = (\tilde{M}, \tilde{K})^{(2)}/4; \]
\[ \kappa_1(\tilde{a}) = (\tilde{M}, C_1)^{(2)}; \]
\[ \kappa_2(\tilde{a}) = (D_2, C_0)^{(1)}; \]
\[ \tilde{N}(a, x, y) = \tilde{K}(a, x, y) + \tilde{H}(a, x, y); \]
\[ \theta(\tilde{a}) = -(\tilde{N}, \tilde{N})^{(2)}/2 \equiv \text{Discrim} \ (N(\tilde{a}, x, y)); \]
\[ \theta_1(\tilde{a}) = 16\eta(\tilde{a}) + \kappa(\tilde{a}); \]
\[ \theta_2(\tilde{a}) = (C_1, \tilde{N})^{(2)}/16; \]
\[ \theta_3(\tilde{a}) = \left(2(\tilde{F}, \tilde{N})^{(2)} - ((\tilde{D}, \tilde{H})^{(2)}, D_2)^{(1)}\right)/32; \]
\[ \theta_4(\tilde{a}) = ((C_2, \tilde{E})^{(2)}, D_2)^{(1)}; \]
\[ \theta_5(\tilde{a}, x, y) = 2C_2(T_6, T_7)^{(1)} - (T_5 + 2D_2C_1)(C_1, D_2^2)^{(2)}; \]
\[ \theta_6(\tilde{a}, x, y) = C_1T_8 - 2C_2T_6, \]

where

\[ \tilde{E} = \left[ D_1(2T_9 - T_8) - 3(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2) \right]/72, \]
\[ \tilde{F} = \left[ 6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0(D_2, T_9)^{(1)} - 9D_2^2T_4 + 288D_1\tilde{E} \right. \]
\[ - 24 \left( C_2, \tilde{D} \right)^{(2)} + 120 \left( D_2, \tilde{D} \right)^{(1)} - 36C_1(D_2, T_7)^{(1)} + 8D_1(D_2, T_5)^{(1)} \right]/144. \]

The geometrical meaning of the invariant polynomials \( C_2, M \) and \( \eta \) is revealed by the next lemma (see [30]).

**Lemma 1.** The form of the divisor \( D_S(C, Z) \) for systems (2) is determined by the corresponding conditions indicated in Table 1, where we write \( w_1^2 + w_2^2 + w_3 \) if two of the points, i.e. \( w_1^2, w_2^2 \), are complex but not real. Moreover, for each form of the divisor \( D_S(C, Z) \) given in Table 1 the quadratic systems (2) can be brought via a linear transformation to one of the following canonical systems (\( S_I \)) – (\( S_V \)) corresponding to their behavior at infinity.
Table 1

<table>
<thead>
<tr>
<th>Case</th>
<th>Form of $D_S(C, Z)$</th>
<th>Necessary and sufficient conditions on the comitants</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$w_1 + w_2 + w_3$</td>
<td>$\eta &gt; 0$</td>
</tr>
<tr>
<td>2</td>
<td>$w_1^2 + w_2^2 + w_3$</td>
<td>$\eta &lt; 0$</td>
</tr>
<tr>
<td>3</td>
<td>$2w_1 + w_2$</td>
<td>$\eta = 0, M \neq 0$</td>
</tr>
<tr>
<td>4</td>
<td>$3w$</td>
<td>$M = 0, C_2 \neq 0$</td>
</tr>
<tr>
<td>5</td>
<td>$D_S(C, Z)$ undefined</td>
<td>$C_2 = 0$</td>
</tr>
</tbody>
</table>

We base our work here on results obtained in [30] and [33] where integer invariants and invariant polynomials were used to classify globally singularities in the neighborhood of infinity. We integrate here this information, using invariant polynomials and types of divisors on the line at infinity, in a unified Theorem where we replace Figure $j$ to Config. $j$ from $j = 1, \ldots, 46$. This Theorem is stated as follows:

Consider the differential operator $\mathcal{L} = x \cdot L_2 - y \cdot L_1$ acting on $\mathbb{R}[a, x, y]$ constructed in [11], where

$$
L_1 = 2a_{00} \frac{\partial}{\partial a_{10}} + a_{10} \frac{\partial}{\partial a_{20}} + \frac{1}{2} a_{01} \frac{\partial}{\partial a_{11}} + 2b_{00} \frac{\partial}{\partial b_{20}} + b_{10} \frac{\partial}{\partial b_{20}} + \frac{1}{2} b_{01} \frac{\partial}{\partial b_{11}},
$$

$$
L_2 = 2a_{00} \frac{\partial}{\partial a_{01}} + a_{01} \frac{\partial}{\partial a_{02}} + \frac{1}{2} a_{10} \frac{\partial}{\partial a_{11}} + 2b_{00} \frac{\partial}{\partial b_{02}} + b_{01} \frac{\partial}{\partial b_{02}} + \frac{1}{2} b_{10} \frac{\partial}{\partial b_{11}}.
$$

Using this operator and the affine invariant $\mu_0 = \text{Res} (p_2(x, y), q_2(x, y))$ we construct the following polynomials

$$
\mu_i(\tilde{a}, x, y) = \frac{1}{i!} \mathcal{L}^{(i)}(\mu_0), \quad i = 1, \ldots, 4, \quad \text{where} \quad \mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)).
$$

These polynomials are in fact comitants of systems (2) with respect to the group $GL(2, \mathbb{R})$ (see [11]). Their geometrical meaning is revealed in Lemmas 2 and 3 below.

**Lemma 2.** ([10]) The total multiplicity of all finite singularities of a quadratic system (2) equals $k$ if and only if for every $i \in \{0, 1, \ldots, k - 1\}$ we have $\mu_i(\tilde{a}, x, y) = 0$ in $\mathbb{R}[x, y]$ and $\mu_k(\tilde{a}, x, y) \neq 0$. Moreover a system (2) is degenerate (i.e. gcd$(P, Q) \neq \text{constant}$) if and only if $\mu_1(\tilde{a}, x, y) = 0$ in $\mathbb{R}[x, y]$ for every $i = 0, 1, 2, 3, 4$.

**Lemma 3.** ([11]) The point $M_0(0, 0)$ is a singular point of multiplicity $k$ ($1 \leq k \leq 4$) for a quadratic system (2) if and only if for every $i \in \{0, 1, \ldots, k - 1\}$ we have $\mu_{k-i}(\tilde{a}, x, y) = 0$ in $\mathbb{R}[x, y]$ and $\mu_{k-k}(\tilde{a}, x, y) \neq 0$. We base our work here on results obtained in [30] and [33] where integer invariants and invariant polynomials were used to classify globally singularities in the neighborhood of infinity. We integrate here this information, using invariant polynomials and types of divisors on the line at infinity, in a unified Theorem where we replace Figure $j$ to Config. $j$ from $j = 1, \ldots, 46$. This Theorem is stated as follows:
Theorem 1. Consider the family of planar quadratic differential systems. The bifurcation diagram of the phase portraits around infinity in the 12-dimensional parameter space of coefficients is given by using invariant polynomials in Diagram 5-7.

From the proof of Main Theorem (see next section) it follows

**Lemma 4.** There exist exactly 30 topologically distinct phase portraits around infinity for the family of degenerate quadratic systems, given in Figure 5. Moreover necessary and sufficient conditions for the realization of each one of these portraits are given in the Diagrams 1–4. These are the cases occurring for \( \mu_i = 0 \) for every \( i \in \{0, 1, 2, 3, 4\} \).

**6 The proof of the Main Theorem**

As we have to examine the infinite singularities we shall consider step by step each one of the five canonical systems \((S_T) - (S_V)\) (see Lemma 1) which are associated to infinite singularities.

**6.1 The family of systems \((S_T)\)**

For these systems we have \( C_2 = yp_2(x, y) - xq_2(x, y) = xy(x - y) \) and \( \eta > 0 \). Therefore at the infinity we have three real distinct singularities: \( R_1(1, 0, 0) \), \( R_2(0, 1, 0) \) and \( R_3(1, 1, 0) \). Constructing the respective
systems at infinity (possessing the point \(R_i\) \((i = 1, 2, 3)\) at the origin of coordinates) we get respectively:

\[
R_1(1, 0, 0) : \begin{cases}
\dot{u} = u - ez - u^2 + (c - f)uz - bz^2 + du^2z + auz^2, \\
\dot{z} = gz + (h - 1)uz + cz^2 + duz^2 + az^3;
\end{cases}
\]

\[
R_2(0, 1, 0) : \begin{cases}
\dot{u} = v - dz - v^2 + (-c + f)uz - az^2 + evz^2 + bvz^2, \\
\dot{z} = hz + (g - 1)uz + f z^2 + evz^2 + bz^3;
\end{cases}
\]

\[
R_3(1, 1, 0) : \begin{cases}
\dot{u} = u - (c + d - e - f)z + u^2 - (c + 2d - f)uz - (a - b)z^2 - du^2z - az^2, \\
\dot{z} = (1 - g - h)z - (h - 1)uz - (c + d)z^2 - duz^2 - az^3.
\end{cases}
\]

So the respective matrix for these singularities are as follows:

\[
R_1 \Rightarrow \begin{pmatrix} 1 & -e \\ 0 & g \end{pmatrix}; \quad R_2 \Rightarrow \begin{pmatrix} 1 & -d \\ 0 & h \end{pmatrix}; \quad R_3 \Rightarrow \begin{pmatrix} 1 & -c - d + e + f \\ 1 & 1 - g - h \end{pmatrix}.
\]  

Remark 1. The eigenvalues of \(R_1\) (respectively \(R_2; R_3\)) are \(\lambda_1 = 1\) and \(\lambda_2 = g\) (respectively \(\lambda_2 = h; \lambda_2 = 1 - g - h\)). We also denote \(\xi = -e\) (respectively \(\xi = -d; \xi = -c - d + e + f\)) for \(R_1\) (respectively \(R_2; R_3\)). The eigenvalue \(\lambda_1\) is associated to the eigenvector tangent to the line at infinity whereas \(\lambda_2\) is associated to the eigenvector directed towards the affine plane. Thus according to the notation given in
Diagram 6 (cont.): Topological configurations for the case \( \eta = 0, M \neq 0 \).

Section 3 the point \( R_i \) for \( i = 1, 2, 3 \) is a node if \( \lambda_2 > 0 \) and more exactly: when \( \lambda_2 > 1 \) the singular point \( R_i \) is \( N^\infty \) and if \( \lambda_2 < 1 \) it is \( N^f \). Moreover, when \( \lambda_2 = 1 \) the singular point \( R_i \) is a star node (i.e. \( N^* \)) if \( \xi = 0 \) and it is a one direction node (i.e. \( N^d \)) if \( \xi \neq 0 \).

Following Theorem 1 (see the diagram 1) we calculate for systems (Sf) the value of the respective invariant polynomials:

\[
\mu_0 = gh(g+h-1), \quad \kappa = 16(g+h-g^2-gh-h^2).
\]

(6)

6.1.1 The case \( \mu_0 < 0 \)

According to Theorem 1 all three infinite singularities are elemental. Moreover by \([30]\) we have a node and two saddles if \( \kappa < 0 \) and three nodes if \( \kappa > 0 \). We claim that in the first case we have a node \( N^\infty \), whereas in the second case all three nodes are of the type \( N^f \).

Indeed, assume first \( \kappa < 0 \), i.e. we have a node and two saddles. These means that two of the quantities among the three ones \( g, h \) and \( 1 - g - h \) are negative and one positive. Without lost of generality we may assume \( g > 0 \) (i.e. \( R_1 \) is a node), \( h < 0 \) and \( 1 - g - h < 0 \). Then \( g > 1 - h > 1 \) and according to Remark 1 \( R_1 \) is a node \( N^\infty \).

Suppose now \( \kappa > 0 \), i.e. we have three nodes. Therefore according to (4) the relations \( g > 0, h > 0 \) and \( g + h < 1 \) must hold. Hence \( g < 1, h < 1 \) and by Remark 1 all three nodes are \( N^f \). Thus our claim is proved.

6.1.2 The case \( \mu_0 > 0 \)

By Theorem 1 and \([30]\) systems (Sf) possess at infinity one saddle and two nodes. According to Remark 1 the types of the nodes depend on the three quantities \( \lambda_2 - 1 \) with \( \lambda_2 \in \{g, h, 1 - g - h\} \). Moreover, if one such quantity vanishes (for example, \( h - 1 = 0 \)) then in order to distinguish between a star node and a one direction node we need to distinguish if either the respective quantity \( \xi \) (which in this case is \( -d \)) vanishes or not. So for the convenience we set for the singular points \( R_i \) (\( i = 1, 2, 3 \)) the following additional notations:

\[
\tau_1 = g - 1, \quad \tau_2 = h - 1, \quad \tau_3 = (1 - g - h) - 1 = -(g + h);
\]
\[
\xi_1 = -e, \quad \xi_2 = -d, \quad \xi_3 = -c - d + e + f.
\]

(7)
Diagram 7: Topological configurations for the case $M = 0$.

Then for for systems $(S_1)$ we calculate

$$\theta = 8\tau_1\tau_2\tau_3,$$
$$\theta_1 = 16(\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3),$$
$$40\theta_2 = (\tau_1\tau_2\xi_3 + \tau_1\tau_3\xi_2 - \tau_2\tau_3\xi_1),$$
$$\theta_3|_{\{\tau_1=\tau_2=0\}} = -2\xi_1\xi_2, \quad \theta_4|_{\{\tau_1=\tau_2=0\}} = \xi_1 + \xi_2,$$
$$\theta_3|_{\{\tau_1=\tau_3=0\}} = -2\xi_1\xi_3, \quad \theta_4|_{\{\tau_1=\tau_3=0\}} = -(\xi_1 + \xi_3),$$
$$\theta_3|_{\{\tau_2=\tau_3=0\}} = 2\xi_2\xi_3, \quad \theta_4|_{\{\tau_2=\tau_3=0\}} = -(\xi_2 - \xi_3),$$

In order to distinguish the signs of the quantities $\tau_1$, $\tau_2$ and $\tau_3$, using the Vietta’s theorem we construct the equation of degree three possessing these quantities as the roots:

$$z^3 - (\tau_1 + \tau_2 + \tau_3)z^2 + (\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3)z - \tau_1\tau_2\tau_3 = 0.$$  

Considering (7) the above equation is equivalent to

$$F(z) = z^3 + 2z^2 + \frac{\theta_1}{16}z - \frac{\theta}{8} = 0.$$  

We note that the existence of one saddle among the singular points $R_i \ (i = 1, 2, 3)$ implies that one of the roots of the equation (9) is negative.
Figure 4: Topologically distinct local configurations of ISPs ([30],[33])
6.1.2.1 The subcase \( \theta < 0 \). Then the remaining two roots are both of the same sign. Moreover, considering the roots of the function \( F'(z) = 3z^2 + 4z + \theta_1/16 \) we conclude, that besides the negative root of (9) we have two negative roots if \( \theta_1 > 0 \) and two positive ones if \( \theta_1 < 0 \). We note that the conditions \( \mu_0 > 0, \theta < 0 \) and \( \theta_1 = 0 \) are incompatible as it can be seen easily using the respective graphic. 

