INTEGRABLE MAPS VIA ASSOCIATIVE ALGEBRAS AND NUMBER SEQUENCES

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Abstract. A new class of integrable maps defined on a regular lattice of points is introduced. These systems are obtained by means of a discretization procedure that preserves many analytic and algebraic properties of a given continuous model, in particular symmetries and integrability [27]. As a byproduct of the theory, some number theoretical issues are also discussed.

Contents

1. Introduction

The study of physical models on a discrete background is a very active research area. Indeed, they are ubiquitous in theoretical physics and mathematics. The study of quantum physics on a lattice is motivated, for instance, by the need for regularizing divergencies in field theory [1]–[4]. In addition, the existence of an elementary length, related to the Planck scale $l_P = \sqrt{\hbar k} = 10^{-33}$ cm is postulated in several scenarios of quantum gravity [5], [6], as for instance, in the explanation of the finiteness of the Bekenstein–Hawking entropy of black holes [7]. This would imply the discreteness of space–time geometry. Discrete versions of nonrelativistic quantum mechanics have also been proposed (see, for instance, [8]–[11] and references therein).

Integrable discrete systems have been widely investigated as well. For several respects, they seem to be more fundamental objects than the continuous ones. For this reason, many efforts have been devoted to the construction of discrete systems possessing a rich algebro–geometric structure.

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In particular, a challenging issue is to *discretize* the continuous integrable systems in such a way that symmetry and integrability properties are preserved (see the recent interesting monograph [12] for the treatment of the Hamiltonian point of view). Classical examples of discrete integrable systems are the Toda systems and the Ablowitz–Ladik hierarchies [13]–[14]. A rich and profound literature exists on discrete differential geometry and related algebraic aspects [15]–[23]. The problem of how discretize in a physically consistent way has been considered in the context of field theories and Hamiltonian gravity, for instance in [24]–[26].

In the paper [27], an alternative approach, offering a possible solution to this problem has been proposed. This approach preserves both Lie symmetries and Lax pairs associated to a given nonlinear PDE. Also, new hierarchies of Gelfand–Dikii type have been presented.

In the spirit of [27], in the present paper we focus on the discretization of *dynamical systems.*

In our algebraic framework, a class of associative functions algebras $(\mathcal{F}_{\mathcal{L}}, +, *_{\mathcal{Q}})$ is introduced. They are spaces of formal power series over a lattice $\mathcal{L}$, endowed with a difference operator $\mathcal{Q}$ and an associative and commutative product "*". The main feature of this product is that the operators $\mathcal{Q}$ act on *–products of functions as *derivations.* In other words, the *Leibnitz rule* is preserved under their action. This simplifies dramatically the calculus. We remember that a similar idea has been proposed for the first time in the interesting papers [28], [29]. An analog product also has appeared, in a different context, in the theory of linear operators acting on spaces of polynomials [30].

The advantage of our procedure is that the algebraic structure underlying the dynamical models we are dealing with becomes transparent. In this perspective, a differential equation is converted into an abstract operator equation, that can be represented in infinitely many ways in terms of a suitable delta operator $\mathcal{Q}$. Once we discretize on the appropriate function space, where $\mathcal{Q}$ act as a derivation, the symmetry and integrability properties of the continuous model are naturally inherited by the discrete one.

Other approaches existing in the literature deal with the discretization of differential equations on suitable nonlinear lattices, adapted to the symmetries of the problem (see, for instance, the review [23]).

Instead, in this work, we introduce a new family of integrable maps, on a regular lattice of equally spaced points. This kind of lattice is usually more convenient for concrete applications, especially for the implementation of numerical algorithms.

In this context, the notion of integrability means that a large class of exact solutions can be analytically constructed, starting from those of the continuous models we are discretizing. In particular, analytic solutions of the continuous systems are converted into exact solutions of the corresponding systems on the lattice.

