The global topological classification of the Lotka-Volterra quadratic differential systems

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Abstract

The Lotka-Volterra planar quadratic differential systems have numerous applications but the global study of this class proved to be a difficult challenge. Indeed, the four attempts to classify them (in works of: A.Reyn (1987), Wörz-Buserkros (1993), Georgescu (2007) and Cao and Jiang (2008)) have all shortcomings either by missing some phase portraits or by listing as distinct, some topologically equivalent phase portraits or by giving none of the phase portraits (1993). The lack of proper global classification tools and the large number of phase portraits, explain this situation. All Lotka-Volterra systems possess invariant straight lines, each with its own multiplicity. In this article we base our global topological classification of Lotka-Volterra systems on the concept of configuration of invariant lines (including the line at infinity). The class splits according to the types of configurations in much smaller subclasses which makes it easier to have a good control over the phase portraits in each subclass. At the same time the classification becomes more transparent and easier to grasp. We obtain a total of 112 topologically distinct phase portraits: 60 of them non-degenerate with exactly three invariant lines, all simple; 27 portraits with invariant lines with total multiplicity at least four; 5 with the line at infinity filled up with singularities; 20 phase portraits of degenerate systems. We give necessary and sufficient conditions in terms of invariant polynomials for the realization of each phase portrait. We also make a thorough analysis of the results in the last one of the four works mentioned above. We prove that there are some phase portraits which are missing and show that portraits claimed to be distinct are in fact topologically equivalent.

Résumé

Les systèmes différentiels planes quadratiques Lotka-Volterra ont de nombreuses applications mais la classification globale de ces systèmes fut un défi difficilement surmontable. En effet, les quatre tentatives de les classifier topologiquement (dans des travaux de : A.Reyn (1987), Wörz-Buserkros (1993), Georgescu (2007) and Cao and Jiang (2008)) ont tous des défauts soit manquant certains portraits de phases soit donnant comme distincts, portraits topologiquement équivalents soit ne donnant aucun portrait de phase (1993). Cette situation s'explique par le manque d'outils adéquats globaux de classification et par le grand nombre de portrait de cette classe. Les systèmes Lotka-Volterra possèdent des droites invariantes, chacune avec sa propre multiplicité. Dans cet article nous basons la classification globale de ces systèmes sur le concept de configuration de droites invariantes. La classe est divisée en sous-classes suivant les divers types de configuration ce qui permet d'avoir un bien meilleur contrôle des portraits de chaque sous-classe. En même temps la classification devient plus transparente et on peut la saisir plus facilement. Nous obtenons 112 portraits de phases qui sont topologiquement distincts : 60 avec exactement trois droites invariantes, toutes simples ; 27 avec droites invariantes de multiplicité totale au moins quatre ; 5 avec la droite a l'infini remplie de singularités ; 20 portraits de systèmes dégénérés. Nous obtenons des conditions nécessaires et suffisantes, en termes de polynômes invariants pour chaque portrait de phase. Nous faisons aussi une analyse approfondie des résultats dans le dernier des quatre travaux mentionés plus haut. Nous prouvons que certains portraits manquent et que des portraits mentionnées comme topologiquement distincts sont en fait équivalents.

1 Introduction

In this article we consider real autonomous differential systems

$$(S) \qquad \frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1.1}$$

where $p,\ q \in \mathbb{R}[x,y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$ and their associated vector fields

$$\tilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}. \tag{1.2}$$

We call *degree* of a system (1.1) (or of a vector field (1.2)) the integer $m = \max(\deg p,\ \deg q)$. In particular we call *quadratic* a differential system (1.1) with $m = 2$. We denote by **QS** the class of all quadratic differential systems.

The study of quadratic differential systems is motivated in part by their many applications. On the other hand they are also interesting for theoretical reasons. Indeed, hard problems on polynomial differential systems, among them Hilbert's 16th problem, have been open for more than a century even for the quadratic case. These problems are of a global nature and while the global study of the whole quadratic class is not within reach at this time, specific subfamilies of this class have been successfully studied globally. The goal of this article is to give a complete global topological classification of such a subfamily, namely the Lotka-Volterra quadratic differential systems. A system in this class is of the following form:

$$\begin{aligned} \dot{x} &= x(a_0 + a_1 x + a_2 y) \equiv p(x,y), \\ \dot{y} &= y(b_0 + b_1 x + b_2 y) \equiv q(x,y), \end{aligned} \tag{1.3}$$

where $p,\ q$ are polynomials in x, y with real coefficients and $\max(\deg(p), \deg(q)) = 2$.

The systems (1.3) are called Lotka-Volterra as they were proposed independently by Alfred J. Lotka in 1925 [15] and Vito Volterra in 1926 [29]. The scientific literature on this family has been steadily growing due to their many applications (see [21] for references on applications).

It is is estimated that the class of quadratic differential systems will yield more than two thousand topologically distinct phase portraits. The study of subfamilies of the quadratic class and in particular the Lotka-Volterra family, forms a good testing ground for the analogous but much more difficult task of classifying the whole quadratic class. The global study of several families of quadratic vector fields was completely done. Examples of such families are:

- the family of quadratic vector fields possessing a center;
- the family of quadratic vector fields possessing an integrable saddle;
- the family of quadratic Hamiltonian vector fields;
- the family of all quadratic vector fields with invariant straight lines of total multiplicity at least four;

The full description of the phase portraits and of the bifurcation diagrams for the families above was obtained using algebraic tools. Also not surprisingly, the research on the global schemes of infinite and finite singularities of the quadratic family was done by using only algebraic means. The classification of the global schemes of singularities was achieved in [16], [23] for infinite singularities and in [1] for finite singularities. In [10] Coppel wrote:

"Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients. However, attempts in this direction have met with very limited success..."

This task proved to be impossible. Indeed, Dumortier and Fiddelaers [12] and Roussarie [19] exhibited examples of families of quadratic vector fields which have non-algebraic bifurcation sets. It is natural to ask the following questions:

How much of the behavior of quadratic (or more generally polynomial) vector fields can be described by algebraic means? How far can we go in the global theory of these vector fields by using algebraic means?

Modulo the action of the group of affine transformations and time homotheties, the planar Lotka-Volterra class is 3-dimensional while the class of quadratic differential systems modulo the same group action is 5-dimensional. Due to the global result saying that any system in the Lotka-Volterra class has no limit cycle (see [4], [10]), it is possible to draw the bifurcation diagram of this class. We know that the bifurcation set of singularities is algebraic.

Is the bifurcation set of all planar Lotka-Volterra systems an algebraic set in the parameter space? Or do we have bifurcation surfaces which are not algebraic and if so are they analytic?

We would like to answer such questions. The literature on the Lotka-Volterra equations has become quite ample. In particular there were several attempts to give complete classifications of this family [18](1987), [31] (1993), [13] (2007), [9] (2008). A quick check of the references given in the last three papers indicates that none of these authors mentions anyone of the previously published papers so probably they were not aware of them. But did they obtain results which are in agreement? In fact they are not. This was shown for the first three works above in [20]. In this work we shall also discuss the results in [9] and we show that our results are not everywhere in agreement with results in [9]. Indeed, we prove here that several phase portraits are missing in [9] and portraits claimed to be topologically distinct are in fact topologically equivalent. In order to obtain a clear, transparent classification, one needs to have adequate global classifying tools and they are missing in [9]. It is thus not surprising that the classification in this paper has shortcomings. But in our view the main shortcoming is neither the fact that 8 portraits are missing, nor that in the list we have topologically equivalent phase portraits which are claimed to be distinct. The main shortcoming of this classification is that it is not helpful for understanding globally this class. Indeed, the classification is done in terms of inequalities on the coefficients of the normal forms of the systems. Since the classification is done by referring to normal forms we cannot apply directly this classification to other presentation of the systems. Moreover since there are so many phase portraits we end up with pages of inequalities but without any global invariant which could be of help in giving us insight into the global phenomena present in this class.

In contrast we base our work here on the notion of *configuration of invariant lines of a system.* This is an affine invariant. While the topological equivalence relation is distinct and in fact coarser than the affine one, the second one is powerful for computation and turns out to be of great use in this classification. We base the topological classification on the affine classification of the *configurations of invariant lines* of Lotka-Volterra systems obtained by us in [21]. Our results split this class into subclasses according to the possibilities we have for the types of configurations occurring for this class. We then focus our attention on these subclasses, each of which has much fewer phase portraits and thus it is much easier to keep track of them. By using this approach we are also able to give necessary and sufficient conditions in terms of polynomial invariants for the realization of each one of the phase portraits.

The Lotka-Volterra equations have an inherent algebro-geometric structure. We spelled out this algebro-geometric structure in [21]. We used the number of distinct invariant straight lines, as

well as their multiplicities, as a basic global geometric classifying tool. We combine this algebro-geometric information with information involving the real singularities of the systems located on these invariant lines in the concept of configuration of invariant straight lines of a system, introduced in [22] (see Section 2 further below). Clearly, any system which could be brought by affine transformation and time homotheties to a system (1.3) has the same geometric properties as (1.3).

Definition 1.1. *We shall denote by* ***LV*** *the class of all planar differential systems which could be brought by affine transformations and time rescaling to the form* (1.3) *above and the systems in this class will be called* ***LV****-systems.*

In [21] it was shown that the ***LV*** quadratic systems form a subset of an algebraic set. The goal of our work is to give a rigorous and complete topological classification of this class and construct its bifurcation diagram within this algebraic set.

Our Main Theorem is the following:

Main Theorem. *The class of all Lotka-Volterra quadratic differential systems has a total of 112 topologically distinct phase portraits. Among these 60 portraits are for systems with three simple invariant lines; 27 are portraits of systems with invariant lines of total multiplicity at least four; 5 distinct phase portraits are for Lotka-Volterra systems which have the line at infinity filled up with singularities; 20 phase portraits are for the degenerate systems.*

(i) The total number of phase portraits for the configurations with three simple invariant lines is 88. They are given in Fig. 5 and for each one of these portraits we give in Table 5 necessary and sufficient affine invariant conditions for its realization.

(ii) The total number of phase portraits of the quadratic Lotka-Volterra systems with invariant lines of total multiplicity at least four is 60. They are given in Fig. 3 and the necessary and sufficient conditions for the realization of each one of the phase portraits are given in Table 3.

(iii) The total number of phase portraits of the quadratic Lotka-Volterra systems with the line at infinity filled up with singularities is 5. They are given in Fig. 1 and the necessary and sufficient conditions for the realization of each one of the phase portraits are given in Table 3.

(iv) The total number of phase portraits of the degenerate quadratic Lotka-Volterra systems is 20. They are given in the pictures appearing in Fig. 6. For each one of these phase portraits, affine invariant necessary and sufficient conditions for its realization are given in Table 6.

(v) The total number of phase portraits occurring in (i) - (iv) is 173 but only 112 of them are topologically distinct which can be seen in subsection 3.3.

The article is organized as follows: In Section 2 we give the definitions of the global concepts used in this article, such as for example the notion of *configuration of invariant lines*. We also state the theorem proved in [21] classifying the Lotka-Volterra differential systems according to their configurations of invariant lines. Needed results obtained in [27], [24], [25], [21]. [25] are also stated along with a few others. In Section 3 we prove the Main Theorem.

2 Global geometric concepts and preliminary results

Our classification is based on the concept of *configuration of invariant lines* of a differential system and on results obtained in [21].

The concept of invariant algebraic curve of a differential system is due to Darboux ([11]). Roughly speaking these are algebraic curves which are unions of phase curves. The presence of such algebraic invariant curves is an important information about a system. For example if we

have sufficiently many such curves, the system is integrable, i.e. it has a non-constant analytic first integral ([11]). The following is the formal definition due to Darboux.

Definition 2.1. *An affine algebraic invariant curve (or an algebraic particular integral) of a polynomial system* (1.1) *or of a vector field* (1.2) *is a curve* $f(x,y)=0$ *where* $f\in\mathbb{C}[x,y]$, $\deg(f)\geq 1$, *such that there exists* $k(x,y)\in\mathbb{C}[x,y]$ *satisfying* $\tilde{D}f=fk$ *in* $\mathbb{C}[x,y]$. *We call* k *the cofactor of* $f(x,y)$ *with respect to the system.*

We stress the fact that we have $f(x,y)\in\mathbb{C}[x,y]$. This is important because even in the case when we are only interested in integrability of real systems, the complex invariant curves are helpful in the search for a first integral of the systems.

If a planar polynomial differential system has invariant algebraic curves then these curves could have *multiplicities*. Just as a singularity of a system could be a multiple singularity, meaning that in perturbations this singularity splits into two or more singularities, so also algebraic invariant curves could have multiplicities, meaning that in neighboring systems this curve splits into two or more invariant algebraic curves. In [8] the authors define several notions of multiplicity of invariant curves and show that they coincide for irreducible invariant curves.

In this work we shall only need invariant straight lines and their multiplicities. All planar Lotka-Volterra systems possess at least two distinct affine invariant lines ($x=0$ and $y=0$) and the line at infinity is also invariant. We could also have other invariant lines and each invariant line could have multiplicity other than one.

Definition 2.2. Consider a planar quadratic differential system. We call *configuration of invariant lines* of this system, the set of invariant lines (which may have real coefficients) of the system, each endowed with its own multiplicity with the exception of the lines filled up with singularities of the systems, and together with all the real isolated singular points of this system located on these invariant lines, each one endowed with its own multiplicity.

This is a more powerful global classifying concept than anyone used in [18], [13], [9].

If a system has a finite number of invariant lines and each one of them has finite multiplicity, we encode globally the information regarding the multiplicities of the invariant lines of a configuration in the notion of *multiplicity divisor* of invariant lines. Moreover we encode globally the information regarding the multiplicities of the real singularities located on the invariant lines in a configuration in the concept of *zero-cycle of multiplicities of singularities* of a configuration. We have the following formal definitions:

Definition 2.3. We consider an **LV**-system possessing a configuration $\mathcal{C}$ having a finite number of invariant lines each with a finite multiplicity.

(i) We attach to this system the *multiplicity divisor of the projective plane* corresponding to the configuration $\mathcal{C}$. This is defined as the formal sum :

$$D_{\mathbb{C}}(\mathcal{C})=\sum_{L\in\mathcal{C}}M(L)L,$$

where L is a line and $\mathrm{M}(L)$ is the multiplicity of this line.

(ii) We attach to a configuration $\mathcal{C}$ the *multiplicity zero-cycle of the projective plane* which counts the multiplicities of the real singularities of the system which are located on the configuration $\mathcal{C}$. This is the formal sum:

$$D_{\mathbb{C}}(\mathrm{Sing},\mathcal{C})=\sum_{r\in\mathcal{C}}m(r)r,$$

where $m(r)$ is the multiplicity of the isolated singular point r.

(iii) For a system S we encode the multiplicities of isolated singularities at infinity in the *multiplicity divisor on the line at infinity* which is the formal sum

$$D_{\mathbb{C}}(S,Z) = \sum_{r\in\{Z=0\}} m(r)r,$$

where r is a singular point at infinity and $m(r)$ denotes its multiplicity.

Our global classification of the class of ***LV***-systems is based on the classification of these systems according to their configurations of invariant lines.

In addition we use the result which affirms that a quadratic Lotka-Volterra differential system cannot have limit cycles. This theorem was proved by Bautin in [4]. Since this is an important result which enables us to determine all phase portraits of the Lotka-Volterra systems we give here below its proof. Our proof is a modification of Coppel's proof in [10] in order to make the arguments more transparent by using a bit of Darboux theory which enables us to effectively see the calculations.

Theorem 2.1 (**Bautin [4]**). *The unique singular point inside a periodic orbit of a Lotka-Volterra quadratic differential systems is a center. Moreover such a system is integrable and so it has no limit cycle.*

Proof: Let γ be a periodic orbit of a Lotka-Volterra system. Since the two axes are affine invariant we may assume that γ is included in the interior of the first quadrant. Let p be the unique singular point (see [10]) inside γ. The two axes $x = 0$ and $y = 0$ are invariant lines and hence for any α, β in $\mathbb{C}$, $R(x, y) = x^\alpha y^\beta = 0$ is an invariant curve so we have $\tilde{D}R = RK$ for

$$K(x, y) = \alpha(a_0 + a_1x + a_2y) + \beta(b_0x + b_1x + b_2y) \in \mathbb{C}[x, y].$$

To show that p is a center it suffices to show that we can find $\alpha, \beta \in \mathbb{C}$ such that R is an integrating factor of the system, i.e. $\frac{\partial(Rp)}{\partial x} + \frac{\partial(Rq)}{\partial x} = 0$. But

$$\frac{\partial(Rp)}{\partial x} + \frac{\partial(Rq)}{\partial x} = \tilde{D}R + R\mathrm{div}(p, q) = R(K + \mathrm{div}(p, q)) = 0.$$

Hence we search for α, β such that $K + \mathrm{div}(p, q) = 0$. This equation yields the system of equations:

$$\begin{aligned} \alpha a_0 + \beta b_0 &= -a_0 - b_0, \\ \alpha a_1 + \beta b_1 &= -2a_1 - b_1, \\ \alpha a_2 + \beta b_2 &= -a_2 - 2b_2. \end{aligned} \tag{2.1}$$

Since the singular point p is isolated and it is not on the axes p satisfies the equations:

$$a_0 + a_1x + a_2y = 0, \quad b_0 + b_1x + b_2y = 0$$

with $D = a_1b_2 - a_2b_1 \neq 0$. We choose α, β so as to form a solution of the second and third equations in (2.1). This yields

$$\alpha = -1 + b_2(b_1 - a_1)/D, \quad \beta = -1 + a_1(a_2 - b_2)/D \tag{2.2}$$

Replacing this in $K + \mathrm{div(p, q)}$ we obtain

$$K+\mathrm{div(p,q)} = \alpha \mathrm{a_0}+\beta \mathrm{b_0}+\mathrm{a_0}+\mathrm{b_0} = (-1+\mathrm{b_2(b_1-a_1)/D})\mathrm{a_0}+(-1+(\mathrm{a_1(a_2-b_2)})/\mathrm{Db_0}+\mathrm{a_0}+\mathrm{b_0} = \mathrm{g/D}$$

where

$$g = a_0 b_2 (b_1 - a_1) + a_1 b_0 (a_2 - b_2).$$

Hence $\mathrm{div}(\mathrm{Rp}, \mathrm{Rq}) = \mathrm{R}(\mathrm{K} + \mathrm{div}(\mathrm{p}, \mathrm{q})) = \mathrm{Rg/D}$. To show that $\mathrm{div}(\mathrm{Rp}, \mathrm{Rq}) = 0$ it suffices to show that $g = 0$. Since γ is a periodic orbit we have:

$$\int_\gamma (Rqdx - Rpdy) = \int_0^T (Rq\dot{x} - Rp\dot{y})dt = \int_0^T (Rqp - Rpq)dt = 0$$

where T is the period of γ. We now use the formula of Green

$$\int_{\overset{o}{\gamma}} \mathrm{div}(\mathrm{Rp}, \mathrm{Rq})\mathrm{dxdy} = \int_\gamma \mathrm{Rqdx} - \mathrm{Rpdy} = 0,$$

where $\overset{o}{\gamma}$ is the interior set of γ. But calculations give

$$\int_{\overset{o}{\gamma}} \mathrm{div}(\mathrm{Rp}, \mathrm{Rq})\mathrm{dxdy} = (\mathrm{g/D}) \int_{\overset{o}{\gamma}} \mathrm{Rdxdy} = 0.$$

Since $x > 0$ and $y > 0$ we must have $R > 0$ and hence $g = 0$. But this gives $\mathrm{div}(\mathrm{Rp}, \mathrm{Rq}) = \mathrm{Rg/D} = 0$ so R is an integrating factor and therefore p is a center. Furthermore since the system is integrable it has no limit cycle. ∎

In [21] all possible 65 distinct configurations of invariant lines of the ***LV***-systems were listed and necessary and sufficient conditions for the realization of each one of them were given. As we need these results we state them further below.

The study of quadratic systems possessing invariant straight lines was begun in [22] and continued in [25], [24], and [26]. The four works jointly taken cover the full study of quadratic differential systems possessing invariant lines of at least four total multiplicity. Among these systems some but not all, belong to the class ***LV*** and for these systems we therefore already have their topological classification. We also have the topological classification of all ***LV*** systems with the line at infinity filled up with singularities in [27]. One of our goals is to give a topological classification of the whole class ***LV*** which is transparent and easy to understand. The algebro-geometric structure of the ***LV***-systems based on the concept of configuration of invariant lines is helpful and this geometric structure is also important for questions regarding integrability of systems in the class ***LV***.

According to the papers [21], [25], [26] and [27] there are 65 distinct configurations of planar ***LV***-quadratic differential systems, given in Fig. 1. The systems split into six distinct classes according to the multiplicity of their invariant lines (including the line at infinity). The necessary and sufficient conditions for the realization of each one of the configurations are expressed in [21] in terms of invariant polynomials, with respect to the action of the affine group and time homotheties.

2.1 Group actions on polynomial systems

Consider real planar polynomial differential systems (1.1). We denote by **PS** the set of all planar polynomial systems (1.1) of a fixed degree n. On the set **PS** acts the group $\mathrm{Aff}(2, \mathbb{R})$ of affine transformations on the plane:

$$\begin{array}{rcl} \mathrm{Aff}(2,\mathbb{R}) \times \mathbf{PS} & \longrightarrow & \mathbf{PS} \\ (g, \quad S) & \longrightarrow & \tilde{S} = gS \end{array} \tag{2.3}$$

This action is defined as follows:

Consider an affine transformation $g \in \mathrm{Aff}\,(2,\mathbb{R})$, $g:\ \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. For this transformation we have:

$$g:\ \begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = M\begin{pmatrix}x\\ y\end{pmatrix} + B; \qquad g^{-1}:\ \begin{pmatrix}x\\ y\end{pmatrix} = M^{-1}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} - M^{-1}B.$$

where $M = ||M_{ij}||$ is a 2×2 nonsingular matrix and B is a 2×1 matrix over $\mathbb{R}$. For every $S \in \mathbf{PS}$ we can form its induced transformed system $\tilde{S} = gS$:

$$\frac{d\tilde{x}}{dt} = \tilde{p}(\tilde{x},\tilde{y}), \qquad \frac{d\tilde{y}}{dt} = \tilde{q}(\tilde{x},\tilde{y}), \tag{$\tilde{S}$}$$

where

$$\begin{pmatrix}\tilde{p}(\tilde{x},\tilde{y})\\ \tilde{q}(\tilde{x},\tilde{y})\end{pmatrix} = M\begin{pmatrix}(p\circ g^{-1})(\tilde{x},\tilde{y})\\ (q\circ g^{-1})(\tilde{x},\tilde{y})\end{pmatrix}.$$

The map (2.3) verifies the axioms for a left group action. For every subgroup $G \subseteq \mathrm{Aff}\,(2,\mathbb{R})$ we have an induced action of G on $\mathbf{PS}$.

Definition 2.4. *Consider a subset $\mathcal{A}$ of $\mathbf{PS}$ and a subgroup G of $\mathrm{Aff}\,(2,\mathbb{R})$. We say that the subset $\mathcal{A}$ is invariant with respect to the group G if for every g in G and for every system (S) in $\mathcal{A}$ the transformed system (gS) is also in $\mathcal{A}$.*

We can identify the set of systems in $\mathbf{PS}$ with a subset of $\mathbb{R}^m$ where $m = (n+1)(n+2)$, via the embedding $\mathbf{PS} \hookrightarrow \mathbb{R}^m$ which associates to each system (S) in $\mathbf{PS}$ the m-tuple $(\boldsymbol{a}_{00},\ldots,\boldsymbol{b}_{0n})$ of its coefficients. We denote by $\mathbb{R}^m_{\mathcal{A}}$ the image of the subset $\mathcal{A}$ of $\mathbf{PS}$ under the embedding $\mathbf{PS} \hookrightarrow \mathbb{R}^m$.

For every $g \in \mathrm{Aff}\,(m,\mathbb{R})$ let $r_g:\ \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be the map which corresponds to g via this action. We know (cf. [28]) that r_g is linear and that the map $r:\ \mathrm{Aff}\,(m,\mathbb{R}) \longrightarrow GL(m,\mathbb{R})$ thus obtained is a group homomorphism. For every subgroup G of $\mathrm{Aff}\,(m,\mathbb{R})$, r induces a representation of G onto a subgroup $\mathcal{G}$ of $GL(m,\mathbb{R})$.

The group $\mathrm{Aff}\,(2,\mathbb{R})$ acts on $\mathbf{QS}$ and this yields an action of this group on $\mathbb{R}^{12}$. For every subgroup G of $\mathrm{Aff}\,(2,\mathbb{R})$, r induces a representation of G onto a subgroup $\mathcal{G}$ of $GL(12,\mathbb{R})$.

2.2 Definitions of invariant polynomials

Definition 2.5. *A polynomial $U(a\,,x,y) \in \mathbb{R}[a,x,y]$ is called a comitant with respect to $(\mathcal{A},G)$, where $\mathcal{A}$ is an affine invariant subset of $\mathbf{PS}$ and G is a subgroup of $\mathrm{Aff}\,(2,\mathbb{R})$, if there exists $\chi \in \mathbb{Z}$ such that for every $(g,\,\boldsymbol{a}) \in G \times \mathbb{R}^m_{\mathcal{A}}$ the following identity holds in $\mathbb{R}[x,y]$:*

$$U(r_g(\boldsymbol{a}),\ g(x,y)\,) \equiv\ (\det\,g)^{-\chi}\,U(\boldsymbol{a},x,y),$$

where $\det g = \det M$. If the polynomial U does not explicitly depend on x and y then it is called invariant. The number $\chi \in \mathbb{Z}$ is called the weight of the comitant $U(a,x,y)$. If $G = GL(2,\mathbb{R})$ (or $G = Aff(2,\mathbb{R})$) and $\mathcal{A} = \mathbf{PS}$ then the comitant $U(a,x,y)$ is called GL-comitant (respectively, affine comitant).

Definition 2.6. *A subset $X \subset \mathbb{R}^m$ will be called G-invariant, if for every $g \in G$ we have $r_g(X) \subseteq X$.*

Let $T(2,\mathbb{R})$ be the subgroup of $\mathrm{Aff}\,(2,\mathbb{R})$ formed by translations. Consider the linear representation of $T(2,\mathbb{R})$ into its corresponding subgroup $\mathcal{T} \subset GL(m,\mathbb{R})$, i.e. for every $\tau \in T(2,\mathbb{R})$, $\tau:\ x = \tilde{x}+\alpha, y = \tilde{y}+\beta$ we consider as above $r_\tau:\ \mathbb{R}^m \longrightarrow \mathbb{R}^m$.

Definition 2.7. *A comitant $U(a,x,y)$ with respect to $(\mathcal{A},G)$ is called a T-comitant if for every $(\tau,\,\boldsymbol{a}) \in T(2,\mathbb{R}) \times \mathbb{R}^m_{\mathcal{A}}$ the identity $U(r_\tau\cdot\boldsymbol{a},\ \tilde{x},\,\tilde{y}) = U(\boldsymbol{a},\ \tilde{x},\,\tilde{y})$ holds in $\mathbb{R}[\tilde{x},\tilde{y}]$.*

Definition 2.8. *The polynomial $U(a,x,y) \in \mathbb{R}[a,x,y]$ has well determined sign on $V \subset \mathbb{R}^m$ with respect to x, y if for every fixed $\boldsymbol{a} \in V$, the polynomial function $U(\boldsymbol{a},x,y)$ is not identically zero on V and has constant sign outside its set of zeroes on V.*

Observation 2.1. *We draw attention to the fact, that if a T-comitant $U(a,x,y)$ with respect to $(\mathcal{A},G)$ of even weight is a binary form in x, y, of even degree in the coefficients of (1.1) and has well determined sign on the affine invariant algebraic subset $\mathbb{R}^m_{\mathcal{A}}$ then this property is conserved by any affine transformation and the sign is conserved.*

2.3 The main invariant polynomials associated to LV-systems

Consider real quadratic systems, i.e. systems of the form:

$$(S) \qquad \begin{cases} \dot{x} &= p_0 + p_1(a,x,y) + p_2(a,x,y) \equiv p(a,x,y), \\ \dot{y} &= q_0 + q_1(a,x,y) + q_2(a,x,y) \equiv q(a,x,y) \end{cases} \tag{2.4}$$

with $\max(\deg(p),\deg(q)) = 2$ and

$$\begin{aligned} p_0 &= a_{00}, \quad p_1(a,x,y) = a_{10}x + a_{01}y, \quad p_2(a,x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, \quad q_1(a,x,y) = b_{10}x + b_{01}y, \quad q_2(a,x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2, \end{aligned}$$

where $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ is the 12-tuple of the coefficients of system (2.4) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$.

Notation 2.2. *We denote by $\boldsymbol{a} = (\boldsymbol{a}_{00}, \boldsymbol{a}_{10} \ldots, \boldsymbol{b}_{02})$ a specific point in $\mathbb{R}^{12}$ and we keep a_{ij} and b_{ij} as parameters. Each particular system (2.4) yields an ordered 12-tuple $\boldsymbol{a}$ of its coefficients.*

Let us consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y], \ i = 0,1,2, \\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y], \ i = 1,2. \end{aligned} \tag{2.5}$$

As it was shown in [28] the polynomials

$$\{\ C_0(a,x,y), \quad C_1(a,x,y), \quad C_2(a,x,y), \quad D_1(a), \quad D_2(a,x,y)\ \} \tag{2.6}$$

of degree one in the coefficients of systems (2.4) are GL-comitants of these systems.

Notation 2.3. *Let f, $g \in \mathbb{R}[a,x,y]$ and*

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h} \ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}. \tag{2.7}$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called the transvectant of index k of (f,g) (cf. [14], [17])

Theorem 2.2 (see [30])**.** *Any GL-comitant of systems (2.4) can be constructed from the elements of the set (2.6) by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(*,*)^{(k)}$.*

Remark 2.4. *We point out that the elements of the set (2.6) generate the whole set of GL-comitants and hence also the set of affine comitants as well as of set of the T-comitants.*

Notation 2.5. *Consider the polynomial* $\Phi_{\alpha,\beta} = \alpha P + \beta Q \in \mathbb{R}[a, X, Y, Z, \alpha, \beta]$ *where* $P = Z^2 p(X/Z, Y/Z)$, $Q = Z^2 q(X/Z, Y/Z)$, $p,\ q \in \mathbb{R}[a, x, y]$ *and* $\max(\deg_{(x,y)} p, \deg_{(x,y)} q) = 2$. *Then*

$$\Phi_{\alpha,\beta} = c_{11}(a,\alpha,\beta)X^2 + 2c_{12}(a,\alpha,\beta)XY + c_{22}(a,\alpha,\beta)Y^2 + 2c_{13}(a,\alpha,\beta)XZ + 2c_{23}(a,\alpha,\beta)YZ + c_{33}(a,\alpha,\beta)Z^2, \qquad \Delta(a,\alpha,\beta) = \det ||c_{ij}(a,\alpha,\beta)||_{i,j\in\{1,2,3\}}$$

and we denote

$$D(a,x,y) = 4\Delta(a,y,-x), \qquad H(a,x,y) = 4\big[\det ||c_{ij}(a,y,-x)||_{i,j\in\{1,2\}}\big]. \tag{2.8}$$

We construct the following T-comitants:

Notation 2.6.

$$\begin{aligned} B_3(a,x,y) &= (C_2, D)^{(1)} = Jacob\,(C_2, D)\,, \\ B_2(a,x,y) &= (B_3, B_3)^{(2)} - 6B_3(C_2, D)^{(3)}, \\ B_1(a) &= \operatorname{Res}_x (C_2, D)\,/y^9 = -2^{-9}3^{-8}\,(B_2, B_3)^{(4)}\,. \end{aligned} \tag{2.9}$$

Lemma 2.1 (see [22])**.** *For the existence of invariant straight lines in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that* $B_1 = 0$ *(respectively* $B_2 = 0$*;* $B_3 = 0$*).*

Let us consider the following GL-comitants of systems (2.4):

Notation 2.7.

$$\begin{aligned} M(a,x,y) &= 2\,\text{Hessian}\big(C_2(x,y)\big), & \eta(a) &= \text{Discrim}\big(C_2(x,y)\big), \\ K(a,x,y) &= \det\Big[\text{Jacobian}\big(p_2(x,y), q_2(x,y)\big)\Big], & \mu_0(a) &= \text{Discrim}\big(K(a,x,y)\big)/16, \\ N(a,x,y) &= K(a,x,y) + H(a,x,y), & \theta(a) &= \text{Discrim}\big(N(a,x,y)\big). \end{aligned} \tag{2.10}$$

Remark 2.8. *We note that by the discriminant of the cubic form* $C_2(a,x,y)$ *we mean the expression given in Maple via the function "*$\text{discrim}(C_2, x)/y^6$*".*

The geometrical meaning of these invariant polynomials is revealed by the next 3 lemmas (see [22]).

