MULTIPLE–SCALE ANALYSIS OF DYNAMICAL SYSTEMS ON THE LATTICE

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Abstract. We propose a new approach to the multiple–scale analysis of difference equations, in the context of the finite operator calculus. We derive the transformation formulae that map any given dynamical system, continuous or discrete, into a rescaled discrete system, by generalizing a classical result due to Jordan. Under suitable analytical hypotheses on the function space we consider, the rescaled equations are of finite order. Our results are applied to the study of multiple–scale reductions of dynamical systems, and in particular to the case of the Duffing equation.

Contents

1. Introduction

Scaling and self–similarity are very useful mathematical concepts, having many applications in science and technology. Scaling laws provide evidence of the existence of hidden structures for dynamical systems; self–similarity manifests itself in the study of fractals, chaotic systems, incompressible fluids, etc. Apart from being a way to get exact solutions, scaling symmetries are essential for defining the correct variables necessary to describe the universal features of many models, for instance their asymptotic behaviour, as happens in the renormalization of quantum theories or in the multiple–scale expansions [3].

In the last years, new physical and mathematical motivations have emerged for the study of dynamical models on a lattice. Many systems of interest in nonlinear optics [21] or in relation with DNA are indeed discrete. Also, awareness is growing that space–time may possess an intrinsic discreteness.

Key words and phrases. Finite operator theory, multiple–scale expansion, difference equations.

For continuous systems, Lie group theory is the most efficient approach to get self–similar solutions [22]. A direct extension of Lie theory to lattice equations [18] has not been yet as fruitful, as its continuous counterpart, in the study of scaling symmetries. For lattice models, we can only define very few group transformations, i.e. we can add a constant (finite translation) or multiply or divide a discrete variable by an integer multiple (finite dilation). These are fundamentally discrete symmetries. In principle, the infinitesimal counterparts of these transformations do not exist, although they can be formally introduced.

Among the perturbation methods used to study asymptotics of problems involving a small parameter, very important ones are the reductive multiple–scale expansions [4]. They are relevant when ordinary perturbation methods fail to give accurate uniform approximations of solutions. One introduces multiple scales to avoid secularity terms and obtain uniform asymptotics. In the case of continuous systems, this procedure has been intensively investigated for the wideness of physical and engineering applications [4].

Theoretically, the multiple–scale approach has given important results, especially in the study of the integrability of nonlinear partial differential equations. Indeed, starting from an integrable system, performing multiple–scale expansion one gets other integrable systems [27, 31]. In [6, 7], Calogero and Eckhaus have proven that a necessary condition for an equation to be integrable is that its multiscale reduction be integrable as well. In [9], higher order corrections have been also investigated: it has been shown that, under suitable hypotheses, the leading amplitude modulations of one–dimensional strongly dispersive waves satisfy the nonlinear Schrödinger hierarchy.

The general problem of the multiple scale analysis for discrete systems seems to be technically much more tricking [19], [15], [13], [1]. In [19], a discretization of the standard reductive method has been proposed, relying on the definition of a large grid of points, indexed by slow variables, and on the comparison of the different physical observable between the original and the large scale grid. This procedure has been further studied in [15] and in [13], where an integrability test based on discrete multiple–scale analysis has been proposed for a $\mathbb{Z}^2$–lattice. We observe that in general the process of rescaling is not univocally defined, since the discrete derivative can be defined in infinitely many different ways. More important, a multiple–scale expansion usually leads to discrete equations of infinite order.

The aim of this work is to contribute to the construction of a general approach to the multiple–scale analysis of discrete dynamical systems, or more generally of evolution equations, defined in an equally spaced lattice of points. Our approach is based on the finite operator theory, developed by [26], [25] and many other authors (see [5] for an update bibliography). This has the advantage of presenting a unified formulation, valid for a large class of discrete operators, which has probed to be useful to encompass some of the typical technical problems appearing in the treatment of these symmetries [16], [17], [29]. It allows to consider on the lattice a large class of multiple–scale expansions, involving bounded functions. We will show how, in this framework, the problem of the description of physical observables defined on different lattices can be completely solved in a natural way.

We remind that a key ingredient, both in the determination of the group action over a difference equation and in the multiple scale analysis, are the transformation formulae of the dependent variables, calculated in terms of shifted points of the rescaled lattice. An example of this formulae, derived by means of combinatorial methods, can be found in the book by Jordan [14]. In Jordan's formula, a function on a lattice is expanded, in a new rescaled lattice, in terms of an infinite number of contributions: differences at all orders will appear. In the continuum limit, this formula simply reproduces the first derivative. Moreover, on the lattice one can define an infinite number of different representations of the continuous derivatives [16]. Jordan formula considers just its simplest representation, i.e. the standard forward difference operator. Other representations can be more important, since they might be adapted to intrinsical symmetries contained in the physical model at study. Observe that the forward discrete derivative is not a self adjoint operator. When constructing a discrete theory of quantum mechanics, the use of a symmetric difference operator is more suitable.

In this paper, in particular, we prove theorem for the transformation of any discrete operator in terms of another one. Consequently, we will derive the equivalent of Jordan formula for arbitrary difference operators. As a rule, these transformations lead to infinite divergent series. In order to encompass this problem, we introduce suitable restrictions, that we call the *slow–varyness conditions*, on the space of functions, which reduce *de facto* the order of the difference equations emerging from the scaling procedure to a finite one. These conditions were already proposed in [15, 13], but only the case of the forward difference operator is analyzed, making use of the classical Jordan formula. In our formulation, the slow variation of the dynamical observables is proposed on a completely general algebraic setting.