Thus beside the saddle we have at infinity two nodes \( N^\infty, N^\infty \) if \( \theta_1 < 0 \) and \( N^f, N^f \) if \( \theta_1 > 0 \).

6.1.2.2 The subcase \( \theta > 0 \). Then the remaining two roots are of the opposite signs and hence, beside the saddle we have at infinity the nodes \( N^f, N^\infty \).

6.1.2.3 The subcase \( \theta = 0 \). In this case one of the roots of (9) vanishes and hence at infinity we have a node with two coinciding eigenvalues.

6.1.2.3.1 Assume first \( \theta_1 \neq 0 \), i.e. other two roots do not vanish. More exactly, as one of the nonzero roots is negative, the second one is positive if \( \theta_1 < 0 \) and it is negative if \( \theta_1 > 0 \).

It remains to distinguish whether the node with two coinciding eigenvalues is a \( N^d \) or it is a \( N^* \). We may assume that such a node is \( R_1 \) (otherwise we can apply a linear transformation). Therefore the
condition $g = 1$ (i.e. $\tau_1 = 0$) holds and considering (8) we obtain $\theta_1 = 16\tau_2\tau_3 \neq 0$ and $\theta_2 = -\tau_2\tau_3\xi_1/4$. So due to $\theta_1 \neq 0$ the condition $\xi_1 = 0$ is equivalent to $\theta_2 = 0$.

Thus in the case $\theta = 0$ and $\theta_1 \neq 0$ we arrive at the following configurations, respectively:

$$
\begin{align*}
\theta_1 < 0, \theta_2 \neq 0 & \Rightarrow S, N^\infty, N^d; \\
\theta_1 < 0, \theta_2 = 0 & \Rightarrow S, N^\infty, N^*; \\
\theta_1 > 0, \theta_2 \neq 0 & \Rightarrow S, N^J, N^d; \\
\theta_1 > 0, \theta_2 = 0 & \Rightarrow S, N^J, N^*.
\end{align*}
$$

6.1.3 The case $\mu_0 = 0$, $\mu_1 \neq 0$.

In this case exactly one finite point has gone to infinity. Considering (6) we have $gh(g + h - 1) = 0$ and we may assume $g = 0$ due to a linear transformation (which replaces the respective lines defined by the factors of $C_2 = xy(x - y)$). So the singular point $R_1$ becomes a semi-elemental saddle-node and for systems $(S_I)$ we calculate

$$
\mu_0 = 0, \quad \mu_1 = (1 - h)h(e - e + eh)y \neq 0, \quad \kappa = 16h(1 - h).
$$

(10)

Remark 2. If $\kappa \neq 0$ (i.e. $h(h - 1) \neq 0$) then considering (5) and $g = 0$ we conclude that $R_2$ and $R_3$ are elemental infinite singularities. Moreover, we have a saddle and a node $N^\infty$ if $\kappa < 0$ and there are two nodes $N^J$, $N^J$ if $\kappa > 0$.

The condition $\mu_1 \neq 0$ implies $\kappa \neq 0$ and by the above remark besides the saddle-node at infinity we have the singular points $S, N^\infty$ if $\kappa < 0$ and $N^J, N^J$ if $\kappa > 0$.

6.1.4 The case $\mu_0 = \mu_1 = 0$.

We shall consider two geometrically distinct situations (see Theorem 1): $\kappa \neq 0$ (when systems $(S_I)$ have at infinity only one multiple singularity) and $\kappa = 0$ (when at infinity there are two multiple singularities).

6.1.4.1 The subcase $\kappa \neq 0$. Then non-degenerate systems $(S_I)$ (i.e. systems with $\sum_{i=0}^4 \mu_i^2 \neq 0$) possess at infinity only one multiple singularity (in this case the point $R_1$). Moreover its multiplicity is three (respectively four; five) if $\mu_2 \neq 0$ (respectively $\mu_2 = 0, \mu_3 \neq 0; \mu_2 = \mu_3 = 0, \mu_4 \neq 0$). It is clear that this point is a semi-elemental saddle-node in the case of even multiplicity and it is either a saddle or a node if its multiplicity is an odd number. Considering Theorem 1, Remark 2 and [30] for non-degenerate systems $(S_I)$ in the case $\mu_0 = \mu_1 = 0$ and $\kappa \neq 0$ we obtain the following configurations of singularities:

$$
\begin{align*}
\mu_2 < 0, \kappa < 0 & \Rightarrow (7)S, S, N^\infty; \\
\mu_2 < 0, \kappa > 0 & \Rightarrow (7)N, N^J, N^J; \\
\mu_2 > 0, \kappa < 0 & \Rightarrow (7)N, S, N^\infty; \\
\mu_2 > 0, \kappa > 0 & \Rightarrow (7)S, N^J, N^J; \\
\mu_2 = 0, \mu_3 \neq 0, \kappa < 0 & \Rightarrow (9)SN, S, N^\infty; \\
\mu_2 = 0, \mu_3 \neq 0, \kappa > 0 & \Rightarrow (9)SN, N^J, N^J; \\
\mu_2 = \mu_3 = 0, \mu_4 \neq 0, \kappa < 0 & \Rightarrow (9)N, S, N^\infty; \\
\mu_2 = \mu_3 = 0, \mu_4 \neq 0, \kappa > 0 & \Rightarrow (9)S, N^J, N^J.
\end{align*}
$$

It remains to examine the case of degenerate systems $(S_I)$, i.e. when the conditions $\mu_i = 0$ for each $i = 0, 1, \ldots, 4$ hold. We shall construct the canonical form of such systems in the case $\kappa \neq 0$. By (10) the
condition $\mu_1 = 0$ implies $c - e + eh = 0$. Moreover, as $g = 0$ via a translation we may assume $e = f = 0$. Then $c = 0$ and systems (S_I) become
\[ \dot{x} = a + dy + (h - 1)xy, \quad \dot{y} = b - xy + hy^2, \]
and for these systems we calculate
\[ \mu_0 = \mu_1 = 0, \quad \mu_2 = h(h - 1)(a - b + bh)y^2, \quad \kappa = 16h(1 - h). \]
So since $\kappa \neq 0$ the condition $\mu_2 = 0$ gives $a = b(1 - h)$ and then we calculate
\[ \mu_3 = bd(1 - h)hy^3, \quad \mu_4 = -by^3(d^2x - d^2hy - bh^2y + 2bh^3y - bh^4y). \]
Clearly the condition $\mu_3 = \mu_4 = 0$ is equivalent to $b = 0$. Therefore we arrive at the degenerate systems
\[ \dot{x} = y(d - x + hx), \quad \dot{y} = -y(x - hy), \tag{11} \]
possessing the invariant singular line $y = 0$ and the respective linear systems have the matrix \[ \begin{pmatrix} h - 1 & 0 \\ -1 & h \end{pmatrix}. \]

As $\kappa \neq 0$ then considering the notation of singularities (see Section 3) we obtain the following configurations of infinite singularities of quadratic systems (11): $N^\infty$, $S$, $(\Theta [[]; 0])$ if $\kappa < 0$ and $N^f$, $N^f$, $(\Theta [[]; 0)$ if $\kappa > 0$.

On the other hand we observe that the behavior of the trajectories at infinity in this case is topologically equivalent to the portraits $[QD_{1}^\infty]$ if $\kappa < 0$ and $[QD_{2}^\infty]$ if $\kappa > 0$ (see Figure 5).

6.1.4.2 The subcase $\kappa = 0$. Then $h(h - 1) = 0$ and without loss of generality we may consider $h = 0$ in the systems (S_I) with $g = 0$ (due to a linear transformation which keeps the line $y = 0$ and replaces the lines $y = x$ with $x = 0$). Moreover since $g = h = 0$ we may assume $d = e = 0$ and systems (S_I) become
\[ \dot{x} = a + cx - xy, \quad \dot{y} = b + fy - xy, \tag{12} \]
which possess at infinity two semi-elemental singular points $R_1(1, 0, 0)$ and $R_2(0, 1, 0)$ and the elemental singular point $R_3(1, 1, 0)$. For the last point we have the respective linear matrix (see (5)) \[ \begin{pmatrix} 1 & -c + f \\ 0 & 1 \end{pmatrix}. \]
Therefore $R_3(1, 1, 0)$ is a node of the type either $N^d$ if $f - c \neq 0$, or $N^*\kappa$ if $f - c = 0$.

On the other hand for systems (12) we calculate:
\[ \mu_0 = \mu_1 = 0, \quad \mu_2 = cfxy, \quad \tilde{L} = 8xy, \quad \theta_2 = (f - c)/4, \quad K_1 = -xy(cx - fy), \]
and therefore we arrive at the next result.

Remark 3. The elemental singular point $R_3(1, 1, 0)$ is a node of the type $N^d$ if $\theta_2 \neq 0$ and $N^*\kappa$ if $\theta_2 = 0$.

6.1.4.2.1 Assume $\mu_2 \neq 0$ Then by Theorem 1 the singularities $R_1(1, 0, 0)$ and $R_2(0, 1, 0)$ are both of multiplicity 2 and hence they are semi-elemental saddle-nodes. Considering [30] we conclude that in the case $\mu_2 \neq 0$ we have the following configurations of infinite singularities:
\[ \mu_2 \tilde{L} < 0 \Rightarrow (\frac{1}{1})SN, (\frac{1}{1})SN, N^d; \]
\[ \mu_2 \tilde{L} > 0, \theta_2 \neq 0 \Rightarrow (\frac{1}{1})SN, (\frac{1}{1})NS, N^d; \]
\[ \mu_2 \tilde{L} > 0, \theta_2 = 0 \Rightarrow (\frac{1}{1})SN, (\frac{1}{1})NS, N^*. \]
We notice that the condition $\mu_2 \tilde{L} < 0$ (i.e. $cf < 0$) implies $\theta_2 \neq 0$. 28
6.1.4.2.2 Admit now $\mu_2 = 0$. Then $cf = 0$ and we may assume $f = 0$ since the change $(x, y, a, b, c, f) \mapsto (y, x, b, a, f, c)$ keeps systems (12). Then the semi-elemental singular point $R_2(0,1,0)$ becomes of the multiplicity $\geq 3$. Moreover, according to Theorem 1 the multiplicities of the semi-elemental singularities are governed in the case $\kappa = 0$ by the invariant polynomials $\mu_3$, $\mu_4$ and $K_1$.

1) Assume first $K_1 \neq 0$. For systems (12) in the case $f = 0$ we have

$$
\mu_3 = (b-a)cx^2y, \quad \mu_4 = -bc^2x^3y + (a-b)^2x^2y^2, \quad K_1 = -cx^2y, \quad \theta_2 = -c/4.
$$

Therefore the condition $K_1 \neq 0$ implies $\theta_2 \neq 0$.

a) If $\mu_3 \neq 0$ by Theorem 1, Remark 3 and [30] we get the configuration $\bar{(7)}N, \bar{(1)}SN, N^d$ if $\mu_3K_1 < 0$ and $\bar{(1)}S, \bar{(1)}SN, N^d$ if $\mu_3K_1 > 0$.

b) Assume $\mu_3 = 0$. Since $K_1 \neq 0$ (i.e. $c \neq 0$) we obtain $b = a$ and then $\mu_4 = -ac^2x^3y$ and $\bar{L} = xy$.

So if $\mu_4 \neq 0$ (i.e. systems (12) are non-degenerate) at infinity we have an elemental singularity (which is $N^d$ by Remark 3), and two semi-elemental saddle-nodes: $R_1(1,0,0)$ of multiplicity two and $R_2(0,1,0)$ of multiplicity four. Considering [30] we obtain the configuration $\bar{(7)}SN, \bar{(1)}SN, N^d$ if $\mu_4\bar{L} < 0$ and $\bar{(1)}SN, \bar{(1)}NS, N^d$ if $\mu_4\bar{L} > 0$.

Assuming $\mu_4 = 0$ (i.e. $a = 0$) since $c \neq 0$ we may take $c = 1$ due to a rescaling and hence, we get the degenerate system $\dot{x} = x(1-y), \quad \dot{y} = -xy$, possessing the invariant singular line $x = 0$. Clearly the singular point $R_2(0,1,0)$ at infinity becomes a non-isolated singularity for above system. So applying our notations (see Section 3) in the case of degenerate system and $\kappa = 0$ and $K_1 \neq 0$ we get the configuration $\bar{(1)}SN, N^d, (\bar{5}|]|0)$; $\emptyset$. On the other hand we observe that the behavior of the trajectories at infinity in this case is topologically equivalent to the portrait $[QD_3^\infty]$ (see Figure 5).

2) Suppose now $K_1 = 0$. Then for systems (12) with $f = 0$ considering (13) we obtain $c = 0$ and then

$$
\mu_3 = 0, \quad \mu_4 = (a-b)^2x^2y^2, \quad \theta_2 = 0.
$$

If $\mu_4 \neq 0$ by Theorem 1 we have at infinity an elemental singularity (which is a star node by Remark 3) and two semi-elemental singular points both of multiplicity three. Considering [30] one of them is a node and another one is a saddle. Thus we get $\bar{(1)}S, \bar{(1)}N, N^*$.