Lemma 2.2. *Let* $S \in \mathbf{QS}$ *and let* $\boldsymbol{a} \in \mathbb{R}^{12}$ *be its 12-tuple of coefficients. The common points of* $P = 0$ *and* $Q = 0$ *(*P, Q *are the homogenizations of* p, q*) on the line* $Z = 0$ *are given by the common linear factors over* $\mathbb{C}$ *of* p_2 *and* q_2*. This yields the geometrical meaning of the comitants* μ_0*,* K *and* H*:*

$$\gcd(p_2(x,y), q_2(x,y)) = \begin{cases} \text{constant} & \Leftrightarrow \ \mu_0(\boldsymbol{a}) \neq 0; \\ bx + cy & \Leftrightarrow \ \mu_0(\boldsymbol{a}) = 0 \text{ and } K(\boldsymbol{a},x,y) \neq 0; \\ (bx+cy)(dx+ey) & \Leftrightarrow \ \mu_0(\boldsymbol{a}) = 0, K(\boldsymbol{a},x,y) = 0 \text{ and } H(\boldsymbol{a},x,y) \neq 0; \\ (bx+cy)^2 & \Leftrightarrow \ \mu_0(\boldsymbol{a}) = 0, K(\boldsymbol{a},x,y) = 0 \text{ and } H(\boldsymbol{a},x,y) = 0, \end{cases}$$

where $bx + cy,\ dx + ey \in \mathbb{C}[x, y]$ *are some linear forms and* $be - cd \neq 0$*.*

Lemma 2.3. *A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a system* (2.4) *corresponding to* $\boldsymbol{a} \in \mathbb{R}^{12}$ *is the condition* $\theta(\boldsymbol{a}) = 0$ *(respectively,* $N(\boldsymbol{a}, x, y) = 0$*).*

Lemma 2.4. *The form of the divisor* $D_S(C, Z)$ *for systems* (2.4) *is determined by the corresponding conditions indicated in Table 1, where we write* $\omega_1^c + \omega_2^c + \omega_3$ *if two of the points, i.e.* ω_1^c, ω_2^c*, are complex but not real.*

Table 1

Case	Form of $D_S(C,Z)$	Necessary and sufficient conditions on the comitants
1	$\omega_1+\omega_2+\omega_3$	$\eta>0$
2	$\omega_1^c+\omega_2^c+\omega_3$	$\eta<0$
3	$2\omega_1+\omega_2$	$\eta=0, \quad M\neq 0$
4	3ω	$M=0, \quad C_2\neq 0$
5	$D_S(C,Z)$ undefined	$C_2=0$

In order to construct other necessary invariant polynomials let us consider the differential operator $\mathcal{L}=x\cdot\mathbf{L}_2-y\cdot\mathbf{L}_1$ acting on $\mathbb{R}[a,x,y]$ constructed in [2] (see also [3]), where

$$\mathbf{L}_1=2a_{00}\frac{\partial}{\partial a_{10}}+a_{10}\frac{\partial}{\partial a_{20}}+\frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}}+2b_{00}\frac{\partial}{\partial b_{10}}+b_{10}\frac{\partial}{\partial b_{20}}+\frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}};$$
$$\mathbf{L}_2=2a_{00}\frac{\partial}{\partial a_{01}}+a_{01}\frac{\partial}{\partial a_{02}}+\frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}}+2b_{00}\frac{\partial}{\partial b_{01}}+b_{01}\frac{\partial}{\partial b_{02}}+\frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.$$

In [2] it is shown that if a polynomial $U\in\mathbb{R}[a,x,y]$ is a comitant of system (2.4) with respect to the group $GL(2,\mathbb{R})$ then $\mathcal{L}(U)$ is also a GL-comitant.

So, by using this operator and the GL-comitant $\mu_0(a)=\operatorname{Res}_x(p_2(x,y),q_2(x,y))/y^4$ we construct the following polynomials:

$$\mu_i(a,x,y)=\frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i=1,..,4, \qquad \text{where} \quad \mathcal{L}^{(i)}(\mu_0)=\mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)). \tag{2.11}$$

These polynomials are in fact comitants of systems (2.4) with respect to the group $GL(2,\mathbb{R})$. The geometrical meaning of the comitants $\mu_i(a,x,y),\ i=0,1,\dots,4$ is revealed by the next 2 lemmas (see [23]).

Lemma 2.5. *The system $P(X,Y,Z)=Q(X,Y,Z)=0$ possesses m $(1\le m\le 4)$ solutions $[X_i:Y_i:Z_i]$ with $Z_i=0$ $(i=1,\dots,m)$ (considered with multiplicities) if and only if for every $i\in\{0,1,\dots,m-1\}$ we have $\mu_i(a,x,y)=0$ in $\mathbb{R}[x,y]$ and $\mu_m(a,x,y)\neq 0$.*

Lemma 2.6. *A quadratic system* (2.4) *is degenerate (i.e.* $\gcd(p,q)\neq$ constant *) if and only if $\mu_i(a,x,y)=0$ in $\mathbb{R}[x,y]$ for every $i=0,1,2,3,4$.*

Using the transvectant differential operator (2.7) and the invariant polynomials (2.5), (2.8) and (2.10) constructed earlier, we define the following invariant polynomials which will be needed later (see also [24], [26]):

$$\begin{aligned}
H_1(a)&=-\big((C_2,C_2)^{(2)},C_2)^{(1)},D\big)^{(3)};\\
H_2(a,x,y)&=(C_1,\ 2H-N)^{(1)}-2D_1N;\\
H_3(a,x,y)&=(C_2,D)^{(2)};\\
H_4(a)&=\big((C_2,D)^{(2)},(C_2,D_2)^{(1)}\big)^{(2)};\\
H_5(a)&=\big((C_2,C_2)^{(2)},(D,D)^{(2)}\big)^{(2)}+8\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)};\\
H_6(a,x,y)&=16N^2(C_2,D)^{(2)}+H_2^2(C_2,C_2)^{(2)};
\end{aligned}$$

$$
\begin{aligned}
H_7(a) &= (N, C_1)^{(2)};\\
H_8(a) &= 9\big((C_2, D)^{(2)}, (D, D_2)^{(1)}\big)^{(2)} + 2\Big[(C_2, D)^{(3)}\Big]^2;\\
H_9(a) &= -\Big(\big((D, D)^{(2)}, D\big)^{(1)}, D\Big)^{(3)};\\
H_{10}(a) &= \big((N, D)^{(2)},\ D_2\big)^{(1)};\\
H_{11}(a, x, y) &= 8H\big[(C_2, D)^{(2)} + 8(D, D_2)^{(1)}\big] + 3H_2^2;\\
H_{12}(a, x, y) &= (D, D)^{(2)} \equiv \text{Hessian}\,(D);\\
H_{13}(a, x, y) &= 2(\tilde{B}, C_2)^{(3)} + \big((C_2, D)^{(2)} + (D_2, D)^{(1)}, \tilde{E}\big)^{(2)};\\
H_{14}(a, x, y) &= 96(D, C_2)^{(3)}(9\mu_0 + \eta) - 4\Big(\Big(\big((B_3, D_2)^{(1)}, D_2\big)^{(1)}, D_2\Big)^{(1)}, D_2\Big)^{(1)} -\\
&\quad - 54\big((H, \tilde{F})^{(1)}, K\big)^{(2)} - 9\Big[\Big(\big(2(C_2, D)^{(2)} + 11(D_2, D)^{(1)}, H\big)^{(1)}, K\Big)^{(2)}\Big];\\
N_1(a, x, y) &= C_1(C_2, C_2)^{(2)} - 2C_2(C_1, C_2)^{(2)};\\
N_2(a, x, y) &= D_1(C_1, C_2)^{(2)} - \big((C_2, C_2)^{(2)}, C_0\big)^{(1)};\\
N_5(a, x, y) &= \big[(D_2, C_1)^{(1)} + D_1 D_2\big]^2 - 4\big(C_2, C_2\big)^{(2)}\big(C_0, D_2\big)^{(1)},\\
\mathcal{G}_2(a) &= 8H_8 - 9H_5,\\
\mathcal{G}_3(a) &= (\mu_0 - \eta)H_1 - 6\eta(H_4 + 12H_{10}),
\end{aligned}
$$

where $\tilde{B}(a, x, y)$, $\tilde{E}(a, x, y)$ and $\tilde{F}(a, x, y)$ are defined on the page 13 below.

Apart from the invariant polynomials constructed above, which in fact are responsible for the configurations of the invariant lines for the family ***LV***-systems, we also need the polynomials for distinguishing the respective phase portraits.

First we construct the following GL—comitants of the second degree with respect to the coefficients of the initial system

$$
\begin{array}{lll}
T_1 = (C_0, C_1)^{(1)}, & T_2 = (C_0, C_2)^{(1)}, & T_3 = (C_0, D_2)^{(1)},\\
T_4 = (C_1, C_1)^{(2)}, & T_5 = (C_1, C_2)^{(1)}, & T_6 = (C_1, C_2)^{(2)},\\
T_7 = (C_1, D_2)^{(1)}, & T_8 = (C_2, C_2)^{(2)}, & T_9 = (C_2, D_2)^{(1)}.
\end{array}
\tag{2.12}
$$

Then we define a family of T–comitants expressed through C_i $(i = 0, 1, 2)$ and D_j $(j = 1, 2)$ (see [6]):

$$
\begin{aligned}
\tilde{A} &= \big(C_1, T_8 - 2T_9 + D_2^2\big)^{(2)}/144,\\
\widetilde{B} &= \Big\{16D_1\,(D_2, T_8)^{(1)}\,(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0\,(D_2, T_9)^{(1)}\,(3D_1D_2 - 5T_6 + 9T_7)\\
&\quad + 2\,(D_2, T_9)^{(1)}\,\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\,(C_0, T_5)^{(1)}\big)\\
&\quad + 6\,(D_2, T_7)^{(1)}\,[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) + C_2(9T_4 + 96T_3)]\\
&\quad + 6\,(D_2, T_6)^{(1)}\,[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2] + 48D_2\,(D_2, T_1)^{(1)}\,\big(2D_2^2 - T_8\big)\\
&\quad - 32D_1T_8\,(D_2, T_2)^{(1)} + 9D_2^2T_4\,(T_6 - 2T_7) - 16D_1\,(C_2, T_8)^{(1)}\,\big(D_1^2 + 4T_3\big)\\
&\quad + 12D_1\,(C_1, T_8)^{(2)}\,(C_1D_2 - 2C_2D_1) + 6D_1D_2T_4\,\big(T_8 - 7D_2^2 - 42T_9\big)\\
&\quad + 12D_1\,(C_1, T_8)^{(1)}\,(T_7 + 2D_1D_2) + 96D_2^2\,\Big[D_1\,(C_1, T_6)^{(1)} + D_2\,(C_0, T_6)^{(1)}\Big]\\
&\quad - 16D_1D_2T_3\,\big(2D_2^2 + 3T_8\big) - 4D_1^3D_2\,\big(D_2^2 + 3T_8 + 6T_9\big) + 6D_1^2D_2^2\,(7T_6 + 2T_7)\\
&\quad - 252D_1D_2T_4T_9\Big\}/(2^8 3^3),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{D} =& \Big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2\Big]/36,\\
\widetilde{E} =& \Big[D_1(2T_9-T_8)-3\,(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\Big]/72,\\
\widetilde{F} =& \big[6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)\ +48C_0\,(D_2,T_9)^{(1)}-9D_2^2T_4+288D_1\widetilde{E}\\
& -24\left(C_2,\widetilde{D}\right)^{(2)}+120\left(D_2,\widetilde{D}\right)^{(1)}-36C_1\,(D_2,T_7)^{(1)}+8D_1\,(D_2,T_5)^{(1)}\Big]/144,\\
\widetilde{K} =& (T_8+4T_9+4D_2^2)/72 \quad \equiv \Big(p_2(x,y),q_2(x,y)\Big)^{(1)}/4,\\
\widetilde{H} =& (-T_8+8T_9+2D_2^2)/72.
\end{aligned}
$$

These polynomials in addition with (2.5) and (2.12) will serve as bricks in constructing affine algebraic invariants for systems (2.4). Using these bricks the minimal polynomial basis of affine invariants up to degree 12, containing 42 elements $A_1,\ldots,A_{42}$, was constructed in [6]. The following are the elements of this polynomial basis:

$$
\begin{aligned}
A_1 &= \tilde{A}, & A_{22} &= \big[C_2,\widetilde{D})^{(1)},D_2\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}D_2\big)^{(1)}/1152,\\
A_2 &= (C_2,\widetilde{D})^{(3)}/12, & A_{23} &= \big[\widetilde{F},\widetilde{H})^{(1)},\widetilde{K}\big)^{(2)}/8,\\
A_3 &= \big[C_2,D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_{24} &= \big[C_2,\widetilde{D})^{(2)},\widetilde{K}\big)^{(1)},\widetilde{H}\big)^{(2)}/32,\\
A_4 &= (\widetilde{H},\widetilde{H})^{(2)}, & A_{25} &= \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{E}\big)^{(2)}/16,\\
A_5 &= (\widetilde{H},\widetilde{K})^{(2)}/2, & A_{26} &= (\widetilde{B},\widetilde{D})^{(3)}/36,\\
A_6 &= (\widetilde{E},\widetilde{H})^{(2)}/2, & A_{27} &= \big[\widetilde{B},D_2)^{(1)},\widetilde{H}\big)^{(2)}/24,\\
A_7 &= \big[C_2,\widetilde{E})^{(2)},D_2\big)^{(1)}/8, & A_{28} &= \big[C_2,\widetilde{K})^{(2)},\widetilde{D}\big)^{(1)},\widetilde{E}\big)^{(2)}/16,\\
A_8 &= \big[\widetilde{D},\widetilde{H})^{(2)},D_2\big)^{(1)}/8, & A_{29} &= \big[\widetilde{D},\widetilde{F})^{(1)},\widetilde{D}\big)^{(3)}/96,\\
A_9 &= \big[\widetilde{D},D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_{30} &= \big[C_2,\widetilde{D})^{(2)},\widetilde{D}\big)^{(1)},\widetilde{D}\big)^{(3)}/288,\\
A_{10} &= \big[\widetilde{D},\widetilde{K})^{(2)},D_2\big)^{(1)}/8, & A_{31} &= \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{K}\big)^{(1)},\widetilde{H}\big)^{(2)}/64,\\
A_{11} &= (\widetilde{F},\widetilde{K})^{(2)}/4, & A_{32} &= \big[\widetilde{D},\widetilde{D})^{(2)},D_2\big)^{(1)},\widetilde{H}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{12} &= (\widetilde{F},\widetilde{H})^{(2)}/4, & A_{33} &= \big[\widetilde{D},D_2)^{(1)},\widetilde{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{13} &= \big[C_2,\widetilde{H})^{(1)},\widetilde{H}\big)^{(2)},D_2\big)^{(1)}/24, & A_{34} &= \big[\widetilde{D},\widetilde{D})^{(2)},D_2\big)^{(1)},\widetilde{K}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{14} &= (\widetilde{B},C_2)^{(3)}/36, & A_{35} &= \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{E}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{15} &= (\widetilde{E},\widetilde{F})^{(2)}/4, & A_{36} &= \big[\widetilde{D},\widetilde{E})^{(2)},\widetilde{D}\big)^{(1)},\widetilde{H}\big)^{(2)}/16,\\
A_{16} &= \big[\widetilde{E},D_2)^{(1)},C_2\big)^{(1)},\widetilde{K}\big)^{(2)}/16, & A_{37} &= \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{D}\big)^{(1)},\widetilde{D}\big)^{(3)}/576,\\
A_{17} &= \big[\widetilde{D},\widetilde{D})^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}/64, & A_{38} &= \big[C_2,\widetilde{D})^{(2)},\widetilde{D}\big)^{(2)},\widetilde{D}\big)^{(1)},\widetilde{H}\big)^{(2)}/64,\\
A_{18} &= \big[\widetilde{D},\widetilde{F})^{(2)},D_2\big)^{(1)}/16, & A_{39} &= \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{F}\big)^{(1)},\widetilde{H}\big)^{(2)}/64,\\
A_{19} &= \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{H}\big)^{(2)}/16, & A_{40} &= \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{F}\big)^{(1)},\widetilde{K}\big)^{(2)}/64,\\
A_{20} &= \big[C_2,\widetilde{D})^{(2)},\widetilde{F}\big)^{(2)}/16, & A_{41} &= \big[C_2,\widetilde{D})^{(2)},\widetilde{D}\big)^{(2)},\widetilde{F}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{21} &= \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{K}\big)^{(2)}/16, & A_{42} &= \big[\widetilde{D},\widetilde{F})^{(2)},\widetilde{F}\big)^{(1)},D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "[" is used in order to avoid placing the otherwise necessary up to five parenthesizes "(".

Finally we construct the needed affine invariants (see also [1]):

$$
\begin{aligned}
U_1(a) =& A_1(A_1A_2-A_{14}-A_{15}),\\
U_2(a) =& -2A_2^2-2A_{17}-3A_{19}+6A_{21},\\
U_3(a) =& 6A_1^2-3A_8+A_{10}+A_{11}-3A_{12},\\
U_4(a) =& A_{30}, \quad G_9(a)=(A_4+2A_5)/4,
\end{aligned}
$$

$$
\begin{aligned}
W_3(a) =& \big[9A_1^2(36A_{18} - 19A_2^2 + 134A_{17} + 165A_{19}) + 3A_{11}(42A_{18} - 102A_{17} + 195A_{19}) \\
&+ 2A_2^2(A_{10} + 3A_{11}) + 102A_3(3A_{30} - 14A_{29}) - 63A_6(17A_{25} + 30A_{26}) \\
&+ 3A_{10}(14A_{18} - 118A_{17} + 153A_{19} + 120A_{21}) + 6A_7(329A_{25} - 108A_{26}) \\
&+ 3A_8(164A_{18} + 153A_{19} - 442A_{17}) + 9A_{12}(2A_{20} - 160A_{17} - 2A_{18} - 59A_{19}) \\
&+ 3A_1(77A_2A_{14} + 235A_2A_{15} - 54A_{36}) + 18A_{21}(21A_9 - 5A_{11}) + 302A_2A_{34} \\
&- 366A_{14}^2 - 12A_{15}(71A_{14} + 80A_{15})\big]/9, \\
W_4(a) =& \big[1512A_1^2(A_{30} - 2A_{29}) - 648A_{15}A_{26} + 72A_1A_2(49A_{25} + 39A_{26}) \\
&+ 6A_2^2(23A_{21} - 1093A_{19}) - 87A_2^4 + 4A_2^2(61A_{17} + 52A_{18} + 11A_{20}) \\
&- 6A_{37}(352A_3 + 939A_4 - 1578A_5) - 36A_8(396A_{29} + 265A_{30}) \\
&+ 72A_{29}(17A_{12} - 38A_9 - 109A_{11}) + 12A_{30}(76A_9 - 189A_{10} - 273A_{11} - 651A_{12}) \\
&- 648A_{14}(23A_{25} + 5A_{26}) - 24A_{18}(3A_{20} + 31A_{17}) + 36A_{19}(63A_{20} + 478A_{21}) \\
&+ 18A_{21}(2A_{20} + 137A_{21}) - 4A_{17}(158A_{17} + 30A_{20} + 87A_{21}) \\
&- 18A_{19}(238A_{17} + 669A_{19})\big]/81,
\end{aligned}
$$

2.4 Preliminary results involving the use of polynomial invariants

We consider the family of real quadratic systems (2.4). We shall use the following lemma, which gives the conditions on the coefficients of the systems (2.4) so that the origin of coordinates be a center. To do this we present the systems (2.4) with $a_{00} = b_{00} = 0$ in the following tensorial form (see [28]):

$$
\begin{aligned}
&\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta, \quad a^1_1 = a_{10},\, a^1_2 = a_{01},\, a^1_{11} = a_{20},\, a^1_{22} = a_{02}, \\
&(j, \alpha, \beta = 1, 2); \qquad\qquad a^2_1 = b_{10},\, a^2_2 = b_{01},\, a^2_{11} = b_{20},\, a^2_{22} = b_{02}, \\
&\qquad\qquad\qquad\qquad\qquad a^1_{12} = a^1_{21} = a_{11},\ a^2_{12} = a^2_{21} = b_{11}.
\end{aligned} \tag{2.13}
$$

Lemma 2.7. [28] *The singular point* $(0, 0)$ *of a quadratic system* (2.13) *is a center if and only if* $I_2 < 0$, $I_1 = I_6 = 0$ *and one of the following sets of conditions holds:*

$$
1)\ I_3 = 0; \quad 2)\ I_{13} = 0; \quad 3)\ 5I_3 - 2I_4 = 13I_3 - 10I_5 = 0,
$$

where

$$
I_1 = a^\alpha_\alpha, \quad I_2 = a^\alpha_\beta a^\beta_\alpha, \quad I_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma} \varepsilon^{pq}, \quad I_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma} \varepsilon^{pq},
$$

$$
I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta} \varepsilon^{pq}, \quad I_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta} \varepsilon^{pq}, \quad I_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu} \varepsilon^{pq} \varepsilon^{rs}.
$$

and the unit bi-vector ε^{pq} *has the coordinates:* $\varepsilon^{12} = -\varepsilon^{21} = 1, \quad \varepsilon^{11} = \varepsilon^{22} = 0.$

To obtain the global classification of the class of ***LV***-systems, we use results in [21] on the classification of these systems according to their configurations of invariant lines. Following this paper we denote by $\mathbf{QSL_i}$ the family of all non-degenerate quadratic differential systems possessing invariant straight lines (including the line at infinity) of total multiplicity i with $i \in \{3, 4, 5, 6\}$

The following is a corollary of Lemma 2.1.

Corollary 2.9. *A necessary condition for a quadratic system* (2.4) *to be in the class* ***LV*** *(i.e. to possess two intersecting real invariant lines) is that the condition* $B_2(\boldsymbol{a}, x, y) = 0$ *be verified in* $\mathbb{R}[x, y]$.

According to [22] and [24] we have:

Lemma 2.8. *If a quadratic system (S) corresponding to a point $\boldsymbol{a} \in \mathbb{R}^{12}$ belongs to the class $\mathbf{QSL_4} \cup \mathbf{QSL_5} \cup \mathbf{QSL_6}$, then for this system one of the following sets of conditions are satisfied in $\mathbb{R}[x, y]$, respectively:*

$$\begin{array}{lll} (S) \in \mathbf{QSL_4} & \Rightarrow & \textit{either } \theta(\boldsymbol{a}) \neq 0 \textit{ and } B_3(\boldsymbol{a}, x, y) = 0, \quad \textit{or} \quad \theta(\boldsymbol{a}) = 0 = B_2(\boldsymbol{a}, x, y); \\ (S) \in \mathbf{QSL_5} & \Rightarrow & \textit{either } \theta(\boldsymbol{a}) = 0 = B_3(\boldsymbol{a}, x, y), \quad \textit{or} \quad N(\boldsymbol{a}, x, y) = 0 = B_2(\boldsymbol{a}, x, y); \\ (S) \in \mathbf{QSL_6} & \Rightarrow & N(\boldsymbol{a}, x, y) = 0 = B_3(\boldsymbol{a}, x, y). \end{array}$$

The next theorem sums up several results in [21], [25], [26] and [27].

Theorem 2.3. *There are 65 distinct configurations of planar quadratic differential **LV**-systems, given in Fig. 4 and Fig. 2. The systems split into six distinct classes according to the multiplicities of their invariant lines (including the line at infinity) and to the presence of lines filled up with singularities, as follows:*

*(i) The **LV**-systems with exactly three invariant straight lines which are all simple. These have 13 configurations Config. 3.j, $j = 1, 2, \ldots, 13$. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as its respective representative are indicated in Table 2.*

*(ii) The **LV**-systems with four invariant straight lines counted with multiplicity. These have 19 configurations Config. 4.j with $j \in \{1, 3, 4, 5, 9, 10, 11, 12, 16, \ldots, 26\}$. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as the additional conditions for the respective phase portraits given in Fig. 3 are indicated in Table 3.*

*(iii) The **LV**-systems with five invariant straight lines counted with multiplicity. These have 11 configurations Config. 5.j with $j \in \{1, 3, 7, 8, 11, 12, 13, 14, 17, 18, 19\}$. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as the additional conditions for the respective phase portraits given in Fig. 3 are indicated in Table 3.*

*(iv) The **LV**-systems with six invariant straight lines counted with multiplicity. These have four configurations Config. 6.j with $j \in \{1, 5, 7, 8\}$. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as the additional conditions for the respective phase portraits given in Fig. 3 are indicated in Table 3.*

*(v) The non-degenerate **LV**-systems with a line of singularities at infinity. For these systems the condition $C_2 = 0$ holds and they have four configurations Config $C_2.j$ with $j \in \{1, 3, 5, 7\}$. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as the additional conditions for the respective five phase portraits given in Fig. 1 are indicated in Table 3.*

*(vi) The degenerate **LV**-systems defined by the conditions $\mu_i = 0$, $i = 0, 1, \ldots, 4$ possessing at least one and at most two affine lines filled with singularities. These have 14 configurations Config. $LV_d.j$ with $j \in \{1, \ldots, 14\}$. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as its respective representative of systems are indicated in Table 4.*

Table 2

Orbit representative	Necessary and sufficient conditions	Configuration
(III.1) $\begin{cases} \dot{x} = x[1 + gx + (h-1)y], \\ \dot{y} = y[f + (g-1)x + hy], \\ f, g, h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_1) \end{cases}$	$\eta > 0,\ \mu_0 B_3 H_9 \neq 0$ and either $\theta \neq 0$ or $(\theta = 0\ \&\ NH_7 \neq 0)$	Config. 3.1
(III.2) $\begin{cases} \dot{x} = x[1 + gx + (h-1)y], \\ \dot{y} = y[(g-1)x + hy], \\ g, h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_2) \end{cases}$	$\eta > 0,\ \mu_0 B_3 \neq 0,\ H_9 = H_{13} = 0$ and either $\theta \neq 0$ or $(\theta = 0\ \&\ NH_7 \neq 0)$	Config. 3.2
(III.3) $\begin{cases} \dot{x} = x[g + gx + (h-1)y], \\ \dot{y} = y[g - 1 + (g-1)x + hy], \\ g, h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_2) \end{cases}$	$\eta > 0,\ \mu_0 B_3 H_{13} \neq 0,\ H_9 = 0$ and either $\theta \neq 0$ or $(\theta = 0\ \&\ NH_7 \neq 0)$	Config. 3.3
(III.4) $\begin{cases} \dot{x} = x[1 + (h-1)y], \\ \dot{y} = y(f - x + hy), \\ f, h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_3) \end{cases}$	$\eta > 0, \theta B_3 H_9 \neq 0, \mu_0 = H_{14} = 0$	Config. 3.4
(III.5) $\begin{cases} \dot{x} = x[1 + (1-h)(x-y)], \\ \dot{y} = y(f - hx + hy), \\ f, h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_3) \end{cases}$	$\eta > 0,\ \theta B_3 H_9 H_{14} \neq 0,\ \mu_0 = 0$	Config. 3.5
(III.6) $\begin{cases} \dot{x} = x[1 + (h-1)y], \\ \dot{y} = y(-x + hy), \\ h \in \mathbb{R},\ h(h-1) \neq 0 \end{cases}$	$\eta > 0,\ \theta B_3 \neq 0,\ \mu_0 = H_9 = 0,$ $H_{13} = H_{14} = 0$	Config. 3.6
(III.7) $\begin{cases} \dot{x} = x[h - 1 + (h-1)y], \\ \dot{y} = y(h - x + hy), \\ h \in \mathbb{R},\ h(h-1) \neq 0 \end{cases}$	$\eta > 0,\ \theta B_3 H_{13} \neq 0,$ $\mu_0 = H_9 = H_{14} = 0$	Config. 3.7
(III.8) $\begin{cases} \dot{x} = x[1 + (1-h)(x-y)], \\ \dot{y} = hy(y - x), \\ h \in \mathbb{R},\ h(h-1) \neq 0 \end{cases}$	$\eta > 0, \theta B_3 H_{14} \neq 0, \mu_0 = H_9 = 0$	Config. 3.8
(III.9) $\begin{cases} \dot{x} = x(1 + gx + y), \\ \dot{y} = y(f - x + gx + y), \\ f, g \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_4) \end{cases}$	$\eta = 0,\ \theta H_4 B_3 \mu_0 H_9 \neq 0$	Config. 3.9
(III.10) $\begin{cases} \dot{x} = x(g + gx + y), \\ \dot{y} = y[g - 1 + (g-1)x + y], \\ g \in \mathbb{R}, \quad g(g-1) \neq 0 \end{cases}$	$\eta = 0, \theta H_4 B_3 \mu_0 H_{13} \neq 0, H_9 = 0$	Config. 3.10
(III.11) $\begin{cases} \dot{x} = x(1 + gx + y), \\ \dot{y} = y(-x + gx + y), \\ g \in \mathbb{R}, \quad g(g-1) \neq 0 \end{cases}$	$\eta = 0,\ \theta H_4 B_3 \mu_0 \neq 0,$ $H_9 = H_{13} = 0$	Config. 3.11
(III.12) $\begin{cases} \dot{x} = x(1 + y), \\ \dot{y} = y(f + x + y), \\ f \in \mathbb{R},\ f(f-1) \neq 0 \end{cases}$	$\eta = 0,\ \theta H_4 B_3 H_9 \neq 0,\ \mu_0 = 0$	Config. 3.12
(III.13) $\begin{cases} \dot{x} = x(1 + y), \\ \dot{y} = y(x + y), \end{cases}$	$\eta = 0,\ \theta H_4 B_3 \neq 0,\ \mu_0 = H_9 = 0$	Config. 3.13

$$gh(g+h-1)(g-1)(h-1)f(f-1)(fg+h)(1-g+fg)(f+h-fh) \neq 0; \qquad (\mathcal{A}_1)$$

$$gh(g+h-1)(g-1)(h-1) \neq 0; \qquad (\mathcal{A}_2)$$

$$h(h-1)f(f-1)(f+h-fh) \neq 0. \qquad (\mathcal{A}_3)$$

$$g(g-1)f(f-1)(1-g+fg) \neq 0. \qquad (\mathcal{A}_4)$$

Table 3

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*	*Phase portrait*
Config. 4.1	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,\ H_7 \neq 0$	$\mu_0 > 0$	*Picture 4.1(a)*
		$\mu_0 < 0,\ K < 0$	*Picture 4.1(b)*
		$\mu_0 < 0,\ K > 0$	*Picture 4.1(c)*
Config. 4.3	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ H_1 \neq 0,\ \mu_0 \neq 0$	$\mu_0 > 0$	*Picture 4.3(a)*
		$\mu_0 < 0,\ K < 0$	*Picture 4.3(b)*
		$\mu_0 < 0,\ K > 0$	*Picture 4.3(c)*
Config. 4.4	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ H_1 \neq 0,\ \mu_0 = 0$	$K < 0$	*Picture 4.4(a)*
		$K > 0$	*Picture 4.4(b)*
Config. 4.5	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ H_1 = 0$	$\mu_0 > 0$	*Picture 4.5(a)*
		$\mu_0 < 0,\ K < 0$	*Picture 4.5(b)*
		$\mu_0 < 0,\ K > 0$	*Picture 4.5(c)*
Config. 4.9	$\eta > 0,\ B_2 = \theta = H_7 = 0,$ $\mu_0 B_3 H_4 H_9 \neq 0$ *and either* $H_{10} N > 0$ *or* $N = 0,\ H_8 > 0$	$\mathcal{G}_2 > 0, H_4 > 0, \mathcal{G}_3 < 0$	*Picture 4.9(a)*
		$\mathcal{G}_2 < 0$	*Picture 4.9(b)*
		$\mathcal{G}_2 > 0, H_4 < 0$	
		$\mathcal{G}_2 > 0, H_4 > 0, \mathcal{G}_3 > 0$	*Picture 4.9(c)*
Config. 4.10	$\eta > 0,\ B_3 \neq 0,\ B_2 = \theta = 0,$ $\mu_0 \neq 0, H_7 = H_9 = 0, H_{10} N > 0$	$H_4 > 0, \mathcal{G}_3 > 0$	*Picture 4.10(a)*
		$H_4 < 0$	*Picture 4.10(b)*
		$H_4 > 0, \mathcal{G}_3 < 0$	*Picture 4.10(c)*
	$\eta > 0, B_3 H_4 \neq 0,$ $B_2 = N = H_9 = 0, H_8 > 0$	–	
Config. 4.11	$\eta = 0,\ M B_3 \neq 0,\ B_2 = \theta = 0,$ $H_7 = 0,\ \mu_0 \neq 0,\ H_{10} > 0$	$H_4 > 0$	*Picture 4.11(a)*
		$H_4 < 0$	*Picture 4.11(b)*
Config. 4.12	$\eta = 0,\ M \neq 0,\ B_3 = \theta = 0,$ $K H_6 \neq 0, H_7 = \mu_0 = 0, H_{11} > 0$	$\mu_2 > 0,\ L > 0$	*Picture 4.12(a)*
		$\mu_2 > 0,\ L < 0$	*Picture 4.12(b)*
		$\mu_2 < 0,\ K < 0$	*Picture 4.12(c)*
		$\mu_2 < 0, K > 0, L > 0$	*Picture 4.12(d)*
		$\mu_2 < 0, K > 0, L < 0$	*Picture 4.12(e)*
Config. 4.16	$\eta > 0,\ B_3 \neq 0,\ B_2 = \theta = 0,$ $\mu_0 = H_7 = 0,\ H_9 \neq 0$	$\mathcal{G}_2 > 0$	*Portrait 4.16(a)*
		$\mathcal{G}_2 < 0$	*Portrait 4.16(b)*
Config. 4.17	$\eta > 0,\ B_3 \neq 0,\ B_2 = \theta = 0,$ $\mu_0 = H_7 = H_9 = 0,\ H_{10} \neq 0$	–	*Picture 4.17*
Config. 4.18	$\eta > 0,\ B_3 = \theta = 0,$ $\mu_0 = 0,\ H_7 \neq 0$	$\mu_2 L > 0$	*Picture 4.18(a)*
		$\mu_2 L < 0$	*Picture 4.18(b)*
Config. 4.19	$\eta = 0, M \neq 0, B_3 = \theta = K = 0,$ $N H_6 \neq 0,\ \mu_0 = H_7 = 0,\ H_{11} \neq 0$	$\mu_3 K_1 < 0$	*Picture 4.19(a)*
		$\mu_3 K_1 > 0$	*Picture 4.19(b)*
Config. 4.20	$\eta = 0,\ M \neq 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ D = 0$	$\mu_0 > 0$	*Picture 4.20(a)*
		$\mu_0 < 0$	*Picture 4.20(b)*