In the present work, we have been inspired by the relevance of the operator methods in the theory of nonlinear PDEs [20], [8]. The main idea of these methods is to represent a differential equation as an abstract operator equation, defined in a suitable Banach algebra. Then, to each representation of the algebra it would correspond a different equation. In perspective, our aim is to develop an analog theory for difference equations, based on the formalism of delta operators. Here we focus on the algebraic structure of multiple–scale analysis for difference equations. An algebraic formulation of the space of solutions in a suitable Banach space of operators has to be done. Some results have been already obtained in [16], [17], [29].

Observe that Rota's theory in general is formulated on the space of formal power series, and this is also the natural space where our approach works. However, convergence problems usually arise in concrete applications. A regularization procedure should be advocated in order to handle these difficulties. This is a challenging aspect, deserving further attention.

The paper is organized as follows. In Section 2, we present a concise introduction to the finite operator calculus. In Section 3, we will propose an analysis of scaling lattice transformations on discrete models. Section 4 is devoted to the study of antidifference operators and the general definition of the slow–varyness condition. The application of the resulting formulas to the multiple–scale expansion of a the Duffing equation is presented in the final Section 5. With a non–obvious choice of the discrete derivative and of the function space, we have been able to find, on the lattice, a non–secular first order approximation to the solution.

A crucial problem remains open for future investigation, i.e. if the rescaled finite difference equations may preserve, on the lattice, the same mathematical properties and physical behaviour of the original continuous system.

2. Finite difference theory: Rota's approach

2.1. **Basic notations. The umbral algebra.** We review the basic features of finite difference operator theory, necessary for the subsequent discussion. Some new results are also presented. An extensive and elegant treatment of this topic can be found in the monographs [25, 26].

Let us fix some notation. The relevant space in which the theory is settled is the algebra $\mathcal{F} \equiv \mathcal{F}[[t]]$ of formal power series in one variable t, endowed with the operations of sum and multiplication of series. An element $F \in \mathcal{F}$ is expressed by the formal power series

$$F(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k. \tag{1}$$

This choice is motivated by the fact that convergence properties are not considered at this stage. Any element of $\mathcal{F}$ can be thought of as a formal power series, as an operator and as a linear functional, according to the following considerations.

Let $\mathcal{P} \equiv K[x]$ be the algebra of polynomials in one space variable $x \in \mathbb{R}$ over a field K, usually identified with $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{P}^*$ be its dual. We will denote by $\{p_n(x)\}_{n\in\mathbb{N}}$ a sequences of polynomials in x of degree n.

We will reserve for the action of a linear functional L on a polynomial $p_n(x) \in \mathcal{P}$ the notation $< L \mid p_n(x) >$.

Let us describe first the connection between formal power series and linear operators on $\mathcal{P}$. We denote the derivative operator D^k with t^k. Hence, its action on a polynomial $p(x)$ is given by

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}.$$

This action can be extended by linearity. Hence, a formal power series $F(t)$ can also be seen as an operator acting on $\mathcal{P}$:

$$F(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} \longleftrightarrow F(D) = \sum_{k=0}^{\infty} \frac{a_k D^k}{k!}. \tag{2}$$

The third interpretation of $\mathcal{F}$, as a space of linear functionals, is motivated by the following construction. For any polynomial $p(x)$ we can define the functional

$$\left\langle t^k \middle| \, x^n \right\rangle = n! \delta_{n,k} \tag{3}$$

where $\delta_{n,k}$ is the Kronecher delta function. Consequently, we deduce that

$$\left\langle t^k \middle| \, p(x) \right\rangle = p^{(k)}(0) = \frac{d^k p(x)}{dx^k}\Big|_{x=0}, \tag{4}$$

where $p^{(k)}(0)$ denotes the k–derivative of $p(x)$ evaluated at $x = 0$. It follows that the power series $F(t)$ defines as well a linear functional, according to the prescription

$$\left\langle F(t) \middle| \, x^n \right\rangle = a_n. \tag{5}$$

One can prove that there exists an isomorphism between the spaces $\mathcal{P}^*$ and $\mathcal{F}$ (see e.g. [25]). The algebra $\mathcal{F}$ of formal power series, once identified with both that of linear operators and of linear functionals on $\mathcal{P}$, is usually called the *umbral algebra*.

2.2. **Delta operators: a short review.** In the following, all operators considered will act on the algebra of (sufficiently regular) functions $\mathcal{F}_{\mathcal{L}}$ defined on a regular equally spaced lattice of points, denoted by $\mathcal{L}$ and indexed by $x = n\sigma$, with $n \in \mathbb{N}$, $\sigma \in R$.

Also, T will denote the shift operator on $\mathcal{L}$, whose action on a function $f \in \mathcal{F}_{\mathcal{L}}$ is given by $Tf(x) = f(x+\sigma)$. As usual, the operator T can also be represented in terms of differential operators as $T = e^{\sigma D} \equiv e^{\sigma t}$.

Definition 1. *An operator S is said to be shift–invariant if commutes with T. A shift–invariant operator Q is called a delta operator if $Qx = const \neq 0$.*

Directly for the definition 2.2, we deduce the following property.

Corollary 2. *For every constant $c \in \mathbb{R}$, $Qc = 0$.*

The most common example of delta operator is provided by the derivative D. As has been proved in [25], there is an isomorphism between the ring of formal power series in a variable t and the ring of shift–invariant operators, carrying eq. (1) into $\sum_{k=0}^{\infty} \frac{a_k Q^k}{k!}$. Formula (2) is a particular example of this isomorphism, since D is a delta operator.