Precisely, we focus on discrete versions of the following models:

$$\frac{d}{dt}z = z^N, \qquad N \in \mathbb{N} \qquad c \in \mathbb{R}. \tag{1}$$

Their discrete analogs by construction admit an interesting class of solutions. Indeed, the general solutions of (1) for every N are converted into *exact solutions* of the discrete corresponding models.

Observe that the "*" product converts our theory into a *nonlocal* one. It means that the value of a dynamical observable on the lattice depends on several points of the lattice. This is actually a common feature of discrete integrable models. When the number of points is infinite, the procedure still holds, but on a formal sense. This means that the value of an observable in a point would depend on infinitely many contributions, i.e. on a formal power series. A regularization procedure should be adopted afterwards, in order to give a physical content to this case. This reminds very much what happens in nonlocal theories when dealing with operator product expansions: all infinite contributions of every operator should be taken into account.

However, in several cases, this number is *finite*, and the discretization is *effective*. It means that the solutions obtained possess the general form

$$z(n) = \sum_{k=0}^{n} a_k f(k) \tag{2}$$

where $a_k \in \mathbb{R}$ or $\mathbb{C}$ and $f(k)$ is a function of the point $k \in \mathbb{N}$ on the lattice only. In this work, we well consider only the case of effective discretizations, i.e. our physical observables are expressed in terms of finite series.

An interesting byproduct of our approach is the following. Suppose we have a recurrence relation involving several points on a lattice. This recurrence relation can be considered as associated to a discrete dynamical system in an auxiliary space. Once we solve the dynamical system, we have obtained a solution (usually a particular one) of the original recurrence relation.

In Section II, a simple introduction to the algebraic techniques relevant in the discretization procedure is proposed. In Section III, a concrete example of discrete dynamical system obtained via our approach is discussed. It is further generalized in Section IV, where some number theoretical aspects are also presented. Some conclusions are drawn in Section V.

2. Integrability preserving discretizations: a general framework

2.1. **Basic definitions.** In order to make the paper self–consistent, in this Section we review some basic notions concerning the algebraic theory of polynomial sequences and of finite difference operators, much in the spirit of the monographs [31]–[33]. We will keep the discussion to an elementary level.

Let us denote by $p_n(x)$, $n = 0, 1, 2, \ldots$, a sequence of polynomials in x of order n. Let $\mathcal{F}$ denote the algebra of formal power series in one variable x, endowed with the operations of sum and multiplication of series. An element of $\mathcal{F}$ is expressed by the formal power series

$$f(x) = \sum_{k=0}^{\infty} b_k x^k. \tag{3}$$

Let T be the shift operator on the lattice, whose action on a function is given by $Tf(x) = f(x+\sigma)$. The operator T can also be represented in terms of a differential operators as $T = e^{\sigma D}$, where D denotes the standard derivative.

Definition 1. *An operator S is said to be shift–invariant if it commutes with T. A shift–invariant operator Q is called a delta operator if $Qx = const \neq 0$.*

Definition 2. *A polynomial sequence $p_n(x)$ is called a sequence of basic polynomials for a delta operator Q if it satisfies the following conditions:*

$$\begin{aligned} &1) \quad p_0(x) = 1; \\ &2) \quad p_n(0) = 0 \text{ for all } n > 0; \\ &3) \quad Qp_n(x) = np_{n-1}(x). \end{aligned}$$

It is possible to prove that every delta operator Q has a unique sequence of associated basic polynomials.

Apart the standard derivative operator ∂_x, the most common examples of Δ operators are

$$\Delta^+ = \frac{T-1}{\sigma}, \ \Delta^- = \frac{1-T^{-1}}{\sigma}; \quad \Delta^s = \frac{T-T^{-1}}{2\sigma}, \tag{4}$$

respectively the first two of order $p = 1$ and the third of order $p = 2$. Other examples will be considered in the final Section.

2.2. **Associative algebras and nonlocal functional products.** Let $\mathcal{Q}$ be a delta operator, and $\{p_n(x)\}_{n\in\mathbb{N}}$ be the associated basic sequence.