Table 3(*continued*)

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*	*Phase portrait*
Config. 4.21	$\eta = 0,\ M \neq 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ D \neq 0,\ \mu_0 \neq 0$	$\mu_0 > 0$	*Picture 4.21(a)*
		$\mu_0 < 0$	*Picture 4.21(b)*
Config. 4.22	$\eta > 0,\ B_3 \neq 0,\ B_2 = \theta = 0,$ $\mu_0 \neq 0,\ N \neq 0,\ H_7 = H_{10} = 0$	$H_1 > 0$	*Picture 4.22(a)*
		$H_1 < 0$	*Picture 4.22(b)*
	$\eta > 0, B_3 H_4 \neq 0, B_2 = \theta = N = H_8 = 0$	–	
Config. 4.23	$\eta = 0,\ MB_3 \neq 0,\ B_2 = \theta = 0,$ $\mu_0 \neq 0,\ H_7 = H_{10} = 0$	–	*Picture 4.23*
Config. 4.24	$\eta = 0, M \neq 0, B_3 = \theta = 0,$ $KH_6 \neq 0,\ \mu_0 = H_7 = H_{11} = 0$	$L > 0$	*Picture 4.24(a)*
		$L < 0$	*Picture 4.24(b)*
Config. 4.25	$\eta = 0,\ M \neq 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 \neq 0$	$\mu_0 > 0$	*Picture 4.25(a)*
		$\mu_0 < 0$	*Picture 4.25(b)*
Config. 4.26	$\eta = 0,\ M \neq 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ D \neq 0,\ \mu_0 = 0$	–	*Picture 4.26*
Config. 5.1	$\eta > 0,\ B_3 = \theta = 0,$ $N \neq 0,\ \mu_0 \neq 0,\ H_1 \neq 0$	–	*Picture 5.1*
Config. 5.3	$\eta > 0,\ B_2 = N = 0,\ B_3 \neq 0,$ $H_1 > 0,\ H_4 = 0,\ H_5 > 0$	–	*Picture 5.3*
Config. 5.7	$\eta > 0,\ B_3 = \theta = 0,$ $N \neq 0,\ \mu_0 = H_6 = 0$	–	*Picture 5.7*
Config. 5.8	$\eta > 0,\ B_3 = \theta = 0,$ $N \neq 0,\ \mu_0 \neq 0,\ H_1 = 0$	–	*Picture 5.8*
Config. 5.11	$\eta = 0,\ M \neq 0,\ B_3 = \theta = 0,$ $\mu_0 \neq 0,\ N \neq 0,\ D \neq 0$	–	*Picture 5.11*
Config. 5.12	$\eta > 0,\ B_2 = N = 0,\ B_3 \neq 0,$ $H_1 > 0,\ H_4 = H_5 = 0$	–	*Picture 5.12*
Config. 5.13	$\eta = 0,\ M \neq 0, B_3 = N = 0,$ $H = N_1 = 0,\ N_2 D \neq 0,\ N_5 > 0$	–	*Picture 5.13*
Config. 5.14	$\eta < 0,\ B_3 = \theta = 0,$ $N \neq 0,\ H_7 \neq 0$	$L > 0$	*Picture 5.14(a)*
		$L < 0$	*Picture 5.14(b)*
Config. 5.17	$\eta = 0,\ M \neq 0, B_3 = N = 0,$ $H = N_1 = N_5 = 0,\ N_2 D \neq 0$	–	*Picture 5.17*
Config. 5.18	$\eta = 0,\ M \neq 0, B_3 = \theta = 0,$ $N \neq 0,\ \mu_0 = K = H_6 = 0$	–	*Picture 5.18*
Config. 5.19	$\eta = 0,\ M \neq 0, B_3 = \theta = 0,$ $\mu_0 \neq 0,\ N \neq 0,\ D = 0$	–	*Picture 5.19*
Config. 6.1	$\eta > 0,\ B_3 = N = 0,\ H_1 > 0$	–	*Picture 6.1*
Config. 6.5	$\eta > 0,\ B_3 = N = \ H_1 = 0$	–	*Picture 6.5*
Config. 6.7	$MD \neq 0, \eta = \ B_3 = N = 0,$ $H = N_1 = N_2 = 0$	–	*Picture 6.7*

Table 3(*continued*)

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*	*Phase portrait*
Config. 6.8	$MH \neq 0, \quad \eta = B_3 = N = 0,$ $H_2 = 0, H_3 > 0$	–	*Picture 6.8*
Config. $C_2.1$	$C_2 = 0,\ H_{10} \neq 0,\ H_9 < 0$	–	*Picture* $C_2.1$
Config. $C_2.3$	$C_2 = 0,\ H_{10} \neq 0, H_9 = 0, H_{12} \neq 0$	–	*Picture* $C_2.3$
Config. $C_2.5$	$C_2 = 0,\ H_{10} = 0, H_{12} \neq 0, H_{11} > 0$	$\mu_2 < 0$	*Picture* $C_2.5(a)$
		$\mu_2 > 0$	*Picture* $C_2.5(b)$
Config. $C_2.7$	$C_2 = 0,\ H_{10} = 0, H_{12} \neq 0, H_{11} = 0$	–	*Picture* $C_2.7$

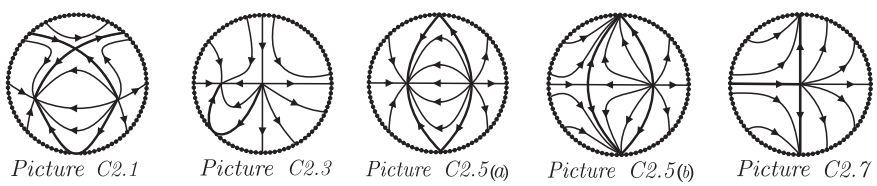

Fig. 1: **The phase portraits of *LV*-systems with infinite degenerate line**

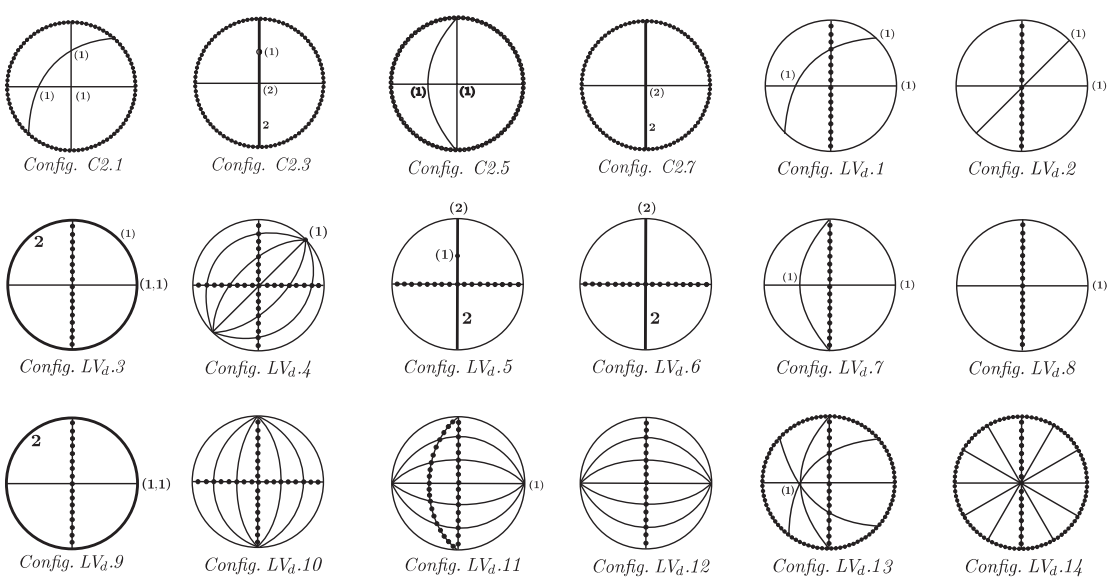

Fig. 2: **Configurations of *LV*-systems with infinite number of singularities**

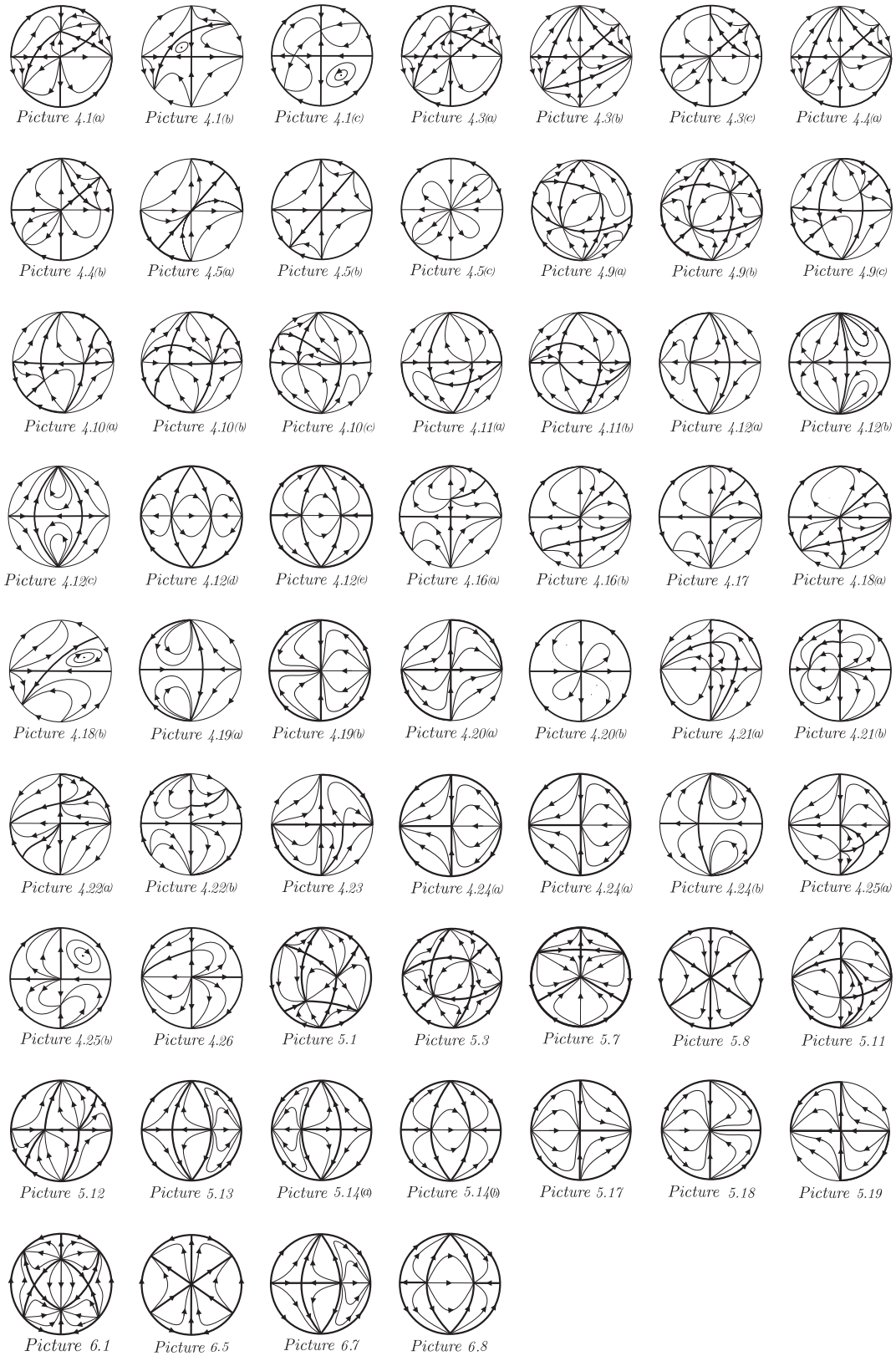

Fig. 3: **The phase portraits of *LV*-systems with at least 4 invariant lines**

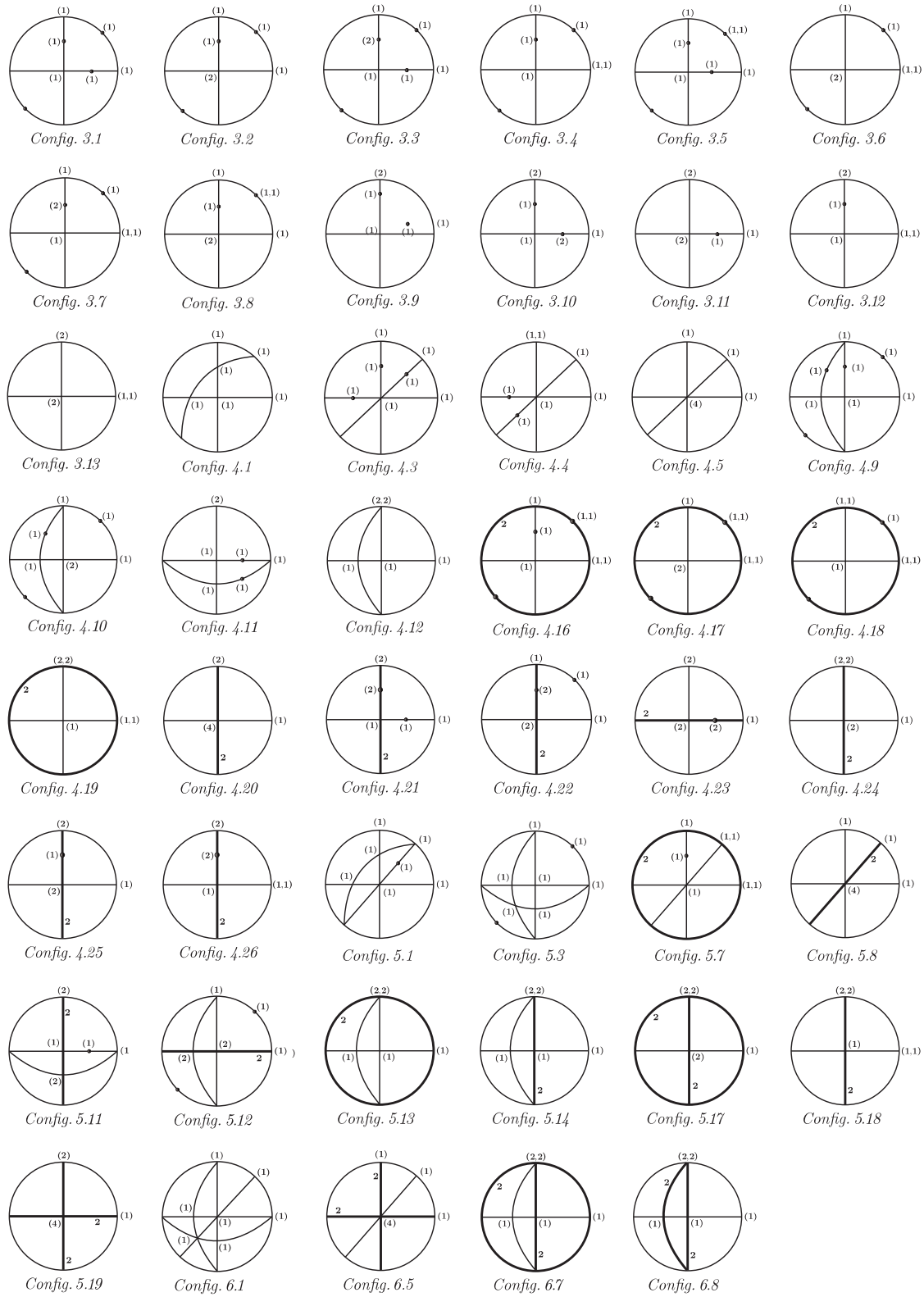

Fig. 4: **Configurations of *LV*-systems with finite number of singularities**

Table 4

Orbit representative	Necessary and sufficient conditions	Configuration
$(LV_d.1)$ $\begin{cases} \dot{x} = x(1+gy-y), \\ \dot{y} = (g-1)xy, \\ g(g-1) \neq 0 \end{cases}$	$\eta > 0,\ \mu_{0,1,2,3,4} = 0,\ \theta \neq 0,\ H_7 \neq 0$	Config. $\mathrm{LV}_d.1$
$(LV_d.2)$ $\begin{cases} \dot{x} = x(gx-y), \\ \dot{y} = (g-1)xy, \\ g(g-1) \neq 0 \end{cases}$	$\eta > 0,\ \mu_{0,1,2,3,4} = 0,\ \theta \neq 0,\ H_7 = 0$	Config. $\mathrm{LV}_d.2$
$(LV_d.3)$ $\begin{cases} \dot{x} = x(1+y), \\ \dot{y} = xy, \end{cases}$	$\eta > 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $H_4 = 0,\ H_7 \neq 0$	Config. $\mathrm{LV}_d.3$
$(LV_d.4)$ $\begin{cases} \dot{x} = xy, \\ \dot{y} = xy, \end{cases}$	$\eta > 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $H_4 = 0,\ H_7 = 0$	Config. $\mathrm{LV}_d.4$
$(LV_d.5)$ $\begin{cases} \dot{x} = xy, \\ \dot{y} = y(1-x+y), \end{cases}$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta \neq 0,\ H_7 \neq 0$	Config. $\mathrm{LV}_d.5$
$(LV_d.6)$ $\begin{cases} \dot{x} = xy, \\ \dot{y} = y(-x+y), \end{cases}$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta \neq 0,\ H_7 = 0$	Config. $\mathrm{LV}_d.6$
$(LV_d.7)$ $\begin{cases} \dot{x} = x(1+gx), \\ \dot{y} = (g-1)xy, \\ g(g-1) \neq 0 \end{cases}$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $K \neq 0,\ H_2 \neq 0$	Config. $\mathrm{LV}_d.7$
$(LV_d.8)$ $\begin{cases} \dot{x} = gx^2, \\ \dot{y} = (g-1)xy, \\ g(g-1) \neq 0 \end{cases}$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $K \neq 0,\ H_2 = 0$	Config. $\mathrm{LV}_d.8$
$(LV_d.9)$ $\begin{cases} \dot{x} = x, \\ \dot{y} = xy, \end{cases}$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $K = 0,\ N \neq 0,\ H_7 = 0,\ H_2 \neq 0$	Config. $\mathrm{LV}_d.9$
$(LV_d.10)$ $\begin{cases} \dot{x} = 0, \\ \dot{y} = xy, \end{cases}$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,\ K = 0,$ $N \neq 0,\ H_7 = 0,\ H_2 = 0,\ D = 0$	Config. $\mathrm{LV}_d.10$
$(LV_d.11)$ $\begin{cases} \dot{x} = x(x+2), \\ \dot{y} = 0, \end{cases}$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $K = 0,\ N = 0,\ D = N_1 = 0,\ N_5 > 0$	Config. $\mathrm{LV}_d.11$
$(LV_d.12)$ $\begin{cases} \dot{x} = x^2, \\ \dot{y} = 0, \end{cases}$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $K = 0,\ N = 0,\ D = N_1 = 0,\ N_5 = 0$	Config. $\mathrm{LV}_d.12$
$(LV_d.13)$ $\begin{cases} \dot{x} = x(1+x), \\ \dot{y} = xy, \end{cases}$	$C_2 = 0,\ \mu_{0,1,2,3,4} = 0,\ H_2 \neq 0$	Config. $\mathrm{LV}_d.13$
$(LV_d.14)$ $\begin{cases} \dot{x} = x^2, \\ \dot{y} = xy, \end{cases}$	$C_2 = 0,\ \mu_{0,1,2,3,4} = 0,\ H_2 = 0$	Config. $\mathrm{LV}_d.14$

3 Proof of the Main Theorem

We first prove two Lemmas (3.1 and 3.2) which will be needed later.

We shall denote by 's' (respectively 'n'; 'f'; 'c'; 'sn') a singular point of *saddle* (respectively *node*; *focus*; *center*; *saddle-node*) type.

Lemma 3.1. *Assume that a quadratic system belongs to the family* ***LV****. Then it possesses a center only if the condition $B_3 = 0$ is fulfilled. Moreover, if this system possesses a focus, then this focus could only be a strong one.*

Proof: Assume that a quadratic system is of Lotka-Volterra type. Then due to an affine transformation we can assume that this system belongs to the family of the systems

$$\dot{x} = x(c + gx + hy), \quad \dot{y} = y(f + mx + ny), \tag{3.1}$$

which could possess the following finite singularities:

$$M_1(0,0), \quad M_2(-c/g, 0), \quad M_3(0, -f/n), \quad M_4\left(\frac{cn - fh}{hm - gn}, \frac{fg - cm}{hm - gn}\right). \tag{3.2}$$

We shall denote by ρ_i (respectively Δ_i) the trace (respectively the determinant) of the respective linear matrix for the singularity M_i $(i = 1, 2, 3, 4)$; and by δ_i we denote the discriminant of the respective equation for the eigenvalues corresponding to the point M_i $(i = 1, 2, 3, 4)$.

Since the points M_1, M_2 and M_3 are placed on the invariant lines it is clear that only the point M_4 (which exists if $hm - gh \neq 0$) could be of the focus-center type. As it is known this occurs only if the discriminant δ_4) of the equation corresponding to the point M_4 is negative. Moreover it could be a center only if the respective trace $\rho_4 = 0$, where

$$\rho_4 = \big[n(cm - fg) + g(fh - cn))\big]/(gn - hm),$$
$$\delta_4 = \Big\{\big[n(cm - fg) - g(fh - cn))\big]^2 + 4hm(cm - fg)(fh - cn)\Big\}/(gn - hm)^2.$$

On the other hand for systems (3.1) calculation yields

$$B_3 = 3(f - c)\big[n(cm - fg) + g(fh - cn))\big]x^2y^2 = 3(f - c)\rho_4 x^2 y^2$$

and therefore the condition $\rho_4 = 0$ implies $B_3 = 0$, i.e. the last condition is necessary for the existence of a center.

Suppose now that a system from the family **LV**-systems could possess a weak focus. For this it is necessary $\delta_4 < 0$ and $\rho_4 = 0$. Replacing the singular point M_4 at the origin of coordinates we get the systems

$$\dot{x} = \Big(\frac{cn - fh}{hm - gn} + x\Big)(gx + hy), \quad \dot{y} = \Big(\frac{fg - cm}{hm - gn} + y\Big)(mx + ny). \tag{3.3}$$

Considering Lemma 2.7 for these systems calculations yield

$$\begin{aligned} I_1 &= \frac{n(fg - cm) + g(cn - fh)}{hm - gn}, \quad 2I_2 = I_1^2 + \delta_4, \quad I_3 = I_1(mn - gh)/2, \\ I_6 &= \frac{I_1}{4(hm - gn)} W(c, f, g, h, m, n), \end{aligned} \tag{3.4}$$

where $W(c, f, g, h, m, n)$ is a polynomial in the coefficients of systems. According to Lemma 2.7 the condition $I_1 = 0$ is necessary for $(0, 0)$ to be a center. Then $I_6 = I_3 = 0$ and the condition

$\delta_4 < 0$ implies $I_2 < 0$. Thus the conditions of Lemma 2.7 are satisfied and hence, $(0,0)$ of systems (3.3) and, consequently, the singular point M_4 of systems (3.1) is a center.

Thus a Lotka-Volterra quadratic system could not possess a weak focus. ∎

As it will follow from the proof of the Main Theorem the following assertion is valid:

Lemma 3.2. *Assume that a quadratic system belongs to the class* ***LV****. Then it could possess only one of the following configurations of the finite singularities:*

(a) *s,s,s,n;*	$(\overset{*}{a})$ *s,s,s,f;*	(b) *s,n,n,n;*	$(\overset{*}{b})$ *s,n,n,f;*	(c) *s,s,n,n;*	$(\overset{*}{c})$ *s,s,n,f;*
(d) *sn,s,s;*	(e) *sn,n,n;*	$(\overset{*}{e})$ *sn,n,f;*	(f) *sn,s,n;*	$(\overset{*}{f})$ *sn,s,f;*	(g) *s,s,n;*
$(\overset{*}{g})$ *s,s,f;*	(h) *s,n,n;*	$(\overset{*}{h})$ *s,n,f;*	(k) *sn,s;*	(l) *sn,n;*	$(\overset{*}{l})$ *sn,f.*

Remark 3.1. *For the notations of the phase portraits corresponding to Configs. 3.j ($j = 1, 2, \ldots, 13$) we shall use the respective number of the configuration and the corresponding additional letter (i) (or $(\overset{*}{i})$) with $i \in \{a, b, c, d, e, f, g, h, k, l\}$ depending of the configuration of its finite singularities indicated by Lemma 3.2. For example, the notation Picture 3.3(e2) denotes one of the phase portraits associated to Configs. 3.3 having the finite singularities sn,n,n. We note that we keep the letter only adding a star (i.e. $(\overset{*}{i})$) in the case when one node is substituted by a focus (which are locally topologically equivalent.*

Proof of the Main Theorem. In order to prove the Main Theorem we first notice that we can split the class of all ***LV***- systems into six distinct subclasses: (i) the class of all those ***LV***-system having exactly three simple real invariant lines; (ii) the three classes of ***LV***- systems possessing invariant lines of total multiplicity 4, respectively 5 and 6; (iii) the class of all those ***LV***systems with the line at infinity filled up with singularities; (iv) the class of all ***LV***systems which are degenerate. We note that in [25] and [26] the phase portraits of the quadratic systems with invariant lines of total multiplicity at least four are constructed. Moreover in [27] the topological classification of the whole family of quadratic systems with the infinite line filled up with singularities (the case $C_2 = 0$) is done and hence the phase portraits for the cases (ii), (iii) are already done. So it only remains to examine the cases (i) which have the configurations given by *Configs. 3.j* with $j = 1, 2, \ldots, 13$ and also the case (iv) of the degenerate ***LV***-systems with the configurations given by *Configs.* $LV_d.j$ with $j = 1, 2, \ldots, 14$.

3.1 Phase portraits of LV-systems with exactly three simple real invariant straight lines

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The final result concerning this class is encapsulated in Table 2. In this Table we observe that for any system with the configuration of invariant lines given by *Configs. 3.j* ($j = 1, 2, \ldots, 13$) the condition $B_3 \neq 0$ holds. Therefore by Lemma 3.1 this system could not have a center.

We shall consider step by step each one of the configurations *Configs. 3.j* ($j = 1, 2, \ldots, 13$) examining its respective representative given by Table 2.

Theorem 3.1. *The phase portraits of Lotka-Volterra quadratic differential systems possessing three simple real invariant straight lines are defined by 13 Configurations leading to a total of 88 phase portraits, 60 of which are topologically distinct. In Table 2 are listed in columns 2 and 3 the necessary and sufficient conditions for the realization of each one of the portraits appearing in column four.*

Table 5

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*			*Phase portrait*
Config. 3.1	$\eta > 0,\ \mu_0 B_3 H_9 \neq 0$ *and either* $\theta \neq 0$ *or* $(\theta = 0\ \&\ NH_7 \neq 0)$	$\mu_0 < 0,$ $K < 0$	$W_4 \geq 0$	$B_3U_1 < 0, U_2 < 0$	*Picture 3.1(a1)*
				$B_3U_1 < 0, U_2 > 0$	*Picture 3.1(a2)*
				$B_3U_1 > 0$	*Picture 3.1(a3)*
			$W_4 < 0$	$-$	*Picture 3.1($\overset{*}{a}$2)*
		$\mu_0 < 0,$ $K > 0$	$W_4 > 0$ *or* $W_4 = 0$ & $W_3 \geq 0$	$B_3U_1 < 0$	*Picture 3.1(b1)*
				$B_3U_1 > 0, U_2 > 0,$ $U_4 > 0, U_3 > 0$	
				$B_3U_1 > 0, U_2 < 0,$ $B_3H_{14} > 0$	*Picture 3.1(b2)*
				$B_3U_1 > 0, U_2 > 0,$ $U_4 > 0, U_3 < 0$	
				$B_3U_1 > 0, U_2 < 0,$ $B_3H_{14} < 0$	*Picture 3.1(b3)*
				$B_3U_1 > 0, U_2 > 0,$ $U_4 < 0$	
			$W_4 < 0$ *or* $W_4 = 0$ & $W_3 < 0$	$B_3U_1 < 0, U_2 < 0$	*Picture 3.1($\overset{*}{b}$1)*
				$B_3U_1 > 0, U_2 > 0,$ $U_4 > 0, U_3 > 0$	
				$B_3U_1 > 0, U_2 < 0$	*Picture 3.1($\overset{*}{b}$2)*
				$B_3U_1 > 0, U_2 > 0,$ $U_4 > 0, U_3 < 0$	
				$B_3U_1 > 0, U_2 > 0,$ $U_4 < 0$	*Picture 3.1($\overset{*}{b}$3)*
				$B_3U_1 < 0, U_2 > 0$	*Picture 3.1($\overset{*}{b}$4)*
		$\mu_0 > 0$	$W_4 > 0$ *or* $W_4 = 0$ & $W_3 \geq 0$	$U_2 < 0, B_3H_{14} < 0$	*Picture 3.1(c1)*
				$U_2 > 0, U_4 > 0,$ $B_3U_1 > 0$	
				$U_2 < 0, B_3H_{14} > 0$ $B_3U_1 < 0$	*Picture 3.1(c2)*
				$U_2 < 0, B_3H_{14} > 0$ $B_3U_1 > 0$	*Picture 3.1(c3)*
				$U_2 > 0, U_4 < 0$	
				$U_2 > 0, U_4 > 0,$ $B_3U_1 < 0$	*Picture 3.1(c4)*
			$W_4 < 0$ *or* $W_4 = 0$ & $W_3 < 0$	$B_3U_1 > 0, U_2 < 0$	*Picture 3.1($\overset{*}{c}$1)*
				$B_3U_1 < 0, U_2 < 0$	*Picture 3.1($\overset{*}{c}$2)*
				$B_3U_1 > 0, U_2 > 0$	*Picture 3.1($\overset{*}{c}$3)*
				$B_3U_1 < 0, U_2 > 0$	*Picture 3.1($\overset{*}{c}$4)*

Proof: We shall consider step by step each one of the configurations *Configs. 3.j* ($j = 1, 2, \ldots, 13$) examining its respective representative given by Table 2.