Definition 3. *Given a delta operator Q, a polynomial sequence $\{p_n(x)\}_{n\in\mathbb{N}}$ will be said to be the sequence of basic polynomials for Q if the following conditions are satisfied:*

$$\begin{aligned} &1) \quad p_0(x) = 1; \\ &2) \quad p_n(0) = 0 \text{ for all } n > 0; \\ &3) \quad Qp_n(x) = np_{n-1}(x). \end{aligned}$$

Notice that, for a given delta operator Q the sequence of associated basic polynomials is unique.

Definition 4. *Given a shift invariant operator S, we call the function (or a formal power series) $s(t)$, associated with S under the isomorphism (2) the indicator of S.*

With a slight abuse of notation, in the following we often identify an operator (especially a delta operator) with its indicator.

Definition 5. *An Appell sequence is a sequence of polynomials $p_n(x)$ such that $Dp_n(x) = np_{n-1}(x)$.*

The most important examples of Appell sequences, apart from $\{x^n\}_{n\in\mathbb{N}}$, are the classical Bernoulli and Euler polynomials and their generalizations (see, for example, [28]).

We now consider the *expansion theorem for basic sequences* [25].

Theorem 6. *If $p_n(x)$ is the basic sequence for $q(t)$, then, for any $h(t) \in \mathcal{F}$ the following formula will be valid*

$$h(t) = \sum_{k=0}^{\infty} \frac{< h(t)|p_k(x) >}{k!} q^k(t). \tag{6}$$

As a consequence of this result, any shift invariant operator S can be expanded into a formal power series in terms of a delta operator Q:

$$S = \sum_{k\geq 0} \frac{a_k}{k!} Q^k,$$

with $a_k =< s(t)|p_k(x) >$, where $p_k(x)$ is the basic polynomial of order k associated with Q and $q(t)$ is the indicator of Q.

Eq. (6) reproduces the standard Taylor expansion formula when $h(t) = e^{at}$.

If instead we choose $q(t) = (e^{at} - 1)/a$ (i.e. the indicator of Δ^+, whose basic sequence is $(x)_n = x(x-a)\ldots(x-(n-1)a)$), and apply it to a function $F(x)$ we get:

$$F(x+a) = \sum_{k=0}^{\infty} \frac{\langle e^{at} \mid (x)_k \rangle}{k!} \Delta^k F(x) =$$

$$= \quad F(x) + a\frac{F(x+a) - F(x)}{2!} + a(a-1)\frac{F(x+2a) - 2F(x+a) + F(x)}{3!} + \ldots$$

This formula, with a proper choice of the constant a, is Jordan's formula.

To study scaling transformations for equations on the lattice, we will consider delta operators expressed as finite Laurent series in shifts [16]:

$$\Delta_r = \frac{1}{\sigma}\sum_{k=l}^{m} a_k T^k, \quad l, m \in \mathbb{Z}, \quad l < m, \quad m - l = r, \tag{7}$$

where a_k are constants such that

$$\sum_{k=l}^{m} a_k = 0, \quad \sum_{k=l}^{m} k a_k = c. \tag{8}$$

and $a_m \neq 0$, $a_l \neq 0$. We choose $c = 1$, to reproduce the derivative D in the continuum limit, when the lattice spacing goes to zero. We will say that a difference operator of the form (7), which satisfies equations (8), is a delta operator of order r, if it approximates the continuous derivative up to terms of order σ^r.

As eq. (7) involves $m-l+1$ constants a_k, subject to just the two conditions (8), we can fix all constants a_k by choosing $m-l-1$ further conditions. A possible choice is, for instance, to set $\sum_{k=l}^{m} k^{\ell} a_k = 0$, $\ell = 2, 3, \ldots, m-l$. Useful examples of Δ_r operators are

$$\Delta^+ = \frac{T-1}{\sigma}, \; \Delta^- = \frac{1-T^{-1}}{\sigma}; \quad \Delta^s = \frac{T - T^{-1}}{2\sigma}, \tag{9}$$

respectively the first two of order $p = 1$ and the third of order $p = 2$.

Eq. (6) solves the problem of expanding the function $h(t)$ in terms of a delta operator $q(t)$. To construct explicitly the expansion (6), we just need to know the basic polynomials $p_n(x)$ associated to $q(t)$. Eq. (6) is very general as is valid for any operator t and any function $h(t)$ and $q(t)$. In particular, it will be true for $t = D$, $h(t) = T^{\pm 1} = \exp(\pm t)$, and $q(t)$ given by any of the delta operator presented above.

We have several ways to construct the basic polynomials $p_k(x)$ corresponding to a delta operator $Q = q(t)$. The most direct one is based on Definition 3. The application of the delta operator Q on an arbitrary first order polynomial in x, satisfying conditions (1) and (2), gives us $p_1(x)$. Iteratively, we can construct $p_n(x)$, for any $n \in \mathbb{N}$. Another way to get sequences of basic polynomials is to apply the so called Transfer Formulae [25]. If $p_n(x)$ is a sequence of basic polynomials for the delta operator $q(t)$, then

$$p_n(x) = x\left(\frac{q(D)}{D}\right)^{-n-1} x^{n-1}. \tag{10}$$

However, the use of eq. (10) is as complicate as the iterative procedure described before, since it requires a series expansion at each order n and it does not provide a closed formula in most of the cases.

In what follows, a more efficient approach, based on [16], is presented.