Let $\mathcal{L}$ be a lattice of equally spaced points on the line, indexed by an integer variable x.

Denote by $\mathcal{F}_\mathcal{L}$ the vector space of the formal power series on $\mathcal{L}$. Since the basic polynomials $p_n(x)$ for every $\mathcal{Q}$ provide a basis of $\mathcal{F}_\mathcal{L}$, then f can be expanded into a series of the form

$$f = \sum_{n=0}^{\infty} a_n p_n(x). \tag{5}$$

We can endow the space $\mathcal{F}_\mathcal{L}$ with the structure of an algebra. Also, denote by $\mathcal{P}$ the space of polynomials in one variable $x \in \mathcal{L}$.

In [27], the following result has been proved.

Theorem 3. *For any delta operator $\mathcal{Q}$, whose associated sequence is $\{p_n(x)\}_{n\in\mathbb{N}}$, the product*

$$* : \mathcal{P} \times \mathcal{P} \longrightarrow \mathbb{R},$$

defined as

$$p_n(x) * p_m(x) := p_{n+m}(x)$$

and extended by linearity on the lattice, is associative, commutative and satisfies the Leibnitz rule:

$$\mathcal{Q}f(x) * g(x) = \mathcal{Q}f(x) * g(x) + f(x) * \mathcal{Q}g(x) \tag{6}$$

where $f(x), g(x)$ are functions on the lattice.

As an immediate consequence of the previous theorem, we have the following result.

Proposition 4. *For any choice of $\mathcal{Q}$, the space $(\mathcal{F}_\mathcal{L}, +, *_\mathcal{Q})$ is an associative algebra.*

The main feature of the $*$ product, as defined in Theorem 3, is the fact that it is nonlocal. Indeed, the product $(f * g)(x)$ does depend on the values of f and g at distinct points of the lattice, with the exception of the standard pointwise product of functions, that corresponds to the choice $\mathcal{Q} = \partial$. The simplest version of this product, i.e. that associated with the forward difference operator Δ, has been proposed in [28].

Observe that the operators $\mathcal{Q}$ act as *derivations* on the associated function algebra $(\mathcal{F}_{\mathcal{L}}, +, *_{\mathcal{Q}})$. However, if the lattice consists of a finite number of points, in general they are not defined on the whole $\mathcal{F}_{\mathcal{L}}$, since there could be several points where the action is meaningless. For instance, the shift operator T is not defined on the extreme point N of the interval $[0, N]$.

In the sequel, we will suppose that the lattice is either infinite, or that could contain sufficiently many points for the delta operators (38)–(39) to be well defined at least on a close subinterval $I \subset N$. In general, the algebras $(\mathcal{F}_{\mathcal{L}}, +, *_{\mathcal{Q}})$ are infinite–dimensional. Nevertheless, the expansion (5) can also truncate in some cases, that are of special interest for the applications.

3. A new discrete dynamical system

Let $z = z(t)$ a continuous function of time t. Consider the simple dynamical model

$$\frac{d}{dt} z = cz^2, \qquad c \in \mathbb{R}. \tag{7}$$

Without loss of generality, from now on we will put $c = 1$. We wish to discretize this models in such a way that their solutions are inherited and easily deduced from the solution of the original continuous models. Following the procedure presented in [27], we get the discrete operator version

$$\Delta z = z * z. \tag{8}$$

Here $z \in \mathfrak{F}(+, \cdot, *_{\Delta})$. Let us focus on the simplest case when $\Delta = \Delta^+$. We consider an equally spaced regular lattice. Now, $t = n\sigma$. For sake of simplicity, we put $\sigma = 1$. Therefore, eq. (8) becomes on the lattice

$$z_{n+1} - z_n = (z * z)_n. \tag{9}$$

By expanding $z(t)$ in terms of the basis $p_n(t)$, we get

$$z(t) = \sum_{n=0}^{\infty} \widehat{z}_n p_n(t) \tag{10}$$

where $p_n(t)$ are the lower factorial polynomials. It might be considered as an analog of the Fourier transform. It follows that

$$p_n(k) = \begin{cases} 0 & if \quad k < n \\ \frac{k!}{(k-n)!} & if \quad k \geq n. \end{cases} \tag{11}$$