Assume that $\mu_4 = 0$. Then for systems (12) the condition $\mu_2 = \mu_3 = \mu_4 = K_1 = 0$ give us $f = c = 0$ and $b = a$, i.e. we get the degenerate systems $\dot{x} = a-xy, \quad \dot{y} = a-xy$. These systems possess a singular invariant hyperbola $xy - a = 0$, which splits in two lines if $a = 0$. As for the above systems we have $L_1 = 12a(x-y)^2$, considering the notations in Section 3 we obtain $N^*, (\emptyset |]|0)$ if $L_1 \neq 0$ and $N^*$, $\emptyset$ if $L_1 = 0$. On the other hand we observe that the behavior of the trajectories at infinity in both the cases is topologically equivalent to the portrait $[QD_3^\infty]$ (see Figure 5).

As all the cases are examined, Main Theorem is proved for the family of systems $(S_I)$.

6.2 The family of systems $(S_{II})$

For these systems we have $C_2 = y p_2(x,y) - x q_2(x,y) = x(x^2 + y^2)$. Therefore clearly at the infinity we have one real singular point $R_2(0,1,0)$ and two imaginary $R_{1,3}(1,\pm i,0)$. Constructing the respective systems at infinity (possessing the real point at the origin of coordinates) we get the family of systems:

$$
R_2(0,1,0) : \begin{cases}
\dot{v} = -v - dz + (-c + f) vz - az^2 - v^3 + ev^2z + bwz^2, \\
\dot{z} = hz + gvz + f z^2 - v^2z + evz^2 + b v z^3
\end{cases}
$$

with the respective linear matrix

$$
\begin{pmatrix}
-1 & -d \\
0 & h
\end{pmatrix}
$$

Considering Remark 1 we arrive at the next result.

Remark 4. If $R_2(0,1,0)$ is an elemental singular point (i.e. $h \neq 0$) then it is a saddle if $h > 0$; a node $N_f$ if $-1 < h < 0$; a node $N_d$ if $h = -1$ and $d \neq 0$; a node $N^*$ if $h = -1$ and $d = 0$; and it is a node $N^\infty$ if $h < -1$.  

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On the other hand for systems \((S_H)\) we have:
\[
\begin{align*}
\mu_0 &= -h[(h + 1)^2 + g^2], \\
\kappa &= -16[g^2 + (h + 1)(1 - 3h)], \\
\theta &= 8(h + 1)[(h - 1)^2 + g^2], \\
\theta_2 \big|_{\mu = 1} &= d(4 + g^2)/4,
\end{align*}
\]  
(14)

6.2.1 The case \(\mu_0 \neq 0\)

In this case we obtain \(\text{sign}(\mu_0) = -\text{sign}(h)\) and by Theorem 1 and [30] and taking into account the above remark we get the following configurations of infinite singularities:

\[
\begin{align*}
\mu_0 < 0 & \quad \Rightarrow S, C, C; \\
\mu_0 > 0, \theta < 0 & \quad \Rightarrow N^\infty, C, C; \\
\mu_0 > 0, \theta > 0 & \quad \Rightarrow N^f, C, C; \\
\mu_0 > 0, \theta = 0, \theta_2 \neq 0 & \quad \Rightarrow N^d, C, C; \\
\mu_0 > 0, \theta = 0, \theta_2 = 0 & \quad \Rightarrow N^*, C, C.
\end{align*}
\]

We notice that in the case \(\mu_0 > 0\) we get \(h < 0\) and then by (14) the condition \(\theta = 0\) is equivalent to \(h = -1\).

6.2.2 The case \(\mu_0 = 0, \mu_1 \neq 0\)

According to Lemma (2) in this case only one finite point has gone to infinity and evidently it must be a real one. So \(R_2\) becomes a semi-elemental double singular point and clearly we get the configuration \(\overline{1)}SN, C, C\).

6.2.3 The case \(\mu_0 = \mu_1 = 0\).

Considering Theorem 1 we shall distinguish again two geometrically different situations: when only the real infinite singular point increases its multiplicity (then \(\kappa \neq 0\)) and when the complex points become multiple, too (then \(\kappa = 0\)).

6.2.3.1 The subcase \(\kappa \neq 0\). Then for non-degenerate systems \((S_H)\) the conditions \(\mu_0 = 0\) and \(\kappa \neq 0\) imply \(h = 0\). In this case we may assume \(c = d = 0\) (doing a translation) and then the condition \(\mu_1 = -f(1 + g^2)x = 0\) gives \(f = 0\). Therefore we get the systems
\[
\begin{align*}
\dot{x} &= a + gx^2 + xy, \\
\dot{y} &= b + ey - x^2 + gxy
\end{align*}
\]  
(15)

for which calculations yield
\[
\begin{align*}
\mu_0 &= \mu_1 = 0, \\
\mu_2 &= (ag - b)(1 + g^2)x^2, \\
\kappa &= -16(1 + g^2).
\end{align*}
\]  
(16)

By Theorem 1 only the real infinite point is a multiple singularity for these systems (in this case the point \(R_2\)). Moreover its multiplicity is three (respectively four; five) if \(\mu_2 \neq 0\) (respectively \(\mu_2 = 0, \mu_3 \neq 0; \mu_2 = \mu_3 = 0, \mu_4 \neq 0\)). Clearly, this point is a semi-elemental saddle-node in the case of even multiplicity and it is either a saddle or a node if its multiplicity is an odd number. Considering Theorem 1, Remark 2 and [30] for non-degenerate systems \((S_H)\) in the case \(\mu_0 = \mu_1 = 0\) and \(\kappa \neq 0\) we obtain the following configurations of infinite singularities:

\[
\begin{align*}
\mu_2 < 0 & \quad \Rightarrow \overline{1)}S, C, C; \\
\mu_2 > 0 & \quad \Rightarrow \overline{1)}N, C, C; \\
\mu_2 = 0, \mu_3 \neq 0 & \quad \Rightarrow \overline{1)}SN, C, C; \\
\mu_2 = \mu_3 = 0, \mu_4 \neq 0 & \quad \Rightarrow \overline{1)}N, C, C.
\end{align*}
\]

Consider now the case of degenerate systems \((S_H)\) when \(\kappa \neq 0\). Therefore we have to impose the conditions \(\mu_2 = \mu_3 = \mu_4 = 0\) for systems (15). By (16) the condition \(\mu_2 = 0\) yields \(b = ag\) and then we obtain
\[
\begin{align*}
\mu_3 &= ae(1 + g^2)x^3, \\
\mu_4 &= a[a(1 + g^2)^2 + e^2g]x^4 + ae^2x^3y.
\end{align*}
\]
Evidently the condition \( \mu_3 = \mu_4 = 0 \) is equivalent to \( a = 0 \) and we arrive at the systems

\[
\dot{x} = (gx + y), \quad \dot{y} = x(e - x + gy)
\]

possessing the invariant singular line \( x = 0 \). The respective linear systems have the complex infinite points and hence, according to our notations (see Section 3) for degenerate systems (15) we get the configuration \( \mathbb{C} \odot, (\mathbb{C} \odot) \). In this case the behavior of the trajectories at infinity is topologically equivalent to the portrait \( QD_5^{\infty} \) (see Figure 5).

### 6.2.3.2 The subcase \( \kappa = 0 \)

As \( \mu_0 = 0 \) then considering (14) we have \( h + 1 = g = 0 \) and then we may assume \( e = f = 0 \) doing a translation. So we get the family of systems

\[
\dot{x} = a + cx + dy, \quad \dot{y} = b - x^2 - y^2
\]

for which we calculate

\[
\kappa = \theta = \mu_0 = \mu_1 = 0, \quad \mu_2 = (c^2 + d^2)(x^2 + y^2), \quad \theta_2 = d.
\]

If \( \mu_2 \neq 0 \) then by Theorem 1 the complex singularities \( R_{1,3}(1, \pm i, 0) \) are both of multiplicity 2. As the condition \( d = 0 \) is equivalent to \( \theta_2 = 0 \), considering Remark 4 we obtain the configuration \( N^d, (\mathbb{C}) \odot, (\mathbb{C}) \odot \) if \( \theta_2 \neq 0 \) and \( N^\ast, (\mathbb{C}) \odot, (\mathbb{C}) \odot \) if \( \theta_2 = 0 \).

Assuming \( \mu_2 = 0 \) we have \( c = d = 0 \) and then for systems (17) we obtain

\[
\mu_2 = \mu_3 = \theta = \theta_2 = 0, \quad \mu_4 = a^2(x^2 + y^2)^2.
\]

For non-degenerate systems we have \( \mu_4 \neq 0 \) (i.e. \( a \neq 0 \)) and by Theorem 1 and Remark 4 we obtain the configuration \( N^\ast, (\mathbb{C}) \odot, (\mathbb{C}) \odot \).

It remains to examine the case \( \mu_4 = 0 \), i.e. when \( \kappa = 0 \) and systems \((\mathbf{S}_H)\) are degenerate. So setting \( \mu_4 = 0 \) (i.e. \( a = 0 \)) in systems (17) with \( c = d = 0 \) we get the systems \( \dot{x} = 0, \quad \dot{y} = b - x^2 - y^2 \) possessing the singular invariant conic \( x^2 + y^2 = b \). For these systems we have \( L_1 = -48bx^2 \), i.e. \( \text{sign}(b) = -\text{sign}(L_1) \) if \( b \neq 0 \). Therefore considering the notations in Section 3 and Remark 4 we obtain the following configurations:

\[
\begin{align*}
L_1 < 0 & \Rightarrow N^\ast, (\mathbb{C} [\theta]; \emptyset, \emptyset); \\
L_1 > 0 & \Rightarrow N^\ast, (\mathbb{C} \odot; \emptyset, \emptyset); \\
L_1 = 0 & \Rightarrow N^\ast, (\mathbb{C} ; \emptyset, \emptyset).
\end{align*}
\]

On the other hand we observe that the behavior of the trajectories at infinity in all three cases is topologically equivalent to the portrait \( QD_5^{\infty} \) (see Figure 5). This complete the proof of Main Theorem in the case of the family of systems \((\mathbf{S}_H)\).

### 6.3 The family of systems \((\mathbf{S}_{III})\)

For these systems we have \( \eta = 0, \ M \neq 0 \) and according to Lemma 1 at infinity we have two distinct real singularities. As \( C_2 = yq_2(x, y) - xq_2(x, y) = x^2y \) these singularities are \( R_1(1, 0, 0) \) and \( R_2(0, 1, 0) \). We note that by Theorem 1 the divisor encoding the multiplicity of infinite points has the form \((i)u + (j)v\) with \( i + j \in \{0, 1, \ldots, 4\} \). Constructing the respective systems at infinity (possessing the point \( R_i \) \( (i = 1, 2) \) at the origin of coordinates) we get respectively:

\[
\begin{align*}
R_1(1, 0, 0) : & \quad \begin{cases}
\dot{u} = u - ez + (c - f)uz - bz^2 + du^2z + au^2z, \\
\dot{z} = gz + huz + cz^2 + du^2z + az^3;
\end{cases} \\
R_2(0, 1, 0) : & \quad \begin{cases}
\dot{v} = -dz - v^2 - (c - f)vz - az^2 + ev^2z + bvz^2, \\
\dot{z} = hz + (g - 1)vz + f^2z + ev^2z + bz^3.
\end{cases}
\end{align*}
\]

So the respective matrix for these singularities are as follows:

\[
R_1 \Rightarrow \begin{pmatrix} 1 & -e \\ 0 & g \end{pmatrix}; \quad R_2 \Rightarrow \begin{pmatrix} 0 & -d \\ 0 & h \end{pmatrix}
\]

and therefore \( R_1 \) is an elemental singular point if \( g \neq 0 \) and \( R_2 \) is a semi-elemental singularity if \( h \neq 0 \).
Remark 5. If \( R_1(1,0,0) \) is an elemental singular point (i.e. \( g \neq 0 \)) then it is a saddle if \( g < 0 \); a node \( N_f \) if \( 0 < g < 1 \); a node \( N_0 \) if \( g = 1 \) and \( e \neq 0 \); a node \( N^* \) if \( g = 1 \) and \( e = 0 \); and it is a node \( N^\infty \) if \( g > 1 \).

On the other hand for systems \((S_{III})\) calculations yield

\[
\begin{align*}
\mu_0 &= gh^2, \\
\theta &= -8h^2(g-1), \quad \theta_2|_{(g=1)} = -eh^2/4,
\end{align*}
\]

(20)

6.3.1 The case \( \mu_0 \neq 0 \)

In this case we obtain \( \text{sign}(\mu_0) = \text{sign}(g) \) and \( \text{sign}(\theta) = -\text{sign}(g-1) \). Therefore by Theorem 1 and [30] and taking into account the above remark we get the following configurations of infinite singularities:

- \( \mu_0 < 0 \) \( \Rightarrow \frac{1}{2}\)SN, S;
- \( \mu_0 > 0, \theta < 0 \) \( \Rightarrow \frac{1}{2}\)SN, \( N^\infty \);
- \( \mu_0 > 0, \theta > 0 \) \( \Rightarrow \frac{1}{2}\)SN, \( N_f \);
- \( \mu_0 > 0, \theta = 0, \theta_2 \neq 0 \) \( \Rightarrow \frac{1}{2}\)SN, \( N^d \);
- \( \mu_0 > 0, \theta = 0, \theta_2 = 0 \) \( \Rightarrow \frac{1}{2}\)SN, \( N^* \).

6.3.2 The case \( \mu_0 = 0 \)

According to Theorem 1 at least one finite singular point has gone to infinity. Moreover this point has coalesced either with \( R_1(1,0,0) \) or with \( R_2(0,1,0) \) and these two possibility are governed by invariant polynomial \( \kappa \).