Definition 7. *Let Q be a delta operator. Its Pincherle derivative Q' is defined by $Q' = [Q, x]$.*

Let us denote by $\beta(t)$ the inverse of Q'. We can prove the following result [16].

Proposition 8. *The basic sequence associated with the delta operator Q is given by*

$$p_n(x) = (x\beta)^n \cdot 1 \tag{11}$$

It is worthwhile to show that we can always re–express the basic sequence associated with a given Δ^p in terms of a generalization of the Stirling numbers, as in [28]. Following the notation in [28], let us denote by $(x)_n^p$ the basic polynomials for $\Delta^p : \Delta_p(x)_n^p = n(x)_{n-1}^p$. This notation is collective, as, for any p, there could be more than one delta operator of the same order p. For instance, Δ^+ and Δ^- are both operators of the first order.

Definition 9. *The generalized Stirling numbers of the first kind and order r associated with the operators (7, 8), and denoted by $s_r(n,k)$ are defined by*

$$(x)_n^r = \sum_{k=0}^{n} s_r(n,k)x^k. \tag{12}$$

The generalized Stirling numbers of the second kind and order r, denoted by $S_r(n,k)$ are defined by

$$x^n = \sum_{k=0}^{n} S_r(n,k)(x)_k^r. \tag{13}$$

One can prove that the generalized Stirling numbers of second kind admit a nice representation in terms of the following generating functions:

$$\sum_{n=k}^{\infty} S_r(n,k)\frac{t^n}{n!} = \frac{[\Delta_r(t)]^k}{k!}, \tag{14}$$

where $\Delta_r(t) = \sum_j a_j e^{jt}$ is the indicator of the discrete operator Δ_r. We also have an interesting orthogonality relation:

$$\delta_{mn} = \sum_k s_r(n,k)S_r(k,m) = \sum_k S_r(n,k)s_r(k,m). \tag{15}$$

2.3. **Examples of basic sequences. Generalized antidifference operators.** The case of the polynomials associated to the operators Δ^+, Δ^- and Δ^s are particularly relevant for the applications. The case of Δ^+ has been discussed above. For Δ^-, the basic polynomials are

$$p_n(x) = x(x+\sigma)(x+2\sigma)...(x+(n-1)\sigma). \tag{16}$$

We propose now a new representation for the basic sequences associated to the delta operator Δ^s. Let us denote it by $\{[x]_n\}_{n\in\mathbb{N}}$.

Recall that the Gould polynomials are defined by [25]

$$G_n\left(x\,;a,b\right)=\frac{x}{x-an}\binom{\frac{x-an}{b}}{n}, \tag{17}$$

where here $(x)_n$ is the basic sequence associated to the operator $q(t)=e^{at}(e^{bt}-1)$. We propose the following simple result, which to our knowledge is new.

Lemma 10. *The basic sequence* $[x]_n$ *for* $\Delta^s=\frac{T-T^{-1}}{2\sigma}$ *is given by*

$$[x]_n=2^n\frac{x}{x+\sigma n}\binom{\frac{x+\sigma n}{2\sigma}}{n}=2^nG_n\left(x\,;-\sigma,2\sigma\right). \tag{18}$$

The generating function of the polynomials $[x]_n$ *is*

$$\sum_{n=0}^{\infty}\frac{[x]_n}{n!}w^n=e^{x\arcsin(\sigma w)}=\sum_{j=0}^{\infty}\frac{[x\arcsin(\sigma w)]^j}{j!}. \tag{19}$$

Proof. □

An equivalent expression for the polynomials $[x]_n$ is provided by

$$[x]_n=x\prod_{k=1}^{n-1}(x+n\sigma-2k\sigma),\qquad n>1. \tag{20}$$

with $[x]_0=1$, $[x]_1=x$, $[x]_2=x^2$. The structure of these polynomials becomes more transparent for our later use once we distinguish the cases n even and n odd. Precisely, we have:

$$[x]_{2n}=x^2\prod_{k=1}^{n-1}[x^2-(2k\sigma)^2], \tag{21}$$

$$[x]_{2n+1}=x^2\prod_{k=1}^{n-1}[x^2-(2k-1)^2\sigma^2]. \tag{22}$$

By differentiating eq. (19) with respect to x we get the following result.

Corollary 11. *the generating function of the numbers* $p_k^{(j)}(0)$ *are provided by*

$$\sum_{k=0}^{\infty}\frac{p_k^{(j)}(0)}{k!}w^k=(\arcsin\sigma w)^j. \tag{23}$$

Moreover, for $j=1$, we have the remarkable relation $p_k^{(1)}(0)=(k-1)!L_{k-1}(0)$, where $L_k(x)$ are the Legendre polynomials of order k.

We introduce now the notion of *antidifference delta operators*. They can be considered as a discrete analog of the indefinite integration. Apart from the case of the operator $(\Delta^+)^{-1}$, already studied by Elaydi [11], the general definition seems to be new.

Definition 12. *Let* Q *be a delta operator acting on the lattice* $\mathcal{L}$, *and* $F(x)$ *a function on the lattice. Let* $f(x)$ *be such that* $QF(x)=f(x)$. *The generalized antidifference operator* $Q^{(-1)}$ *is defined by*

$$Q^{(-1)}f(x)=F(x)+c \tag{24}$$

for some arbitrary constant c.

Hence

$$QQ^{(-1)}f(x) = f(x) \tag{25}$$

$$Q^{(-1)}QF(x) = F(x) + c \tag{26}$$

This notion will be useful in Section 4, in the study of the slow–varyness conditions for functions defined on the lattice.