It is straightforward to prove that

$$z_n = \sum_{l=0}^{n} \frac{n!}{(n-l)!} \widehat{z}_l \tag{12}$$

and, for its inverse transform,

$$\widehat{z}_n = \sum_{l=0}^{n} (-1)^{n-l} \frac{1}{l!(n-l)!} z_l. \tag{13}$$

The r.h.s. of eq. (9) can be computed explicitly as follows.

$$(z * z)_n = \sum_{l_1,l_2=0}^{\infty} \widehat{z}_{l_1}\widehat{z}_{l_2} P_{l_1+l_2}(n) =$$

$$\sum_{l_1=0}^{n}\sum_{l_2=0}^{n}\sum_{k_1=0}^{l_1}\sum_{k_2=0}^{l_2}(-1)^{l_1-k_1+l_2-k_2}[\frac{z_{k_1}z_{k_2}}{k_1!(l_1-k_1)!k_2!(l_2-k_2)!}$$

$$\cdot\frac{n!}{(n-l_1-l_2)!}] = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\frac{(-1)^{k_1+k_2}}{k_1!k_2!}z_{k_1}z_{k_2}\mathrm{K}_{n,k_1,k_2},$$

where we have introduced the kernel

$$\mathrm{K}_{n,k_1,k_2} = \sum_{l_1=0}^{n}\sum_{l_2=0}^{n}(-1)^{l_1+l_2}\frac{1}{(l_1-k_1)!}\cdot\frac{1}{(l_2-k_2)!}\frac{n!}{(n-l_1-l_2)!}. \tag{14}$$

This expression, after some algebraic manipulations, reduces to

$$\mathrm{K}_{n,k_1,k_2} = \sum_{l=k_1+k_2}^{n}\frac{(-1)^l n!}{(n-l)!}\frac{2^{l-k_1-k_2}}{(l-k_1-k_2)!}. \tag{15}$$

Putting $s = l - k_1 - k_2$, and summing over s, we arrive at the final expression for the kernel (14):

$$\mathrm{K}_{n,k_1,k_2} = \frac{(-1)^n n!}{(n-k_1-k_2)!}. \tag{16}$$

The discrete dynamical system corresponding to eq. (9) is

$$z_{n+1} - z_n = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\frac{(-1)^{k_1+k_2+n}}{k_1!k_2!}z_{k_1}z_{k_2}\frac{n!}{(n-k_1-k_2)!}. \tag{17}$$

This system possesses an interesting *alter ego* in the auxiliary space $\mathcal{A}$ of the variables $\widehat{z}$. Observe that

$$\Delta^+ z_n = \sum_{k=0}^{\infty}\widehat{z}_l l p_{l-1}(n) = \sum_{l=0}^{n}\frac{n!}{(n-l)!}(l+1)\widehat{z}_{l+1} \tag{18}$$

and

$$(z * z)_n = \sum_{l_1,l_2=0}^{\infty}\widehat{z}_{l_1}\widehat{z}_{l_2}P_{l_1+l_2} = \sum_{l=0}^{n}\frac{n!}{(n-l)!}\sum_{l_1=0}^{l}\widehat{z}_{l_1}\widehat{z}_{l-l_1}. \tag{19}$$

We conclude that the following recurrence holds.

$$(l+1)\widehat{z}_{l+1} = \sum_{l'=0}^{l}\widehat{z}_{l'}\widehat{z}_{l-l'} \tag{20}$$

This recurrence can be solved by using the ansatz

$$\widehat{z}_l = z_0^{l+1} \tag{21}$$

We deduce the solution

$$z_n = \sum_{l=0}^{n}\frac{n!}{(n-l)!}z_0^{l+1}, \tag{22}$$

which can be also expressed in the compact form

$$z_n = z_0 \cdot {}_2 F_0(1, n; z_0). \tag{23}$$

This proves that the discretization procedure we adopted is *effective*. Also, it preserves integrability. It means that a large class of solutions is inherited by the discrete counterpart of the continuous model (7). Precisely, if we write the general solution of (7) in the functional space $\mathfrak{F}(+,\cdot,*_\Delta)$, we get the corresponding solution of the discrete model (17).