6.3.2.1 The subcase \( \kappa \neq 0 \). Then for systems \((S_{III})\) the conditions \( \mu_0 = 0 \) and \( \kappa \neq 0 \) imply \( g = 0 \), i.e. the finite singular point has coalesced with \( R_1(1,0,0) \). We note that by Theorem 1 if \( \kappa \neq 0 \) then all the finite singularities which have gone to infinity, has coalesced only with the point which multiplicity is \( (\frac{1}{2}) \), i.e. in this case with \( R_1 \). Moreover this point remains a semi-elemental singularity which multiplicity \( i+1 \) depends of the number of the vanishing invariant polynomials \( \mu_j \) \( (j \in \{0,1,\ldots,3\}) \) (see Lemma 2). It is clear that this point is a semi-elemental saddle-node in the case of even multiplicity and it is either a saddle or a node if its multiplicity is an odd number.

Therefore considering Theorem 1 and [30], in the case \( \mu_0 = 0 \) and \( \kappa \neq 0 \) for non-degenerate systems \((S_{III})\) we obtain the following configurations of infinite singularities:

- \( \mu_1 \neq 0 \) \( \Rightarrow \frac{1}{2}\)SN, \( (\frac{1}{4})\)SN;
- \( \mu_1 = 0, \mu_2 < 0 \) \( \Rightarrow \frac{1}{2}\)SN, \( (\frac{1}{2})\)S;
- \( \mu_1 = 0, \mu_2 > 0 \) \( \Rightarrow \frac{1}{2}\)SN, \( (\frac{1}{2})\)N;
- \( \mu_1 = \mu_2 = 0, \mu_3 \neq 0 \) \( \Rightarrow \frac{1}{2}\)SN, \( (\frac{1}{4})\)SN;
- \( \mu_1 = \mu_2 = \mu_3 = 0, \mu_4 \neq 0 \) \( \Rightarrow \frac{1}{2}\)SN, \( (\frac{1}{4})\)N.

In order to end the case \( \kappa \neq 0 \) we consider the degenerate systems \((S_{III})\), i.e. by Lemma 2 the conditions \( \mu_i = 0 \) must hold for each \( i = 0,1,\ldots,4 \).

As \( \kappa \neq 0 \) we have \( h \neq 0 \) and we may assume \( h = 1 \) and \( c = d = 0 \) due to the affine transformation \( x_1 = x + d/h, \ y_1 = hy + (ch - 2dg)/h \). It was mentioned above that the conditions \( \mu_0 = 0 \) and \( \kappa \neq 0 \) yield \( g = 0 \) and then \( \mu_1 = -ey = 0 \) implies \( e = 0 \). Thus we obtain the systems

\[
\dot{x} = a + xy, \quad \dot{y} = b + fy - xy + y^2
\]

for which we have \( \mu_2 = (a+b)y^2 \). So \( \mu_2 = 0 \) gives \( b = -a \) and then we calculate

\[
\mu_3 = afy^3, \quad \mu_4 = ay^3(f^2x + ay).
\]

Evidently the condition \( \mu_3 = \mu_4 = 0 \) is equivalent to \( a = 0 \) and then we obtain the degenerate systems

\[
\dot{x} = xy, \quad \dot{y} = y(f - x + y).
\]
with \( f \in \{0, 1\} \) doing a rescaling. These systems possess invariant singular line \( y = 0 \) and the respective linear systems possess a double point at infinity which corresponds to the point \( R_2(0, 1, 0) \) of quadratic systems. So using the notations given in Section 3 we arrive at the configuration \((\overrightarrow{2})SN, (\emptyset[[; \emptyset])\). On the other hand we observe that the behavior of the trajectories at infinity in this case is topologically equivalent to the portrait \([QD_6^\infty] \) (see Figure 5).

6.3.2.2 The subcase \( \kappa = 0 \). Then by (20) we get \( h = 0 \) and this implies \( \mu_0 = 0 \). We observe that in this case the singular point \( R_2(0, 1, 0) \) becomes either a nilpotent or intricate one and for systems \((S_{III})\) we calculate

\[
\mu_1 = dg(g - 1)^2x, \quad \tilde{K} = 2g(g - 1)x^2, \quad \tilde{L} = 8gx^2, \quad \kappa_1 = -32d, \quad \tilde{N} = (g^2 - 1)x^2. \tag{21}
\]

If \( \mu_1 \neq 0 \) then \( \tilde{L}\tilde{K} \neq 0 \), \( \text{sign}(g) = \text{sign}(\tilde{L}) \) and if \( \tilde{L} > 0 \) then \( \text{sign}(g - 1) = \text{sign}(\tilde{K}) \). Therefore considering (21) we arrive at the next result.

The condition \( \mu_1 \neq 0 \) implies \( d \neq 0 \) and hence the second singular point \( R_2(0, 1, 0) \) is nilpotent of the third multiplicity. As \( d(g - 1) \neq 0 \) then for systems \((S_{III})\) with \( h = 0 \) we may assume \( e = f = 0 \) and \( d = 1 \) (doing a translation and a rescaling). So considering (18) we have the following systems

\[
\dot{v} = -z - v^2 - cvz - az^2 + bvz^2, \quad \dot{z} = (g - 1)vz + bz^3 \tag{22}
\]

which possess \( R_2 \) at the origin of coordinates. The behavior of the trajectories in the neighborhood of this point depends on the parameter \( g \). More exactly, applying a blow-up (see [7]) and using our notation from Section 3 we obtain the following types of the vicinity of \( R_2 \) (depending on \( g \)):

\[
\begin{align*}
g < -1 & \quad \Rightarrow \quad \left(\frac{1}{2}\right) \dot{\hat{P}}_{\lambda} H \dot{\hat{P}}_{\lambda} - E; \\
-1 < g < 0 & \quad \Rightarrow \quad \left(\frac{1}{2}\right) \dot{\hat{P}}_{\lambda} E \dot{\hat{P}}_{\lambda} - H; \\
g > 1 & \quad \Rightarrow \quad \left(\frac{1}{2}\right) H_{\lambda} H_{\lambda} - H.
\end{align*}
\]

We observe that the above intervals for the parameter \( g \) are completely defined by the invariant polynomials \( \tilde{K}, \tilde{L} \) and \( \tilde{N} \) given in (21). So considering Remark 5 in the case \( \kappa = 0 \) and \( \mu_1 \neq 0 \) we obtain the following configurations of infinite singularities:

\[
\begin{align*}
\tilde{K} < 0 & \quad \Rightarrow \quad \left(\frac{1}{2}\right) \dot{\hat{P}}_{\lambda} E \dot{\hat{P}}_{\lambda} - H, \quad N_f; \\
\tilde{K} > 0, \tilde{L} < 0, \tilde{N} < 0 & \quad \Rightarrow \quad \left(\frac{1}{2}\right) \dot{\hat{P}}_{\lambda} E \dot{\hat{P}}_{\lambda} - H, \quad S; \\
\tilde{K} > 0, \tilde{L} < 0, \tilde{N} = 0 & \quad \Rightarrow \quad \left(\frac{1}{2}\right) \tilde{H} - E, \quad S; \\
\tilde{K} > 0, \tilde{L} < 0, \tilde{N} > 0 & \quad \Rightarrow \quad \left(\frac{1}{2}\right) \dot{\hat{P}}_{\lambda} H \dot{\hat{P}}_{\lambda} - E, \quad S; \\
\tilde{K} > 0, \tilde{L} > 0 & \quad \Rightarrow \quad \left(\frac{1}{2}\right) H_{\lambda} H_{\lambda} - H, \quad N^\infty.
\end{align*}
\]

In what follows we assume \( \mu_1 = 0 \) and we shall consider two cases: \( \tilde{K} \neq 0 \) and \( \tilde{K} = 0 \).

6.3.2.2.1 Assume first \( \tilde{K} \neq 0 \). Considering (21) we have \( g(g - 1) \neq 0 \) and therefore the condition \( \mu_1 = 0 \) gives \( d = 0 \). In this case we may assume \( e = f = 0 \) (due to a translation) and we get the systems

\[
\begin{align*}
\dot{x} = a + cx + gx^2, \quad \dot{y} = b + (g - 1)xy
\end{align*}
\]

(23)

for which we have

\[
\begin{align*}
\mu_0 = \mu_1 = \kappa_1 = 0, \quad \mu_2 = ag(g - 1)^2x^2, \quad \tilde{K} = 2g(g - 1)x^2, \\
\tilde{L} = 8gx^2, \quad K_2 = 48(c^2 - 4ag)(2 - g + g^2)x^2.
\end{align*}
\]

(24)

The condition \( \tilde{K} \neq 0 \) implies \( \tilde{L} \neq 0 \) and the infinite singular point \( R_1(1, 0, 0) \) of the above systems is elemental. Its type is described by Remark 5.
On the other hand since \( h = d = 0 \), by (19) the second infinite singularity \( R_2(0,1,0) \) becomes an intricate singular point the multiplicity of which by Lemma 2 depends on the number of the vanishing invariant polynomials \( \mu_i \) \( (i = 2, 3) \).

1) Assume \( \mu_2 \neq 0 \). Then \( R_2(0,1,0) \) has the multiplicity four: two infinite and two finite singularities have coalesced all together. The respective systems (18) in this case are the systems

\[
\dot{v} = -v^2 - cvz - az^2 + bvz^2, \quad \dot{z} = (g - 1) vz + bz^3, \quad ag(g - 1) \neq 0,
\]

having \( R_2 \) at the origin of coordinates. Applying a blow-up (see [7]) we determine that the behavior of the trajectories in the neighborhood of this point depends on the parameters \( a, c \) and \( g \). More exactly, using our notation from Section 3 we obtain the following types of the vicinity of \( R_2 \):

\[
\begin{align*}
ag < 0, \ g < 0 \quad &\Rightarrow \ (2)\ PH \ PH - \ PH \ PH; \\
ag < 0, \ 0 < g < 1 \quad &\Rightarrow \ (2)\ PE \ PE - \ PE \ PE; \\
ag < 0, \ g > 1 \quad &\Rightarrow \ (2)\ HHH - HHH; \\
ag > 0, \ g < 0, \ c^2 - 4ag < 0 \quad &\Rightarrow \ (2)\ E - E; \\
ag > 0, \ g < 0, \ c^2 - 4ag = 0 \quad &\Rightarrow \ (2)\ PE - PE; \\
ag > 0, \ g < 0, \ c^2 - 4ag > 0 \quad &\Rightarrow \ (2)\ PE \ PE - \ PE \ PE; \\
ag > 0, \ g > 0, \ c^2 - 4ag < 0 \quad &\Rightarrow \ (2)\ H - H; \\
ag > 0, \ g > 0, \ c^2 - 4ag = 0 \quad &\Rightarrow \ (2)\ PH \ PH - \ PH \ PH; \\
ag > 0, \ g > 0, \ c^2 - 4ag > 0 \quad &\Rightarrow \ (2)\ PH \ PH - \ PH \ PH.
\end{align*}
\]

According to (24) if \( \mu_2K_2 \tilde{K} \neq 0 \) then we have

\[
\text{sign}(ag) = \text{sign}(\mu_2), \quad \text{sign}(c^2 - 4ag) = \text{sign}(K_2), \quad \text{sign}(g) = \text{sign}(\tilde{L}).
\]

Moreover, as by Remark 5 the type of the simple node \( R_1 \) depends on the sign\( (g - 1) \) we notice that

\[
\text{sign}(g - 1) = \text{sign}(\tilde{L}K_1).
\]

Thus considering the types of the intricate singular point \( R_2 \) (described above) and Remark 5 in the case \( \kappa = \mu_1 = 0 \) and \( \tilde{K}\mu_2 \neq 0 \) we obtain the following configurations of infinite singularities:

\[
\begin{align*}
\mu_2 < 0, \ \tilde{K} < 0 \quad &\Rightarrow \ (2)\ PE \ PE - \ PE \ PE, \ N_f; \\
\mu_2 < 0, \ \tilde{K} > 0, \ \tilde{L} < 0 \quad &\Rightarrow \ (2)\ PH \ PH - \ PH \ PH, \ S; \\
\mu_2 < 0, \ \tilde{K} > 0, \ \tilde{L} > 0 \quad &\Rightarrow \ (2)\ HHH - HHH, \ N_\infty; \\
\mu_2 > 0, \ \tilde{K} < 0, \ K_2 < 0 \quad &\Rightarrow \ (2)\ H - H, \ N_f; \\
\mu_2 > 0, \ \tilde{K} < 0, \ K_2 > 0 \quad &\Rightarrow \ (2)\ PH \ PH - \ PH \ PH, \ N_f; \\
\mu_2 > 0, \ \tilde{K} < 0, \ K_2 = 0 \quad &\Rightarrow \ (2)\ PH \ PH - \ PH \ PH, \ N_f; \\
\mu_2 > 0, \ \tilde{K} > 0, \ \tilde{L} < 0, \ K_2 < 0 \quad &\Rightarrow \ (2)\ E - E, \ S; \\
\mu_2 > 0, \ \tilde{K} > 0, \ \tilde{L} < 0, \ K_2 > 0 \quad &\Rightarrow \ (2)\ PH \ PH - \ PH \ PH, \ S; \\
\mu_2 > 0, \ \tilde{K} > 0, \ \tilde{L} < 0, \ K_2 = 0 \quad &\Rightarrow \ (2)\ PE - PE, \ S; \\
\mu_2 > 0, \ \tilde{K} > 0, \ \tilde{L} > 0, \ K_2 < 0 \quad &\Rightarrow \ (2)\ H - H, \ N_\infty; \\
\mu_2 > 0, \ \tilde{K} > 0, \ \tilde{L} > 0, \ K_2 > 0 \quad &\Rightarrow \ (2)\ PH \ PH - \ PH \ PH, \ N_\infty; \\
\mu_2 > 0, \ \tilde{K} > 0, \ \tilde{L} > 0, \ K_2 = 0 \quad &\Rightarrow \ (2)\ PH \ PH - \ PH \ PH, \ N_\infty.
\end{align*}
\]

2) Suppose now \( \mu_2 = 0 \) and \( \mu_3 \neq 0 \). Considering (24) we have \( a = 0 \) and then

\[
\mu_3 = -bcg(g - 1)x^3, \quad \mu_4 = bx^3[gx^2 + c^2(g - 1)y], \quad \tilde{K} = 2g(g - 1)x^2.
\]