3. Dilation transformations for evolution equations on the lattice

Consider an ordinary difference equation $\mathcal{E}(\{u_{n\pm j}\}) = 0$ for a dependent variable u_n defined on an equally spaced lattice, whose points are labeled by the index n. The function u can be thought as a continuous function, sampled on the discrete values $x = n\sigma$, where σ is the lattice spacing. The equation $\mathcal{E} = 0$, which involves the function u evaluated at the point n and at one or more of its neighboring points $\{n \pm j\}$, can also be interpreted as a functional equation for u, calculated in x and in its neighboring points $\{x \pm j\sigma\}$. The functional equation we get $\mathcal{E}(\{u(x \pm j\sigma)\}) = 0$ can be written in terms of operators $T_x^{\pm j}$, which shift the variable x by $\pm j\sigma$. In turn, the shift operators can be formally expressed as functions of the derivative operator:

$$T_x^{\pm j} = e^{\pm j\sigma D_x} = \sum_{k=0}^{\infty} \frac{(\pm j\sigma D_x)^k}{k!}, \tag{27}$$

Let us perform a dilation of the lattice variable, i.e. let us pass from the index n to a new integer index m such that $n = Nm$, where $N \gg 1$ is an integer. At the same time, we can consider a change of the independent variable x into a new variable $y = \varepsilon x$, where ε is small, and

$$y = m\sigma$$
$$x = n\sigma$$

Therefore, we get the relation $\epsilon N = 1$. We can ask ourselves how the equation changes under this transformation of the independent variable, i.e. how the equation looks like in the new lattice variable of index m. Let $l \in \mathbb{Z}$. The relation between the powers of the shift operators in the two lattices reads

$$T_m = T_n^N \quad \rightarrow \quad T_n^l = T_m^{\frac{l}{N}}. \tag{28}$$

For powers of the shift operators, Theorem 6 can be reformulated as follows.

Theorem 13. *The shift operator $T_n = e^{\sigma D_x}$ on the n–lattice can be expanded in terms of any given delta operator, represented by the function $q(D_y)$ on the m–lattice, with associated basic sequence $p_n(x)$, $n = 0, 1, 2, \dots$. We have:*

(29)

$$T_n^l = e^{\epsilon\sigma l D_y} = \sum_{k=0}^{\infty} \frac{\left\langle e^{\epsilon\sigma l D_y} \mid p_k(y)\right\rangle}{k!} q(D_y)^k = \sum_{k=0}^{\infty} \frac{p_k(\epsilon\sigma l)}{k!} q(D_y)^k \equiv \sum_{k=0}^{\infty} \frac{p_k(\epsilon\sigma l)}{k!} \Delta_p^k,$$

In eq. (29), we have taken into account the fact that $\langle e^{yt} \mid p_k(x)\rangle = p_k(y)$, as one can infer from eq. (5).

Observe that, in the expansion (29), every polynomial $p_k(z)$, with $z = \epsilon\sigma l$, contributes to the coefficient z^p, for $k \geq p$. Even the first order term in z is represented, a priori, by an infinite series.

An interesting aspect is that the previous expansions can also be used to map continuous operators into discrete ones. The following result, aiming to clarify this point, is also a consequence of Theorem 6.

Lemma 14. *Let $q(t)$ be a difference delta operator, and $p_k(x), k = 0, 1, 2, \dots$ the associated basic polynomials. The expansion of the j–th derivative $(D_x)^j$ in terms of the chosen difference operator is given by*

$$(D_x)^j = \sum_{k=0}^{\infty} \frac{\langle (D_x)^j \mid p_k(x) \rangle}{k!} q(D_x)^k = \sum_{k=0}^{\infty} \frac{p_k^{(j)}(0)}{k!} q(D_x)^k \equiv \sum_{k=0}^{\infty} \frac{p_k^{(j)}(0)}{k!} \Delta_p^k, \tag{30}$$

where $p_k^{(j)}(x)$ denotes the j–th derivative of $p_k(x)$.

The relevance of Lemma 14 and Theorem 13 is that they allow us to map a continuous or a discrete dynamical system into a new one, expressed in terms of other discrete operators. Nevertheless, in order to get a difference equation of finite order, the series in formula (29) should truncate. This will be true if there exists an integer k_0 such that the polynomials $p_k(\epsilon\sigma l)$ vanish for $k \geq k_0$. It turns out that, at least in the cases of Δ^+ and Δ^-, the truncation of the expansions is possible only in a trivial sense.

The operator Δ^s is self–adjoint and its basic polynomials do admit zeroes on the whole line, as is clear from formula (20). This observation gives a special rôle to the discrete operators symmetric in T and T^{-1} (see also [30] for the relevance of symmetric operators for the integrability of differential–difference models). However, from eqs. (21) and (22) it emerges that it is not possible to annihilate all polynomials $p_n(x)$ at the same time. Instead, one can find a zero for the polynomials of even index or, separately, for those of odd index. It means that the series (29), when written in terms of Δ^s, cannot be truncated by this kind of choices.

The derivatives $\frac{d^j p_k(x)}{dx^j}$ of the basic polynomials associated with $\Delta^{\pm}$ or Δ^s do possess the same qualitative behavior as the polynomials $p_k(x)$, i.e. the structure of the zeroes is essentially the same, due to the property 3) in Definition 2. Therefore, the same conclusions will apply also for the development proposed in formula (29).

We also mention the following result.