Table 5 *(continued)*

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*			*Phase portrait*
Config. 3.2	$\eta > 0,\ \mu_0 B_3 \neq 0$ $H_9 = 0, H_{13} = 0$ *and either* $\theta \neq 0$ *or* $(\theta = 0\ \&\ NH_7 \neq 0)$	$\mu_0 < 0,\ K < 0$			*Picture 3.2(d1)*
		$\mu_0 < 0,$ $K > 0$	$W_4 \geq 0$	$H_1 > 0$	*Picture 3.2(e1)*
				$H_1 < 0$	*Picture 3.2(e2)*
			$W_4 < 0$	$H_1 > 0$	*Picture 3.2($\overset{*}{e}$1)*
				$H_1 < 0, H_5 < 0$	*Picture 3.2($\overset{*}{e}$2)*
				$H_1 < 0, H_5 > 0$	*Picture 3.2($\overset{*}{e}$3)*
		$\mu_0 > 0$	$W_4 \geq 0$	$B_3 H_{14} < 0, H_5 < 0$	*Picture 3.2(f1)*
				$B_3 H_{14} < 0, H_5 > 0, H_1 < 0$	*Picture 3.2(f2)*
				$B_3 H_{14} < 0, H_5 > 0, H_1 > 0$	*Picture 3.2(f3)*
				$B_3 H_{14} > 0, H_5 < 0$	*Picture 3.2(f4)*
				$B_3 H_{14} > 0, H_5 > 0$	*Picture 3.2(f5)*
			$W_4 < 0$	$H_1 < 0$	*Picture 3.2($\overset{*}{f}$2)*
				$H_1 > 0$	*Picture 3.2($\overset{*}{f}$5)*
Config. 3.3	$\eta > 0,\ \mu_0 B_3 \neq 0$ $H_9 = 0, H_{13} \neq 0$ *and either* $\theta \neq 0$ *or* $(\theta = 0\ \&\ NH_7 \neq 0)$	$\mu_0 < 0,\ K < 0$			*Picture 3.3(d1)*
		$\mu_0 < 0,$ $K > 0$	$H_5 < 0$		*Picture 3.3(e1)*
			$H_5 > 0$		*Picture 3.3(e2)*
		$\mu_0 > 0$	$B_3 H_{14} < 0, H_5 < 0$		*Picture 3.3(f1)*
			$B_3 H_{14} < 0, H_5 > 0, H_1 < 0$		*Picture 3.3(f2)*
			$B_3 H_{14} < 0, H_5 > 0, H_1 > 0$		*Picture 3.3(f3)*
			$B_3 H_{14} > 0, H_5 < 0$		*Picture 3.3(f4)*
			$B_3 H_{14} > 0, H_5 > 0$		*Picture 3.3(f5)*
Config. 3.4	$\eta > 0,\ \theta B_3 H_9 \neq 0,$ $\mu_0 = H_{14} = 0$	$K < 0$	$W_4 \geq 0$	$B_3 U_1 < 0, U_2 < 0$	*Picture 3.4(g1)*
				$B_3 U_1 < 0, U_2 > 0$	*Picture 3.4(g2)*
				$B_3 U_1 > 0$	*Picture 3.4(g3)*
			$W_4 < 0$		*Picture 3.4($\overset{*}{g}$2)*
		$K > 0$	$W_4 > 0$ *or* $W_4 = 0$ & $W_3 \geq 0$	$B_3 U_1 < 0$	*Picture 3.4(h1)*
				$B_3 U_1 > 0, U_2 < 0, H_5 < 0$	*Picture 3.4(h2)*
				$B_3 U_1 > 0, U_2 > 0$	
				$B_3 U_1 > 0, U_2 < 0, H_5 > 0$	*Picture 3.4(h3)*
			$W_4 < 0$ *or* $W_4 = 0$ & $W_3 < 0$	$B_3 U_1 < 0,\ U_2 < 0$	*Picture 3.4($\overset{*}{h}$1)*
				$B_3 U_1 > 0,\ U_2 > 0$	*Picture 3.4($\overset{*}{h}$2)*
				$B_3 U_1 > 0,\ U_2 < 0$	*Picture 3.4($\overset{*}{h}$3)*
				$B_3 U_1 < 0,\ U_2 > 0$	*Picture 3.4($\overset{*}{h}$4)*
Config. 3.5	$\eta > 0,\ \theta B_3 H_9 H_{14} \neq 0,$ $\mu_0 = 0$	$K < 0$	$B_3 U_1 < 0$		*Picture 3.5(g1)*
			$B_3 U_1 > 0$		*Picture 3.5(g2)*
		$K > 0$	$B_3 U_1 < 0$		*Picture 3.5(h1)*
			$B_3 U_1 > 0,\ U_3 > 0$		
			$B_3 U_1 > 0,\ U_3 < 0$		*Picture 3.5(h2)*

Table 5 *(continued)*

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*			*Phase portrait*
Config. 3.6	$\eta > 0,\ \theta B_3 \neq 0,$ $\mu_0 = H_{14} = 0,$ $H_9 = H_{13} = 0$	$K < 0$			*Picture 3.6(k1)*
		$K > 0$	$W_4 > 0$		*Picture 3.6(l2)*
			$W_4 < 0$	$H_1 < 0$	*Picture 3.6($\overset{*}{l}$1)*
				$H_1 > 0$	*Picture 3.6($\overset{*}{l}$2)*
Config. 3.7	$\eta > 0,\ \theta B_3 H_{13} \neq 0,$ $\mu_0 = H_{14} = H_9 = 0$	$K < 0$			*Picture 3.7(k1)*
		$K > 0$	$H_5 > 0$		*Picture 3.7(l1)*
			$H_5 < 0$		*Picture 3.7(l2)*
Config. 3.8	$\eta > 0,\ \theta B_3 H_{14} \neq 0,$ $\mu_0 = H_9 = 0$	$K < 0$			*Picture 3.8(k1)*
		$K > 0$	$H_5 < 0$		*Picture 3.8(l1)*
			$H_5 > 0$		*Picture 3.8(l2)*
Config. 3.9	$\eta = 0,\ \theta \mu_0 B_3 H_9 \neq 0$	$\mu_0 < 0$	$W_4 > 0$ *or* $W_4 = 0$ & $W_3 \geq 0$	$U_4 < 0,\ H_4 < 0$	*Picture 3.9(b1)*
				$U_4 < 0,\ H_4 > 0$	*Picture 3.9(b2)*
				$U_4 > 0$	*Picture 3.9(b3)*
			$W_4 < 0$ *or* $W_4 = 0$ & $W_3 < 0$	$B_3 U_1 > 0,\ U_4 < 0$	*Picture 3.9($\overset{*}{b}$2)*
				$B_3 U_1 > 0,\ U_4 > 0$	*Picture 3.9($\overset{*}{b}$3)*
				$B_3 U_1 < 0$	*Picture 3.9($\overset{*}{b}$4)*
		$\mu_0 > 0$	$W_4 \geq 0$	$B_3 U_1 < 0,\ U_4 < 0$	*Picture 3.9(c1)*
				$B_3 U_1 > 0,\ H_4 > 0$	
				$B_3 U_1 < 0,\ U_4 > 0$	*Picture 3.9(c2)*
				$B_3 U_1 > 0,\ H_4 < 0$	*Picture 3.9(c3)*
			$W_4 < 0$	$B_3 U_1 > 0$	*Picture 3.9($\overset{*}{c}$1)*
				$B_3 U_1 < 0$	*Picture 3.9($\overset{*}{c}$2)*
Config. 3.10	$\eta = 0,\ H_9 = 0,$ $\theta B_3 \mu_0 H_{13} \neq 0$	$\mu_0 < 0$			*Picture 3.10(e1)*
		$\mu_0 > 0$	$B_3 U_1 < 0$		*Picture 3.10(f1)*
			$B_3 U_1 > 0$		*Picture 3.10(f2)*
Config. 3.11	$\eta = 0,\ \theta B_3 \mu_0 \neq 0,$ $H_9 = H_{13} = 0$	$\mu_0 < 0$			*Picture 3.11($\overset{*}{e}$1)*
		$\mu_0 > 0$	$H_5 > 0$	$W_4 \geq 0$	*Picture 3.11(f1)*
				$W_4 < 0$	*Picture 3.11($\overset{*}{f}$1)*
			$H_5 < 0$		*Picture 3.11(f2)*
Config. 3.12	$\eta = 0,\ \theta B_3 H_9 \neq 0,$ $\mu_0 = 0$	$W_4 > 0$	$H_5 < 0$		*Picture 3.12(h1)*
			$H_5 > 0$		*Picture 3.12(h2)*
		$W_4 < 0$	$B_3 U_1 > 0$		*Picture 3.12($\overset{*}{h}$2)*
			$B_3 U_1 < 0$		*Picture 3.12($\overset{*}{h}$3)*
Config. 3.13	$\eta = 0,\ \theta B_3 \neq 0,$ $\mu_0 = H_9 = 0$	–			*Picture 3.13($\overset{*}{l}$1)*

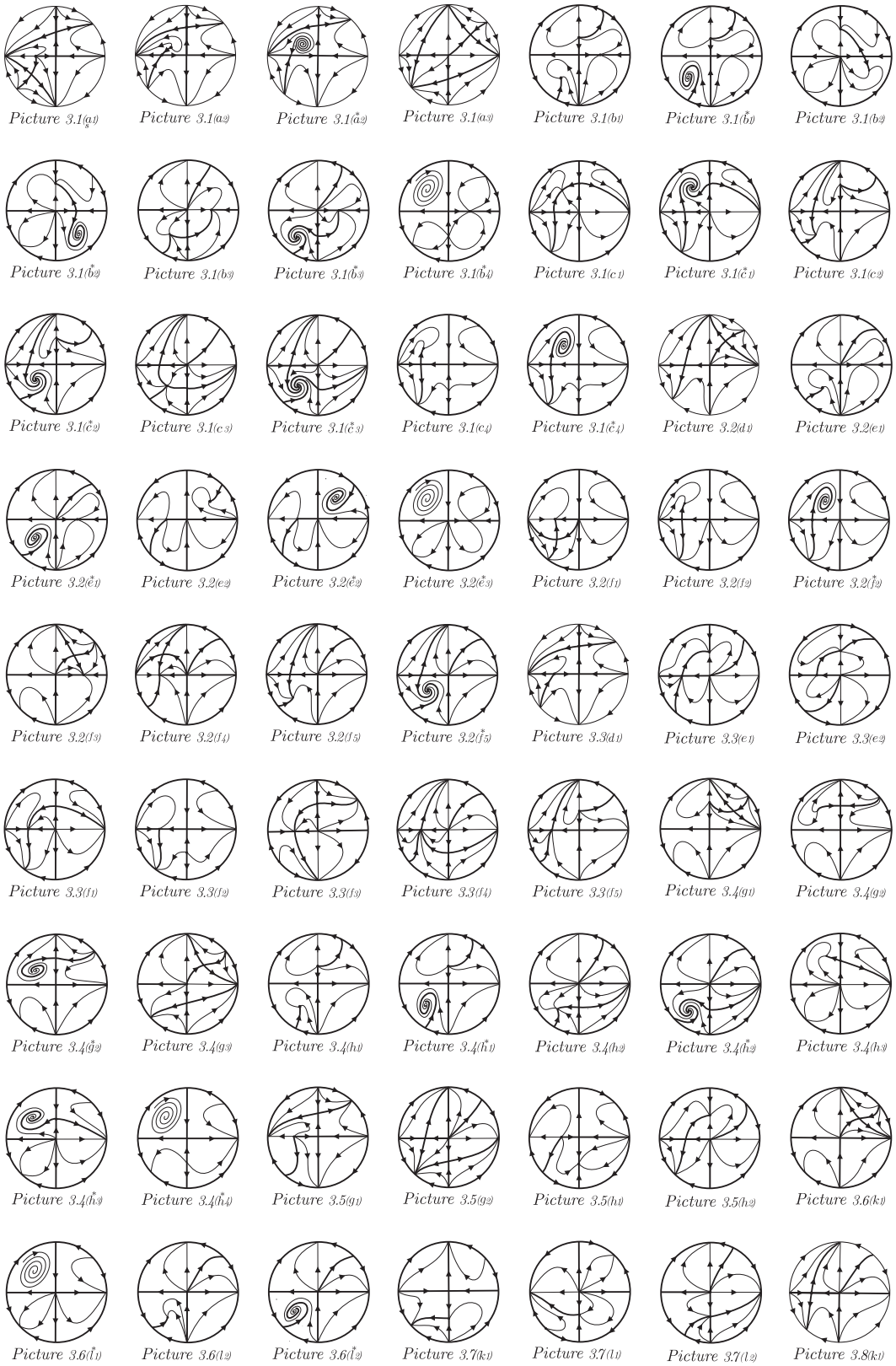

Fig. 5: **Phase portraits of the family of *LV*-systems with exactly three invariant lines**

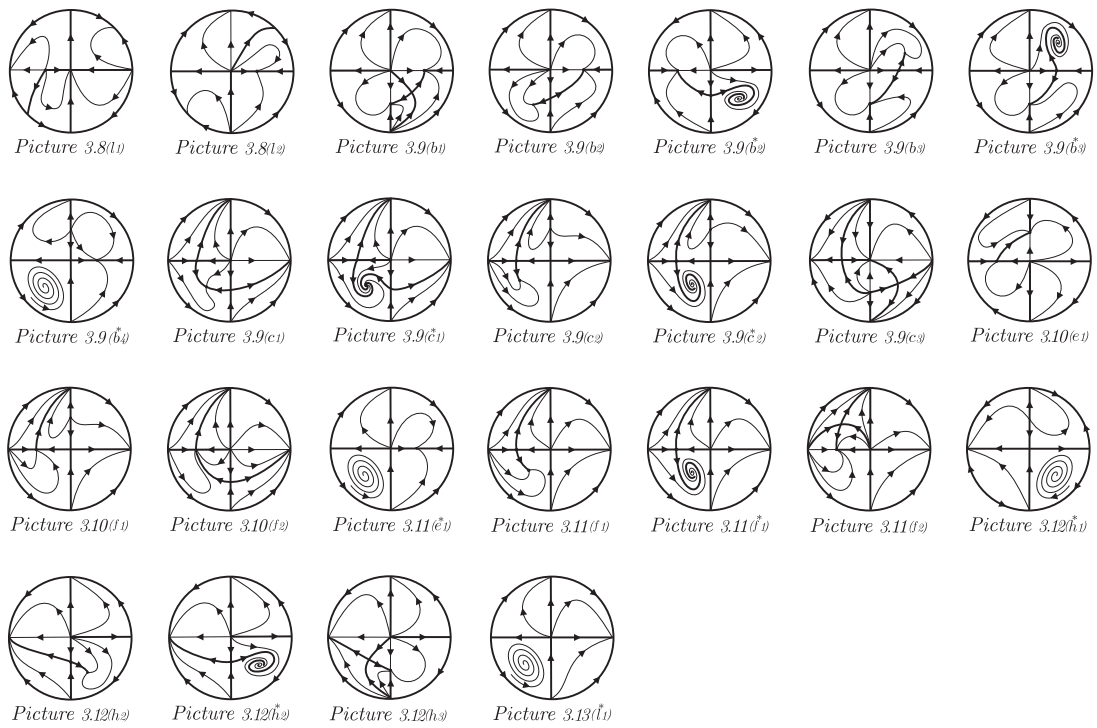

Fig. 5: (*continued*)

3.1.1 The phase portraits associated to *Config. 3.1*

According to Table 2 we shall consider the family of systems

$$\dot{x} = x[1 + gx + (h-1)y], \quad \dot{y} = y[f + (g-1)x + hy], \tag{3.5}$$

for which the condition

$$gh(g+h-1)(g-1)(h-1)f(f-1)(fg+h)(g-1-fg)(fh-f-h) \neq 0 \tag{3.6}$$

holds. For the all four distinct finite singularities of systems (3.5) with the condition (3.6) we have

$$\begin{aligned} M_1(0,0): &\quad \Delta_1 = f, \quad \rho_1 = f+1, \quad \delta_1 = (f-1)^2; \\ M_2(-1/g,0): &\quad \Delta_2 = (g-1-fg)/g, \quad \delta_2 = (1+fg)^2/g^2; \\ M_3(0,-f/h): &\quad \Delta_3 = f(fh-f-h)/h, \quad \delta_3 = (f+h)^2/h^2; \\ M_4\left(\frac{fh-f-h}{g+h-1}, \frac{g-1-fg}{g+h-1}\right): &\quad \Delta_4 = \frac{(fh-f-h)(g-1-fg)}{g+h-1}, \\ &\quad \rho_4 = \frac{(fg+h)}{1-g-h}, \quad \delta_4 = \rho_4^2 - 4\Delta_4. \end{aligned} \tag{3.7}$$

and for the three infinite singular points we obtain

$$R_1(1,1,0): \ \widetilde{\Delta}_1 = 1-g-h; \quad R_2(1,0,0): \ \widetilde{\Delta}_2 = g; \quad R_3(0,1,0): \ \widetilde{\Delta}_3 = h. \tag{3.8}$$

Observation 3.2. *For our convenience we denote the infinite singular point $R_2(1,0,0)$ (respectively $R_3(0,1,0)$) with the same lower index as the singular point M_2 (respectively M_3) located on the same line.*

Remark 3.3. *We observe that the substitution* $(x, y, t, f, g, h) \mapsto (fy, fx, t/f, 1/f, h, g)$ *keeps systems* (3.5) *and interchanges the two invariant lines (and consequently the respective singularities).*

Considering (3.7) the condition (3.6) is equivalent to the condition

$$\Delta_1\Delta_2\Delta_3\Delta_4\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3(\widetilde{\Delta}_1+\widetilde{\Delta}_2)(\widetilde{\Delta}_1+\widetilde{\Delta}_3)\delta_1\rho_4 \neq 0 \tag{3.9}$$

Taking into account (3.7) and (3.8) we evaluate for systems (3.5) the needed invariant polynomials:

$$\begin{aligned}
\mu_0 &= gh(g+h-1) = -\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3,\\
K &= 2g(g-1)x^2 + 4ghxy + 2h(h-1)y^2,\\
W_4 &= \mu_0^2\,\delta_1\delta_2\delta_3\delta_4, \quad W_3 = \mu_0^2 \sum_{1\le i<j<l\le 4} \delta_i\,\delta_j\,\delta_l,\\
H_9 &= -576f^2(1-g+fg)^2(f+h-fh)^2 = -576\Delta_2^2\Delta_3^2,\\
H_{14} &= 30(1-f)gh(fg+h) = 30\rho_4\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3(1-f),\\
B_3 &= 3(1-f)(fg+h)x^2y^2 = 3\widetilde{\Delta}_1\rho_4(1-f)x^2y^2,\\
U_1 &= \frac{1}{8}f(1-f)(g-1)^2(h-1)^2(fg+h) = \frac{1}{8}\Delta_1\widetilde{\Delta}_1\rho_4(1-f)(g-1)^2(h-1)^2,\\
U_2 &= 3f(g-1-fg)(fh-f-h) = 3\Delta_2\Delta_3\widetilde{\Delta}_2\widetilde{\Delta}_3,\\
U_3 &= \frac{1}{2}(1+f)(g-1)(h-1)(fg+h) = \frac{1}{2}\widetilde{\Delta}_1\rho_1\rho_4(g-1)(h-1),\\
U_4 &= f(g-1-fg)(fh-f-h)(fg-2f-f^2g-h+fh).
\end{aligned} \tag{3.10}$$

Denoting $\mathcal{A} = f(g-1-fg) = \Delta_1\Delta_2\widetilde{\Delta}_2$ and $\mathcal{B} = (fh-f-h) = \Delta_3\widetilde{\Delta}_3/\Delta_1$ the invariants U_2 and U_4 could be represented in the following forms, respectively:

$$U_2 = 3\mathcal{A}\mathcal{B}, \quad U_4 = \mathcal{A}\mathcal{B}(\mathcal{A}+\mathcal{B}).$$

So, taking into consideration (3.10) we make the next remark.

Remark 3.4. *Provided the condition* (3.6) *is satisfied, the following relations hold:*

- $B_3U_1 = \frac{3}{8}\Delta_1\rho_4^2\widetilde{\Delta}_1^2(g-1)^2(h-1)^2(f-1)^2x^2y^2 \quad \Rightarrow \quad \operatorname{sign}(B_3U_1) = \operatorname{sign}(\Delta_1)$*;*
- $\operatorname{sign}(U_2) = \operatorname{sign}(\Delta_2\Delta_3\widetilde{\Delta}_2\widetilde{\Delta}_3)$*;*
- *If* $U_2 > 0 \quad \Rightarrow \quad \operatorname{sign}(U_4) = \operatorname{sign}(\Delta_1\Delta_2\widetilde{\Delta}_2) = \operatorname{sign}(\Delta_1\Delta_3\widetilde{\Delta}_3)$*;*
- $B_3H_{14} = 90\rho_4^2\widetilde{\Delta}_2\widetilde{\Delta}(f-1)^2\widetilde{\Delta}_1^2x^2y^2 \quad \Rightarrow \quad \operatorname{sign}(B_3H_{14}) = \operatorname{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3)$.

3.1.1.1 The case $\mu_0 < 0$. As $\mu_0 = \operatorname{Discrim}(K)/16$ we conclude that $K(x, y)$ ie a binary form with well defined sign and we shall consider two subcases: $K < 0$ and $K > 0$.

3.1.1.1.1 The subcase $K < 0$. Then according to [1, Table 1] on the finite part of the phase plane of systems (3.5) there are three saddles and one anti-saddle. Moreover the anti-saddle is a node if $W_4 \geq 0$ and it is of the center-focus type if $W_4 < 0$. However by Lemma 3.1 in the second case we have a strong focus.

On the other hand, as at infinity there exist three real distinct singularities, according to the index theory all of them must be nodes.

I. Assume first $W_4 \geq 0$. We claim that in this case the phase portraits of systems (3.5)

correspond to one of those indicated below if and only if the following conditions are respectively satisfied:

$$\begin{array}{lcl} \textit{Picture 3.1(a1)} & \Leftrightarrow & B_3U_1 < 0,\ U_2 < 0; \\ \textit{Picture 3.1(a2)} & \Leftrightarrow & B_3U_1 < 0,\ U_2 > 0; \\ \textit{Picture 3.1(a3)} & \Leftrightarrow & B_3U_1 > 0. \end{array}$$

Indeed, as by Remark 3.4 we have $\operatorname{sign}(\Delta_1) = \operatorname{sign}(B_3U_1)$ we shall consider both possibilities.

1) If $B_3U_1 < 0$ then $\Delta_1 < 0$ and hence the singular point $M_1(0,0)$ is a saddle. On the other hand as all infinite singular points are nodes this implies $\widetilde{\Delta}_i > 0$, $i = 1,2,3$. Then by Remark 3.4 we get $\operatorname{sign}(U_2) = \operatorname{sign}(\Delta_2\Delta_3)$.

Thus, if $U_2 < 0$ then $\Delta_2\Delta_3 < 0$ and one of the points M_2 or M_3 is a node and the remaining points are saddles. This univocally leads to *Picture 3.1(a1)*.

Assuming $U_2 > 0$ we obtain $\Delta_2\Delta_3 > 0$, i.e. both points are saddles and the point M_4 is a node. In this case we clearly have *Picture 3.1(a2)*.

2) Suppose now $B_3U_1 > 0$. Then $\Delta_1 > 0$ and hence the singular point $M_1(0,0)$ is a node, whereas the remaining tree finite singularities are saddles. As the infinite singular points are nodes we get *Picture 3.1(a3)*. This completes the proof of our claim.

II. Assume $W_4 < 0$. Then on the phase plane, apart from the three saddles there exists a focus which clearly could only be the singular point M_4. It is known that this point must be located inside the triangle formed by other three saddle points (see for instance [5]).

We claim that there could not be a separatrix connection M_2M_3. Indeed, suppose the contrary, that such a connection exists. Then evidently on the segment of the straight line M_2M_3 there must be at least one point of contact. Therefore according to [7, Theorem 2.5] the straight line passing through these singular points must be an invariant line. However this contradicts the fact that *Config. 3.1* contains only two simple invariant affine lines. These arguments lead univocally to the phase portrait which is topologically equivalent to *Picture 3.1($\overset{*}{a}$2)*.

3.1.1.1.2 The subcase $K > 0$ Then according to [1, Table 1] systems (3.5) possess one saddle and three anti-saddles. Clearly only one anti-saddle could be a focus and considering [1, Table 1] besides the saddle we have three nodes if either $W_4 > 0$ or $W_4 = 0$ and $W_3 \geq 0$; and we have two nodes and a focus if either $W_4 < 0$ or $W_4 = 0$ and $W_3 < 0$.

I. Assume first $W_4 > 0$ *or* ($W_4 = 0$,$W_3 \geq 0$). We claim that in this case the phase portraits of systems (3.5) correspond to one of the ones indicated below if and only if the following conditions are satisfied, respectively:

$$\begin{array}{lcl} \textit{Picture 3.1(b1)} & \Leftrightarrow & \text{either } B_3U_1 < 0, \text{ or } B_3U_1 > 0,\ U_2 > 0,\ U_4 > 0,\ U_3 < 0; \\ \textit{Picture 3.1(b2)} & \Leftrightarrow & B_3U_1 > 0 \text{ and either } U_2 < 0,\ B_3H_{14} > 0, \text{ or } U_2 > 0,\ U_4 > 0,\ U_3 > 0; \\ \textit{Picture 3.1(b3)} & \Leftrightarrow & B_3U_1 > 0 \text{ and either } U_2 < 0,\ B_3H_{14} < 0, \text{ or } U_2 > 0,\ U_4 < 0. \end{array}$$

Indeed, first of all we observe that at infinity we must have one node and two saddles (the sum of the indexes must be -1). Then due to Remark 3.3 without loss of generality we may assume that $R_2(1,0,0)$ is a saddle, i.e. $\widetilde{\Delta}_2 < 0$.

As it was mentioned previously, the type of the singular point $M_1(0,0)$ depends on the sign of the invariant polynomial B_3U_1. So we shall consider the respective two cases.

1) If $B_3U_1 < 0$ then $\Delta_1 < 0$ and hence the singular point $M_1(0,0)$ is a saddle and the other three singularities are nodes. As $R_2(1,0,0)$ is a saddle by 3.8 we have $g < 0$ and this fixes the

position of the finite node $M_2(-1/g, 0)$. We claim that in this case the singular point $R_3(0, 1, 0)$ could not be a saddle. Indeed, assuming the contrary we have $\widetilde{\Delta}_3 < 0$ (i.e. $h < 0$) and since $f < 0$ we get the domain in the second quadrant bordered by the three invariant lines (one being the line at infinity) and having on its border only three singularities, which are saddles. Moreover the singular point M_4 inside this domain is forced in this case to be a focus or a center. So we obtain a contradiction which proves our claim.

Thus $R_3(0, 1, 0)$ is a node (i.e. $h > 0$) and we get univocally the phase portrait topologically equivalent to *Picture 3.1(b1)*.

2) Assume now $B_3U_1 > 0$, i.e. the singular point $M_1(0, 0)$ is a node. As by our assumption $R_2(1, 0, 0)$ is a saddle (i.e. $\widetilde{\Delta}_2 < 0$) considering Remark 3.4 we obtain sign $(U_2) = -$sign $(\Delta_2\Delta_3\widetilde{\Delta}_3)$.

a) If $U_2 < 0$ then we obtain $\Delta_2\Delta_3\widetilde{\Delta}_3 > 0$. On the other hand by Remark 3.4 we have sign $(B_3H_{14}) =$ sign $(\widetilde{\Delta}_2\widetilde{\Delta}_3)$ and we shall consider two subcases: $B_3H_{14} > 0$ and $B_3H_{14} < 0$.

***α*)** Suppose first $B_3H_{14} > 0$. Since $\widetilde{\Delta}_2 < 0$ we obtain $\widetilde{\Delta}_3 < 0$, i.e. $h < 0$. Therefore the condition $U_2 < 0$ implies $\Delta_2\Delta_3 < 0$, i.e. one of the singularities $M_2(-1/g, 0)$ or $M_3(0, -f/h)$ is a saddle and other one is a node. As both infinite singular points $R_2(1, 0, 0)$ and $R_3(0, 1, 0)$ are of the same types (saddles), due to Remark 3.3 without loss of generality we may assume that $M_2(-1/g, 0)$ is a saddle. Taking into account that the finite saddle must be inside the triangle formed by three finite nodes we obviously arrive to the phase portrait topologically equivalent to *Picture 3.1(b2)*.

***β*)** In the case $B_3H_{14} < 0$ we obtain $\widetilde{\Delta}_3 > 0$, i.e. $h > 0$ and hence, besides the saddle R_2 at infinity we have the node R_3 and the saddle R_1. In this case considering the condition $U_2 < 0$ we get $\Delta_2\Delta_3 > 0$ and this means that both the singular points M_2 and M_3 are nodes. Clearly the remaining point M_4 is a saddle. Then considering the location of the singularities we get the phase portrait which is topologically equivalent to *Picture 3.1(b3)*.

b) Suppose now $U_2 > 0$. Then due to the Remark 3.4 and $\widetilde{\Delta}_2 < 0$ we get $\Delta_2\Delta_3\widetilde{\Delta}_3 < 0$. Moreover, as $U_2 > 0$ we have sign $(U_4) =$ sign $(\Delta_1\Delta_3\widetilde{\Delta}_3)$ and since $\Delta_1 > 0$ then clearly we obtain sign $(U_4) =$ sign $(\Delta_3\widetilde{\Delta}_3) = -$sign (Δ_2).