As \( \mu_3 \neq 0 \) the intricate singular point \( R_2(0,1,0) \) has the multiplicity five. The respective systems (18) in this case are of the form

\[
\dot{v} = -v^2 - cvz + bvz^2, \quad \dot{z} = (g - 1) vz + bz^3, \quad bcg(g - 1) \neq 0,
\]

\[34\]
having $R_2$ at the origin of coordinates. Applying again a blow-up (see [7]) we determine that the behavior of the trajectories in the neighborhood of this point depends only on the parameter $g$. More exactly, using our notation from Section 3 we obtain the following types of the vicinity of $R_2$:
\[
\begin{align*}
g < 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} H \hat{P} - \hat{P} \hat{P} E; \\
0 < g < 1 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} E \hat{P} - \hat{P} \hat{P} H; \\
g > 1 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) H \hat{P} \hat{P} - H H H.
\end{align*}
\]

Therefore considering Remark 5 in the case $\kappa = \mu_1 = \mu_2 = 0$ and $\tilde{K} \mu_3 \neq 0$ we obtain the following configurations of infinite singularities:
\[
\begin{align*}
\tilde{K} < 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} \hat{P} E - E \hat{P} \hat{P}, \quad N^f; \\
\tilde{K} > 0, \tilde{L} < 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} H \hat{P} - \hat{P} \hat{P} E, \quad S; \\
\tilde{K} > 0, \tilde{L} > 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) H \hat{P} \hat{P} - H H H, \quad N^\infty.
\end{align*}
\]

3) If $\mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$ considering (25) we get $c = 0$ and $bg(g - 1) \neq 0$. In this case the intricate singular point $R_2(0, 1, 0)$ becomes of multiplicity six. The respective systems (18) in this case are of the form
\[
\begin{align*}
\hat{v} = -v^2 + b v z^2, \\
\hat{z} = (g - 1) v z + b z^3, \\
bg(g - 1) \neq 0
\end{align*}
\]
and we need to examine the point $(0, 0)$ of these systems. In the same manner as before, applying a blow-up (see [7]) we determine that the behavior of the trajectories in the neighborhood of this point depends only on the parameter $g$. More exactly, using our notation from Section 3 we obtain the following types of the vicinity of $R_2$:
\[
\begin{align*}
g < 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} \hat{P} E - E \hat{P} \hat{P}; \\
g = 1/2 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} H - H \hat{P}; \\
g > 1 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} \hat{P} H - H \hat{P} \hat{P}.
\end{align*}
\]

We note that for systems ($S_{III}$) in the case under examination we have $\tilde{R} = 8g(2g - 1)x^2$ and hence in the case $g > 0$ we have $\text{sign}(2g - 1) = \text{sign}(\tilde{R})$.

Thus considering the types of the intricate singular point $R_2$ (described above) and Remark 5 in the case $\kappa = \mu_1 = \mu_2 = \mu_3 = 0$ and $\tilde{K} \mu_4 \neq 0$ we obtain the following configurations of infinite singularities:
\[
\begin{align*}
\tilde{K} < 0, \tilde{R} < 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} \hat{P} E - E \hat{P} \hat{P}, \quad N^f; \\
\tilde{K} < 0, \tilde{R} > 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} \hat{P} H - E \hat{P} \hat{P}, \quad N^f; \\
\tilde{K} < 0, \tilde{R} = 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} H - H \hat{P}, \quad N^f; \\
\tilde{K} > 0, \tilde{L} < 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} \hat{P} E - E \hat{P} \hat{P}, \quad S; \\
\tilde{K} > 0, \tilde{L} > 0 & \quad \Rightarrow \quad \left(\frac{3}{2}\right) \hat{P} \hat{P} H - H \hat{P} \hat{P}, \quad N^\infty.
\end{align*}
\]

4) Assuming $\mu_2 = \mu_3 = \mu_4 = 0$, i.e. the degenerate systems ($S_{III}$) in the case $\kappa = 0$ and $\tilde{K} \neq 0$.

Considering (25) we observe that the condition $\mu_3 = \mu_4 = 0$ is equivalent to $b = 0$. Therefore systems (23) with $a = 0$ become degenerate of the form
\[
\begin{align*}
\dot{x} = x(c + gx), \\
\dot{y} = (g - 1)xy
\end{align*}
\]
possessing invariant singular line $x = 0$. The respective linear systems possess two infinite singularities $R_1(1, 0, 0)$ and $R_2(0, 1, 0)$. The respective matrices for these singularities are as follows:
\[
R_1 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}; \quad R_2 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 - g \end{pmatrix}.
\]

As $\tilde{K} = 2g(g - 1)x^2 \neq 0$ we conclude that both singularities are elemental. Moreover their types are governed by the parameter $g$ as follows:
\[ g < 0 \quad \Rightarrow \quad R_1 \to S, \quad R_2 \to N^\infty; \]
\[ 0 < g < 1 \quad \Rightarrow \quad R_1 \to N^J, \quad R_2 \to N^J; \]
\[ g > 1 \quad \Rightarrow \quad R_1 \to N^\infty, \quad R_2 \to S. \]

We observe that the singular invariant line of systems (27) coincides with invariant line of the respective linear systems if and only if \( c = 0 \). Therefore as \( \tilde{L} = 8g x^2 \) and \( \tilde{K}_2 = 48 c^2 (2 - g - g^2) x^2 \) then considering our notations (see Section 3) for the degenerate systems \((S_{HI})\) in the case \( \kappa = 0 \) and \( \tilde{K} \neq 0 \) we obtain the following configurations of infinite singularities and the respective topological behavior at the infinity (see Figure 5):
\[
\begin{align*}
\tilde{K} < 0, \, \tilde{K}_2 \neq 0 & \quad \Rightarrow \quad N^J, \, (\Theta [\!\! | ] ; N^J_1) \quad [Q D^\infty_2]; \\
\tilde{K} < 0, \, \tilde{K}_2 = 0 & \quad \Rightarrow \quad N^J, \, (\Theta [\!\! | ] ; N^J_2) \quad [Q D^\infty_j]; \\
\tilde{K} > 0, \, \tilde{L} < 0, \, \tilde{K}_2 \neq 0 & \quad \Rightarrow \quad S, \, (\Theta [\!\! | ] ; N^\infty_2) \quad [Q D^\infty_9]; \\
\tilde{K} > 0, \, \tilde{L} < 0, \, \tilde{K}_2 = 0 & \quad \Rightarrow \quad S, \, (\Theta [\!\! | ] ; N^\infty_3) \quad [Q D^\infty_{10}]; \\
\tilde{K} > 0, \, \tilde{L} > 0, \, \tilde{K}_2 \neq 0 & \quad \Rightarrow \quad N^\infty, \, (\Theta [\!\! | ] ; S_1) \quad [Q D^\infty_{11}]; \\
\tilde{K} > 0, \, \tilde{L} > 0, \, \tilde{K}_2 = 0 & \quad \Rightarrow \quad N^\infty, \, (\Theta [\!\! | ] ; S_2) \quad [Q D^\infty_{12}].
\end{align*}
\]

### 6.3.2.2.2 Suppose now \( \tilde{K} = 0 \).

Then by (21) we get \( g(g - 1) = 0 \) and we shall consider two subcases: \( \tilde{L} \neq 0 \) and \( \tilde{L} = 0 \).

1) Assume first \( \tilde{L} \neq 0 \). Then \( g = 1 \) and we may assume \( c = 0 \) due to a translation. So we get the systems
\[
\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + fy
\]
for which we have
\[
\begin{align*}
\mu_0 = \mu_1 &= \tilde{K} = 0, \quad \mu_2 = f^2 x^2, \quad \kappa_1 = -d, \quad \tilde{L} = 8 x^2, \\
\theta_5 &= 96 d e x^3, \quad \theta_6|_{d=0} = 8 e x^4, \quad K_2|_{d=0} = -384 a x^2.
\end{align*}
\]

### a) Suppose \( \mu_2 \neq 0 \), i.e. \( f \neq 0 \). Then via a rescaling we may assume \( f = 1 \). As \( g = 1 \) by Remark 5 we obtain that the singular point \( R_1(1, 0, 0) \) is a node \( N^d \) if \( e \neq 0 \) and it is a star node if \( e = 0 \). As regard the singularity \( R_2(0, 1, 0) \), it has the multiplicity four: two infinite and two finite singularities have coalesced all together. Moreover it is a nilpotent singularity if \( d = 0 \) and it is an intricate singular point if \( d = 0 \).

The respective systems (18) in this case are the systems
\[
\dot{v} = -dz - v^2 + vz + vex^2 z - ax^2 + bvz^2, \quad \dot{z} = z^2 + evz^2 + bz^3,
\]
having \( R_2 \) at the origin of coordinates. Applying a blow-up (see [7]) we determine that the behavior of the trajectories in the neighborhood of this point depends on the parameters \( a \) and \( d \). More exactly, using our notation from Section 3 we obtain the following types of the vicinity of \( R_2 \):
\[
\begin{align*}
d \neq 0 & \quad \Rightarrow \quad (\gamma_2) \tilde{P}_x \tilde{P} H_x - \tilde{H}; \\
\quad d = 0, \quad a < 0 & \quad \Rightarrow \quad (\gamma_2) \tilde{P} \tilde{P} H - \tilde{P} \tilde{P} H; \\
\quad d = 0, \quad a = 0 & \quad \Rightarrow \quad (\gamma_2) \tilde{P} H - \tilde{P} H; \\
\quad d = 0, \quad a > 0 & \quad \Rightarrow \quad (\gamma_2) H - H.
\end{align*}
\]

Therefore considering the types of the elemental singular point \( R_1 \) and (29), in the case \( \kappa = \tilde{K} = 0 \) and \( \mu_2 \tilde{L} \neq 0 \) we obtain the following configurations of infinite singularities:
\[
\begin{align*}
\kappa_1 &\neq 0, \, \theta_5 \neq 0 \quad \Rightarrow \quad (\gamma_2) \tilde{P}_x \tilde{P} H_x - \tilde{H}, \quad N^d; \\
\kappa_1 &\neq 0, \, \theta_5 = 0 \quad \Rightarrow \quad (\gamma_2) \tilde{P}_x \tilde{P} H_x - \tilde{H}, \quad N^*; \\
\kappa_1 & = 0, \, K_2 < 0, \, \theta_6 \neq 0 \quad \Rightarrow \quad (\gamma_2) \tilde{P} \tilde{P} H - \tilde{P} \tilde{P} H, \quad N^d; \\
\kappa_1 & = 0, \, K_2 < 0, \, \theta_6 = 0 \quad \Rightarrow \quad (\gamma_2) \tilde{P} \tilde{P} H - \tilde{P} \tilde{P} H, \quad N^*; \\
\kappa_1 & = 0, \, K_2 > 0, \, \theta_6 \neq 0 \quad \Rightarrow \quad (\gamma_2) \tilde{P} \tilde{P} H - \tilde{P} \tilde{P} H, \quad N^d; \\
\kappa_1 & = 0, \, K_2 > 0, \, \theta_6 = 0 \quad \Rightarrow \quad (\gamma_2) \tilde{P} \tilde{P} H - \tilde{P} \tilde{P} H, \quad N^*. 
\end{align*}
\]
b) Assume \( \mu_2 = 0 \) and \( \mu_3 \neq 0 \). In this case for systems (28) we obtain \( f = 0 \) and then

\[
\mu_3 = de^2 x^3, \quad \mu_4 = (b^2 + a e^2)x^4 - b d e x^3 y, \quad \kappa_1 = -32d, \quad K_1 = -ex^3. \tag{30}
\]

The condition \( \mu_3 \neq 0 \) implies \( de \neq 0 \) and we may assume \( d = 1 \) and \( a = 0 \) due to a rescaling and a translation. Therefore the singular point \( R_1(1,0,0) \) is a node \( N^d \) and \( R_2(0,1,0) \) is a nilpotent singularity of multiplicity five. In order to examine the vicinity of the second point we consider the respective systems (see (18))

\[
\dot{v} = -z - v^2 + ev^2 z + bv z^2, \quad \dot{z} = ev^2 + bz^3,
\]

having \( R_2 \) at the origin of coordinates. Applying a blow-up (see [7]) we determine that the behavior of the trajectories in the neighborhood of this point depends on the sign of the parameter \( e \). More precisely we obtain the following types of the vicinity of \( R_2 \):

\[
e < 0 \implies (\frac{3}{2}) H_\lambda HH_\lambda - H; \quad e > 0 \implies (\frac{3}{2}) \hat{P}_\lambda E \hat{P}_\lambda - H.
\]

As by (30) in the case \( d = 1 \) we have sign(\( \mu_3 K_1 \)) = -sign(\( e \)), in the case under consideration we get the following configurations of the infinite singularities:

\[
\mu_3 K_1 < 0 \implies (\frac{3}{2}) \hat{P}_\lambda E \hat{P}_\lambda - H, N^d; \quad \mu_3 K_1 > 0 \implies (\frac{3}{2}) H_\lambda HH_\lambda - H, N^d.
\]

**c) Admit now that \( \mu_3 = 0 \).** Then \( de = 0 \) and we shall consider two subcases: \( d = 0 \) and \( d \neq 0 \). Evidently these cases are distinguished by invariant polynomial \( \kappa_1 \) (see (30)).

\[\alpha\] Assume first \( \kappa_1 \neq 0 \), i.e. \( d \neq 0 \) and \( e = 0 \). As it was mentioned above we can take \( d = 1 \) and \( a = 0 \) and therefore we get the systems

\[
\dot{x} = y + x^2, \quad \dot{y} = b
\]

for which \( \mu_4 = b^2 x^4 \). If \( \mu_4 \neq 0 \) then according to Remark 5 the elemental singular point \( R_1(1,0,0) \) is a star node, whereas \( R_2(0,1,0) \) is a nilpotent singularity of the multiplicity six: two infinite and four finite singularities have coalesced all together. The respective systems (18) in this case are the systems

\[
\dot{v} = -z - v^2 + bv z^2, \quad \dot{z} = bv z^3, \quad b \neq 0,
\]

having \( R_2 \) at the origin of coordinates. Applying a blow-up (see [7]) we found out that the behavior of the trajectories in the neighborhood of this point is determined in unique mode: \( (\frac{1}{2}) \hat{P}_\lambda \hat{P}_H - H \).