Corollary 15. *The shift operator T_n on the n–lattice can also be expanded, on the m–lattice, in the form*

$$T_n^l = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{s^r(k,j)(\epsilon\sigma l)^j}{k!} \Delta_r^k. \tag{31}$$

The previous Theorem 13 and Corollary 15 can be interpreted as a generalization of a classical expansion formula due to Jordan [14], which states that

$$\Delta_H^k g(n_1) = \sum_{i=k}^{\infty} \frac{k!}{i!} \mathcal{K}(i,k) \Delta_h^i f(n), \tag{32}$$

where

$$\mathcal{K}(i,j) = \sum_{\alpha=j}^{i} \left(\frac{H}{h}\right)^{\alpha} S_i^{\alpha} \mathfrak{S}_{\alpha}^j.$$

Here $H = N\sigma$ and $h = \sigma$ are the spacing of the two lattices; n_1 and n are two integers indexing the two lattices. Also, $f(n)$ is a function defined on the first lattice and $g(n_1)$ the corresponding one on the second lattice. Finally, $S_i^{\alpha}, \mathfrak{S}_{\alpha}^{j}$ are the Stirling numbers of first and second kind, respectively. The transformation formula (32) can be easily reproduced, by taking into account that $\Delta^{+} = \frac{T_n - 1}{\sigma}$.

Remark 16. *Theorem 13 allows to determine the transformation rule of a difference delta operator of the form (7,8) between two different lattices. For instance, if we want to expand them in terms of Δ^{+}, we obtain*

$$\Delta_r = \sum_{l=r}^{q} a_k T_n^l = \sum_{k=0}^{\infty} \sum_{l=p}^{q} a_k \frac{(\varepsilon l)_k}{k!} (\Delta^{+})^k, \quad r,\, q \in \mathbb{Z}, \quad r < q. \tag{33}$$

Here $(\varepsilon l)_k$ denotes the lower factorial polynomial of its argument, with $\mid \varepsilon L \mid < 1$, where $L = max(\mid p \mid, \mid q \mid)$ As a consequence of the above considerations, we can perform a scaling transformation, mapping a dynamical system described by a differential or a difference equation into a new system, represented by a difference equation of a distinct order, usually infinite.

For continuous models, it suffices to use eq. (30) by expanding in terms of $\Delta^{\pm}$. For discrete models, one has to rewrite first the difference equation under investigation in such a way that only positive shift (resp. negative shift) do appear. This can be done in all cases, without loss of generality. Then, by using Lemma 8 and the previous analysis, one can get another difference model on a different lattice.

To overcome the drawback represented by the infinite order expansions emerging from this theory, a possibility is to work in the space of slow–varying functions [15, 13]. In the next section, we look for those conditions on the scaling transformations under which the series expansions *do truncate.*

4. Slow–varyness conditions for higher–order difference operators

In this Section, we introduce a condition of summability for functions defined on a regular lattice. We assume that $x = n\sigma$.

Definition 17. *Let Q be a delta operator. A slow–varying function (with respect to Q) is a function $f \in \mathcal{F}_{\mathcal{L}}$, such that the following properties hold:*

(i) there exists an integer N such that

$$Q^N f(x) \neq 0; \tag{34}$$

(ii) the exists a sequence of numbers $\{a_k\}_{k\in\mathbb{N}}$ such that the relation

$$Q^{N+k} f(x) = Q^k f(x) \cdot a_k, \tag{35}$$

is satisfied, as well as the convergence requirement

$$\sum_{k=1}^{\infty} a_k = s \in \mathbb{R}. \tag{36}$$

In particular, when $a_k = 0$, $k = 1, 2, \ldots$, and $Q \equiv \Delta^{+}$, this definition specializes into the requirement of [15], [13] that there exists an integer N such that

$$Q^{N+1} f(x) = 0. \tag{37}$$

Observe that the more restrictive assumption (37) implies that $f(x)$ is nothing but a polynomial in x of order N. This can be ascertained by using the antidifference operator associated with Q, and the following statement.

Lemma 18. *Let $\{p_n(x)_{n\in\mathbb{N}}\}$ be the family of basic polynomials associated to Q, and $p_n(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0$ the basic polynomial of order n. The following properties hold:*

$$(i) \qquad Q^{(-1)}p_n(x) = \frac{p^{(n+1)}}{n+1} + c \tag{38}$$

$$(ii) \qquad Q^{(-n)}1 = \frac{p_n(x)}{n!} + c_1 p_{n-1}(x) + c_2 p_{n-2}(x) + \ldots + c_{n-1}p_1(x) + c_n, \tag{39}$$

where $c, c_1, c_2, \ldots, c_n$ are arbitrary constants.

Proof. The result *(i)* follows directly from formula (3) of Definition 3 and from Corollary 2. To prove the result *(ii)*, it suffices to observe that $Q^{(n)}p_n(x) = b_n n!$, $Q^{(n+i)}p_n(x) = 0$ for $i \geq 1$ and to apply $Q^{(n)}$ to both sides of formula *(ii)*. □

Finally, we can represent a given function $f(x) \in \mathcal{F}_{\mathcal{L}}$ as a linear combination of the basic polynomials associated to Q:

$$f(x) = \sum_{k=0}^{\infty} a_k p_k(x). \tag{40}$$

By applying the condition (37), the polynomial structure of $f(x)$ is evident. Instead, if we use the more general definition (34), the polynomiality restriction no longer holds.