***α*)** Admit first $U_4 > 0$, i.e. $\Delta_3\widetilde{\Delta}_3 > 0$. So, we have $\Delta_2 < 0$ (i.e. M_2 is a saddle) and then M_3 and M_4 are nodes. Hence $\Delta_3 > 0$ and this implies $\widetilde{\Delta}_3 > 0$, i.e. R_3 is a node. Taking into consideration the node R_3 and the saddles R_2 and R_1 at infinity, we arrive at the two possibilities given by the phase portraits presented in *Figures 1* and *2*. We observe that for the first (respectively second) phase portraits the nodes M_1 and M_4 have different (respectively the same) stability. As $\rho_1 = 1 + f > 0$ we conclude that the realization of each portrait depends on the sign of ρ_4. We point out that the singularity M_4 changes its stability when ρ_4 change the sign (passing through zero) and since $\rho_4 \neq 0$ (due to $B_4 \neq 0$) we could not have a separatrix connection M_2R_1.

On the other hand, as R_1 is a saddle (i.e. $1 - g - h < 0$) considering $g < 0$ we get $h > 1 - g > 1$. Therefore according to (3.10) we obtain sign $(U_3) =$ sign $(\rho_1\rho_4)$. Thus, we get *Figure 1* if $U_3 < 0$ and *Figure 2* if $U_3 > 0$. It remains to note, that the phase portrait given by *Figure 1* (respectively by *Figure 2*) is topologically equivalent to *Picture 3.1(b2)* (respectively to *Picture 3.1(b1)*).

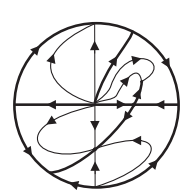

Figure 1

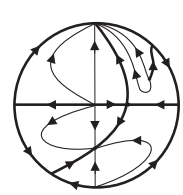

Figure 2

***β*)** Assuming $U_4 < 0$ we have $\Delta_2 > 0$ (i.e. M_2 is a node). So we have two possibilities: either M_3 is a saddle and M_4 is a node, or M_3 is a node and M_4 is a saddle. In the first (respectively the second) case due to $\Delta_3\widetilde{\Delta}_3 < 0$ we obtain that R_3 is a node (respectively a saddle) and hence R_1 is a saddle (respectively a node). So in both the cases we get the phase portraits which are topologically equivalent to *Picture 3.1(b3)*.

II. Suppose now $W_4 < 0$ or ($W_4 = 0$,$W_3 < 0$). Then it was mentioned earlier that systems (3.5) possess as finite singularities a saddle, two nodes and a focus. We claim that in this case the phase portrait of these systems corresponds to one of the portraits indicated below if and only if the corresponding conditions given below are satisfied, respectively:

$$\begin{array}{lcl}
\textit{Picture 3.1}(\overset{*}{b}1) & \Leftrightarrow & \text{either } B_3U_1 < 0,\ U_2 < 0, \text{ or } B_3U_1 > 0,\ U_2 > 0,\ U_4 > 0,\ U_3 > 0; \\
\textit{Picture 3.1}(\overset{*}{b}2) & \Leftrightarrow & B_3U_1 > 0 \text{ and either } U_2 < 0, \text{ or } U_2 > 0,\ U_4 > 0,\ U_3 < 0; \\
\textit{Picture 3.1}(\overset{*}{b}3) & \Leftrightarrow & B_3U_1 > 0,\ U_2 > 0,\ U_4 < 0; \\
\textit{Picture 3.1}(\overset{*}{b}4) & \Leftrightarrow & B_3U_1 < 0,\ U_2 > 0.
\end{array}$$

Indeed, first of all we recall that the singular point of the focus type could be only M_4.

1) *The subcase $B_3U_1 < 0$.* Then by Remark 3.4 we have $\Delta_1 < 0$ (yielding $f < 0$), i.e. $M_1(0,0)$ is a saddle and M_2 and M_3 are nodes (M_4 being a focus). At infinity we have the same singularities: two saddles and one node. Hence as in the previous case, due to Remark 3.3 we may assume that $R_2(1,0,0)$ is a saddle. So one of the remaining infinite points R_3 and R_1 is a saddle and other one is a node. Moreover, since $\Delta_2 > 0$ and $\Delta_3 > 0$, by Remark 3.4 we find out that $\operatorname{sign}(U_2) = -\operatorname{sign}(\widetilde{\Delta}_3)$. So we shall examine two cases: $U_2 < 0$ and $U_2 > 0$.

a) If $U_2 < 0$ then $\widetilde{\Delta}_3 > 0$ and hence R_3 is node and R_1 is a saddle. Considering the focus $M_4(x_4, y_4)$ we univocally arrive to the *Picture 3.1*($\overset{*}{b}$*1)*.

b) Admit now $U_2 > 0$. Then $\widetilde{\Delta}_3 < 0$ and therefore the singular point R_3 is a saddle and R_1 is a node. Taking into account the location of all the singularities in this case we get *Picture 3.1*($\overset{*}{b}$*4)*.

2) *The subcase $B_3U_1 > 0$.* Considering Remark 3.4 we have $\Delta_1 > 0$ (i.e. $f > 0$) and therefore $M_1(0,0)$ is a node. Hence $\Delta_2\Delta_3 < 0$ (since M_4 is a focus) and as $\tilde{\Delta}_2 < 0$ according to Remark 3.4 we obtain $\operatorname{sign}(U_2) = \operatorname{sign}(\widetilde{\Delta}_3)$.

a) Suppose $U_2 < 0$. Then $\widetilde{\Delta}_3 < 0$ and hence $R_3(0,1,0)$ is a saddle. Therefore since the infinite singularity $R_2(1,0,0)$ is also a saddle, then without loss of generality we may assume that the point $M_2(-1/g, 0)$ is a saddle due Remark 3.3. Then considering the positions of the focus M_3 and of the node R_1 we univocally arrive to the *Picture 3.1*($\overset{*}{b}$*2)*.

b) Assume now $U_2 > 0$. Then we get $\widetilde{\Delta}_3 > 0$ (i.e. $h > 0$) and in this case the singular point R_3 is a node and the third infinite point R_1 is a saddle. Hence according to Remark 3.4 we have $\operatorname{sign}(U_4) = \operatorname{sign}(\Delta_1\Delta_3\widetilde{\Delta}_3) = \operatorname{sign}(\Delta_3)$.

***α*)** Admit first $U_4 < 0$. Then $\Delta_3 < 0$ (i.e. M_3 is a saddle) and therefore M_2 is a node. Taking into consideration the location of all the singularities of a system (3.5) in the case under consideration we obtain univocally the phase portrait given by *Picture 3.1*($\overset{*}{b}$*3)*.

***β*)** If $U_4 > 0$ then $\Delta_3 > 0$, i.e. M_3 is a node and hence M_2 is a saddle. So, in the same manner as in the previous case (more precisely in the case with one saddle and three nodes) we obtain two different phase portraits given by *Figure 1* and *Figure 2* (see page 33) but with a unique difference: instead of the node M_4 we have now the focus M_4. Moreover, it is clear, that the stabilities of the node M_1 and of the focus M_4 distinguish these two phase portraits. More exactly, we obtain *Figure 1* if $\rho_1\rho_4 < 0$ and *Figure 2* if $\rho_1\rho_4 > 0$. It remains to remark, that *Figure 1* (respectively *Figure 2*) is topologically equivalent to *Picture 3.1*($\overset{*}{b}$*2)* (respectively *Picture 3.1*($\overset{*}{b}$*1)*) and according to (3.10), in this case we have $\operatorname{sign}(\rho_1\rho_4) = \operatorname{sign}(U_3)$.

3.1.1.2 The case $\mu_0 > 0$. Then according to [1, Table 1] on the finite part of the phase plane, for systems (3.5) there are two saddles and two anti-saddles. Moreover as for systems in this family

one anti-saddle is always a node, by [1] we conclude that both anti-saddles are nodes if either $W_4 > 0$ or $W_4 = 0$ and $W_3 \geq 0$; and one of them is a focus if either $W_4 < 0$ or $W_4 = 0$ and $W_3 < 0$.

On the other hand since the three singular points at infinity are simple, then by the index theory we must have two nodes and a saddle. So due to Remark 3.3 we may assume that the point $R_2(1,0,0)$ is a node, i.e. $\tilde{\Delta}_2 > 0$ (this yields $g > 0$).

3.1.1.2.1 *Assume first $W_4 > 0$ or ($W_4 = 0$,$W_3 \geq 0$).* So systems (3.5) have two saddles and two nodes. We claim that the phase portrait of a system in this family is given by one of the portraits indicated below if and only if the corresponding conditions are satisfied, respectively:

$$\begin{array}{lcl}
\textit{Picture 3.1(c1)} & \Leftrightarrow & \text{either } U_2 < 0,\ B_3H_{14} < 0, \ \text{ or } \ U_2 > 0,\ U_4 > 0,\ B_3U_1 > 0; \\
\textit{Picture 3.1(c2)} & \Leftrightarrow & U_2 < 0,\ B_3H_{14} > 0,\ B_3U_1 < 0; \\
\textit{Picture 3.1(c3)} & \Leftrightarrow & \text{either } U_2 < 0,\ B_3H_{14} > 0,\ B_3U_1 > 0, \ \text{ or } \ U_2 > 0,\ U_4 < 0; \\
\textit{Picture 3.1(c4)} & \Leftrightarrow & U_2 > 0,\ U_4 > 0,\ B_3U_1 < 0.
\end{array}$$

Indeed first we recall that the type of the singular point $M_1(0,0)$ is governed by the invariant polynomial B_3U_1.

I. The subcase $B_3U_1 < 0$. Then by Remark 3.4 it follows $\Delta_1 < 0$ (yielding $f < 0$), i.e. $M_1(0,0)$ is a saddle.

1) *Suppose first $U_2 < 0$.* Then by Remark 3.4 we get $\Delta_2\Delta_3\widetilde{\Delta}_2\widetilde{\Delta}_3 < 0$ and $\operatorname{sign}(B_3H_{14}) = \operatorname{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3)$.

a) If $B_3H_{14} < 0$ then $\widetilde{\Delta}_2\widetilde{\Delta}_3 < 0$ and this implies $\Delta_2\Delta_3 > 0$. Therefore (as M_1 is a saddle) both the points M_2 and M_3 are nodes. On the other hand since $\widetilde{\Delta}_2 > 0$ we obtain $\widetilde{\Delta}_3 < 0$, i.e. R_3 is a saddle. So taking into considerations the locations of the finite singularities we arrive univocally at the phase portrait which is topologically equivalent to *Picture 3.1(c1)*.

b) In the case $B_3H_{14} > 0$ we obtain $\widetilde{\Delta}_2\widetilde{\Delta}_3 > 0$ (i.e. R_2 and R_3 are both nodes) and this implies $\Delta_2\Delta_3 < 0$. Therefore one of the points M_2 or M_3 is a saddle and without loss of generality due to Remark 3.3 we may assume that such a saddle is M_2. So in this case we get univocally the *Picture 3.1(c2)*.

2) *Assume now $U_2 > 0$.* Then $\Delta_2\Delta_3\widetilde{\Delta}_2\widetilde{\Delta}_3 > 0$ and according to Remark 3.4 due to the condition $\Delta_1 < 0$ we obtain $\operatorname{sign}(U_4) = -\operatorname{sign}(\Delta_2\widetilde{\Delta}_2) = -\operatorname{sign}(\Delta_3\widetilde{\Delta}_3)$.

a) If $U_4 < 0$ then we obtain $\Delta_2\widetilde{\Delta}_2 > 0$ and $\Delta_3\widetilde{\Delta}_3 > 0$. As by assumption $\widetilde{\Delta}_2 > 0$ we obtain $\Delta_2 > 0$, i.e. the singular point M_2 is a node. We claim that in this case the singular point R_3 could not be a saddle. Indeed, supposing the contrary, we obtain $\widetilde{\Delta}_3 = h < 0$ and since $f < 0$ this implies $\Delta_3 = f(fh - f - g)/h > 0$. However this contradicts $\Delta_3\widetilde{\Delta}_3 > 0$ and our claim is proved.

Thus we get the conditions $\Delta_2 > 0$, $\Delta_3 > 0$, $\widetilde{\Delta}_2 > 0$ and $\widetilde{\Delta}_3 > 0$. In other words all the singularities M_2, M_3, R_2 and R_3 are nodes. Considering the position of the saddles M_4 and R_1 we univocally get a phase portrait which is topologically equivalent to *Picture 3.1(c3)*.

b) Assuming $U_4 > 0$ we obtain $\Delta_2\widetilde{\Delta}_2 < 0$ and $\Delta_3\widetilde{\Delta}_3 < 0$. As $\widetilde{\Delta}_2 > 0$ we obtain $\Delta_2 < 0$, i.e. the singular point M_2 is a saddle. Therefore the other two finite singularities are nodes. Hence $\Delta_3 > 0$ and this implies $\widetilde{\Delta}_3 < 0$, i.e. the infinite point R_3 is a saddle. i.e. the singular point M_2 is a node. Considering the location of the singularities we obtain univocally *Picture 3.1(c4)*.

II. The subcase $B_3U_1 > 0$. Considering Remark 3.4 we obtain $\Delta_1 > 0$ (i.e. $M_1(0,0)$ is a node). As by assumption R_2 is a node (i.e. $\widetilde{\Delta}_2 > 0$), by Remark 3.4 we get $\operatorname{sign}(U_2) = \operatorname{sign}(\Delta_2\Delta_3\widetilde{\Delta}_3)$ and $\operatorname{sign}(B_3H_{14}) = \operatorname{sign}(\widetilde{\Delta}_3)$.

1) *Assume first $U_2 < 0$.* Then $\Delta_2\Delta_3\widetilde{\Delta}_3 < 0$ and we shall consider two subcases: $B_3H_{14} < 0$

and $B_3H_{14} > 0$.

a) If $B_3H_{14} < 0$ then we have $\widetilde{\Delta}_3 < 0$, i.e. R_3 is a saddle and then R_1 is a node. In this case the condition $U_2 < 0$ implies $\Delta_2\Delta_3 > 0$ and as M_1 is a node, both singular points M_2 and M_3 must be saddles. Therefore the fourth point M_4 is a node. So taking into considerations the location of the singular points and their respective types we obtain univocally *Picture 3.1(c1)*.

b) Suppose now $B_3H_{14} > 0$, i.e. $\widetilde{\Delta}_3 > 0$. Hence both points R_2 and R_3 are nodes and R_1 is a saddle and the condition $U_2 < 0$ yields $\Delta_2\Delta_3 < 0$, i.e. one of points M_2 or M_3 is a node and another one is a saddle. As R_2 and R_3 are nodes due to a substitution (see Remark 3.3) we may consider that M_2 is a saddle. So M_4 is a saddle and we arrive univocally at a phase portrait topologically equivalent to *Picture 3.1(c3)*.

2) *Admit now* $U_2 > 0$. Then $\Delta_2\Delta_3\widetilde{\Delta}_3 > 0$ and according to Remark 3.4 we obtain $\text{sign}\,(U_4) = \text{sign}\,(\Delta_1\Delta_2\widetilde{\Delta}_2)$ and as $\Delta_1 > 0$ and $\widetilde{\Delta}_2 > 0$ we get $\text{sign}\,(U_4) = \text{sign}\,(\Delta_2)$.

a) If $U_4 < 0$ then we have $\Delta_2 < 0$ and $\Delta_3\widetilde{\Delta}_3 < 0$. So we could have two possibilities: *(i)* $\Delta_3 < 0$ (and then systems (3.5) possess the singularities: saddles M_2, M_3, R_1 and nodes M_1, M_4, R_2, R_3) and *(ii)* $\Delta_3 > 0$ (and then these systems possess the saddles M_2, M_4, R_3 and the nodes M_1, M_3, R_1, R_2). Considering the location of these singularities and their types, in both cases we arrive at phase portraits which are topologically equivalent to *Picture 3.1(c3)*.

b) Assuming $U_4 > 0$ we obtain $\Delta_2 > 0$ and $\Delta_3\widetilde{\Delta}_3 > 0$. So M_2 is a node and then M_3 and M_4 are saddles. This implies $\Delta_3 < 0$ and then $\widetilde{\Delta}_3 < 0$, i.e. R_3 is a saddle and the remaining infinite singular point R_1 will be a node. In the same manner as above, considering the types and the location of the singularities we arrive at a phase portrait which is topologically equivalent to *Picture 3.1(c1)*.

Summarizing the sets of conditions given above for each one of the *Pictures 3.1(c j)*, $j = 1, 2, 3, 4$ we conclude that our claim is proved.

3.1.1.2.2 *Suppose now* $W_4 < 0$ *or* ($W_4 = 0$,$W_3 < 0$). So on the phase plane of systems (3.5) there are two saddles, one node and one focus and two nodes and a saddle at infinity. We assume again that R_2 is a node, i.e. $\widetilde{\Delta}_2 > 0$.

We claim that the phase portrait of a system in this family is given by one of the ones indicated below if and only if the corresponding conditions are satisfied, respectively:

$$\begin{array}{llll} \textit{Picture 3.1}(\overset{*}{c}1) & \Leftrightarrow \quad B_3U_1 > 0,\ U_2 < 0; & \textit{Picture 3.1}(\overset{*}{c}2) & \Leftrightarrow \quad B_3U_1 < 0,\ U_2 < 0; \\ \textit{Picture 3.1}(\overset{*}{c}3) & \Leftrightarrow \quad B_3U_1 > 0,\ U_2 > 0; & \textit{Picture 3.1}(\overset{*}{c}4) & \Leftrightarrow \quad B_3U_1 < 0,\ U_2 > 0. \end{array}$$

Indeed to convince ourselves we shall examine again both cases: $B_3U_1 < 0$ and $B_3U_1 > 0$.

I. The case $B_3U_1 < 0$. Then by Remark 3.4 it follows $\Delta_1 < 0$ (yielding $f < 0$) (i.e. $M_1(0, 0)$ is a saddle).

As $\widetilde{\Delta}_2 > 0$ according to Remark 3.4 we obtain $\text{sign}\,(U_2) = \text{sign}\,(\Delta_2\Delta_3\widetilde{\Delta}_3)$. Moreover, since M_4 is a focus then either M_2 or M_3 must be a saddle, i.e. $\Delta_2\Delta_3 < 0$ and this implies $\text{sign}\,(U_2) = -\text{sign}\,(\widetilde{\Delta}_3)$.

1) *Assume first* $U_2 < 0$. Then R_3 is a node and R_1 is a saddle. Taking into account that R_2 and R_3 are both nodes then due to Remark 3.3 without loss of generality we may assume that M_2 is a saddle (then M_3 is a node). So considering the types and the location of all the singularities we arrive at *Picture 3.1*($\overset{*}{c}$*2)*.

2) *Admit now* $U_2 > 0$. In this case $\widetilde{\Delta}_3 < 0$ ($h < 0$), i.e. R_3 is a saddle and hence R_1 is a node. We observe that the conditions $f < 0$ and $h < 0$ imply $fh - f - h > 0$ and therefore

$\Delta_3 = f(fh - f - h)/h > 0$, i.e. M_3 is a node. Consequently M_2 is a saddle and as M_4 is a focus we arrive to and to *Picture 3.1($\overset{*}{c}$4)*.

II. The case $B_3U_1 > 0$. Herein by Remark 3.4 it follows that $\Delta_1 > 0$ ($f > 0$), i.e. $M_1(0,0)$ is a node. Since M_4 is a focus then clearly M_2 and M_3 should be saddles, i.e. $\Delta_2 < 0$ and $\Delta_3 < 0$. Hence taking into account our assumption (i.e. that R_2 is a node) according to Remark 3.4 in this case we have $\operatorname{sign}(U_2) = \operatorname{sign}(\widetilde{\Delta}_3)$.

If $U_2 < 0$ we get $\widetilde{\Delta}_3 < 0$ (i.e. R_3 is a saddle) and then R_1 is a node. This leads to *Picture 3.1($\overset{*}{c}$1)*.

The condition $U_2 > 0$ implies that both points R_2 and R_3 are nodes (then R_1 is a saddle) and we get *Picture 3.1($\overset{*}{c}$3)*. This completes the proof of our claim and thus all the phase portraits associated to *Config. 3.1* as well as the respective invariant criteria for the realization of each of them are determined.

3.1.2 The phase portraits associated to *Config. 3.2*

According to Table 2 we shall consider the family of systems

$$\dot{x} = x[1 + gx + (h-1)y], \quad \dot{y} = y[(g-1)x + hy], \tag{3.11}$$

for which the condition

$$gh(g+h-1)(g-1)(h-1) \neq 0 \tag{3.12}$$

holds. We observe that this family of systems is a particular case of the family (3.5) when the parameter f equals zero (and in this case the point M_3 sticks together with $M_1(0,0)$). We shall keep the same notations for the singularities of systems (3.11). So these systems have one double singular point $M_3 \equiv M_1(0,0)$ and two simple ones: $M_2(-1/g, 0)$ and $M_4\left(\frac{-h}{g+h-1}, \frac{g-1}{g+h-1}\right)$. For these finite singularities of systems (3.11) with the condition (3.12) we have

$$\begin{aligned} M_3 \equiv M_1(0,0): &\quad \Delta_{1,3} = 0, \quad \rho_{1,3} = 1, \quad \delta_{1,3} = 1; \\ M_2(-1/g, 0): &\quad \Delta_2 = (g-1)/g, \quad \rho_2 = (1-2g)/g, \quad \delta_2 = 1/g^2; \\ M_4\left(\frac{h}{1-g-h}, \frac{1-g}{1-g-h}\right): &\quad \Delta_4 = \frac{-h(g-1)}{g+h-1}, \quad \rho_4 = \frac{h}{1-g-h}, \quad \delta_4 = \rho_4^2 - 4\Delta_4. \end{aligned} \tag{3.13}$$

and for the three infinite singular points we have again

$$R_1(1,1,0): \ \widetilde{\Delta}_1 = 1 - g - h; \quad R_2(1,0,0): \ \widetilde{\Delta}_2 = g; \quad R_3(0,1,0): \ \widetilde{\Delta}_3 = h. \tag{3.14}$$

We observe that due to the relation $\rho_1 = 1$ (i.e. only one of the respective eigenvalues vanishes) the double singular point $M_1(0,0)$ is a saddle-node.

Taking into account (3.13) and (3.14) we evaluate for systems (3.11) the needed invariant polynomials:

$$\begin{aligned} \mu_0 &= gh(g+h-1) = -\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3, \\ K &= 2g(g-1)x^2 + 4ghxy + 2h(h-1)y^2, \\ W_4 &= h^3[4(g-1)^2 - 3h + 4gh] = \mu_0^2\,\delta_1\delta_2\delta_3\delta_4, \\ H_1 &= 288h = 288\widetilde{\Delta}_3, \\ H_5 &= 384(1-g)h^2 = -384\Delta_2\widetilde{\Delta}_2\widetilde{\Delta}_3^2, \\ H_{14} &= 30gh^2 = 30\widetilde{\Delta}_2\widetilde{\Delta}_3^2, \\ B_3 &= 3hx^2y^2 = 3\widetilde{\Delta}_1\rho_4x^2y^2 = 3\widetilde{\Delta}_3x^2y^2. \end{aligned} \tag{3.15}$$

3.1.2.1 The case $\mu_0 < 0$. In the same manner as in the previous section (in the case of *Config. 3.1*) we shall consider two subcases: $K < 0$ and $K > 0$.

3.1.2.1.1 The subcase $K < 0$. According to [1, Table 1] on the finite part of the phase plane of systems (3.11) besides the saddle-node there are two saddles. We claim that in this case we obtain a unique phase portrait which corresponds to *Picture 3.2(d1)*.

Indeed, due to the index theory evidently all three simple infinite singularities (3.14) are nodes. And considering the position of the saddles M_2 and M_4 and of the saddle-node M_1 we get univocally the phase portrait given by *Picture 3.2(d1)*.

3.1.2.1.2 The subcase $K > 0$. Following [1, Table 1] we find that besides the saddle-node there are two anti-saddles. Moreover, by (3.13) and (3.15) we observe that the relation $W_4 = 0$ holds if and only if $\delta_4 = 0$. So according to [1, Table 1] we have two nodes if $W_4 \geq 0$ and we have one node and one focus if $W_4 < 0$.

On the other hand in both cases at infinity we have two saddles and a node.

I. Assume first $W_4 \geq 0$. According to (3.15) we have $\operatorname{sign}(H_1) = \operatorname{sign}(\widetilde{\Delta}_3)$.

1) If $H_1 < 0$ then $\widetilde{\Delta}_3 < 0$ (i.e. $h < 0$) and hence the singular point R_3 is a saddle. We claim that the condition $W_4 \geq 0$ implies $\widetilde{\Delta}_2 > 0$, i.e. $g > 0$. Indeed assuming the contrary we have $g < 0$ and due to $h < 0$ we obtain $h[4(g-1)^2 - 3h + 4gh] < 0$ which contradicts $W_4 \geq 0$ (see (3.15)).

Thus $g > 0$ (i.e. $\widetilde{\Delta}_2 > 0$) and then R_2 is a node and consequently R_3 will be a saddle. Considering the position and the types of the singularities we arrive at *Picture 3.2(e2)*.

2) Suppose now $H_1 > 0$. Then $\widetilde{\Delta}_3 > 0$ (i.e. $h > 0$) and hence the singular point R_3 is a node and consequently the other two infinite points are saddles. Obviously we get *Picture 3.2(e1)*.

II. Admit now $W_4 < 0$. Then the point M_4 is a focus and M_2 is a node, i.e. $\Delta_2 > 0$.

1) If $H_1 < 0$ then $\widetilde{\Delta}_3 < 0$ (i.e. $h < 0$) and hence the singular point R_3 is a saddle. According to (3.15) due to $\Delta_2 > 0$ we obtain $\operatorname{sign}(H_5) = -\operatorname{sign}(\widetilde{\Delta}_2)$.

a) Assume $H_5 < 0$. Then $\widetilde{\Delta}_2 > 0$ and hence R_2 is a node. So the remaining infinite point R_1 will be a saddle. As $M_1(0,0)$ is a saddle-node and M_2 is a node we get *Picture 3.2($\overset{*}{e}$2)*.

b) If $H_5 > 0$ then $\widetilde{\Delta}_2 < 0$ and the singular point R_2 is a saddle, whereas R_1 is a node. As $g < 0$, $h < 0$ and $1 - g - h > 0$ considering (3.13) we obtain

$$\rho_2\rho_4 = \frac{(1-2g)h}{g(1-g-h)} > 0.$$

Hence we conclude that the node M_2 and the focus M_4 have the same stability and we univocally obtain *Picture 3.2($\overset{*}{e}$3)*.

2) Assume now $H_1 > 0$. Then we have $\widetilde{\Delta}_3 > 0$ (i.e. $h > 0$) and the singular point R_3 is a node. Therefore the remaining two infinite singular points are saddles and considering the location of the singularities we obtain univocally *Picture 3.2($\overset{*}{e}$1)*.

3.1.2.2 The case $\mu_0 > 0$. According to [1, Table 1] besides the saddle-node systems (3.11) possess one saddle and either one node if $W_4 \geq 0$, or one focus if $W_4 < 0$. On the other hand at infinity we have a saddle and two nodes.

3.1.2.2.1 The subcase $W_4 \geq 0$. We claim that in this case the phase portrait of systems (3.11) corresponds to one of the indicated below if and only if the following conditions are satisfied,

respectively:

$$\begin{array}{lcl} \textit{Picture 3.2(f1)} & \Leftrightarrow & B_3H_{14}<0,\ H_5<0; \\ \textit{Picture 3.2(f2)} & \Leftrightarrow & B_3H_{14}<0,\ H_5>0,\ H_1<0; \\ \textit{Picture 3.2(f3)} & \Leftrightarrow & B_3H_{14}<0,\ H_5>0,\ H_1>0; \\ \textit{Picture 3.2(f4)} & \Leftrightarrow & B_3H_{14}>0,\ H_5<0; \\ \textit{Picture 3.2(f5)} & \Leftrightarrow & B_3H_{14}>0,\ H_5>0. \end{array}$$

Indeed, first of all we observe that according to (3.15) we have:

$$\mathrm{sign}\,(B_3H_{14}) = \mathrm{sign}\,(\widetilde{\Delta}_2\widetilde{\Delta}_3), \quad \mathrm{sign}\,(H_1) = \mathrm{sign}\,(\widetilde{\Delta}_3), \quad \mathrm{sign}\,(H_5) = -\mathrm{sign}\,(\Delta_2\widetilde{\Delta}_2), \tag{3.16}$$

so we can control the signs of each of the determinants $\widetilde{\Delta}_2$, $\widetilde{\Delta}_3$ and Δ_2.

I. The possibility $B_3H_{14}<0$. Then $\widetilde{\Delta}_2\widetilde{\Delta}_3<0$ and we shall consider two subcases: $H_5<0$ and $H_5>0$.

1) If $H_5<0$ then considering (3.16) and (3.13) we obtain $g^2(g-1)>0$, i.e. $g>1$. Consequently $\widetilde{\Delta}_2>0$ and $\widetilde{\Delta}_3<0$. Hence the singular points R_2, R_1 and M_2 are nodes, whereas R_3 and M_4 are saddles. Considering the position of these singularities we univocally get *Picture 3.2(f1)*.

2) Assume $H_5>0$. Then $\Delta_2\widetilde{\Delta}_2<0$ and we consider two possibilities: $H_1<0$ and $H_1>0$.

a) Assume first $H_1<0$. According to (3.16) we have $\widetilde{\Delta}_3<0$ and hence we obtain $\widetilde{\Delta}_2>0$ which implies $\Delta_2<0$. So besides the saddle-node $M_1(0,0)$ systems (3.11) has the nodes R_2, R_1 and M_4 and the saddles R_3 and M_2. This obviously leads to the phase portrait given by *Picture 3.2(f2)*.

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b) If $H_1>0$, similarly as above, we get the nodes R_3, R_1 and M_2 and the saddles R_2 and M_4. Considering the location of these singularities we univocally obtain *Picture 3.2-2(f3)*.

II. The possibility $B_3H_{14}>0$. In this case we have $\widetilde{\Delta}_2\widetilde{\Delta}_3>0$ and as there could not be two saddles at infinity, both points R_2 and R_3 are nodes (i.e. $\widetilde{\Delta}_2>0$ and $\widetilde{\Delta}_3>0$), whereas the point R_1 is a saddle.

1) If $H_5<0$ then by (3.16) we obtain $\Delta_2>0$ (then $g>1$) and therefore M_2 is a node and M_4 is a saddle. Considering the location of all the singularities we univocally obtain *Picture 3.2(f4)*.

2) Assume $H_5>0$. This implies $\Delta_2<0$ (then $0<g<1$) and hence M_2 is a saddle and M_4 is a node. Thus we get *Picture 3.2(f5)*.

3.1.2.2.2 The subcase $W_4\geq 0$. As it was mentioned above, in this case besides the saddle-node systems (3.11) have a saddle and a focus. Clearly a focus could only be the singularity M_4 and hance M_2 is a saddle. Therefore we have $\Delta_2<0$ that considering (3.13) implies $0<g<1$. Consequently $\widetilde{\Delta}_2>0$, i.e. R_2 is a node. It remains to distinguish the possibilities when R_3 is a node or a saddle. According to (3.16), these situations are governed by the invariant polynomial H_1. More precisely, we obtain the saddle R_3 and the node R_1 if $H_1<0$, and we have the saddle R_1 and the node R_3 if $H_1>0$. Considering the location of the singularities in the first case we obtain *Picture 3.2($\overset{*}{f}$2)*, whereas in the second we have *Picture 3.2($\overset{*}{f}$5)*.