To prove this, observe that the general solution of eq. (7) is provided by

$$z(t) = \frac{z_0}{1 - z_0 t}. \tag{24}$$

We get

$$z(t) = z_0 + z_0^2 t + z_0^3 t^2 + \dots \tag{25}$$

and, by virtue of the correspondence $t^n \longrightarrow p_n(t)$, taking into account (11), we obtain immediately the solution (22).

4. A more general family of dynamical systems

Consider the more general dynamical system

$$\frac{d}{dt} z = c z^N, \qquad N \in \mathbb{N} \qquad c \in \mathbb{R}. \tag{26}$$

In order to discretize it, let us introduce the notation

$$z^{[N]} = \underbrace{z * \dots * z} \qquad N \quad times. \tag{27}$$

The third system of the hierarchy is the dynamical system

$$\Delta z = c z^{[3]}. \tag{28}$$

Its realization in the space $\mathfrak{F}(+,\cdot,*_{\Delta+})$ can be performed in a similar way. Therefore, eq. (28) becomes on the lattice

$$\Delta z = z_{n+1} - z_n =$$
$$= c \sum_{k_1,k_2,k_3=0}^{n} \frac{(-1)^{k_1+k_2+k_3+n}}{k_1! k_2! k_3!} z_{k_1} z_{k_2} z_{k_3} \frac{n!}{(n-k_1-k_2-k_3)!}.$$

This system, as before, can be written in terms of the transformed variables (13). It has the form of a recurrence relation

$$(l+1)\widehat{z}_{l+1} = c \sum_{\substack{l_1,l_2=0 \\ l_1+l_2 \le l}} \widehat{z}_{l_1} \widehat{z}_{l_2} \widehat{z}_{l-l_1-l_2}. \tag{29}$$

The solution of system (1), for $N = 3$, is provided (up to a sign) by

$$z(t) = \frac{1}{\sqrt{2}\sqrt{-ct + c_0}}, \tag{30}$$

where c_0 is an arbitrary constant.

According to the fact that our discretization preserves integrability, we get the following series expression for the solution of the system

$$z_n = \sum_{k=0}^{n} \frac{a_k}{\sqrt{2}} \frac{n!}{(n-k)!} \frac{c^k}{c_0^{(2k+1)/2}}, \qquad n = 0, 1, 2, \ldots \tag{31}$$

where the first terms of the sequence $\{a_k\}_{k\in\mathbb{N}}$ are

$$a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{3}{8}, a_3 = \frac{5}{16},$$

$$a_4 = \frac{35}{128}, \quad a_5 = \frac{63}{256}, \quad etc. \tag{32}$$

The generating function of this sequence is given by

$$a_n = \frac{1}{n!}\left[\frac{d^n}{dx^n}\frac{1}{\sqrt{1-x}}\right]|_{x=0} \, . \tag{33}$$

The procedure can be easily generalized to any $N \in \mathbb{N}$. To each value of N, it naturally corresponds a different sequence of rational numbers, as in (33).