The definition (34) of slow–varying functions generalizes that one given in [15, 13] in two respects: it allows to consider a larger class of functions, (not necessarily polynomials), and also is valid for every delta operator Q (not just in the case of Δ^+). The main reason to consider the class of slow-varying functions (34) is that, by working on this space, all of the expansions that we considered in Section 3 are *convergent*. Indeed

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{p_k(\tau l)}{k!} Q^k f(x) &= \sum_{k=0}^{(N-1)} \frac{p_k(\tau l)}{k!} Q^k f(x) + \sum_{j=0}^{\infty} \frac{p_{j+N}(\tau l)}{(j+N)!} Q^k f(x) = \\ &= \sum_{k=0}^{N} \frac{p_k(\tau l)}{k!} Q^k f(x) + Q^N f(x) \cdot \sum_{j=1}^{\infty} \frac{a_j}{(j+N)!} p_{j+N}(\tau l), \end{aligned} \tag{41}$$

where the last series is a power series, having a priori a nonzero radius of convergence, due to the convergence of a_j.

The constraint (34), being specific of a given operator Q, does not restrict any other expansion related to a distinct delta operator $\overline{Q}$. This is a consequence of the uniqueness of the sequence of basic polynomials associated with a delta operator.

In summary, in the last two Sections, we have shown the following.

(1) One can rescale a discrete dynamical system, and obtain another difference equation involving discrete derivatives arbitrarily chosen; the same procedure can be applied to map a differential equation into an a priori infinite family of discrete equations, each for any possible choice of Q.

(2) By a proper choice of the function space we work in, i.e. by imposing a slow–varyness condition, one can obtain a dynamical system of *finite order* on the lattice.

Remark 19. *It is evident that the hypothesis (34) can be further relaxed, for instance by requiring a weaker condition of the form $Q^{N+k}f(x) \simeq Q^k f(x) \cdot a_k$. A priori, the divergent case can also be relevant. It would be interesting to perform an analysis of divergencies by using, for instance, a Borel or Abel summation–type approach, Gevrey analysis or other modern techniques* [2, 12].

5. Application: multiple–scale analysis of the Duffing equation

A standard example of application of multiple–scale expansions, that can be found in many books devoted to asymptotic analysis of differential equations (see for instance [4]) is provided by the *Duffing equation*, the nonlinear extension of the harmonic oscillator:

$$\frac{d^2y(t)}{dt^2} + y(t) + \epsilon y^3(t) = 0, \qquad y(0) = 1, \; \frac{dy(t)}{dt}|_{t=0} = 0, \tag{42}$$

where ϵ is a small parameter. By introducing a set of time variables, $t_0 = t$, $t_1 = \epsilon t$, etc., and assuming a perturbation expansion of the form

$$y(t) = Y_0(t_0, t_1) + \epsilon Y_1(t_0, t_1) + \cdots \tag{43}$$

eq. (42) reduces to

$$\frac{d^2Y_0}{dt_0^2} + Y_0 + \epsilon\{\frac{d^2Y_1}{dt_0^2} + Y_1 + Y_0^3 + 2\frac{d^2Y_0}{dt_0 dt_1}\} + \mathcal{O}(\epsilon^2) = 0. \tag{44}$$

The most general solution of the ϵ independent part of eq. (44) is

$$Y_0(t_0, t_1) = A(t_1)e^{it_0} + \bar{A}(t_1)e^{-it_0}, \tag{45}$$

where $A(t_1)$ is an arbitrary complex function of its argument and $\bar{A}(t_1)$ is its complex conjugate. $A(t_1)$ is determined by the condition that no secular term appear in the first order terms in ϵ of eq. (44), and that the initial conditions (42) be satisfied. In this way, we get

$$A(t_1) = \frac{1}{2}e^{3it_1/8}. \tag{46}$$

When we replace eq. (46) into eq. (45), we obtain a solution of the Duffing equation valid up to second order in the perturbation parameter. This procedure relays on the introduction of the rescaled variable t_1 to remove the secular effects.

Let us consider the equivalent discrete equation

$$(\Delta_n^+)^2 y_{n-1} + y_n + \epsilon y_n^3 = 0, \tag{47}$$

with Δ^+ given as in eq. (9). The linear part is chosen in such a way that admits an oscillatory solution equivalent to (45). The perturbation expansion of y_n reads

$$y_n = Y^0_{n_0,n_1} + \epsilon Y^1_{n_0,n_1} + \cdots , \tag{48}$$

where $n_0 = n$, $n_1 = \epsilon n$, etc. with $\epsilon = 1/N$. From now on, we will denote by Δ_i, $i = 0, 1$ the delta operator acting in the space corresponding to the discrete variables n_0 and n_1. Taking into account the theory developed in the previous sections, from eq. (31) we get

$$y_{n+1} = T_n y_n = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{s_p(k,j)\epsilon^j\sigma^j}{j!}(\Delta_1)^k[Y^0_{n_0+1,n_1} + \epsilon Y^1_{n_0+1,n_1} + \cdots], \tag{49}$$

$$y_{n-1} = T_n^{-1} y_n = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{s_p(k,j)\epsilon^j\sigma^j}{j!}(\Delta_1)^k[Y^0_{n_0-1,n_1} + \epsilon Y^1_{n_0-1,n_1} + \cdots]. \tag{50}$$

where Δ_1 is a delta operator of order p and $s_p(k,j)$ are the corresponding generalized Stirling numbers of the first kind. Since, from Definition 3, $s_p(0,0) = 1$, while $s_p(k,1) \neq 0$ for any k, the lowest order terms of the expansions (49, 50) read:

$$y_{n+1} = Y^0_{n_0+1,n_1} + \epsilon[Y^1_{n_0+1,n_1} + \sigma\sum_{k=1}^{\infty} s_p(k,1)(\Delta_1)^k Y^0_{n_0+1,n_1}], \tag{51}$$

$$y_{n-1} = Y^0_{n_0-1,n_1} + \epsilon[Y^1_{n_0-1,n_1} + \sigma\sum_{k=1}^{\infty} s_p(k,1)(\Delta_1)^k Y^0_{n_0-1,n_1}], \tag{52}$$

where σ is the lattice spacing. The explicit values of $s_p(k,1)$ depend on the delta operator we choose.