3.1.3 The phase portraits associated to *Config. 3.3*

In accordance with Table 2 we shall consider the family of systems

$$\dot{x} = x[g+gx+(h-1)y], \quad \dot{y} = y[g-1+(g-1)x+hy], \tag{3.17}$$

for which the condition

$$gh(g+h-1)(g-1)(h-1) \neq 0 \tag{3.18}$$

holds. As the quadratic parts of these systems coincides with the corresponding parts of systems (3.5) and (3.11) we shall keep the same notations for the singularities. For systems (3.17) we have:

$$\begin{aligned} M_1(0,0): \Delta_1 = g(g-1);\ \delta_1 = g^2; \\ M_4 \equiv M_2(-1,0):\ \Delta_2 = 0,\ \rho_2 = -g,\ \ \delta_2 = g^2; \\ M_3\big(0,(1-g)/h\big):\ \Delta_3 = (g-1)(1-g-h)/h,\ \ \delta_3 = (gh-g-1)^2/h^2 \end{aligned} \tag{3.19}$$

for the finite singularities and

$$R_1(1,1,0):\ \widetilde{\Delta}_1 = 1-g-h; \quad R_2(1,0,0):\ \widetilde{\Delta}_2 = g; \quad R_3(0,1,0):\ \widetilde{\Delta}_3 = h. \tag{3.20}$$

for infinite ones. We observe that due to the condition (3.18) we have $\rho_2 = -g \neq 0$ (i.e. only one of the respective eigenvalues vanishes) and hence the double singular point $M_1(0,0)$ is a saddle-node.

Taking into consideration the expressions above for systems (3.17) we obtain

$$\begin{aligned} \mu_0 &= gh(g+h-1) = -\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3, \\ K &= 2g(g-1)x^2 + 4ghxy + 2h(h-1)y^2, \\ B_3 &= 3g(g+h-1)x^2y^2 = -3\widetilde{\Delta}_1\widetilde{\Delta}_2x^2y^2, \\ H_1 &= 288g(g+h-1) = -288\widetilde{\Delta}_1\widetilde{\Delta}_2, \\ H_5 &= 384(1-g)g^2(1-g-h)^2 = -384\Delta_1\widetilde{\Delta}_2\widetilde{\Delta}_1^2. \\ H_{14} &= g^2h(g+h-1) = -30\widetilde{\Delta}_1\widetilde{\Delta}_2^2\widetilde{\Delta}_3, \end{aligned} \tag{3.21}$$

Clearly the next relations hold:

$$\operatorname{sign}(B_3H_{14}) = \operatorname{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3), \quad \operatorname{sign}(H_1) = -\operatorname{sign}(\widetilde{\Delta}_1\widetilde{\Delta}_2), \quad \operatorname{sign}(H_5) = -\operatorname{sign}(\Delta_1\widetilde{\Delta}_2). \tag{3.22}$$

So we can control the signs of each one of the determinants $\widetilde{\Delta}_2$, $\widetilde{\Delta}_3$ and Δ_1.

As in this case we have one double (saddle-node) and two simple finite singularities, the types of these points are determined by the same conditions as indicated in the previous section (for systems (3.11)). So we shall consider the same cases as for *Config. 3.2*.

3.1.3.1 The case $\mu_0 < 0$. According to [1, Table 1] the simple singular points M_1 and M_3 are of the same type.

3.1.3.1.1 The subcase $K < 0$. Then M_1 and M_3 are saddles and all the infinite points are nodes and this univocally leads to the phase portrait given by *Picture 3.3(d1)*

3.1.3.1.2 The subcase $K > 0$. Then M_1 and M_3 are nodes and at infinity we have two saddles and one node. In order to distinguish which one among the infinite points is a node, we shall apply the invariant polynomial H_5 considering its sign given in (3.22).

I. Assume first $H_5 < 0$. Then due to $\Delta_1 > 0$ (as M_1 is a node) we obtain $\widetilde{\Delta}_2 > 0$ and hence R_2 is a node. Consequently R_3 and R_1 are saddles and we univocally obtain *Picture 3.3(e1)*.

II. Suppose now $H_5 > 0$. In this case $\widetilde{\Delta}_2 < 0$ (i.e. $g < 0$) and the singular point R_2 is a saddle. So there exist two possibilities: $h < 0$ (when R_3 is a saddle and R_1 is a node) and $h > 0$ (when R_3 is a node and R_1 is a saddle). It easily could be determine that in both cases we obtain phase portraits both topologically equivalent to *Picture 3.3(e2)*.

3.1.3.2 The case $\mu_0 > 0$. As both singular points M_1 and M_3 are located on the invariant lines we conclude that one of them is a saddle and another one is a node. Regarding the infinite singular points, clearly by the index theory there should be a saddle and two nodes. We claim that in this case the phase portrait of a system (3.17) is given by one of the ones indicated below if and only if the corresponding conditions are satisfied:

$$\begin{array}{lcl} \textit{Picture 3.3(f1)} & \Leftrightarrow & B_3H_{14} < 0,\ H_5 < 0; \\ \textit{Picture 3.3(f2)} & \Leftrightarrow & B_3H_{14} < 0,\ H_5 > 0,\ H_1 < 0; \\ \textit{Picture 3.3(f3)} & \Leftrightarrow & B_3H_{14} < 0,\ H_5 > 0,\ H_1 > 0; \\ \textit{Picture 3.3(f4)} & \Leftrightarrow & B_3H_{14} > 0,\ H_5 < 0; \\ \textit{Picture 3.3(f5)} & \Leftrightarrow & B_3H_{14} > 0,\ H_5 > 0. \end{array}$$

Indeed in order to prove this claim, considering (3.22) we shall examine

3.1.3.2.1 The subcase $B_3H_{14} < 0$. Then $\widetilde{\Delta}_2\widetilde{\Delta}_3 < 0$ and as $\mu_0 > 0$ according to (3.21) it follows $\widetilde{\Delta}_1 > 0$, i.e. R_1 is a node.

I. Assume first $H_5 < 0$. In this case by (3.22) we obtain $\Delta_1\widetilde{\Delta}_2 > 0$ and this implies $g^2(g-1) > 0$. Therefore $g > 1$ and consequently $\widetilde{\Delta}_1 > 0$ and $\widetilde{\Delta}_3 < 0$. Hence the singular points R_2, R_1 and M_1 are nodes, whereas R_3 and M_3 are saddles. Considering the location of these singularities we univocally get *Picture 3.3(f1)*.

II. Suppose now $H_5 > 0$. Then $g-1 < 0$ and we consider two possibilities: $H_1 < 0$ and $H_1 > 0$.

1) If $H_1 < 0$, then considering (3.22) we obtain $\operatorname{sign}(B_3H_{14}H_1) = \operatorname{sign}(\mu_0\widetilde{\Delta}_2) < 0$ and as $\mu_0 > 0$ (i.e. $\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3 < 0$) we get $\widetilde{\Delta}_2 > 0$. Therefore the condition $H_1 < 0$ yields $\widetilde{\Delta}_2 > 0$ and the condition $H_5 > 0$ implies $\Delta_1 < 0$. Thus besides the saddle-node $M_2(0,0)$, systems (3.17) have the nodes R_2, R_1 and M_3 and the saddles R_3 and M_1. This obviously leads to the phase portrait given by *Picture 3.3(f2)*.

2) If $H_1 > 0$ then in a similar way as above we get the nodes R_3, R_1 and M_1 and the saddles R_2 and M_3. Considering the locations of these singularities we univocally obtain *Picture 3.3(f3)*.

3.1.3.2.2 The subcase $B_3H_{14} > 0$. Then $\widetilde{\Delta}_2\widetilde{\Delta}_3 > 0$ and as at infinity there could not be two saddles we obtain that both points R_2 and R_3 are nodes (i.e. $\widetilde{\Delta}_2 > 0$ and $\widetilde{\Delta}_3 > 0$), whereas the point R_1 is a saddle. Therefore considering (3.22) we have $\operatorname{sign}(H_5) = -\operatorname{sign}(\Delta_1)$, i.e. H_5 governs the types of the finite singularities. It not to difficult to determine that we get *Picture 3.3(f4)* if $H_5 < 0$ and *Picture 3.3(f5)* if $H_5 > 0$.

3.1.4 The phase portraits associated to *Config. 3.4*

Considering Table 2 we shall examine the family of systems

$$\dot{x} = x[1 + (h-1)y], \quad \dot{y} = y(f - x + hy), \tag{3.23}$$

for which the condition

$$h(h-1)f(f-1)(f+h-fh) \neq 0 \tag{3.24}$$

holds. We observe that this family of systems is a particular case of the family (3.5) when the parameter g equals zero and in this case the point M_2 has gone to infinity and stick together with $R_2(1,0,0)$. We shall keep the same notations for the singularities of systems (3.23). So these

systems possess three simple finite singularity and three infinite singularities (one of which being double). For all the singularities of these systems with the condition (3.24) we have .

$$\begin{aligned} M_1(0,0) &: \Delta_1 = f, \quad \delta_1 = (1-f)^2; \\ M_3(0,-f/h) &: \Delta_3 = f(fh-f-h)/h, \quad \delta_3 = (f+h)^2/h^2; \\ M_4\left(\frac{f+h-fh}{1-h}, \frac{1}{1-h}\right) &: \Delta_4 = \frac{fh-f-h}{1-h}, \quad \rho_4 = \frac{h}{1-h}, \quad \delta_4 = \rho_4^2 - 4\Delta_4 \end{aligned} \tag{3.25}$$

for the finite singularities and

$$R_1(1,1,0): \ \widetilde{\Delta}_1 = 1-h; \quad R_2(1,0,0): \ \widetilde{\Delta}_2 = 0, \ \rho_2 = 1; \quad R_3(0,1,0): \ \widetilde{\Delta}_3 = h \tag{3.26}$$

for the infinite ones. As $\rho_2 \neq 0$ the singular point $R_2(1,0,0)$ is a saddle-node (both saddle-sectors being on the same part of the infinite line).

Taking into consideration (3.25) and (3.26) we evaluate for systems (3.23) the needed invariant polynomials:

$$\begin{aligned} &\mu_0 = 0, \quad K = 2h(h-1)y^2, \quad \eta = 1, \\ &B_3 = 3h(1-f)x^2y^2 = 3(1-f)\widetilde{\Delta}_3 x^2y^2, \\ &W_3 = \widetilde{\Delta}_1^2\widetilde{\Delta}_3^2(\delta_1\,\delta_3 + \delta_1\,\delta_4 + \delta_3\delta_4), \\ &W_4 = \widetilde{\Delta}_1^2\widetilde{\Delta}_3^2\,\delta_1\delta_3\delta_4, \\ &U_1 = \frac{1}{8}fh(1-f)(1-h)^2 = \frac{1}{8}\Delta_1\widetilde{\Delta}_1^2\widetilde{\Delta}_3(1-f), \\ &U_2 = 3f(f+h-fh) = -3\Delta_3\widetilde{\Delta}_3, \\ &H_5 = -384\big[h(fh-f-h) + f^2(h-1)\big], \\ &G_9 = h(h-1)/8 = -\widetilde{\Delta}_1\widetilde{\Delta}_3/8. \end{aligned} \tag{3.27}$$

From this, considering (3.25) and (3.26) we get the following relations:

$$\operatorname{sign}(B_3U_1) = \operatorname{sign}(\Delta_1); \quad \operatorname{sign}(U_2) = -\operatorname{sign}(\Delta_3\widetilde{\Delta}_3). \tag{3.28}$$

For systems (3.23) with three finite simple singularities we have $\mu_0 = 0$ and $G_9 = h(h-1)/8$. Therefore considering the fact that we could not have two foci, according to [1, Table 1] the types of the finite singularities are determined by the following affine invariant conditions, respectively:

$$\begin{aligned} &(g)\ s,s,n \quad \Leftrightarrow \quad K<0,\ W_4 \geq 0; \qquad (\overset{*}{g})\ s,s,f \quad \Leftrightarrow \quad K<0,\ W_4<0; \\ &(h)\ s,n,n \quad \Leftrightarrow \quad K>0 \ \textit{and either}\ W_4>0,\ \textit{or}\ W_4=0, W_3 \geq 0; \\ &(\overset{*}{h})\ s,n,f \quad \Leftrightarrow \quad K>0 \ \textit{and either}\ W_4<0,\ \textit{or}\ W_4=0, W_3<0; \end{aligned} \tag{3.29}$$

So we shall consider the two cases: $K < 0$ and $K > 0$.

3.1.4.1 The case $K < 0$. Then by (3.27) we have $0 < h < 1$ and considering (3.26) we get $\widetilde{\Delta}_1 > 0$ and $\widetilde{\Delta}_3 > 0$. So apart from the saddle-node $R_2(1,0,0)$, systems (3.23) possess at the infinity two nodes.

3.1.4.1.1 The subcase $W_4 \geq 0$. According to (3.29) we have two saddles and one node. We claim, that in this case the phase portrait of a system (3.23) corresponds to one of those

indicated below if and only if the corresponding conditions are satisfied, respectively:

$$\begin{array}{lcl} \textit{Picture 3.4(g1)} & \Leftrightarrow & B_3U_1 < 0,\ U_2 < 0; \\ \textit{Picture 3.4(g2)} & \Leftrightarrow & B_3U_1 < 0,\ U_2 > 0; \\ \textit{Picture 3.4(g3)} & \Leftrightarrow & B_3U_1 > 0. \end{array}$$

Indeed, considering (3.28) we shall examine two possibilities: $B_3U_1 < 0$ and $B_3U_1 > 0$.

I. The possibility $B_3U_1 < 0$. Then $\Delta_1 < 0$ and the singular point $M_1(0,0)$ is a saddle.

If $U_2 < 0$ then considering (3.28) and the relation $\widetilde{\Delta}_3 > 0$ we obtain $\Delta_3 > 0$. Hence M_3 is a node and consequently M_4 is a saddle and this leads univocally to *Picture 3.4(g1)*.

Assume now $U_2 > 0$. In this case we get $\Delta_3 < 0$ and then M_3 is a saddle whereas M_4 is a node. In this case we univocally get the phase portrait given by *Picture 3.4(g2)*.

II. The possibility $B_3U_1 > 0$. In this case we obtain that the singular point $M_1(0,0)$ is a node (as $\Delta_1 > 0$). Hence the others two singularities are saddles and considering the infinite singularities this leads to*Picture 3.4(g3)*.

3.1.4.1.2 The subcase $W_4 < 0$. As a focus could only be at the singular point M_4 we obtain that M_1 and M_3 are saddles and considering the nodes R_1 and R_3 at infinity this univocally leads to *Picture 3.4($\overset{*}{g}$2)*.

3.1.4.2 The case $K > 0$. Then $h(h-1) > 0$ and considering (3.26) we get $\widetilde{\Delta}_1\widetilde{\Delta}_3 < 0$, i.e. at infinity besides the saddle-node we have one saddle and one node.

On the other hand according to (3.29), on the phase plane of systems (3.23) there exist one saddle and two anti-saddles and these two possibilities are distinguished by the invariant polynomials W_4 and W_3 as it is indicate in (3.29).

3.1.4.2.1 The subcase $W_4 > 0$ or $W_4 = 0$ and $W_3 \geq 0$. Then we have one saddle and two nodes. We claim that in this case the phase portrait of a system (3.23) is given by one of those indicated below if and only if the corresponding conditions are satisfied, respectively:

$$\textit{Picture 3.4(h1)} \Leftrightarrow \quad B_3U_1 < 0; \quad \textit{Picture 3.4(h3)} \Leftrightarrow B_3U_1 > 0, U_2 < 0, H_5 > 0;$$
$$\textit{Picture 3.4(h2)} \Leftrightarrow B_3U_1 > 0 \textit{ and either } U_2 > 0, \textit{ or } U_2 < 0, H_5 < 0.$$

To prove this claim we shall consider again two possibilities: $B_3U_1 < 0$ and $B_3U_1 > 0$.

I. The possibility $B_3U_1 < 0$. Then $\Delta_1 < 0$ (i.e. $f < 0$) and M_1 is a saddle whereas the other two points are nodes. We observe that due to $\delta_4 \geq 0$ the condition $K > 0$ (i.e. $h(h-1) > 0$) implies $h > 1$. Indeed since $f < 0$, supposing $h < 0$ we clearly obtain a contradiction: $\delta_4 = \frac{4f(h-1)^2 - 3h^2 + 4h}{(h-1)^2} < 0$.

Thus, $h > 1$ and then R_3 is a node and R_1 is a saddle. This immediately leads to the *Picture 3.4(h1)*.

II. The possibility $B_3U_1 > 0$. In this case we obtain $\Delta_1 > 0$ (i.e. $f > 0$) and M_1 is a node. As by (3.28) the invariant polynomial U_2 governs the sign of the product $\Delta_3\widetilde{\Delta}_3$ we shall examine two subcases: $U_2 < 0$ and $U_2 > 0$.

1) Assume first $U_2 > 0$. Then by (3.28) we obtain $\Delta_3\widetilde{\Delta}_3 < 0$, i.e. the singular points M_3 and R_3 are of the different types. Fixing first $\Delta_3 > 0$ and then $\Delta_3 < 0$ the types of all the singularities,

as well as their location, become well determined. In both cases we get phase portraits which are topologically equivalent to *Picture 3.4(h2)*.

2) Admit now $U_2 < 0$. Then we obtain $\Delta_3\widetilde{\Delta}_3 > 0$ and we have to distinguish via invariant polynomials when these determinants are both negative and when they are positive. We observe that due to the condition $f > 0$ and considering (3.27) and (3.25) we obtain

$$\begin{aligned} &\text{sign}\,(\Delta_3) = \text{sign}\,\big(h(fh - f - h)\big) = \text{sign}\,(\widetilde{\Delta}_3) = -\text{sign}\,(\widetilde{\Delta}_1) = \text{sign}\,\big(f^2(h-1)\big) \\ \Rightarrow \quad &\text{sign}\,(\Delta_3) = \text{sign}\,\big(h(fh - f - h) + f^2(h-1)\big) = -\text{sign}\,(H_5). \end{aligned}$$

So if $H_5 < 0$ then $\Delta_3 > 0$, $\widetilde{\Delta}_3 > 0$ and $\widetilde{\Delta}_1 < 0$. Hence systems (3.23) possess the nodes M_1, M_3 and R_3 and the saddles M_4 and R_1 (R_2 being a saddle-node). Therefore we get the phase portrait which is topologically equivalent to *Picture 3.4(h2)*.

In the case $H_5 > 0$ in the same manner as above we get the nodes M_1, M_4 and R_1 and the saddles M_3 and R_3. This leads univocally to *Picture 3.4(h3)*.

3.1.4.2.2 The subcase $W_4 < 0$ or $W_4 = 0$ and $W_3 < 0$. Then systems (3.23) have one saddle, one node and one focus. We claim that the phase portrait of a system in this family corresponds to one of those indicated below if and only if the following conditions are satisfied, respectively:

$$\begin{aligned} &\textit{Picture 3.4}(\overset{*}{h}1) \Leftrightarrow B_3U_1 < 0,\ U_2 < 0; \quad \textit{Picture 3.4}(\overset{*}{h}2) \Leftrightarrow B_3U_1 > 0,\ U_2 > 0; \\ &\textit{Picture 3.4}(\overset{*}{h}3) \Leftrightarrow B_3U_1 > 0,\ U_2 < 0; \quad \textit{Picture 3.4}(\overset{*}{h}4) \Leftrightarrow B_3U_1 < 0,\ U_2 > 0. \end{aligned} \tag{3.30}$$

Indeed, first we mention that the focus could be only the singularity M_4 and that the type of the singularities M_1 is governed again by the invariant polynomial B_3U_1. Moreover, we observe that in the case under examination we have $\Delta_1\Delta_3 < 0$ and $\widetilde{\Delta}_1\widetilde{\Delta}_3 < 0$. Therefore considering (3.28) we obtain the relations:

$$\text{sign}\,(\Delta_1) = -\text{sign}\,(\Delta_3) = \text{sign}\,(B_3U_1); \quad \text{sign}\,(\widetilde{\Delta}_3) = -\text{sign}\,(\widetilde{\Delta}_1) = \text{sign}\,(U_2B_3U_1).$$

Herein it is easy to convince ourselves that provided the conditions indicated in (3.30) are satisfied, we get the respective phase portrait. It remains to point out that in the case of the portrait *Picture 3.4*$(\overset{*}{h}4)$ it is necessary to take into account that in this case for the focus M_4 and the node M_3 of systems (3.23) calculations yield: $\rho_3\rho_4 = \frac{f+h-2fh}{1-h}$. Therefore in the case $h < 0$ and $f < 0$ we obtain $\rho_3\rho_4 < 0$, i.e. these singularities have different stabilities. This completes the proof of the claim.

3.1.5 The phase portraits associated to *Config. 3.5*

According to Table 2 we shall consider the family of systems

$$\dot{x} = x[1 + (1-h)(x-y)], \quad \dot{y} = y(f - hx + hy), \tag{3.31}$$

for which the condition

$$h(h-1)f(f-1)(f+h-fh) \neq 0 \tag{3.32}$$

holds. We observe that this family of systems is a particular case of the family (3.5) when we have $g = 1 - h$ and in this case the singular point M_4 has gone to infinity and sticks together with $R_1(1,1,0)$. We shall keep the same notations for the singularities of systems (3.31). So these

systems possess three simple finite singularities and three infinite singularities (one of which being double). So for all the singularities of these systems with the condition (3.24) we have for the finite singularities :

$$\begin{aligned} M_1(0,0) &: \Delta_1 = f, \quad \delta_1 = (1-f)^2; \\ M_2(1/(h-1),0) &: \Delta_2 = (fh-f-h)/(1-h), \quad \delta_3 = (fh-f-1)^2/(h-1)^2; \\ M_3(0,-f/h) &: \Delta_3 = f(fh-f-h)/h, \quad \delta_3 = (f+h)^2/h^2 \end{aligned} \tag{3.33}$$

and

$$R_1(1,1,0): \ \widetilde{\Delta}_1 = 0, \ \widetilde{\rho}_1 = -1; R_2(1,0,0): \ \widetilde{\Delta}_2 = 1-h; \quad R_3(0,1,0): \ \widetilde{\Delta}_3 = h. \tag{3.34}$$

for the infinite ones. As $\widetilde{\rho}_1 \neq 0$ the singular point $R_1(1,1,0)$ is a saddle-node (both saddle-sectors being on the same part of the infinite line).

Evaluating for systems (3.31) the needed invariant polynomials, we obtain:

$$\begin{aligned} &\mu_0 = 0, \quad K = 2h(h-1)(x-y)^2 = 2\widetilde{\Delta}_2\widetilde{\Delta}_3(x-y)^2, \\ &B_3 = 3(f-1)(fh-f-h)x^2y^2 = 3(f-1)\Delta_2\widetilde{\Delta}_2x^2y^2, \\ &U_1 = \frac{1}{8}f(f-1)(fh-f-h)h^2(1-h)^2 = \frac{1}{8}(f-1)\Delta_1\Delta_2\widetilde{\Delta}_2^3\widetilde{\Delta}_3^2, \\ &U_3 = \frac{1}{2}(f+1)h(h-1)(fh-f-h) = -\frac{1}{2}(f+1)\Delta_2\widetilde{\Delta}_2^2\widetilde{\Delta}_3. \end{aligned} \tag{3.35}$$

Herein considering (3.33) and (3.34) we get the following relations:

$$\operatorname{sign}(B_3U_1) = \operatorname{sign}(\Delta_1); \quad \text{if} \ \ B_3U_1 > 0 \ \ \Rightarrow \ \ \operatorname{sign}(U_3) = -\operatorname{sign}(\Delta_2\widetilde{\Delta}_3). \tag{3.36}$$

We observe that systems (3.31) possess three finite simple singularities. Therefore considering the fact that these systems possess neither a focus nor a center, according to [1, Table 1] we obtain two saddles and one node if $K < 0$ and on saddle and two nodes if $K > 0$.

On the other hand by (3.35) we have $\operatorname{sign}(K) = \operatorname{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3)$. So clearly besides the saddle-node $R_1(1,1,0)$ systems (3.31) possess at the infinity two nodes if $K < 0$ and they have a node and a saddle if $K > 0$.

Remark 3.5. Without loss of generality we may assume that the infinite singular point $R_2(1,0,0)$ is a node due to the substitution $(x,y,t,f,h) \mapsto (x/f, y/f, ft, 1/f, 1-h)$, which keeps the systems (3.31) but replaces the point R_2 by R_3 and vice versa.

3.1.5.1 The case $K < 0$. Then besides the saddle-node $R_1(1,1,0)$, systems (3.31) possess at the infinity two nodes.

3.1.5.1.1 *The subcase $B_3U_1 < 0$.* Then $\Delta_1 < 0$ (i.e. M_1 is a saddle) and hence one of the singular points M_2 or M_3 is a saddle and another one is a node. Therefore we have either $\Delta_2 > 0$ and $\Delta_3 < 0$ or $\Delta_2 < 0$ and $\Delta_3 > 0$. Considering the relations $f > 0$ and $0 < h < 1$ (which fix the position of the singularities) as well as the two nodes and the saddle-node at infinity in both cases we get the phase portraits topologically equivalent to *Picture 3.5(g1)*.

3.1.5.1.2 The subcase $B_3U_1 > 0$. In this case we have $\Delta_1 > 0$ (i.e. $f > 0$) and M_1 is a node. Then the remaining two finite singularities are saddles an this leads to *Picture 3.5(g2)*.

3.1.5.2 The case $K > 0$. Systems (3.31) possess as finite singularities a saddle and two nodes and at infinity besides the saddle-node there exist a saddle and a node. According to Remark 3.5 we may assume that R_2 is a node and R_3 is a saddle (i.e. $h < 0$).

3.1.5.2.1 The case $B_3U_1 < 0$. We have $\Delta_1 < 0$ (i.e. $f < 0$) and therefore M_1 is a saddle whereas the remaining points are nodes. Considering the saddle R_3 and the node R_2 we obtain the phase portrait given by *Picture 3.5(h1)*.

3.1.5.2.2 The case $B_3U_1 > 0$. In this case we obtain $\Delta_1 > 0$ (i.e. $f > 0$) and hence M_1 is a node. As $\widetilde{\Delta}_3 < 0$ (since R_3 is a saddle) considering (3.35) we obtain $\operatorname{sign}(U_3) = \operatorname{sign}(\Delta_2)$.

I. The subcase $U_3 < 0$. Then $\Delta_2 < 0$, i.e. M_2 is a saddle and M_3 is a node. This leads univocally to the phase portrait given *Picture 3.5(h2)*.

II. The subcase $U_3 > 0$. In this case M_2 is a node and M_3 is a saddle and considering the location of all the singularities we get the phase portrait which is topologically equivalent to *Picture 3.5(h1)*.

3.1.6 The phase portraits associated to *Config. 3.6*

According to Table 2 we shall consider the one-parameter family of systems

$$\dot{x} = x[1 + (h-1)y], \quad \dot{y} = y(-x + hy), \qquad h(h-1) \neq 0. \tag{3.37}$$

We observe that this family of systems is a particular case of the family (3.11) when $g = 0$. Hence in this case we have two pairs of singularities which have stucked together: M_3 with M_1 and M_2 with infinite point $R_2(1,0,0)$. We shall keep the same notations for the finite singularities of systems (3.37). So for the singularities of these systems we have for the finite singularities:

$$\begin{aligned} M_3 \equiv M_1(0,0) &: \Delta_1 = 0, \quad \rho_1 = 1; \\ M_4\Big(\frac{h}{1-h}, \frac{1}{1-h}\Big) &: \Delta_4 = \frac{h}{h-1} = -\rho_4, \quad \delta_4 = \frac{h(4-3h)}{(h-1)^2} \end{aligned} \tag{3.38}$$

and

$$R_1(1,1,0): \ \widetilde{\Delta}_1 = 1-h; \quad R_2(1,0,0): \ \widetilde{\Delta}_2 = 0, \ \widetilde{\rho}_2 = 1; \quad R_3(0,1,0): \ \widetilde{\Delta}_3 = h \tag{3.39}$$

for the infinite ones. Clearly both double points $M_1(0,0)$ and $R_2(1,0,0)$ are saddle-nodes. Moreover, for the second point both saddle-sectors are on the same part of the infinite line.

For systems (3.37) we have:

$$\begin{aligned} K &= 2h(h-1)y^2 = 2\Delta_4\widetilde{\Delta}_1^2 y^2 = -2\widetilde{\Delta}_1\widetilde{\Delta}_3 y^2, \\ H_1 &= 288h = 288\widetilde{\Delta}_3, \\ W_4 &= h^3(4-3h) = \delta_4\widetilde{\Delta}_1^2\widetilde{\Delta}_3^2. \end{aligned} \tag{3.40}$$

Herein we observe that the invariant polynomials above govern the types of the simple singular points of systems (3.37). More precisely the types of the singularities M_4, R_1 and R_3 are determined by the following conditions, respectively:

(i)	$K < 0$	$\Rightarrow$	M_4 – saddle, R_1, R_3 – nodes;
(ii)	$K > 0, H_1 < 0$	$\Rightarrow$	M_4 – focus, R_1 – node, R_3 – saddle;
(iii)	$K > 0, H_1 > 0, W_4 \geq 0$	$\Rightarrow$	M_4 – node, R_1 – saddle, R_3 – node;
(iv)	$K > 0, H_1 > 0, W_4 < 0$	$\Rightarrow$	M_4 – focus, R_1 – saddle, R_3 – node.

Then considering the location of the singular point M_4 in each one of the cases above we arrive at the phase portraits given by: *Picture 3.6(k1)* in the case *(i)*; *Picture 3.6($\overset{*}{l}$1)* in the case *(ii)*; *Picture 3.6(l2)* in the case *(iii)* and *Picture 3.6($\overset{*}{l}$2)* in the case *(iv)*.

We stress that in the case of *Picture 3.6($\overset{*}{l}$1)* the behaviour of the trajectories in the vicinity of the focus M_4 is determined univocally due to the relation $\rho_1\rho_4 = -\Delta_4 < 0$ and this means that the stability of the focus is opposite to the stability of the parabolic sectors of the saddle-node M_1.

3.1.7 The phase portraits associated to *Config. 3.7*

According to Table 2 a system possessing this configuration belongs to the one-parameter family of systems

$$\dot{x} = x[h - 1 + (h - 1)y], \quad \dot{y} = y(h - x + hy), \qquad h(h - 1) \neq 0. \tag{3.41}$$

Comparing the singularities of these systems with them of the systems (3.5) we observe that in this case we have two pairs of singularities which have stuck together: M_4 with M_3 and M_2 with infinite point $R_2(1, 0, 0)$. We shall keep the same notations for the finite singularities of systems (3.41). So for the singularities of these systems we obtain:

$$M_1(0, 0) : \Delta_1 = h(h - 1); \quad M_4 \equiv M_3(0, -1) : \ \Delta_3 = 0, \ \rho_3 = -h \tag{3.42}$$

for the finite singularities, and

$$R_1(1, 1, 0) : \ \widetilde{\Delta}_1 = 1 - h; \ \ R_2(1, 0, 0) : \ \widetilde{\Delta}_2 = 0, \ \widetilde{\rho}_2 = 1; \ \ R_3(0, 1, 0) : \ \widetilde{\Delta}_3 = h \tag{3.43}$$

for the infinite ones. Clearly both double points $M_3(0, 0)$ and $R_2(1, 0, 0)$ are saddle-nodes and the saddle-sectors of the second saddle-node are located on the same part of the infinite line.

For systems (3.41) we calculate:

$$\begin{aligned} K = 2h(h - 1)y^2 = 2\Delta_1 y^2 \quad &\Rightarrow \quad \text{sign}\,(K) = \text{sign}\,(\Delta_1) = -\text{sign}\,(\widetilde{\Delta}_1\widetilde{\Delta}_3); \\ H_5 = 384h^2(1 - h)^3 = 384\widetilde{\Delta}_1^3\widetilde{\Delta}_3^2 \quad &\Rightarrow \quad \text{sign}\,(H_5) = \text{sign}\,(\widetilde{\Delta}_1). \end{aligned} \tag{3.44}$$

Herein we observe that the invariant polynomials K and H_5 govern the types of the simple singular points of systems (3.41). More precisely the types of the singularities M_1, R_1 and R_3 are determined by the following conditions, respectively:

$$\begin{array}{llll} \textit{(i)} & K < 0 & \Rightarrow & M_1 - \text{saddle}, \ R_1, R_3 - \text{nodes}; \\ \textit{(ii)} & K > 0, H_5 < 0 & \Rightarrow & M_1 - \text{node}, \ R_1 - \text{saddle}, \ R_3 - \text{node}; \\ \textit{(iii)} & K > 0, H_5 > 0 & \Rightarrow & M_1 - \text{node}, \ R_1 - \text{node}, \ R_3 - \text{saddle}. \end{array}$$

As the coordinates of all positions of the singularities are determined, in each one of the cases above we arrive univocally at the phase portrait given by: *Picture 3.7(k1)* in the case *(i)*; *Picture 3.7(l2)* in the case *(ii)* and *Picture 3.7(l1)* in the case *(iii)*.