Therefore considering the elemental star node in the case \( \kappa = \bar{K} = \mu_2 = \mu_3 = 0 \) and \( L \kappa_1 \mu_4 \neq 0 \) we obtain the following configuration of infinite singularities: \( (\frac{3}{2}) \hat{P}_\lambda \hat{P}_H - H, N^* \).

Supposing \( \mu_4 = 0 \) we have \( b = 0 \) and from (31) we obtain the degenerate system, possessing the singular invariant parabola \( y = -x^2 \). Considering the notations from Section 3 we get the configuration \( N^* \), (\( \Theta [\lambda]; \emptyset \)). On the other hand we observe that the behavior of the trajectories at infinity in this case is topologically equivalent to the portrait \( [QD_{13}^\infty] \) (see Figure 5).

\[\beta\] Suppose now \( \kappa_1 = 0 \). Then \( d = 0 \) and we obtain the systems

\[
\dot{x} = a + x^2, \quad \dot{y} = b + ex.
\]

for which we have

\[
\kappa_1 = \mu_3 = 0, \quad \mu_4 = (b^2 + ae^2)x^4, \quad K_2 = -384ax^2, \quad \theta_0 = 8ex^4. \tag{33}
\]

By Remark 5 the elemental singular point \( R_1(1,0,0) \) is a node \( N^d \) if \( e \neq 0 \) and it is a star node \( N^* \) if \( e \neq 0 \).

If \( \mu_4 \neq 0 \) then \( R_2(0,1,0) \) is an intricate singularity of the multiplicity six: two infinite and four finite singularities have coalesced all together. To describe its vicinity we need to examine the vicinity of the respective systems

\[
\dot{v} = -v^2 - az^2 + ev^2 z + bv z^2, \quad \dot{z} = ev z^2 + bz^3, \quad b^2 + ae^2 \neq 0,
\]

having \( R_2 \) at the origin of coordinates. Applying a blow-up (see [7]) we found out that the behavior of the trajectories in the neighborhood of this point depends on the parameters \( a, b \) and \( e \). More exactly, using our notation from Section 3 we obtain the following types of the vicinity of \( R_2 \):


\[ a < 0, \ b^2 + ae^2 < 0 \quad \Rightarrow \quad (3) \ \dot{P}E \dot{P} - HHH; \quad a = 0 \quad \Rightarrow \quad (3) \ \dot{P}P_H^3 H - H \dot{P}_\lambda \dot{P}; \]
\[ a < 0, \ b^2 + ae^2 > 0 \quad \Rightarrow \quad (3) \ \dot{P}P_H - H \dot{P} \dot{P}; \quad a > 0 \quad \Rightarrow \quad (3) H - H. \]

As by (33) have \( \text{sign}(a) = -\text{sign}(K_2) \) and \( \text{sign}(b^2 + ae^2) = \text{sign}(\mu_4) \), in the case under consideration we get the following configurations of the infinite singularities:

- \( K_2 < 0, \ \theta_6 \neq 0 \quad \Rightarrow \quad (3) H - H, N^d; \)
- \( K_2 < 0, \ \theta_6 = 0 \quad \Rightarrow \quad (3) H - H, N^*; \)
- \( K_2 > 0, \ \mu_4 < 0, \ \theta_6 \neq 0 \quad \Rightarrow \quad (3) \ \dot{P}E \dot{P} - HHH, N^d; \)
- \( K_2 > 0, \ \mu_4 < 0, \ \theta_6 = 0 \quad \Rightarrow \quad (3) \ \dot{P}E \dot{P} - HHH, N^*; \)
- \( K_2 > 0, \ \mu_4 > 0, \ \theta_6 \neq 0 \quad \Rightarrow \quad (3) \ \dot{P}P_H - H \dot{P} \dot{P}, N^d; \)
- \( K_2 > 0, \ \mu_4 > 0, \ \theta_6 = 0 \quad \Rightarrow \quad (3) \ \dot{P}P_H - H \dot{P} \dot{P}, N^*; \)
- \( K_2 = 0, \ \theta_6 \neq 0 \quad \Rightarrow \quad (3) \ \dot{P}P_H^3 H - H \dot{P}_\lambda \dot{P}, N^d; \)
- \( K_2 = 0, \ \theta_6 = 0 \quad \Rightarrow \quad (3) \ \dot{P}P_H^3 H - H \dot{P}_\lambda \dot{P}, N^*. \)

Assume now \( \mu_4 = 0 \), i.e. \( b^2 + ae^2 = 0 \). We shall consider two subcases: \( e \neq 0 \) and \( e = 0 \) (this condition is governed by the invariant polynomial \( \theta_6 \)).

- **\( \beta_1 \)** If \( \theta_6 \neq 0 \) then \( e \neq 0 \) and we may assume \( e = 1 \) due to a rescaling. Then \( a = -b^2 \) and we obtain the degenerate systems

\[
\dot{x} = (b + x)(x - b), \quad \dot{y} = b + x,
\]

possessing the singular invariant line \( x = -b \). We note that the linear systems \( \dot{x} = -b + x, \ \dot{y} = 1 \) have the invariant line \( x = b \) and two infinite points \( R_1(1,0,0) \) and \( R_2(0,1,0) \) with the following respective matrices:

\[
R_1 \Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}; \quad R_2 \Rightarrow \begin{pmatrix} -1 & b \\ 0 & 0 \end{pmatrix}.
\]

So the elemental singular point \( R_1 \) is a node \( N^d \) and the double singular point \( R_2 \) is semi-elemental saddle-node. Moreover we observe that the singular line is different from the invariant line of linear system if \( b \neq 0 \) and they coincide if \( b = 0 \).

Therefore considering our notations (see Section 3) we clearly get either the configuration \( N^d, (\ominus [1] ; \{ 1 \} S \bar{N}_3) \) (if \( b \neq 0 \)), or \( N^d, (\ominus [1] ; \{ 1 \} S \bar{N}_2) \) (if \( b = 0 \)).

- **\( \beta_2 \)** If \( \theta_6 = 0 \) we have \( e = b = 0 \) and the systems (32) become degenerate as follows

\[
\dot{x} = a + x^2, \quad \dot{y} = 0.
\]

So in this case we have a singular invariant conic which splits in two parallel complex (respectively real) lines if \( a > 0 \) (respectively \( a < 0 \)) and represents a double real line if \( a = 0 \). Clearly at infinity the respective constant system possesses one singular point which is a star node. Therefore considering the notations from Section 3 we arrive at the next configuration of the singular points for the above systems:

\[
\begin{align*}
| a < 0 & \Rightarrow N^*, (\ominus [1]; \emptyset) ; \\
| a = 0 & \Rightarrow N^*, (\ominus [2]; \emptyset) ; \\
| a > 0 & \Rightarrow N^*, (\ominus [3]; \emptyset) .
\end{align*}
\]

On the other hand, for systems (34) (respectively for systems (35)) we have \( K_2 = 384 b^2 x^2 \geq 0 \) (respectively \( K_2 = -384ax^2 \)). Therefore the invariant polynomials \( K_2 \) and \( \theta_6 \) distinguish the configurations of the infinite singularities as well as the behavior of the trajectories at infinity for the degenerate systems (\( S_{III} \)) in the case \( \kappa = \bar{K} = 0 \) and \( \bar{L} \neq 0 \) as follows:

\[
\begin{align*}
K_2 < 0 & \Rightarrow N^*, (\ominus [1] ; \emptyset) , [QD_{\infty}]; \\
K_2 > 0, \ \theta_6 \neq 0 & \Rightarrow N^d, (\ominus [1] ; \{ 1 \} S N_3) , [QD_{\infty}]; \\
K_2 > 0, \ \theta_6 = 0 & \Rightarrow N^*, (\ominus [1] ; \emptyset) , [QD_{\infty}]; \\
K_2 = 0, \ \theta_6 \neq 0 & \Rightarrow N^d, (\ominus [1] ; \{ 1 \} S N_2) , [QD_{\infty}]; \\
K_2 = 0, \ \theta_6 = 0 & \Rightarrow N^*, (\ominus [2] ; \emptyset) , [QD_{\infty}].
\end{align*}
\]

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2) Consider now the case $\tilde{L} = 0$. Then $g = 0$ and we may assume $e = f = 0$ due to a translation. So we get the systems
\[ \dot{x} = a + cx + dy, \quad \dot{y} = b - xy \] (36)
for which we have
\[ \kappa = \mu_0 = \mu_1 = \overline{K} = \tilde{L} = 0, \quad \mu_2 = -cdy, \quad \kappa_1 = -32d. \]

a) Suppose $\mu_2 \neq 0$, i.e. $cd \neq 0$. Then via a rescaling we may assume $d = 1$ and considering (19) and Theorem 1 we deduce that the singular point $R_1(1,0,0)$ is a double semi-elemental saddle-node, whereas $R_2(0,1,0)$ is a nilpotent singularity of multiplicity three: two infinite and one finite singularities have coalesced all together. To determine the geometric structure of the vicinity of $R_2(0,1,0)$ we shall examine the singular point $(0,0)$ of the systems
\[ \dot{v} = -z - v^2 - cvz - az^2 + bvz^2, \quad \dot{z} = -vz + bz^3, \quad c \neq 0. \]

Remark 6. Doing a blow-up we detect a unique configuration of the vicinity of $(0,0)$: $\tilde{P}_1E\tilde{P}_2 - H$, independently of the values of the parameters $a, b$ and $c$.

So considering the saddle-node $R_1(1,0,0)$ we obtain that the systems (36) possess at infinity the configuration of singularities $(\frac{1}{2}) \tilde{P}_1E\tilde{P}_2 - H, (\overline{1})SN$. 

b) Assume $\mu_2 = 0$. Then $cd = 0$ and we shall examine two subcases: $d \neq 0$ and $d = 0$.

\[ \alpha \] Admit first $\kappa_1 \neq 0$, i.e. $d \neq 0$ and $c = 0$. Due to a rescaling we may take $d = 1$ and for systems (36) we calculate:
\[ \mu_2 = 0, \quad \mu_3 = ax^2, \quad \kappa_1 = -32 \neq 0, \quad \kappa_1 = -xy^2. \]

\[ \alpha_1 \] If $\mu_3 \neq 0$ then according to Theorem 1 the singular point $R_1(1,0,0)$ becomes a triple semi-elemental singularity: two finite singularities have coalesced with one infinite. Moreover $R_1$ is a saddle if $a < 0$ and it is a node if $a > 0$. We note that in this case the triple nilpotent point $R_2$ is of the type defined in Remark 6.

Thus taking into account that sign$(a) = -\text{sign}(\mu_3K_1)$, for systems (36) we obtain at infinity $(\frac{1}{2}) \tilde{P}_1E\tilde{P}_2 - H, (\overline{1})N$ if $\mu_3K_1 < 0$ and $(\frac{1}{2}) \tilde{P}_1E\tilde{P}_2 - H, (\overline{1})S$ if $\mu_3K_1 > 0$.

\[ \alpha_2 \] Suppose now $\mu_3 = 0$. Then $a = 0$ and we obtain the systems
\[ \dot{x} = y, \quad \dot{y} = b - xy \] (37)
for which we have $\mu_4 = -bxy^3$. If $\mu_4 \neq 0$ then by Theorem 1 the singularity $R_1$ increases its multiplicity and it becomes a semi-elemental saddle-node of the multiplicity four. On the other hand by Remark 6 the type of the singular point $R_2$ remains the same (as it is indicated in this remark). So we arrive at the following unique configuration of infinite singularities of systems (37): $(\frac{1}{2}) \tilde{P}_1E\tilde{P}_2 - H, (\overline{1})SN$.

Supposing $\mu_4 = 0$ we have $b = 0$ and we get the degenerate system $\dot{x} = y$, $\dot{y} = -xy$, possessing the singular invariant line $y = 0$. The reduced system is linear, having the unique infinite singularity $(0,1,0)$ which is a nilpotent elliptic saddle. More exactly as system is linear we have $\frac{1}{2} E - H$. Considering the singular line $y = 0$, for the degenerate system above we get at infinity $\frac{1}{2} E - H, (\overline{0})$. On the other hand we observe that the behavior of the trajectories at infinity in this case is topologically equivalent to the portrait $[QD_{18}]$ (see Figure 5).

\[ \beta \] In the case $\kappa_1 = 0$ we have $d = 0$ and for systems (36) we calculate:
\[ \mu_2 = 0, \quad \mu_3 = -acx^2y, \quad \kappa_1 = -cx^2y. \] (38)

\[ \beta_1 \] If $\mu_3 \neq 0$ then $ac \neq 0$ and we may assume $c = 1$ due to a rescaling. By Theorem 1 the singular point $R_1(1,0,0)$ is a double semi-elemental saddle-node. At the same time the singular point $R_2(0,1,0)$ is an intricate singularity of multiplicity four: two finite point have coalesced with two infinite ones. To determine the geometric structure of the vicinity of $R_2$ we shall examine the singular point $(0,0)$ of the systems
\[ \dot{v} = -v^2 - vz - az^2 + bvz^2, \quad \dot{z} = -vz + bz^3, \quad a \neq 0. \]
Applying a blow-up (see [7]) we determine that the behavior of the trajectories in the neighborhood of this point depends on the parameter $a$. More exactly, using our notation from Section 3 we obtain the following types of the vicinity of $R_2$:

\[ a < 0 \Rightarrow \left( \frac{3}{2} \right) \hat{P} E - \hat{P} E; \quad a > 0 \Rightarrow \left( \frac{3}{2} \right) \hat{P} H - \hat{P} H. \]

As by (38) we have \( \text{sign}(a) = \text{sign}(\mu_3 K_1) \), considering the singularity $R_1$ we get the following configurations of infinite singularities for systems (36) in the case $d = 0$ (i.e. $\kappa_1 = 0$):

\[
\mu_3 K_1 < 0 \Rightarrow \left( \frac{3}{2} \right) \hat{P} E - \hat{P} E, (\frac{r}{1}) \text{SN}; \quad \mu_3 K_1 > 0 \Rightarrow \left( \frac{3}{2} \right) \hat{P} H - \hat{P} H, (\frac{r}{1}) \text{SN}.
\]

\( \beta_2 \) Supposing $\mu_3 = 0$ by (38) we obtain $ac = 0$ and we shall consider two subcases: $c \neq 0$ and $c = 0$.