By replacing eqs. (48, 51, 52) into eq. (47), we get:

$$\begin{aligned}(\Delta_0^+)^2 Y^0_{n_0-1,n_1} + Y^0_{n_0,n_1} + \epsilon\big\{(\Delta_0^+)^2 Y^1_{n_0-1,n_1} + Y^1_{n_0,n_1} + \\ + \frac{\sum_{k=1}^{\infty} s^1(k,1)}{\sigma}\big[(\Delta_1)^k Y^0_{n_0+1,n_1} - (\Delta_1)^k Y^0_{n_0-1,n_1}\big] + (Y^0_{n_0,n_1})^3\big\} + \cdots = 0.\end{aligned} \tag{53}$$

The equation obtained by equating to zero the terms of order ϵ^0 in eq. (53) possesses the solution

$$Y^0_{n_0,n_1} = A_{n_1}\kappa^{n_0} + \bar{A}_{n_1}\bar{\kappa}^{n_0}, \tag{54}$$

where κ and $\bar{\kappa}$ are the two complex conjugate solutions of the algebraic equation $k^2 + (\sigma^2 - 2)k + 1 = 0$. The function A_{n_1} is a complex function of the discrete variable n_1 only.

By substituting the solution (54) into the equation for the ϵ^1 terms of eq. (53), we get the nonlinear difference equation

$$\begin{aligned}(\Delta_0^+)^2 Y^1_{n_0-1,n_1} + Y^1_{n_0,n_1} = -\big(A_{n_1}\kappa^{n_0} + \bar{A}_{n_1}\bar{\kappa}^{n_0}\big)^3 + \\ -\frac{\sum_{k=1}^{\infty} s_p(k,1)}{\sigma}\big((\Delta_1)^k A_{n_1}\kappa^{n_0+1} + (\Delta_1)^k \bar{A}_{n_1}\bar{\kappa}^{n_0+1} - \\ -(\Delta_1)^k A_{n_1}\kappa^{n_0-1} - (\Delta_1)^k \bar{A}_{n_1}\bar{\kappa}^{n_0-1}\big).\end{aligned}$$

The solution of the previous equation for Y^1 contains secularities unless the complex function A_{n_1} satisfies the following nonlinear difference equation

$$\frac{\sum_{k=1}^{\infty} s_p(k,1)}{\sigma}(\kappa - \bar{\kappa})(\Delta_1)^k A_{n_1} + 3|A_{n_1}|^2 A_{n_1} = 0. \tag{55}$$

Since we are interested in bounded solutions (otherwise the perturbative approach is no more valid), we can look for solutions of the form

$$A_{n_1} = Re^{izn_1}, \quad (R, z) \in \mathbb{R}. \tag{56}$$

With this ansatz for A_{n_1}, and choosing $\Delta_1 = \Delta_1^+ = \frac{1}{\sigma}(T_1 - 1)$, we get an algebraic relation between R and z, $\sum_{k=1}^{\infty} \frac{s_1(k,1)(e^{iz}-1)^k}{(\sigma)^k} = -\frac{3R^2\sigma}{(\kappa-\bar{\kappa})}$ which is not compatible with the request that z be real.

If, instead, we consider $\Delta_1 = \Delta_1^s = \frac{1}{2\sigma}(T_1 - T_1^{-1})$, the algebraic relation becomes

$$(\kappa - \bar{\kappa}) \sum_{k=1}^{\infty} \frac{s_2(k,1) i^k (\sin z)^k}{(\sigma)^k} + 3R^2\sigma = 0. \tag{57}$$

Eq. (57), as the constant $\kappa - \bar{\kappa}$ is purely imaginary, splits into two equations

$$i(\kappa - \bar{\kappa}) \sum_{k=0}^{\infty} \frac{s_2(2k+1,1)(-1)^k (\sin z)^{2k+1}}{(\sigma)^{2k+1}} + 3R^2\sigma = 0, \tag{58}$$

$$\sum_{k=1}^{\infty} \frac{s_2(2k,1)(-1)^k (\sin z)^{2k}}{(\sigma_1)^{2k}} = 0. \tag{59}$$

Eq. (59) is a polynomial equation for $\sin(z)$ of infinite order, whose solution gives the admissible values of z. In turn, eq. (58) provides the corresponding values of R.

A simple way to solve eq. (57) is to assume a slow–varying condition of order $N = 1$. Then, eq. (59) vanishes and the infinite series contained in eq. (58) reduces to a single term. The equation is solved if

$$\sin z = \frac{3i\sigma^2 R^2}{(\kappa - \bar{\kappa})}. \tag{60}$$

where we have taken into account that $s_2(1,1) = 1$. The relation (60) can always be satisfied by a particular choice of R. In this case, z stays arbitrary.

Let us discuss briefly the slow–varying condition we have adopted. By using the ansatz for A_{n_1} (56) and eq. (60), we deduce that

$$\Delta_{n_1} A_{n_1} = A_{n_1} \frac{-3\sigma R^2}{(\kappa - \bar{\kappa})}. \tag{61}$$

Eq. (61) shows that the derivatives of order k of A_{n_1} are $O(\sigma^k)$. This result is coherent with the continuum limit of the model.

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