3.1.8 The phase portraits associated to *Config. 3.8*

According to Table 2 a system possessing this configuration belongs to the one-parameter family of systems

$$\dot{x} = x[1 + (1 - h)(x - y)], \quad \dot{y} = hy(y - x), \qquad h(h - 1) \neq 0. \tag{3.45}$$

Comparing the singularities of these systems with those of the systems (3.5) we observe that in this case we have again two pairs of singularities which have stucked together: M_3 with M_1 and

M_4 with the infinite point $R_1(1,1,0)$. We shall keep the same notations for the finite singularities of systems (3.45). So for the singularities of these systems we obtain:

$$M_3 \equiv M_1(0,0) : \Delta_1 = 0,\ \rho_1 = 1; \quad M_2(1/(h-1),0) :\ \Delta_2 = h/(h-1) \tag{3.46}$$

for the finite singularities and

$$R_1(1,1,0) :\ \widetilde{\Delta}_1 = 0,\ \widetilde{\rho}_1 = -1; \quad R_2(1,0,0) :\ \widetilde{\Delta}_2 = 1-h; \quad R_3(0,1,0) :\ \widetilde{\Delta}_3 = h \tag{3.47}$$

for the infinite ones. Clearly the both double points $M_1(0,0)$ and $R_1(1,0,0)$ are saddle-nodes and the saddle-sectors of the second saddle-node are located on the same part of the infinite line.

For systems (3.45) we calculate:

$$\begin{aligned} K = 2h(h-1)(x-y)^2 = 2\Delta_1\widetilde{\Delta}_2^2(x-y)^2 \ &\Rightarrow\ \mathrm{sign}\,(K) = \mathrm{sign}\,(\Delta_1) = -\mathrm{sign}\,(\widetilde{\Delta}_2\widetilde{\Delta}_3); \\ H_5 = 384h^3 = 384\widetilde{\Delta}_3^3 \ &\Rightarrow\ \mathrm{sign}\,(H_5) = \mathrm{sign}\,(\widetilde{\Delta}_3). \end{aligned} \tag{3.48}$$

So we again obtain that these invariant polynomials determine completely the types of the simple singularities. Thus following the same arguments as above we get for the systems (3.45) the following phase portraits: *Picture 3.8(k1)* if $K < 0$; *Picture 3.8(l1)* if $K > 0$ and $H_5 < 0$; and *Picture 3.8(l2)* if $K > 0$ and $H_5 > 0$.

3.1.9 The phase portraits associated to *Config. 3.9*

According to Table 2 we shall consider the family of systems

$$\dot{x} = x(1+gx+y), \quad \dot{y} = y(f-x+gx+y), \tag{3.49}$$

for which the condition

$$g(g-1)f(f-1)(1-g+fg) \neq 0 \tag{3.50}$$

holds. For the all four distinct finite singularities of systems (3.49) with the condition (3.50) we have

$$\begin{aligned} M_1(0,0) :&\quad \Delta_1 = f, \quad \rho_1 = f+1, \quad \delta_1 = (f-1)^2; \\ M_2(-1/g,0) :&\quad \Delta_2 = (g-1-fg)/g, \quad \delta_2 = (1+fg)^2/g^2; \\ M_3(0,-f) :&\quad \Delta_3 = f(f-1), \quad \delta_3 = 1-2f; \\ M_4(f-1,\, g-1-fg) :&\quad \Delta_4 = (f-1)(g-1-fg), \quad \rho_4 = -1, \quad \delta_4 = 4g(f-1)^2 + 4f - 3. \end{aligned} \tag{3.51}$$

and for the two infinite singular points we obtain

$$R_2(1,0,0) :\ \widetilde{\Delta}_2 = g; \quad R_1 \equiv R_3(0,1,0) :\ \widetilde{\Delta}_3 = 0,\ \widetilde{\rho}_3 = 1. \tag{3.52}$$

We note that in this case the infinite singularity $R_3(0,1,0)$ is a saddle-node for which the infinite line serves as a separatrix for the saddles sectors (i.e. one saddle and parabolic sectors and one parabolic sector are located on the same part of the infinite line).

Taking into consideration (3.51) and (3.52) we evaluate for systems (3.49) the invariant polynomials we need:

$$\begin{aligned} \mu_0 &= g = \widetilde{\Delta}_2, \quad B_3 = 3(1-f)x^2y^2, \\ K &= 2g(g-1)x^2 + 4gxy + 2y^2, \\ U_1 &= \frac{1}{8}f(1-f)(g-1)^2 = \frac{1}{8}\Delta_1(1-f)(g-1)^2, \\ U_4 &= f(1-f)^2(g-1-fg) = \Delta_1\Delta_2\widetilde{\Delta}_2(f-1)^2, \\ H_4 &= 48(1-f) = -48\Delta_3/\Delta_1. \end{aligned} \tag{3.53}$$

Herein considering the condition (3.50) we evidently obtain the relations

$$\begin{gathered}\operatorname{sign}(\mu_0)=\operatorname{sign}(\widetilde{\Delta}_2),\quad \operatorname{sign}(B_3U_1)=\operatorname{sign}(\Delta_1),\\ \operatorname{sign}(U_4)=\operatorname{sign}(\Delta_1\Delta_2\widetilde{\Delta}_2),\quad \operatorname{sign}(H_4)=-\operatorname{sign}(\Delta_1\Delta_3).\end{gathered}\tag{3.54}$$

3.1.9.1 The case $\mu_0<0$. As $\mu_0=\operatorname{Discrim}(K)/16$ by (3.53) we conclude that $K>0$. Therefore according to [1, Table 1] on the finite part of the phase plane of systems (3.49) there are one saddles and three anti-saddle. Clearly only one anti-saddle could be a focus and considering [1, Table 1], apart from the saddle we have three nodes if either $W_4>0$ or $W_4=0$ and $W_3\geq 0$; and we have two nodes and a focus if either $W_4<0$ or $W_4=0$ and $W_3<0$.

On the other hand, by (3.54) we get $\widetilde{\Delta}_2<0$, i.e. the infinite singularity $R_2(1,0,0)$ is a saddle.

Remark 3.6. *We note that in the case $\mu_0<0$ and $\delta_4\geq 0$ (i.e. when M_4 is a node) the singular point M_1 should be a node.*

Indeed suppose that M_1 is a saddle. Considering (3.51) the conditions $g<0$ and $f<0$ impliy $\delta_4<0$, i.e. we get a contradiction.

Herein considering (3.54), we obtain that the types of the finite singularities of systems (3.49) are determined by the following conditions, respectively:

			M_1	M_2	M_3	M_4
(i)	$(W_4>0)\vee(W_4=0,W_3\geq 0),U_4<0,H_4<0$	$\Rightarrow$	n	n	n	s;
(ii)	$(W_4>0)\vee(W_4=0,W_3\geq 0),U_4<0,H_4>0$	$\Rightarrow$	n	n	s	n;
(iii)	$(W_4>0)\vee(W_4=0,W_3\geq 0),U_4>0$	$\Rightarrow$	n	s	n	n;
(iv)	$(W_4<0)\vee(W_4=0,W_3<0),B_3U_1>0,U_4<0$	$\Rightarrow$	n	n	s	f;
(v)	$(W_4<0)\vee(W_4=0,W_3<0),B_3U_1>0,U_4>0$	$\Rightarrow$	n	s	n	f;
(vi)	$(W_4<0)\vee(W_4=0,W_3<0),B_3U_1<0$	$\Rightarrow$	s	n	n	f.

We note that in the case *(vi)* we have $\rho_3\rho_4<0$, i.e. the node M_3 and the focus M_4 are of the opposite stabilities. So considering the infinite singularities $R_2(1,0,0)$ (a saddle) and $R_3(0,1,0)$ (a saddle-node) we arrive in each of the mentioned cases at the phase portrait given by *Picture 3.9(*)*, respectively:

(i) 3.9(b1); *(ii) 3.9(b2)*; *(iii) 3.9(b3)*; *(iv) 3.9($\overset{*}{b}$2)*; *(v) 3.9($\overset{*}{b}$3)*; (vi) *3.9($\overset{*}{b}$4)*.

3.1.9.2 The case $\mu_0>0$. According to [1, Table 1] on the finite part of the phase plane of systems (3.49) there are two saddles and two anti-saddles. Moreover as only one anti-saddle could be a focus, besides the saddles we have two nodes if $W_4\geq 0$ and we have a node and a focus if $W_4<0$.

On the other hand, by (3.54) we get $\widetilde{\Delta}_2>0$, i.e. the infinite singularity $R_2(1,0,0)$ is a node.

Thus considering (3.54) we obtain that the types of the finite singularities of systems (3.49) are determined by the following conditions, respectively:

			M_1	M_2	M_3	M_4
(i)	$W_4\geq 0,B_3U_1<0,U_4<0$	$\Rightarrow$	s	n	n	s;
(ii)	$W_4\geq 0,B_3U_1<0,U_4>0$	$\Rightarrow$	s	s	n	n;
(iii)	$W_4\geq 0,B_3U_1>0,H_4<0$	$\Rightarrow$	n	s	n	s;
(iv)	$W_4\geq 0,B_3U_1>0,H_4>0,U_4<0$	$\Rightarrow$	n	s	s	n;
(v)	$W_4\geq 0,B_3U_1>0,H_4>0,U_4>0$	$\Rightarrow$	n	n	s	s;
(vi)	$W_4<0,B_3U_1>0$	$\Rightarrow$	n	s	n	f;
(vii)	$W_4<0,B_3U_1<0$	$\Rightarrow$	s	s	n	f.

We note that in the cases *(i)*, *(iv)* and *(v)* we obtain the phase portraits which are topologically equivalent to the same portrait, given by *Picture 3.9(c1)*. Therefore this picture occurs if and only if $W_4 \geq 0$ and either $B_3U_1 < 0$ and $U_4 < 0$, or $B_3U_1 > 0$ and $H_4 > 0$.

Examining all the cases above considering the infinite singularities $R_2(1,0,0)$ (a node) and $R_3(0,1,0)$ (a saddle-node) we arrive in each of the remaining cases to the phase portrait given by *Picture 3.9(*)*, respectively:

(ii) 3.9(c2); *(iii) 3.9(c3)*; (vi) *3.9($\overset{*}{c}$1)*; (vii) *3.9($\overset{*}{c}$2)*.

Thus we arrive exactly to the respective conditions given by Table 5 in this case.

3.1.10 The phase portraits associated to *Config. 3.10*

According to Table 2 a system possessing this configuration belongs to the one-parameter family of systems

$$\dot{x} = x(g + gx + y), \quad \dot{y} = y[g - 1 + (g - 1)x + y], \qquad g(g - 1) \neq 0. \tag{3.55}$$

These systems possess three finite singularities (one of them being double) and two infinite (one double). For the finite singularities of systems (3.55) we have

$$\begin{array}{c} M_1(0,0): \ \Delta_1 = g(g-1); \quad M_3(0,1-g): \ \Delta_3 = 1-g; \\ M_4 \equiv M_2(-1,0): \quad \Delta_2 = 0, \quad \rho_2 = -g \end{array} \tag{3.56}$$

and for the two infinite singular points we obtain

$$R_2(1,0,0): \ \widetilde{\Delta}_2 = g; \quad R_1 \equiv R_3(0,1,0): \ \widetilde{\Delta}_3 = 0, \ \widetilde{\rho}_3 = 1. \tag{3.57}$$

For systems (3.55) calculations yield:

$$\mu_0 = g, \quad B_3 = 3gx^2y^2, \quad U_1 = \quad \frac{1}{8}g^2(g-1)^3. \tag{3.58}$$

Herein considering (3.56) and (3.57) we obtain the following relations:

$$\text{sign}\,(\mu_0) = \text{sign}\,(\widetilde{\Delta}_2) = -\text{sign}\,(\Delta_1\Delta_3), \quad \text{sign}\,(B_3U_1) = \text{sign}\,(\Delta_1). \tag{3.59}$$

So we observe that the two invariant polynomials μ_0 and B_3U_1 determine completely the types of the simple singularities. More exactly we obtain that the types of all the singularities of systems (3.55) (for infinite points we denote them by capital letters) and they are determined by the following conditions, respectively:

			M_1	M_2	M_3	R_2	R_3
(i)	$\mu_0 < 0$	$\Rightarrow$	n	sn	n	S	SN;
(ii)	$\mu_0 > 0, B_3U_1 < 0$	$\Rightarrow$	s	sn	n	N	SN;
(iii)	$\mu_0 > 0, B_3U_1 > 0$	$\Rightarrow$	n	sn	s	N	SN.

These types of the singularities univocally lead to the following phase portraits, respectively:

(i) Picture 3.10(e1); *(ii) Picture 3.10(f1)*; *Picture 3.10(f2)*.

3.1.11 The phase portraits associated to *Config. 3.11*

According to Table 2 a system possessing this configuration belongs to the one-parameter family of systems

$$\dot{x} = x(1 + gx + y), \quad \dot{y} = y(-x + gx + y), \qquad g(g-1) \neq 0. \tag{3.60}$$

These systems possess the following finite singularities:

$$\begin{gathered} M_3 \equiv M_1(0,0): \ \Delta_1 = 0, \ \rho_1 = 1; \quad M_2(-1/g, 0): \ \Delta_2 = \frac{g-1}{g}, \ \rho_2 = \frac{1-2g}{g}; \\ M_4(-1, g-1): \quad \Delta_4 = 1-g, \quad \rho_4 = -1, \quad \delta_4 = 4g-3 \end{gathered} \tag{3.61}$$

and infinite ones:

$$R_2(1,0,0): \ \widetilde{\Delta}_2 = g; \quad R_1 \equiv R_3(0,1,0): \ \widetilde{\Delta}_3 = 0, \ \widetilde{\rho}_3 = 1. \tag{3.62}$$

For systems (3.60) calculations yield:

$$\mu_0 = g, \quad W_4 = 4g - 3, \quad H_5 = \ 384(1-g). \tag{3.63}$$

Herein we obtain:

$$\text{sign}\,(\mu_0) = \text{sign}\,(\widetilde{\Delta}_2) = -\text{sign}\,(\Delta_2\Delta_4), \quad \text{sign}\,(H_5) = \text{sign}\,(\Delta_4), \quad \text{sign}\,(W_4) = \text{sign}\,(\delta_4). \tag{3.64}$$

So we obtain that the types of all the singularities of systems (3.60) are determined by the following conditions, respectively:

			M_1	M_2	M_4	R_2	R_3
(i)	$\mu_0 < 0$	$\Rightarrow$	*sn*	*n*	*f*	*S*	*SN*;
(ii)	$\mu_0 > 0, H_5 > 0, W_4 \geq 0$	$\Rightarrow$	*sn*	*s*	*n*	*N*	*SN*;
(iii)	$\mu_0 > 0, H_5 > 0, W_4 < 0$	$\Rightarrow$	*sn*	*s*	*f*	*N*	*SN*;
(iv)	$\mu_0 > 0, H_5 < 0$	$\Rightarrow$	*sn*	*n*	*s*	*N*	*SN*.

We observe that in the case $\mu_0 < 0$ (i.e. $g < 0$) the condition $\rho_2\rho_4 = (2g-1)/g > 0$, i.e. the stabilities of the node M_2 and of the focus M_4 coincide. So considering the types of the singular points above we get univocally the following phase portraits, respectively:

(i) Picture 3.11($\overset{*}{e}$1); (ii) Picture 3.11(f1); (iii) Picture 3.11($\overset{*}{f}$1); (iv) Picture 3.11(f2).

3.1.12 The phase portraits associated to *Config. 3.12*

According to Table 2 we shall consider the family of systems

$$\dot{x} = x(1+y), \quad \dot{y} = y(f + x + y), \quad f(f-1) \neq 0 \tag{3.65}$$

which possess the following five singularities:

$$\begin{aligned} M_1(0,0): &\quad \Delta_1 = f, \quad \rho_1 = f+1; \\ M_3(0,-f): &\quad \Delta_3 = f(f-1), \quad \rho_3 = 1-2f; \\ M_4\,(1-f,\,-1): &\quad \Delta_4 = 1-f, \quad \rho_4 = -1, \quad \delta_4 = 4f-3 \end{aligned} \tag{3.66}$$

and

$$R_1 = R_3(0,1,0): \quad \widetilde{\Delta}_3 = 0 \ \widetilde{\rho}_3 = 1; \quad R_2(1,0,0): \quad \widetilde{\Delta}_2 = 0, \ \widetilde{\rho}_2 = -1. \tag{3.67}$$

Remark 3.7. We observe that both infinite points are double and they are saddle-nodes. However for $R_3(0,1,0)$ the infinite line serves as a separatrix for the saddles sectors, whereas both saddle-sectors of the saddle-node $R_2(1,0,0)$ are located on the same part of the infinite line.

Taking into consideration (3.66) for systems (3.65) we calculate

$$\begin{gathered} B_3 = 3(f-1)x^2y^2 = -3\Delta_4 x^2y^2, \quad U_1 = \frac{1}{8}f(f-1) = -\frac{1}{8}\Delta_1\Delta_4, \\ H_5 = 384(1-f) = 384\Delta_4, \quad W_4 = (f-1)^2(4f-3) = \Delta_4^2\delta_4. \end{gathered} \tag{3.68}$$

Herein we obtain

$$\operatorname{sign}(B_3U_1) = \operatorname{sign}(\Delta_1); \quad \operatorname{sign}(H_5) = \operatorname{sign}(\Delta_4); \quad \operatorname{sign}(W_4) = \operatorname{sign}(\delta_4). \tag{3.69}$$

So these invariant polynomials determine the types of all the finite singularities of systems (3.65) as follows:

			M_1	M_3	M_4
(i)	$B_3U_1 < 0$	$\Rightarrow$	s	n	f;
(ii)	$B_3U_1 > 0, H_5 > 0, W_4 \geq 0$	$\Rightarrow$	n	s	n;
(iii)	$B_3U_1 > 0, H_5 > 0, W_4 < 0$	$\Rightarrow$	n	s	f;
(iv)	$B_3U_1 > 0, H_5 < 0$	$\Rightarrow$	n	n	s.

We observe that in the case $B_3U_1 < 0$ (i.e. $f < 0$) the condition $\rho_3\rho_4 = 2f - 1 < 0$, i.e. the stabilities of the node M_3 and of the focus M_4 are opposite. So considering the types of the singular points above and Remark 3.7 we get univocally the following phase portraits, respectively:

(i) Picture 3.12($\overset{*}{h}$1); *(ii) Picture 3.12(h2)*; *(iii) Picture 3.12($\overset{*}{h}$2)*; *(iv) Picture 3.12(h3)*.

As we have the one-parameter family of systems the conditions above could be simplified. More precisely as the bifurcation value $f = 3/4$ (respectively $f = 0$; $f = 1$) for the parameter f is given by polynomial W_4 (respectively B_3U_1; H_5), we get for the respective phase portraits the conditions given by Table 5.

3.1.13 The phase portraits associated to *Config. 3.13*

According to Table 2 this configuration could only possess the following system

$$\dot{x} = x(1+y), \quad \dot{y} = y(x+y), \tag{3.70}$$

which could be viewed as a special case of systems (3.65), when $f = 0$. So considering (3.66) the singular point M_3 sticks together with $M_1(0,0)$ (becoming a saddle-node) and M_4 in this case becomes a focus. Therefore considering Remark 3.7 we get univocally *Picture 3.13($\overset{*}{l}$1)*.

3.2 Phase portraits of degenerate LV-systems

In this section we shall examine the phase portraits of the degenerate $\boldsymbol{LV}$-systems with the configurations *Configs.* $LV_d.j$ with $j = 1, 2, \ldots, 14$ (see Table 4 and Fig. 2). We remark that in this case the zero-cycle $\mathcal{D}$ counting the number and multiplicities of singularities, is not defined.

Theorem 3.2. *The degenerate* $\boldsymbol{LV}$*-systems have a total of 20 topologically distinct phase portraits which are given in Fig.6. The necessary and sufficient conditions for the realization of each one of these phase portraits are given in columns 2 and 3 of Table 6.*

Table 6

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*		*Phase portrait*
Config. $LV_d.1$	$\eta > 0,\ \mu_{0,1,2,3,4} = 0,$ $\theta \neq 0,\ H_7 \neq 0$	$K < 0$		*Picture* $LV_d.1(a)$
		$K > 0$		*Picture* $LV_d.1(b)$
Config. $LV_d.2$	$\eta > 0,\ \mu_{0,1,2,3,4} = 0,$ $\theta \neq 0,\ H_7 = 0$	$K < 0$		*Picture* $LV_d.2(a)$
		$K > 0$		*Picture* $LV_d.2(b)$
Config. $LV_d.3$	$\eta > 0,\ \mu_{0,1,2,3,4} = 0,$ $\theta = H_4 = 0,\ H_7 \neq 0$	–		*Picture* $LV_d.3$
Config. $LV_d.4$	$\eta > 0,\ \mu_{0,1,2,3,4} = 0,$ $\theta = H_4 = 0,\ H_7 = 0$	–		*Picture* $LV_d.4$
Config. $LV_d.5$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,$ $\theta \neq 0,\ H_7 \neq 0$	–		*Picture* $LV_d.5$
Config. $LV_d.6$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,$ $\theta \neq 0,\ H_7 = 0$	–		*Picture* $LV_d.6$
Config. $LV_d.7$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,$ $\theta = 0, K \neq 0, H_2 \neq 0$	$K < 0$		*Picture* $LV_d.7(a)$
		$K > 0$	$L < 0$	*Picture* $LV_d.7(b)$
			$L > 0$	*Picture* $LV_d.7(c)$
Config. $LV_d.8$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,$ $\theta = 0, K \neq 0, H_2 = 0$	$K < 0$		*Picture* $LV_d.8(a)$
		$K > 0$	$L < 0$	*Picture* $LV_d.8(b)$
			$L > 0$	*Picture* $LV_d.8(c)$
Config. $LV_d.9$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $K = H_7 = 0,\ N \neq 0,\ H_2 \neq 0$	–		*Picture* $LV_d.9$
Config. $LV_d.10$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $K = H_7 = 0,\ N \neq 0,\ H_2 = 0$	–		*Picture* $LV_d.10$
Config. $LV_d.11$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $K = N = D = N_1 = 0,\ N_5 > 0$	–		*Picture* $LV_d.11$
Config. $LV_d.12$	$\eta = 0,\ \mu_{0,1,2,3,4} = 0,\ \theta = 0,$ $K = N = D = N_1 = 0,\ N_5 = 0$	–		*Picture* $LV_d.12$
Config. $LV_d.13$	$C_2 = 0,\ \mu_{0,1,2,3,4} = 0,\ H_2 \neq 0$	–		*Picture* $LV_d.13$
Config. $LV_d.14$	$C_2 = 0,\ \mu_{0,1,2,3,4} = 0,\ H_2 = 0$	–		*Picture* $LV_d.14$

Proof: We shall examine each one of the canonical systems ($LV_d.j$) (j$\in \{1, 2, \ldots, 14)$ given in the Table 6 corresponding to the configurations Config. LV_d.j of the degenerate $\boldsymbol{LV}$-systems.

Clearly a degenerate $\boldsymbol{LV}$-systems could posses only real straight lines filled up with singularities. So the phase portraits can easily be detected and in what follows we will only indicate:

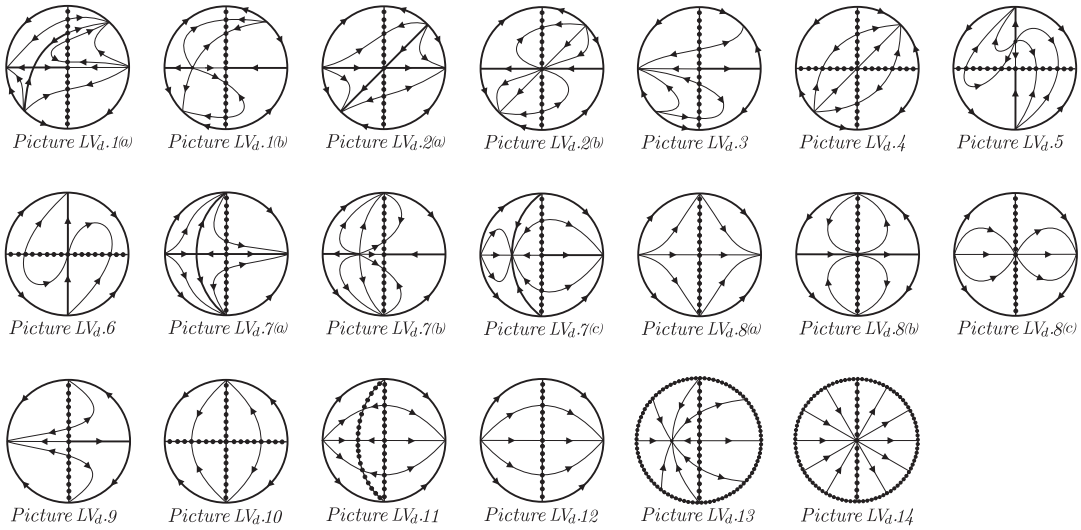

Fig. 6: **Phase portraits of the family of degenerate *LV*-systems**

(i) the invariant lines filled up with singularities; (ii) the corresponding linear (or even constant) systems; (iii) the invariant lines of the linear systems; (iv) the topologically distinct phase portraits of quadratic systems corresponding to the respective Configuration; (v) and the affine invariant polynomials which provide the respective conditions.

3.2.1 The phase portraits associated to *Config. LV_d.1*

(i) Singular line: $x = 0$; ***(ii)*** Corresponding linear systems: $\dot{x} = 1+gx-y,\ \ \dot{y} = (g-1)y,\ \ g(g-1) \neq 0$;

(iii) Invariant lines of the linear systems: $y = 0$ and $g(x - y) + 1 = 0$;

(iv) Phase portraits: *Picture LV_d.1(a)* if $g(g - 1) < 0$ and *Picture LV_d.1(b)* if $g(g - 1) > 0$;

(v) Invariant polynomial: $K = 2g(g - 1)x^2 \Rightarrow \operatorname{sign}(K) = \operatorname{sign}\big(g(g - 1)\big)$.

3.2.2 The phase portraits associated to *Config. LV_d.2*

(i) Singular line: $x = 0$; ***(ii)*** Corresponding linear systems: $\dot{x} = gx-y,\ \ \dot{y} = (g-1)y,\ \ g(g-1) \neq 0$;

(iii) Invariant lines of the linear systems: $y = 0$ and $x - y = 0$;

(iv) Phase portraits: *Picture LV_d.2(a)* if $g(g - 1) < 0$ and *Picture LV_d.2(b)* if $g(g - 1) > 0$;

(v) Invariant polynomial: $K = 2g(g - 1)x^2 \Rightarrow \operatorname{sign}(K) = \operatorname{sign}\big(g(g - 1)\big)$.

3.2.3 The phase portraits associated to *Config. LV_d.3*

(i) Singular line: $x = 0$; ***(ii)*** Corresponding linear system: $\dot{x} = 1 + y,\ \ \dot{y} = y$;

(iii) Invariant lines of the linear system: $y = 0$; ***(iv)*** Phase portrait: *Picture LV_d.3*.

3.2.4 The phase portraits associated to *Config. LV_d.4*

(i) Singular lines: $x = 0$ and $y = 0$; ***(ii)*** Corresponding constant system: $\dot{x} = 1,\ \ \dot{y} = 1$;

(iii) Invariant lines of the constant system: $y = x + C$; ***(iv)*** Phase portrait: *Picture $LV_d.4$.*

3.2.5 The phase portraits associated to *Config. $LV_d.5$*

(i) Singular line: $y = 0$; ***(ii)*** Corresponding linear system: $\dot{x} = x, \quad \dot{y} = 1 - x + y$;
(iii) Invariant lines of the linear system: $x = 0$ (double); ***(iv)*** Phase portrait: *Picture $LV_d.5$.*

3.2.6 The phase portraits associated to *Config. $LV_d.6$*

(i) Singular line: $y = 0$; ***(ii)*** Corresponding linear system: $\dot{x} = x, \quad \dot{y} = -x + y$;
(iii) Invariant lines of the linear system: $x = 0$ (double); ***(iv)*** Phase portrait: *Picture $LV_d.6$.*

3.2.7 The phase portraits associated to *Config. $LV_d.7$*

(i) Singular line: $x = 0$; ***(ii)*** Corresponding linear systems: $\dot{x} = 1+gx, \quad \dot{y} = (g-1)y, \quad g(g-1) \neq 0$;
(iii) Invariant lines of the linear systems: $y = 0$ and $gx + 1 = 0$;
(iv) Phase portraits: *Pictures: $LV_d.7(a)$* if $g(g-1) < 0$; *$LV_d.7(b)$* if $g < 0$; *$LV_d.7(c)$* if $g > 1$;
(v) Invariant polynomials: $\begin{cases} K = 2g(g-1)x^2 & \Rightarrow \operatorname{sign}(K) = \operatorname{sign}\big(g(g-1)\big); \\ L = 8gx^2 & \Rightarrow \operatorname{sign}(L) = \operatorname{sign}(g). \end{cases}$

3.2.8 The phase portraits associated to *Config. $LV_d.8$*

(i) Singular line: $x = 0$; ***(ii)*** Corresponding linear systems: $\dot{x} = gx, \quad \dot{y} = (g-1)y, \quad g(g-1) \neq 0$;
(iii) Invariant lines of the linear systems: $y = 0$ and $x = 0$;
(iv) Phase portraits: *Pictures: $LV_d.8(a)$* if $g(g-1) < 0$; *$LV_d.8(b)$* if $g < 0$; *$LV_d.8(c)$* if $g > 1$;
(v) Invariant polynomials: $\begin{cases} K = 2g(g-1)x^2 & \Rightarrow \operatorname{sign}(K) = \operatorname{sign}\big(g(g-1)\big); \\ L = 8gx^2 & \Rightarrow \operatorname{sign}(L) = \operatorname{sign}(g). \end{cases}$

3.2.9 The phase portraits associated to *Config. $LV_d.9$*

(i) Singular line: $x = 0$; ***(ii)*** Corresponding linear system: $\dot{x} = 1, \quad \dot{y} = y$;
(iii) Invariant lines of the linear system: $y = 0$; ***(iv)*** Phase portrait: *Picture $LV_d.9$.*

3.2.10 The phase portraits associated to *Config. $LV_d.10$*

(i) Singular lines: $x = 0$ and $y = 0$; ***(ii)*** Corresponding constant system: $\dot{x} = 0, \quad \dot{y} = 1$;
(iii) Invariant lines of the constant system: $x = C$, $C \in \mathbb{R}$; ***(iv)*** Phase portrait: *Picture $LV_d.10$.*

3.2.11 The phase portraits associated to *Config. $LV_d.11$*

(i) Singular lines: $x = 0$ and $x + 2 = 0$; ***(ii)*** Corresponding constant system: $\dot{x} = 1, \quad \dot{y} = 0$;
(iii) Invariant lines of the constant system: $y = C$, $C \in \mathbb{R}$; ***(iv)*** Phase portrait: *Picture $LV_d.11$.*

3.2.12 The phase portraits associated to *Config. LV_d.12*

(i) Singular line: $x^2 = 0$; ***(ii)*** Corresponding constant system: $\dot{x} = 1, \quad \dot{y} = 0$;

(iii) Invariant lines of the constant system: $y = C$, $C \in \mathbb{R}$; ***(iv)*** Phase portrait: *Picture LV_d.12*.