(i) If $K_1 \neq 0$ then $c \neq 0$ and by the same reasons as above, we may assume $c = 1$. Then $a = 0$ and we arrive at the systems $\dot{x} = x, \quad \dot{y} = b - xy$ for which we calculate:

\[
\mu_2 = \mu_3 = 0, \quad \mu_4 = -bx^3 y, \quad K_1 = -x^2 y.
\]

If $\mu_4 \neq 0$ then by Theorem 1 besides the double semi-elemental saddle-node $R_1(1,0,0)$ the above systems possess the intricate singular point $R_2(0,1,0)$ of multiplicity five: three finite singularities have coalesced with two infinite ones. In order to determine the geometric structure of the vicinity of $R_2$ we shall examine the singular point $(0,0)$ of the systems

\[
\dot{v} = -v^2 - vz + bvz^2, \quad \dot{z} = -vz + bz^3, \quad b \neq 0.
\]

Applying a blow-up (see [7]) we determine that the behavior of the trajectories in the neighborhood of this point could be described as $\left( \frac{3}{2} \right) E \hat{P} - \hat{P} H$.

Considering the saddle-node $R_1$ in the case under examination we obtain $\left( \frac{3}{2} \right) E \hat{P} - \hat{P} H, (\frac{r}{1}) \text{SN}$.

Assuming $\mu_4 = 0$ we obtain $b = 0$ and this leads to the degenerate system $\dot{x} = x, \dot{y} = -xy$ possessing the singular invariant line $x = 0$. It can be easily determined that the reduced linear system has at infinity two singular points: (i) the semi-elemental saddle-node $R_1(1,0,0)$ (which corresponds to the singular point $R_1$ of degenerate quadratic systems); (ii) the singular point $R_2(0,1,0)$ which is a node $N^d$.

Therefore considering our notations (see Section 3) for degenerate systems ($S_{H_4}$) in the case $\kappa = \tilde{L} = \kappa_1 = 0, K_1 \neq 0$ the configuration of infinite singularities corresponds to $\left( \frac{r}{1} \right) \text{SN}, (\mathcal{S}[\varepsilon]), N^d$, and the behavior of the trajectories at infinity is topologically equivalent to the portrait $[QD_{10}]$ (see Figure 5).

\( ii \) Suppose now $K_1 = 0$, i.e. $c = 0$. Then we obtain the systems

\[
\dot{x} = a, \quad \dot{y} = b - xy
\]

for which we calculate:

\[
\mu_2 = \mu_3 = 0, \quad \mu_4 = a^2 x^2 y^2, \quad \kappa_2 = -a, \quad L_1 = 8ax^2, \quad L_2 = -3b.
\]

If $\mu_4 \neq 0$ then according to Theorem 1 the singular point $R_1(1,0,0)$ is semi-elemental of multiplicity three: two finite singularities have coalesced with one infinite singularity. Moreover it is a node if $\kappa_2 < 0$ (i.e. $a > 0$) and it is a saddle if $\kappa_2 > 0$ (i.e. $a < 0$). At the same time $R_2(0,1,0)$ is an intricate singular point of multiplicity four: two finite singularities have coalesced with two infinite ones. In order to examine the vicinity of the second point we consider the respective systems

\[
\dot{v} = -v^2 - avz^2 + bvz^2, \quad \dot{z} = -vz + bz^3, \quad a \neq 0.
\]

having $R_2$ at the origin of coordinates. Applying a blow-up (see [7]) we determine that the behavior of the trajectories in the neighborhood of this point depends on the sign of the parameter $a$. More precisely we obtain the following types of the vicinity of $R_2$:

\[ a < 0 \Rightarrow \left( \frac{3}{2} \right) E - E; \quad a > 0 \Rightarrow \left( \frac{3}{2} \right) H - H. \]

As \( \text{sign}(a) = \text{sign}(L_1) \), considering semi-elemental singular point $R_1$ in the case under examination we get the following configurations of infinite singularities for systems (39) in the case $a \neq 0$ (i.e. $\mu_4 \neq 0$):
where

\[ L_1 < 0 \Rightarrow \left( \frac{3}{2} \right) E - E, (\overline{\lambda}) S; \quad L_1 > 0 \Rightarrow \left( \frac{3}{2} \right) H - H, (\overline{\lambda}) N. \]

Supposing \( \mu_4 = 0 \) we obtain \( \alpha \) and this leads to the degenerate systems \( \dot{x} = 0, \dot{y} = b - xy \) possessing the singular invariant conic \( xy = b \). Clearly this conic splits in two lines if \( b = 0 \) and this situation is governed by the invariant polynomial \( L_2 \). We observe that both infinite singularities of degenerate systems are non-isolated ones.

Thus applying the respective notations (see Section 3) for degenerate systems (\( S_{III} \)) in the case \( \kappa = \bar{L} = \kappa_1 = K_1 = 0 \) we obtain the following configurations of infinite singularities and the respective topological behavior at the infinity (see Figure 5):

\[
L_2 \neq 0 \Rightarrow (\ominus [\lambda]; N^*, 0) [QD_{26}^{\infty}]; \quad L_2 = 0 \Rightarrow (\ominus [\times]; N^*, 0) [QD_{21}^{\infty}].
\]

As all the cases are examined, Main Theorem is proved for the family of systems (\( S_{III} \)).

### 6.4 The Family of Systems (\( S_{IV} \))

For these systems we have \( \eta = M = 0 \) and \( C_2 \neq 0 \) and according to Lemma 1 at infinity we have one real singularity of multiplicity more than or equal to three. As \( C_2 = y p_2(x, y) - x q_2(x, y) = x^3 \) this singularity is \( R_2(0,1,0) \) and by Theorem 1 the divisor encoding the multiplicity of infinite points has the form \( \left( \frac{4}{3} \right) u \) with \( i \in \{0,1,\ldots,4\} \). Constructing the respective systems at infinity (possessing the point \( R_2 \) at the origin of coordinates) we obtain

\[
\begin{aligned}
\dot{v} &= -dz - (c - f)vz - az^2 - v^3 + ev^2z + bvz^2, \\
\dot{z} &= hz + gvz + f z^2 - v^2z + evz^2 + bz^3,
\end{aligned}
\]

\( \Rightarrow R_2 \Rightarrow \left( \begin{array}{cc} 0 & -d \\ 0 & h \end{array} \right). \quad (40) \)

So \( R_2 \) is a triple semi-elemental singular point if \( h \neq 0 \). For these systems we have \( \mu_0 = -h^3 \) and by Theorem 1 \( R_2 \) is a saddle if \( \mu_0 < 0 \) and it is a node if \( \mu_0 > 0 \).

Thus in the case \( \mu_0 \neq 0 \) the configuration of infinite singularities for systems (\( S_{IV} \)) corresponds to \( \left( \frac{4}{3} \right) S \) if \( \mu_0 < 0 \) and to \( \left( \frac{4}{3} \right) N \) if \( \mu_0 > 0 \).

In what follows we assume \( \mu_0 = 0 \). Then \( h = 0 \) and for systems (\( S_{IV} \)) we calculate

\[
\mu_0 = 0, \quad \mu_1 = dg^2x, \quad \overline{K} = 2g^2x^2. \quad (41)
\]

#### 6.4.1 The Case \( \mu_1 \neq 0 \)

Then \( dg \neq 0 \) and considering (40) and Theorem 1 the singular point \( R_2(0,1,0) \) is a nilpotent singular point of multiplicity four. To determine the behavior of the trajectories in its vicinity we shall consider systems (40) with \( h = 0 \), having \( R_2 \) at the origin of coordinates. Doing a blow-up (see [7]) and using our notation from Section 3 we obtain the unique configuration in this case: \( \left( \frac{4}{3} \right) H, H \overline{\lambda} - \overline{P} \).

#### 6.4.2 The Case \( \mu_1 = 0 \)

In this case we get \( dg = 0 \) and we shall examine two subcases: \( \overline{K} \neq 0 \) and \( \overline{K} = 0 \).

##### 6.4.2.1 The Subcase \( \overline{K} \neq 0 \)

Then by (41) we have \( g \neq 0, d = 0 \) and we may assume \( e = f = 0 \) (doing a translation) and \( g = 1 \) (doing a rescaling). So we get the systems

\[
\dot{x} = a + cx + x^2, \quad \dot{y} = b - x^2 + xy \quad (42)
\]

for which we calculate

\[
\mu_0 = \mu_1 = 0, \quad \mu_2 = ax^2, \quad K_2 = 48(c^2 - 4a)x^2. \quad (43)
\]
6.4.2.1 If $\mu_2 \neq 0$ then by Theorem 1 the singular point $R_2(0,1,0)$ is an intricate singular point of multiplicity five: two finite singularities have coalesced with three infinite ones. In this case in order to determine the geometrical type of this point considering \((40)\) we shall examine the systems

$$
\dot{v} = -cvz - az^2 - v^3 + bvz^2, \quad \dot{z} = vz - v^2z + bz^3
$$

having $R_2$ at the origin of coordinates. Applying a blow-up (see [7]) we determine that the behavior of the trajectories in the neighborhood of this point depends on the sign of the parameter $a$. More precisely we obtain the following types of the vicinity of $R_2$:

- $a < 0 \quad \Rightarrow \quad (\tilde{3}) \ H\ H\ \hat{P} - \hat{P}HH$;
- $a > 0, \ c^2 - 4a < 0 \quad \Rightarrow \quad (\tilde{3}) \ \hat{P} - \hat{P}$;
- $a > 0, \ c^2 - 4a > 0 \quad \Rightarrow \quad (\tilde{3}) \ H\ P\ E - \hat{P} \hat{P} \hat{P}$;
- $a > 0, \ c^2 - 4a = 0 \quad \Rightarrow \quad (\tilde{3}) \ HE - \hat{P} \hat{P} \hat{P}$.

It remains to note that by \((43)\) we obtain $\text{sign}(a) = \text{sign}(\mu_2)$ and $\text{sign}(c^2 - 4a) = \text{sign}(K_3)$. Therefore we arrive at the corresponding conditions indicate in the Diagram 4 (see Main Theorem).

6.4.2.1.2 Assuming $\mu_2 = 0$ by \((43)\) we get $a = 0$ and then we calculate

$$
\mu_0 = \mu_1 = \mu_2 = 0, \quad \mu_3 = -bcx^3, \quad \mu_4 = bx^3(bx - c^2x + c^2y), \quad K_3 = -6bx^6.
$$

1) If $\mu_3 \neq 0$ then $bc \neq 0$ and we may assume $c = 1$ doing a rescaling. In this case the intricate singular point $R_2(0,1,0)$ has multiplicity six and to determine its geometric type we shall consider again the systems \((44)\) with $a = 0$ and $c = 1$. In this case doing a blow-up we found out that the behavior of the trajectories in the neighborhood of $R_2$ depends on the sign of the parameter $b$. And as $\text{sign}(b) = -\text{sign}(K_3)$ we obtain the following types of the vicinity of $R_2$ with the respective conditions:

$$
K_3 < 0 \quad \Rightarrow \quad (\tilde{3}) \ H\ P\ E - \hat{P}HH; \quad K_3 > 0 \quad \Rightarrow \quad (\tilde{3}) \ H\ H\ \hat{P} - \hat{P} \hat{P} \hat{P}.
$$

2) Suppose now $\mu_3 = 0$, i.e. $bc = 0$. If $\mu_4 \neq 0$ considering \((45)\) we obtain $b \neq 0$ and this implies $c = 0$. So we obtain the systems

$$
\dot{x} = x^2, \quad \dot{y} = b - x^2 + xy
$$

for which we have

$$
\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \quad \mu_4 = b^2x^4, \quad K_3 = -6bx^6.
$$

In this case all four finite singularities of systems $(S_N)$ have coalesced with the triple infinite one and hence, the intricate singular point $R_2(0,1,0)$ of the above systems has the multiplicity seven. Doing a blow-up for the singular point $(0,0)$ of the respective systems

$$
\dot{v} = -v^3 + bvz^2, \quad \dot{z} = vz - v^2z + bz^3
$$

determine, that the geometric structure of its vicinity depends again on the parameter $b$. As $\text{sign}(b) = -\text{sign}(K_3)$ we arrive at the configurations

$$
K_3 < 0 \quad \Rightarrow \quad (\tilde{4}) \ E\ \hat{P}_\lambda H - \hat{P}_\lambda E; \quad K_3 > 0 \quad \Rightarrow \quad (\tilde{4}) \ \hat{P}_\lambda \hat{P} - \hat{P}_\lambda \hat{P} \hat{P}.
$$

In order to finish the case $K \neq 0$ it remains to examine the degenerate systems $(S_N)$. Considering \((45)\) we observe that for systems \((42)\) with $a = 0$ the condition $\mu_3 = \mu_4 = 0$ implies $b = 0$. Therefore we obtain the degenerate systems

$$
\dot{x} = x(c + x), \quad \dot{y} = x(y - x)
$$

possessing the singular invariant line $x = 0$. We observe that the reduced systems are linear, having the infinite singular point $R_2(0,1,0)$ which is a double semi-elemental saddle-node. Moreover the reduced systems have the invariant line $x = -c$. Clearly that this line coincide with the singular line of systems \((42)\) if and only if $c = 0$. This condition is governed by the invariant polynomial $K_2$ as for these systems we have $K_2 = 48c^2x^2$. So using the notations given in Section 3 we get the following configurations of infinite singularities and the respective topological behavior at the infinity(see Figure 5) for degenerate systems $(S_N)$ in the case $K \neq 0$:

$$
K_2 \neq 0 \quad \Rightarrow \quad (\Theta \square; (\overline{0})SN_3) \ [QD_2^\infty]; \quad K_2 = 0 \quad \Rightarrow \quad (\Theta \square; (\overline{0})SN_2) \ [QD_2^\infty].
$$
6.4.2.2 The subcase $\widetilde{K} = 0$. Then for systems $(S_{IV})$ with $h = 0$ we have $g = 0$ and assuming $e = 0$ (doing a translation if necessary) we get the systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b + fy - x^2.$$  \hspace{1cm} (46)

We calculate

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = d^2x^2.$$  \hspace{1cm} (46)

6.4.2.2.1 If $\mu_2 \neq 0$ (i.e. $d \neq 0$) by Theorem 1 the singular point $R_2(0,1,0)$ is a nilpotent singular point of multiplicity $(\frac{3}{4})$. Its geometrical type can be determined applying a blow-up for $(0,0)$ of the systems

$$\dot{v} = -dz + (f - c)vz - az^2 - v^3 + bvz^2, \quad \dot{z} = fz^2 - v^2z + bz^3$$

having $R_2$ at the origin of coordinates. We obtain univocally that the vicinity of $R_2$ geometrically is equivalent to $(\frac{3}{4}) \hat{P}_\lambda \hat{P} - \hat{P}_\lambda \hat{P}$.