3.2.13 The phase portraits associated to *Config. LV_d.13*

(i) Singular line: $x = 0$; ***(ii)*** Respective linear system: $\dot{x} = 1 + x, \quad \dot{y} = y$;

(iii) Invariant lines of the constant system: $y = C(x + 1)$, $C \in \mathbb{R}$; ***(iv)*** Phase portrait: *Picture LV_d.13*.

3.2.14 The phase portraits associated to *Config. LV_d.14*

(i) Singular line: $x = 0$; ***(ii)*** Corresponding linear system: $\dot{x} = x, \quad \dot{y} = y$;

(iii) Invariant lines of the constant system: $y = Cx$, $C \in \mathbb{R}$; ***(iv)*** Phase portrait: *Picture LV_d.14*.

3.3 Topologically distinct phase portraits of *LV*-systems

In order to find the exact number of topologically distinct phase portraits of ***LV***-systems, we need some invariants which will help us in distinguishing phase portraits. We list below the topological invariants we need and the notation we use.

I. *Singularities, multiplicities and indexes:*

- $\mathcal{N}$ = total number of all singularities (all real) of the systems;
- $\binom{\mathcal{N}_f}{\mathcal{T}_m}$ = the number $\mathcal{N}_f$ of all distinct finite singularities having a total multiplicity $\mathcal{T}_m$;
- $\deg J$ = the sum of the indexes of all finite singularities of the systems.

II. *Connections of separatrices:*

- $\#SC_s^s$ = total number of finite saddle to finite saddle connections;
- $\#SC_s^S$ = total number of finite saddle to infinite saddle connections;
- $\#SC_s^{\mathcal{SN}}$ = total number of finite saddle to infinite saddle-node connections;
- $\#SC_{sn}^s$ = total number of finite saddle-node to finite saddle connections;
- $\#SC_{sn}^S$ = total number of finite saddle-nodes to infinite saddle-nodes connections;
- $\#SC_{sn}^{\mathcal{SN}}$ = total number of finite saddle-nodes to infinite saddle-nodes connections;
- $\#SC_{sn(hh)}^{\mathcal{S}}$ = total number of separatrices of finite saddle-nodes dividing the two hyperbolic sectors, going to infinite saddles;
- $\#SC_{sn(hh)}^{\mathcal{SN}}$ = total number of separatrices of finite saddle-nodes dividing the two hyperbolic sectors, connecting with separatrices of infinite saddle-nodes.

III. *The maximum number of separatrices ending (for $t \to \pm\infty$) at a singular point:*

- $Msep[n]$ = the maximum number of separatrices ending at one finite node;
- $Msep[sn]$ = the maximum number of separatrices ending at a finite saddle-node.

Using the topological invariants listed above we construct the following global topological invariant $\mathcal{I}$:

$$\mathcal{I} = \Big(\mathcal{N},\ \binom{\mathcal{N}_f}{\mathcal{T}_m},\ \deg J,\ \#SC_s^s,\ \#SC_s^S,\ \#SC_s^{\mathcal{SN}},\ \#SC_{sn}^s,\ \#SC_{sn}^S,\ \#SC_{sn}^{\mathcal{SN}},\ \#SC_{sn(hh)}^{\mathcal{S}},\ Msep[n],\ Msep[sn]\Big)$$

which classifies all ***LV***-systems.

Theorem 3.3. *I. The class of non-degenerate **LV**-systems have a total of 92 topologically distinct phase portraits, distinguished by the topological invariant $\mathcal{I}$. The different phase portraits are contained in the **Global Geometric Diagram** distinguished by the various components of $\mathcal{I}$. In the middle of this diagram there appear all 153 phase portraits given by the classes (i)-(iii) of the Main Theorem, and topological equivalences are listed. On the right side of this diagram the distinct phase portraits are numbered from (1) to (92). Moreover for each phase portrait we indicate on its right side the corresponding phase portraits in the paper [9]. More precisely we have the following three cases:*

(a) to the portrait (i) there corresponds only one portrait in [9];

(b) to the portrait (i) there correspond several portraits claimed to be distinct in [9].

(c) to the portrait (i) with $i \in \{68, 81, 86, 87\}$ there is no corresponding phase portrait in [9]. So from the 92 phase portraits 4 portraits are missing in [9].

*II. The labels of the 20 phase portraits of the class of degenerate **LV**-systems appear on the left side of the Correspondence List (see page 62). In the middle of this List we have their corresponding numbers, i.e. from 93 to 112. On the right side of this List the same three possibilities analogous to (a),(b) and (c) above appear. More precisely for case (c) we have:*

*(c') to the portrait (i) with $i \in \{107, 108, 109, 110\}$ there is no corresponding phase portrait in [9]. So from the 20 phase portraits of degenerate **LV**-systems, 4 portraits are missing in [9].*

Remark 3.8. *We observe that in the Global Geometric Diagram far, on the right hand side we have occasionally a star. For example we have [Fig.1: (24-3)*]. We use the star to indicate those cases where some mistake occurs in that phase portrait, such as for example a wrong orientation of a specific phase curve, or the presence of a phase curve which should not be there or the absence of some separatrices or some other minor error. If such a mistake is corrected, then the resulting phase portrait is equivalent to the corresponding phase portrait in the middle of the Diagram.*

1

*Phase portraits of the family of **LV**-systems: the **Global Geometric Diagram***

1. $\mathcal{N} = 7$

1.1. $\deg J = -2$

1.1.1. $\#SC_s^s = 0 \quad \Rightarrow \quad$ *3.1(a3)* $\simeq$ *4.3(b)*; (1) [Fig.1: (18)]

1.1.2. $\#SC_s^s = 1 \quad \Rightarrow \quad$ *3.1(a1)*; (2) [Fig.1: (21)]

1.1.3. $\#SC_s^s = 2 \quad \Rightarrow \quad$ *3.1(a2)* $\simeq$ *3.1($\overset{*}{a}$2)*; (3) $\begin{cases}\text{[Fig.1: (28-1)]}\\ \text{[Fig.1: (28-2)]}\end{cases}$

1.1.4. $\#SC_s^s = 3 \quad \Rightarrow \quad$ *4.1(b)*; (4) [Fig.1: (35)]

1.2. $\deg J = 0$

1.2.1. $\#SC_s^s = 0$

1.2.1.1 $\#SC_s^{\mathcal{S}} = 0 \quad \Rightarrow \quad \begin{cases} 3.1(c3) \simeq 3.1(\overset{*}{c}\ 3) \simeq \\ 4.1(a) \simeq 4.3(a) \simeq 5.1 \\ \simeq 4.9(b) \simeq 5.3 \simeq 6.1; \end{cases}$ (5) $\begin{cases}\text{[Fig.1: (14-1)]}\\ \text{[Fig.1: (15)]}\\ \text{[Fig.1: (17-3)]}\\ \text{[Fig.1: (25-3)]}\end{cases}$

1.2.1.2 $\#SC_s^{\mathcal{S}} = 1 \quad \Rightarrow \quad$ *3.1(c1)* $\simeq$ *3.1*$(\overset{*}{c}1)$ $\simeq$ *4.9(a)*; (6) $\begin{cases}\text{[Fig.1: (14-2)]}\\ \text{[Fig.1: (20-2)]}\\ \text{[Fig.1: (25-1)]}\end{cases}$

1.2.2. $\#SC_s^{s} = 1$

1.2.2.1 $\#SC_s^{\mathcal{S}} = 0 \quad \Rightarrow \quad$ *3.1(c2)* $\simeq$ *3.1*$(\overset{*}{c}2)$ $\simeq$ *4.9(c)*; (7) [Fig.1: (27-3)]

1.2.2.2 $\#SC_s^{\mathcal{S}} = 1 \quad \Rightarrow \quad$ *3.1(c4)* $\simeq$ *3.1*$(\overset{*}{c}4)$; (8) [Fig.1: (27-1)]

1.3. $\deg J = 2$

1.3.1. $\#SC_s^{\mathcal{S}} = 0 \quad \Rightarrow \quad$ *3.1(b3)* $\simeq$ *3.1*$(\overset{*}{b}3)$ $\simeq$ *4.3(c)*; (9) $\begin{cases}\text{[Fig.1: (19-1)]}\\ \text{[Fig.1: (19-3)]}\\ \text{[Fig.1: (24-3)*]}\end{cases}$

1.3.2. $\#SC_s^{\mathcal{S}} = 1$

1.3.2.1 $\#Sep_s^{n} = 2 \quad \Rightarrow \quad$ *3.1(b1)* $\simeq$ *3.1*$(\overset{*}{b}1)$; (10) $\begin{cases}\text{[Fig.1: (24-5-1)]}\\ \text{[Fig.1: (26)]}\end{cases}$

1.3.2.2 $\#Sep_s^{n} = 3 \quad \Rightarrow \quad$ *3.1(b2)* $\simeq$ *3.1*$(\overset{*}{b}2)$; (11) $\begin{cases}\text{[Fig.1: (24-1)]}\\ \text{[Fig.1: (24-5-2)]}\end{cases}$

1.3.3. $\#SC_s^{\mathcal{S}} = 2$

1.3.3.1 $\#Centers = 0 \quad \Rightarrow \quad$ *3.1*$(\overset{*}{b}4)$; (12) $\begin{cases}\text{[Fig.1: (31-1-1)]}\\ \text{[Fig.1: (31-1-2)]}\end{cases}$

1.3.3.2 $\#Centers = 1 \quad \Rightarrow \quad$ *4.1(c)*; (13) [Fig.1: (33)]

2. $\mathcal{N} = 6$

2.1. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{4}{4}$

2.1.1. $\deg\ J = 0$

2.1.1.1. $\#SC_s^{s} = 0$

2.1.1.1.1 $\#SC_s^{\mathcal{SN}} = 0 \quad \Rightarrow \quad$ *3.9(c3)* $\simeq$ *4.11(b)*; (14) [Fig.1: (17-1)]

2.1.1.1.2 $\#SC_s^{\mathcal{SN}} = 1 \quad \Rightarrow \quad$ *3.9(c1)* $\simeq$ *3.9*$(\overset{*}{c}1)$ $\simeq$ *4.11(a)*; (15) $\begin{cases}\text{[Fig.1: (17-2)]}\\ \text{[Fig.1: (20-1)]}\\ \text{[Fig.1: (25-2)]}\end{cases}$

2.1.1.2. $\#SC_s^{s} = 1 \quad \Rightarrow \quad$ *3.9(c2)* $\simeq$ *3.9*$(\overset{*}{c}2)$; (16) [Fig.1: (27-2)]

2.1.2. $\deg\ J = 2$

2.1.2.1. $\#SC_s^{\mathcal{S}} = 0$

2.1.2.1.1. $\#SC_s^{\mathcal{SN}} = 0 \quad \Rightarrow \quad$ *3.9(b1)*; (17) [Fig.1: (19-2)]

2.1.2.1.2. $\#SC_s^{\mathcal{SN}} = 1 \quad \Rightarrow \quad$ *3.9(b2)* $\simeq$ *3.9*$(\overset{*}{b}2)$; (18) [Fig.1: (24-2)]

2.1.2.2. $\#SC_s^{\mathcal{S}} = 1$

2.1.2.2.1. $\#SC_s^{\mathcal{SN}} = 0 \quad \Rightarrow \quad$ *3.9(b3)* $\simeq$ *3.9*$(\overset{*}{b}3)$; (19) [Fig.1: (22-4)]

2.1.2.2.2. $\#SC_s^{\mathcal{SN}} = 1 \quad \Rightarrow \quad$ *3.9*$(\overset{*}{b}4)$; (20) [Fig.1: (31-2)]

2.2. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{3}{4}$

2.2.1. $\deg J = -2$

2.2.1.1. $\#SC_s^{\mathcal{S}} = 0 \quad \Rightarrow \quad$ *3.2(d1)*; (21) [Fig.1: (23)]

2.2.1.2. $\#SC_s^{\mathcal{S}} = 1 \quad \Rightarrow \quad$ *3.3(d1)*; (22) [Fig.1: (12)]

2.2.2. $\deg J = 0$

2.2.2.1. $\#SC_{sn}^{s} = 0$

2.2.2.1.1. $\#SC^{\mathcal{S}}_{sn} = 0$

2.2.2.1.1.1. $\#SC^{\mathcal{S}}_{s} = 0 \quad \Rightarrow \quad$ *3.2(f4)* $\simeq$ *3.3(f4)* $\simeq$ *4.10(b)*; (23) $\begin{cases}\text{[Fig.1: (9-2)]}\\ \text{[Fig.1: (22-4)]}\end{cases}$

2.2.2.1.1.2. $\#SC^{\mathcal{S}}_{s} = 1 \quad \Rightarrow \quad$ *3.3(f1)*; (24) [Fig.1: (9-4)]

2.2.2.1.2. $\#SC^{\mathcal{S}}_{sn} = 1$

2.2.2.1.1.1. $\#SC^{\mathcal{S}}_{sn(hh)} = 0 \quad \Rightarrow \quad$ *3.2(f3)* $\simeq$ *3.3(f3)* $\simeq$ *4.10(c)*; (25) $\begin{cases}\text{[Fig.1: (9-5)]}\\ \text{[Fig.1: (22-2)]}\end{cases}$

2.2.2.1.1.2. $\#SC^{\mathcal{S}}_{sn(hh)} = 1 \quad \Rightarrow \quad$ *3.2(f1)*; (26) [Fig.1: (22-3)*]

2.2.2.2. $\#SC^{s}_{sn} = 1$

2.2.2.2.1. $\#SC^{\mathcal{S}}_{sn} = 0$

2.2.2.2.1.1. $\#SC^{\mathcal{S}}_{s} = 0 \Rightarrow \begin{cases} 3.2(f5) \simeq 3.2(\overset{*}{f}\ 5) \simeq \\ 3.3(f5) \simeq 4.10(a); \end{cases}$ (27) $\begin{cases}\text{[Fig.1: (11-1)]}\\ \text{[Fig.1: (30-3)]}\end{cases}$

2.2.2.2.1.2. $\#SC^{\mathcal{S}}_{s} = 1 \quad \Rightarrow \quad$ *3.3(f2)*; (28) [Fig.1: (11-3)]

2.2.2.2.2. $\#SC^{\mathcal{S}}_{sn} = 1 \quad \Rightarrow \quad$ *3.2(f2)* $\simeq$ *3.2*($\overset{*}{f}$*2)*; (29) [Fig.1: (30-2)]

2.2.3. $\deg J = 2$

2.2.3.1. $\#SC^{\mathcal{S}}_{sn} = 0 \quad \Rightarrow \quad$ *3.3(e1)*; (30) [Fig.1: (10-5)]

2.2.3.2. $\#SC^{\mathcal{S}}_{sn} = 1$

2.2.3.2.1. $\#SC^{\mathcal{S}}_{sn(hh)} = 0$

2.2.3.2.1.1. $Msep\,(sn) = 3 \quad \Rightarrow \quad$ *3.3(e2)*; (31) $\begin{cases}\text{[Fig.1: (10-1)]}\\ \text{[Fig.1: (10-3)*]}\end{cases}$

2.2.3.2.1.2. $Msep\,(sn) = 4 \quad \Rightarrow \quad$ *3.2(e1)* $\simeq$ *3.2*($\overset{*}{e}$*1)*; (32) [Fig.1: (29-2)]

2.2.3.2.2. $\#SC^{\mathcal{S}}_{sn(hh)} = 1 \quad \Rightarrow \quad$ *3.2(e2)* $\simeq$ *3.2*($\overset{*}{e}$*2)*; (33) [Fig.1: (29-1)]

2.2.3.3. $\#SC^{\mathcal{S}}_{sn} = 2 \quad \Rightarrow \quad$ *3.2*($\overset{*}{e}$*3)*; (34) [Fig.1: (32-2)]

2.3. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{3}{3}$

2.3.1. $\deg J = -1$

2.3.1.1. $\#SC^{s}_{s} = 0$

2.3.1.1.1 $\#SC^{\mathcal{SN}}_{s} = 0 \quad \Rightarrow \quad$ *3.4(g3)* $\simeq$ *3.5(g2)* $\simeq$ *4.4(a)*; (35) $\begin{cases}\text{[Fig.1: (4)]}\\ \text{[Fig.3: (4-1)]}\end{cases}$

2.3.1.1.2 $\#SC^{\mathcal{SN}}_{s} = 1 \quad \Rightarrow \quad$ *3.4(g1)*; (36) [Fig.3:(4-5)]

2.3.1.2. $\#SC^{s}_{s} = 1$

2.3.1.2.1 $\#SC^{\mathcal{SN}}_{s} = 0 \quad \Rightarrow \quad$ *3.5(g1)*; (37) [Fig.1: (6)]

2.3.1.2.2 $\#SC^{\mathcal{SN}}_{s} = 1 \quad \Rightarrow \quad$ *3.4(g2)* $\simeq$ *3.4*($\overset{*}{g}$*2)*; (38) [Fig.3:(4-3)]

2.3.2. $\deg J = 1$

2.3.2.1. $\#SC^{\mathcal{S}}_{s} = 0$

2.3.2.1.1 $\#SC^{\mathcal{SN}}_{s} = 0 \quad \Rightarrow \quad$ *3.4(h2)* $\simeq$ *3.4*($\overset{*}{h}$*2)* $\simeq$ *3.5(h2)* $\simeq$ *4.4(b)*; (39) $\begin{cases}\text{[Fig.1: (3-2)]}\\ \text{[Fig.3: (2-3)]}\\ \text{[Fig.3: (3-1)]}\\ \text{[Fig.3: (3-3)]}\end{cases}$

2.3.2.1.2 $\#SC^{\mathcal{SN}}_{s} = 1 \quad \Rightarrow \quad$ *3.4(h1)* $\simeq$ *3.4*($\overset{*}{h}$*1)*; (40) [Fig.3:(3-5)]

2.3.2.2. $\#SC_s^{\mathcal{S}} = 1$

2.3.2.2.1 $\#SC_s^{\mathcal{SN}} = 0$

2.3.2.2.1.1 $Msep\ [n] = 2 \quad \Rightarrow \quad$ *3.5(h1)*; (41) $\begin{cases}\text{[Fig.1: (3-3)}^*\text{]} \\ \text{[Fig.1: (5-2)]}\end{cases}$

2.3.2.2.1.2 $Msep\ [n] = 3 \quad \Rightarrow \quad$ *3.4(h3)* $\simeq$ *3.4($\overset{*}{h}$3)*; (42) [Fig.3:(2-1)]

2.3.2.2.2 $\#SC_s^{\mathcal{SN}} = 1 \quad \Rightarrow \quad$ *3.4($\overset{*}{h}$4)*; (43) [Fig.3:(2-5)]

3. $\mathcal{N} = 5$

3.1. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{3}{4}$

3.1.1. $\deg J = 0$

3.1.1.1. $\#SC_{sn}^{s} = 0$

3.1.1.1.1. $\#SC_s^{\mathcal{SN}} = 0$

3.1.1.1.1.1. $\#SC_{sn(hh)}^{\mathcal{SN}} = 0 \quad \Rightarrow \quad$ *4.21(a)* $\simeq$ *4.25(a)* $\simeq$ *5.11* ; (44) $\begin{cases}\text{[Fig.1: (9-1)]} \\ \text{[Fig.1: (16)]}\end{cases}$

3.1.1.1.1.2. $\#SC_{sn(hh)}^{\mathcal{SN}} = 1 \quad \Rightarrow \quad$ *3.11(f2)*; (45) [Fig.1: (22-1)*]

3.1.1.1.2. $\#SC_s^{\mathcal{SN}} = 1 \quad \Rightarrow \quad$ *3.10(f2)*; (46) [Fig.1: (9-3)]

3.1.1.2. $\#SC_{sn}^{s} = 1$

3.1.1.2.1. $\#SC_{sn}^{\mathcal{SN}} = 0 \quad \Rightarrow \quad$ *3.10(f1)*; (47) [Fig.1: (11-2)]

3.1.1.2.2. $\#SC_{sn}^{\mathcal{SN}} = 1 \quad \Rightarrow \quad$ *3.11(f1)* $\simeq$ *3.11($\overset{*}{f}$1)*; (48) [Fig.1: (30-1)]

3.1.2. $\deg J = 2$

3.1.2.1. $\#SC_{sn}^{\mathcal{S}} = 0 \quad \Rightarrow \quad$ *4.21(b)*; (49) [Fig.1: (10-4)]

3.1.2.2. $\#SC_{sn}^{\mathcal{S}} = 1$

3.1.2.2.1. $\#SC_{sn}^{\mathcal{SN}} = 0 \quad \Rightarrow \quad$ *3.10(e1)*; (50) [Fig.1: (10-2)]

3.1.2.2.2. $\#SC_{sn}^{\mathcal{SN}} = 1$

3.1.2.2.2.1. $\#Centers = 0 \quad \Rightarrow \quad$ *3.11($\overset{*}{e}$1)*; (51) [Fig.1: (32-1)]

3.1.2.2.2.1. $\#Centers = 1 \quad \Rightarrow \quad$ *4.25(b)*; (52) [Fig.1: (36)]

3.2. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{3}{3}$

3.2.1. $\#SC_s^{\mathcal{SN}} = 0 \quad \Rightarrow \quad$ *3.12(h3)*; (53) [Fig.3: (1-3)]

3.2.2. $\#SC_s^{\mathcal{SN}} = 1 \quad \Rightarrow \quad$ *3.12(h2)* $\simeq$ *3.12($\overset{*}{h}$2)*; (54) [Fig.3: (1-1)]

3.2.3. $\#SC_s^{\mathcal{SN}} = 2 \quad \Rightarrow \quad$ *3.12($\overset{*}{h}$1)*; (55) [Fig.3: (1-5)]

3.3. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{2}{4}$

3.3.1. $\#SC_{sn}^{\mathcal{S}} = 0 \quad \Rightarrow \quad$ *4.22(a)* $\simeq$ *5.12*; (56) [Fig.1: (13-3)]

3.3.2. $\#SC_{sn}^{\mathcal{S}} = 1 \quad \Rightarrow \quad$ *4.22(b)*; (57) [Fig.1: (13-2)]

3.4. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{2}{3}$

3.4.1. $\deg J = -1$

3.4.1.1. $\#SC_{sn}^{s} = 0 \quad \Rightarrow \quad$ *3.6(k1)*; (58) [Fig.4: (1)]

3.4.1.2. $\#SC_{sn}^{s} = 1$

3.4.1.2.1. $\#SC_s^{\mathcal{SN}} = 0 \quad \Rightarrow \quad$ *3.8(k1)*; (59) [Fig.1: (8)]

3.4.1.2.2. $\#SC_s^{\mathcal{SN}} = 1 \quad \Rightarrow \quad$ *3.7(k1)*; (60) [Fig.3: (4-4)]

3.4.2. $\deg J = 1$

3.4.2.1. $\#SC^{\mathcal{S}}_{sn} = 0$

3.4.2.1.1. $\#SC^{\mathcal{SN}}_{sn} = 0 \quad \Rightarrow \quad$ *3.7(l2)*; (61) [Fig.3: (3-2)]

3.4.2.1.2. $\#SC^{\mathcal{SN}}_{sn} = 1 \quad \Rightarrow \quad$ *3.6(l2)* $\simeq$ *3.6($\overset{*}{l}$2)*; (62) [Fig.4: (2)]

3.4.2.2. $\#SC^{\mathcal{S}}_{sn} = 1$

3.4.2.2.1. $\#SC^{\mathcal{SN}}_{sn} = 0$

3.4.2.2.1.1. $\#SC^{\mathcal{S}}_{sn(hh)} = 0$

3.4.2.2.1.1.1 $Msep\ [n] = 2 \quad \Rightarrow \quad$ *3.8(l2)*; (63) [Fig.1: (7-2)]

3.4.2.2.1.1.2 $Msep\ [n] = 3 \quad \Rightarrow \quad$ *3.7(l1)*; (64) [Fig.3:(2-2)]

3.4.2.2.1.2. $\#SC^{\mathcal{S}}_{sn(hh)} = 1 \quad \Rightarrow \quad$ *3.8(l1)*; (65) [Fig.1: (7-3)]

3.4.2.2.2. $\#SC^{\mathcal{SN}}_{sn} = 1 \quad \Rightarrow \quad$ *3.6($\overset{*}{l}$1)*; (66) [Fig.4: (3-2)]

3.5. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{2}{2}$

3.5.1. $\#SC^{\mathcal{SN}}_{s} = 0 \quad \Rightarrow \quad$ *4.16(b)* $\simeq$ *4.18(a)* $\simeq$ *5.7*; (67)[Fig.5: (2-2)]

3.5.2. $\#SC^{\mathcal{SN}}_{s} = 1 \quad \Rightarrow \quad$ *4.16(a)*; **(68)!**

3.5.3. $\#SC^{\mathcal{SN}}_{s} = 2 \quad \Rightarrow \quad$ *4.18(b)*; (69) [Fig.5: (2-1)]

4. $\mathcal{N} = 4$

4.1. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{2}{4} \quad \Rightarrow \quad$ *4.23*; (70)[Fig.1: (13-1)*]

4.2. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{2}{3}$

4.2.1. $\#SC^{\mathcal{SN}}_{sn} = 1 \quad \Rightarrow \quad$ *4.26*; (71) [Fig.3: (1-2)]

4.2.2. $\#SC^{\mathcal{SN}}_{sn} = 2 \quad \Rightarrow \quad$ *3.13($\overset{*}{l}$1)*; (72) [Fig.4: (3-1)]

4.3. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{2}{2}$

4.3.1. $\deg J = -2 \quad \Rightarrow \quad$ *4.12(c)*; (73) [Fig.2:(5-3)]

4.3.3. $\deg J = 0$

4.3.3.1. $\#SC^{\mathcal{S}}_{s} = 0 \quad \Rightarrow \quad$ *4.12(a)* $\simeq$ *5.13* $\simeq$ *5.14(a)* $\simeq$ *6.7*; (74) $\begin{cases} \text{[Fig.2: (4-1)]} \\ \text{[Fig.2: (4-4)]} \\ \text{[Fig.2: (5-1)]} \\ \text{[Fig.2: (5-4)]} \end{cases}$

4.3.3.2. $\#SC^{\mathcal{S}}_{s} = 1 \quad \Rightarrow \quad$ *4.12(b)*; (75) $\begin{cases} \text{[Fig.2: (3-1)]} \\ \text{[Fig.2: (3-3)]} \end{cases}$

4.3.4. $\deg J = 2$

4.3.4.1 $Msep\ [n] = 2 \quad \Rightarrow \quad$ *4.12(d)* ; (76) [Fig.2: (4-2)]

4.3.4.2 $Msep\ [n] = 3 \quad \Rightarrow \quad$ *4.12(e)* $\simeq$ *5.14(b)* $\simeq$ *6.8*; (77) [Fig.2: (3-2)]

4.4. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{1}{4}$

4.4.1. $\deg J = -2 \quad \Rightarrow \quad$ *4.5(b)*; (78) [Fig.10: (3)]

4.4.2. $\deg J = 0 \quad \Rightarrow \quad$ *4.5(a)* $\simeq$ *5.8* $\simeq$ *6.5*; (79) [Fig.10: (4)]

4.4.3. $\deg J = 2 \quad \Rightarrow \quad$ *4.5(c)*; (80) [Fig.10: (5)]

4.5. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{1}{2} \quad \Rightarrow \quad$ *4.17*; **(81)!**

5. $\mathcal{N} = 3$

5.1. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{1}{4}$

5.1.1. $\deg J = 0 \quad \Rightarrow \quad$ *4.20(a)* $\simeq$ *5.19*; (82) [Fig.10: (2)]

5.1.2. $\deg J = 2 \quad \Rightarrow \quad$ *4.20(b)*; (83) [Fig.10: (1)]

5.2. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{1}{2}$

5.2.1. $\#SC^{\mathcal{S}}_{sn} = 0 \quad \Rightarrow \quad$ *4.24(a)* $\simeq$ *5.17*; (84) [Fig.6: (2-1)]

5.2.2. $\#SC^{\mathcal{S}}_{sn} = 1 \quad \Rightarrow \quad$ *4.24(b)*; (85) [Fig.6: (2-2)]

5.3. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{1}{1}$

5.3.1. $\deg J = -1 \quad \Rightarrow \quad$ *4.19(a)*; **(86)!**

5.3.2. $\deg J = 1 \quad \Rightarrow \quad$ *4.19(b)* $\simeq$ *5.18*; **(87)!**

6. $\mathcal{N} = \infty$ (?)

6.1. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{3}{3} \quad \Rightarrow \quad C_2.1$; (88) $\begin{cases}\text{[Fig.1: (3-1)]}\\ \text{[Fig.1: (5-1)]}\end{cases}$

6.2. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{2}{3} \quad \Rightarrow \quad C_2.3$; (89) [Fig.1: (7-1)]

6.3. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{2}{2}$

6.3.1. $\deg J = 0 \quad \Rightarrow \quad C_2.5$(b); (90) $\begin{cases}\text{[Fig.2: (1-2)]}\\ \text{[Fig.2: (1-5)]}\end{cases}$

6.3.2. $\deg J = 2 \quad \Rightarrow \quad C_2.5$(a); (91) [Fig.2: (1-3)]

6.2. $\binom{\mathcal{N}_f}{\mathcal{T}_m} = \binom{1}{2} \quad \Rightarrow \quad C_2.7$; (92) [Fig.6: (1)*]

The ***Correspondence List*** *for degenerate* ***LV****-systems*

1. *Picture* LV_d*.1(a)* $\Rightarrow$ (93) $\begin{cases}\text{[Fig.1: (2)];}\\ \text{[Fig.3: (4-2)];}\end{cases}$
2. *Picture* LV_d*.1(b)* $\Rightarrow$ (94) $\begin{cases}\text{[Fig.1: (1-2)];}\\ \text{[Fig.3: (2-4)]}\\ \text{[Fig.3: (3-4)];}\end{cases}$
3. *Picture* LV_d*.2(a)* $\Rightarrow$ (95) [Fig.9: (4)*];
4. *Picture* LV_d*.2(b)* $\Rightarrow$ (96) $\begin{cases}\text{[Fig.9: (1)];}\\ \text{[Fig.9: (3)];}\end{cases}$
5. *Picture* LV_d*.3* $\Rightarrow$ (97) [Fig.5: (1)]
6. *Picture* LV_d*.4* $\Rightarrow$ (98) [Fig.7];
7. *Picture* LV_d*.5* $\Rightarrow$ (99) [Fig.3: (1-4)];
8. *Picture* LV_d*.6* $\Rightarrow$ (100) [Fig 9: (2)];
9. *Picture* LV_d*.7(a)* $\Rightarrow$ (101) $\begin{cases}\text{[Fig.2: (2-3)];}\\ \text{[Fig.2: (5-2)];}\end{cases}$
10. *Picture* LV_d*.7(b)* $\Rightarrow$ (102) $\begin{cases}\text{[Fig.2: (1-1)];}\\ \text{[Fig.2: (3-3)];}\end{cases}$
11. *Picture* LV_d*.7(c)* $\Rightarrow$ (103) $\begin{cases}\text{[Fig.2: (2-2)];}\\ \text{[Fig.2: (4-3)];}\end{cases}$
12. *Picture* LV_d*.8(a)* $\Rightarrow$ (104) [Fig 8: (4)];
13. *Picture* LV_d*.8(b)* $\Rightarrow$ (105) [Fig 8: (3)];
14. *Picture* LV_d*.8(c)* $\Rightarrow$ (106) [Fig 8: (2)];
15. *Picture* LV_d*.9* $\Rightarrow$ **(107)!**
16. *Picture* LV_d*.10* $\Rightarrow$ **(108)!**
17. *Picture* LV_d*.11* $\Rightarrow$ **(109)!**
18. *Picture* LV_d*.12* $\Rightarrow$ **(110)!**
19. *Picture* LV_d*.13* $\Rightarrow$ (111) $\begin{cases}\text{[Fig.1: (1-1)];}\\ \text{[Fig.2: (1-1)];}\\ \text{[Fig.2: (1-4)];}\end{cases}$
20. *Picture* LV_d*.14* $\Rightarrow$ (112) [Fig.8: (1)].

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