6.4.2.2 Assume $\mu_2 = 0$. Then $d = 0$ and for systems (46) we calculations yield:

$$\mu_3 = -c^2fx^3, \quad K_1 = -cx^3, \quad K_3 = 6(2c - f)x^6.$$  \hspace{1cm} (46)

1) If $\mu_3 \neq 0$ then $c, f \neq 0$ and we may assume $c = 1$ (doing a rescaling) and $b = 0$. In this case the intricate singular point $R_2(0,1,0)$ has the multiplicity six and to determine its geometric type we shall consider the systems

$$\dot{v} = (f - 1)vz - az^2 - v^3, \quad \dot{z} = fz^2 - v^2z$$

having $R_2$ at the origin of coordinates. Using a blow-up we found out that the behavior of the trajectories in the neighborhood of $R_2$ depends on the sign of the parameter $f$. More exactly we obtain the following types of the vicinity of $R_2$:

$$f < 0 \Rightarrow (\frac{3}{4}) \hat{P}_\lambda \hat{P} \hat{P}_\lambda - \hat{P} \hat{P}; \quad 0 < f < 2 \Rightarrow (\frac{3}{4}) \hat{P}_\lambda \hat{P} \hat{P}_\lambda - HH; \quad f = 2 \Rightarrow (\frac{3}{4}) HH - \hat{P} \hat{P}; \quad f > 2 \Rightarrow (\frac{3}{4}) H_\lambda \hat{P} \hat{P} H_\lambda - \hat{P} \hat{P}.$$  \hspace{1cm} (46)

On the other hand we observe that in the case $c = 1$ we have $\text{sign}(f) = \text{sign}(\mu_3K_1)$ and $\text{sign}(f(f - 2)) = -\text{sign}(K_3)$. Moreover as $\mu_3 \neq 0$, the condition $f = 2$ is equivalent to $K_3 = 0$. So we get the following configurations of the infinite singularities for systems $(S_{IV})$ (for $\mu_0 = \mu_1 = \mu_2 = \widetilde{K} = 0, \mu_2 \neq 0$) with the respective conditions:

$$\mu_3K_1 < 0 \Rightarrow (\frac{3}{4}) \hat{P}_\lambda \hat{P} \hat{P}_\lambda - \hat{P} \hat{P}; \quad \mu_3K_1 > 0, K_3 > 0 \Rightarrow (\frac{3}{4}) \hat{P}_\lambda \hat{P} \hat{P}_\lambda - HH; \quad \mu_3K_1 > 0, K_3 = 0 \Rightarrow (\frac{3}{4}) HH - \hat{P} \hat{P}; \quad \mu_3K_1 > 0, K_3 < 0 \Rightarrow (\frac{3}{4}) H_\lambda \hat{P} \hat{P} H_\lambda - \hat{P} \hat{P}.$$  \hspace{1cm} (46)

2) Suppose now $\mu_3 = 0$. Then for systems (46) with $d = 0$ we obtain $cf = 0$ and we shall consider two subcases: $c \neq 0$ and $c = 0$. Clearly these possibilities are distinguished by the invariant polynomial $K_1$.

a) If $K_1 \neq 0$ then $c \neq 0, f = 0$ and we may assume $c = 1$ due to a rescaling. So we get the systems

$$\dot{x} = a + x, \quad \dot{y} = b - x^2$$

for which $\mu_4 = (a^2 - b)x^3$. If $\mu_4 \neq 0$ all four finite singularities of systems $(S_{IV})$ have coalesced with the triple infinite one and hence, the intricate singular point $R_2(0,1,0)$ of the above systems has the multiplicity seven. Doing a blow-up for the singular point $(0,0)$ of the respective systems

$$\dot{v} = -vz - az^2 - v^3 + bvz^2, \quad \dot{z} = vz^2 - v^2z + bz^3$$

we determine, that the geometric structure of its vicinity depends on the sign of the expression $a^2 - b$. As $\text{sign}(a^2 - b) = \text{sign}(\mu_4)$ we arrive at the configurations

$$\mu_4 < 0 \Rightarrow (\frac{4}{3}) \hat{P}_\lambda \hat{P} \hat{P}_\lambda - HH; \quad \mu_4 > 0 \Rightarrow (\frac{4}{3}) \hat{P}_\lambda \hat{P} \hat{P}_\lambda - \hat{P} \hat{P}.$$  \hspace{1cm} (46)
In the case $\mu_4 = 0$ we obtain $b = a^2$ and this leads to the degenerate systems
\[
\dot{x} = a + x, \quad \dot{y} = (a + x)(a - x)
\]
possessing the singular invariant line $x = -a$. It can be easily determined that the reduced linear system possess at infinity one nilpotent singular point of multiplicity three: a finite singular point has coalesced with two infinite ones. In our notations for the above degenerate systems we obtain the configuration \((\Xi |) ; (\frac{1}{2}) E - H\). On the other hand we observe that the behavior of the trajectories at infinity in this case is topologically equivalent to the portrait \([QD^\infty_{24}]\) (see Figure 5).

b) Assume $K_1 = 0$. In this case we have $c = 0$ and we obtain the systems
\[
\dot{x} = a, \quad \dot{y} = b + fy - x^2 \tag{47}
\]
for which we calculate $\mu_4 = a^2 x^4$. If $\mu_4 \neq 0$ then $a \neq 0$ and we may assume $a = 1$ due to a rescaling. As in the previous case we determine that the intricate singular point $R_2(0, 1, 0)$ of the above systems has the multiplicity seven. Doing a blow-up for the singular point $(0, 0)$ of the respective systems
\[
\dot{v} = f vz - z^2 - v^3 + b v z^2, \quad \dot{z} = f z^2 - v^2 z + b z^3
\]
we found out that the geometric structure of its vicinity depends on the parameter $f$. More precisely we have $(\frac{4}{3}) \hat{P}_\lambda EH_\lambda - \hat{P}$ if $f \neq 0$ and $(\frac{4}{3}) \hat{P}_\lambda \hat{P} - \hat{P} \hat{P}_\lambda$ if $f = 0$. On the other hand for systems (47) we have $K_3 = -6 f^2 x^6$. So we obtain the following configurations:
\[
K_3 \neq 0 \Rightarrow (\frac{4}{3}) \hat{P}_\lambda EH_\lambda - \hat{P}; \quad K_3 = 0 \Rightarrow (\frac{4}{3}) \hat{P}_\lambda \hat{P} - \hat{P} \hat{P}_\lambda.
\]
In the case $\mu_4 = 0$ we have $a = 0$ and we get the degenerate systems
\[
\dot{x} = 0, \quad \dot{y} = b + fy - x^2 \tag{49}
\]
possessing the singular invariant conic $b + fy - x^2 = 0$. Clearly the type of this conic depends on the parameters $b$ and $f$. More exactly, if $f \neq 0$ we have a parabola. In the case $f = 0$ this conic splits in two parallel lines, which are real if $b > 0$, they are complex if $b < 0$ and the conic represents a double real line if $b = 0$.

On the other hand for systems (49) we have $K_3 = -6 f^2 x^6$ and in the case $f = 0$ (i.e. $K_3 = 0$) we obtain $L_3 = 4 bx^4$. So evidently the conditions above are governed by these two invariant polynomials. We observe also that for the reduced constant system the infinite singular point $(0, 1, 0)$ is a star node.

Thus for degenerate systems \((S_N)\) in the case $K = K_1 = 0$ we get the following configurations of infinite singularities and the respective topological behavior at the infinity(see Figure 5):
\[
K_3 \neq 0 \Rightarrow (\Xi |) ; \quad K_3 = 0 , L_3 < 0 \Rightarrow (\Xi |) ; \quad K_3 = 0 , L_3 > 0 \Rightarrow (\Xi |) ;
\]
\[
K_3 = 0 , L_3 > 0 \Rightarrow (\Xi |) ; \quad K_3 = 0 \Rightarrow (\Xi |).
\]
As all the cases are examined, Main Theorem is proved for the family of systems \((S_N)\).

6.5 The family of systems \((S_V)\)

For these systems we have $C_2 = 0$ (this implies $\eta = M = 0$) and we may consider $e = f = 0$ due to a translation. So in what follows we shall consider the systems
\[
\dot{x} = a + cx + dy + x^2, \quad \dot{y} = b + xy, \tag{50}
\]
for which we have $\mu_0 = 0$ and $\mu_1 = dx$. The line at infinity of systems (50) is filled up with singularities, and removing the degeneracy in the systems obtained on the local charts at infinity we get the following two systems
\[
\begin{cases}
\dot{u} = cu - bz + du^2 + auz, \\
\dot{z} = 1 + cz + duz + az^2;
\end{cases} \quad \begin{cases}
\dot{v} = -d - cv - az + bvz, \\
\dot{z} = v + bz^2.
\end{cases}
\]
which we call reduced systems. As we could observe on the line $z = 0$ the first systems could not have singular points, whereas the second ones could possess such point (if $d = 0$). So in what follows we shall concentrate our attention under the quadratic systems
\[
\dot{v} = -d - cv - az + bvz, \quad \dot{z} = v + bz^2. \tag{51}
\]
6.5.1 The case $\mu_1 \neq 0$

Then $d \neq 0$ and systems (51) do not have any singular point on the line $z = 0$. This means that after removal of the degeneracy, similarly to what we did above, the systems (50) do not have infinite singularities. According to our notations (see Section 3) we have the configuration $[\infty; \emptyset]$.

6.5.2 The case $\mu_1 = 0$, $\mu_2 \neq 0$

This implies $d = 0$ and then for systems (50) we have $\mu_2 = ax^2 \neq 0$. On the other hand systems (51) possess the singular point $(0, 0)$ with the following linear matrix and the respective eigenvalues:

\[
\begin{pmatrix}
-c & -a \\
1 & 0
\end{pmatrix}, \quad \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4a}}{2}, \quad \lambda_1 \lambda_2 = a \neq 0.
\]

Therefore this singular point is a saddle if $a < 0$; it is a generic node (both directions being transversal to the line $z = 0$) if $a > 0$ and $c^2 - 4a > 0$; it is a node with one direction if $a > 0$ and $c^2 - 4a = 0$; and it is either a focus or a center if $c^2 - 4a < 0$.

On the other hand for systems (50) we have $K_2 = 48(c^2 - 4a)x^2$ and hence we obtain

\[
\text{sign}(a) = \text{sign}(\mu_2), \quad \text{sign}(c^2 - 4a) = \text{sign}(K_2).
\]

So using the notations given in Section 3 we obtain the following configurations of infinite singularities for the systems $(S_V)$ in the case $\mu_1 = 0$ and $\mu_2 \neq 0$:

- $\mu_2 < 0 \quad \Rightarrow \quad [\infty; S]$;
- $\mu_2 > 0, K_2 < 0 \quad \Rightarrow \quad [\infty; C]$;
- $\mu_2 > 0, K_2 > 0 \quad \Rightarrow \quad [\infty; N]$;
- $\mu_2 > 0, K_2 = 0 \quad \Rightarrow \quad [\infty; N^d]$.

6.5.3 The case $\mu_2 = 0$

Then $a = 0$ and systems (50) become

\[
\dot{x} = cx + x^2, \quad \dot{y} = b + xy,
\]

with the respective reduced systems at infinity

\[
\dot{v} = -cv + bvz, \quad \dot{z} = v + bz^2.
\]

Clearly the singular point $(0, 0)$ of the last systems is a semi-elemental double saddle-node if $c \neq 0$ and it is a triple nilpotent point (which is an elliptic saddle) if $c = 0$ and $b \neq 0$. As for systems (52) we have

\[
\mu_1 = \mu_2 = 0, \quad \mu_3 = -bcx^3, \quad \mu_4 = b(2x^2y + cy^2),
\]

we arrive at the following configurations of infinite singularities for systems (52):

\[
\mu_3 \neq 0 \quad \Rightarrow \quad [\infty; (0)SN]; \quad \mu_3 = 0, \mu_4 \neq 0 \quad \Rightarrow \quad [\infty; (0)ES].
\]

Assuming $\mu_4 = 0$ (i.e. $b = 0$) we get the degenerate systems

\[
\dot{x} = x(c + x), \quad \dot{y} = xy,
\]

possessing the singular invariant line $x = 0$. We observe that the behavior of the trajectories on the whole Poincaré disc for the above systems is described in Section 3. More exactly in Figure 3 are indicated the phase portraits of systems (53), which correspond to \((c)\) if $c \neq 0$ and to \((d)\) if $c = 0$. Herein the section is described in details the notations both in the finite part and at infinity. Using the notations for infinite singularities and considering the fact, that for systems (53) we have $K_2 = 48c^2x^2$, we obtain the following configurations of singularities for degenerate systems $(S_V)$ and the respective topological behavior at the infinity (see Figure 5):

\[
K_2 \neq 0 \quad \Rightarrow \quad [\infty; (3); \theta_3] \quad [QD^\infty_{28}]; \quad K_2 = 0 \quad \Rightarrow \quad [\infty; (3); \theta_2] \quad [QD^\infty_{29}].
\]

Thus all the families of the quadratic systems given by Lemma 1 are examined and hence the Main Theorem is completely proved.
References