An interesting aspect of this theory is a sort of inverse approach. Consider a recurrence relation of the form

$$(l+1)\widehat{z}_{l+1} = c \sum_{\substack{l_1, l_2, \ldots, l_N = 0 \\ l_1 + \ldots + l_N \leq l}} \widehat{z}_{l_1}\widehat{z}_{l_2}\cdots\widehat{z}_{l_N}\widehat{z}_{l-l_1-l_2-\cdots l-N}. \tag{34}$$

We wish to find an exact solution of it. A strategy is the following. For every N, we interpret the recurrence (34) as the difference equation defining an abstract dynamical system, in the auxiliary space $\mathcal{A}$. Then, a particular solution of the recurrence can be obtained by associating to it the discrete map

$$\begin{aligned} z_{n+1} - z_n = c \sum_{k_1,\ldots,k_N=0}^{n} & \frac{(-1)^{k_1+\cdots+k_N+n}}{k_1!\cdots!k_N!} z_{k_1}\cdots \\ & \cdot z_{k_N}\frac{n!}{(n-k_1-k_2-\cdots k_N)!}. \end{aligned} \tag{35}$$

obtained by means of the inverse transform (13). Then, we come back to the continuous system (26), of whom we consider the solution

$$z(t) = \left[(1-N)(t+c_0)\right]^{1/(1-N)} . \tag{36}$$

Once we expand this solution and discretize it on the lattice $\mathcal{L}$, exactly in the same way as before, we get a solution of the recurrence (34), which depends on the choice of the initial condition of (26), in terms of a sequence of rational numbers.

Precisely, we get

$$z_n = (1-N)^{\frac{1}{1-N}} c_0^{\frac{1}{1-N}} \cdot \sum_{k=0}^{n}\left[\frac{c}{c_0}\right]^k \frac{1}{(N-1)^k}\prod_{j=0}^{k-2}\left[(j+1)N - j\right]\binom{n}{k}, \tag{37}$$

and from (13) we obtain a solution of the recurrence (34).

It would be interesting to generalize this procedure to the case of recurrences possessing other functional forms, especially to the case of infinite order recurrences. They can be constructed by using alternative discrete derivatives.

First, notice that a general class of difference operators can be defined by [9]

$$\Delta_p = \frac{1}{\sigma}\sum_{k=l}^{m}\alpha_k T^k, \quad l,\, m \in \mathbb{Z}, \quad l < m, \quad m - l = p, \tag{38}$$

where σ is the constant lattice spacing and α_k are constants such that

$$\sum_{k=l}^{m}\alpha_k = 0, \quad \sum_{k=l}^{m} k\alpha_k = c. \tag{39}$$

and $\alpha_m \neq 0$, $\alpha_l \neq 0$. We choose $c = 1$, to reproduce the derivative D in the continuous limit, when the lattice spacing goes to zero.

A difference operator of the form (38), which satisfies equations (39), is said to be a delta operator of order p, if it approximates the continuous derivative up to terms of order σ^p.

As eq. (38) involves $m - l + 1$ constants a_k, subject to just the two conditions (39), we can fix all constants a_k by choosing $m-l-1$ further conditions. A possible choice is, for instance, to set $\sum_{k=l}^{m} k^\ell a_k = 0$, $\ell = 2, 3, ..., m - l$.

Now, if we use discrete derivatives of the form (38)–(39), we get discretizations involving a countable number of points. However, the underlying continuous model still is represented by eq. (26), so in principle its solutions may be used to produce exact solutions of infinite order difference equations.

5. Future perspectives

The example proposed reveal the potential of the method. First, observe that we have worked with the example of the operator Δ^+. The case of a discretization involving Δ^- is very similar, and left to the reader. The basic polynomials associated are defined by

$$p_n(x) = x(x + 1)(x + 2)...(x + (n - 1)). \tag{40}$$

Higher order operators $\mathcal{Q}$, as in (38), starting from the case of Δ^s, a priori would lead to non effective discretizations, if we stay on a regular lattice of point, isomorphous to $\mathbb{N}$ or to a subset $S \in \mathbb{N}$. In this case, the choice of a suitable lattice $\mathcal{L}$ is crucial. The general prescription in order to make effective the procedure is that the lattice we work in should correspond to the set of zeroes of the basic polynomials $\{p_n(x)\}_{n\in\mathbb{N}}$ associated to $\mathcal{Q}$. Therefore, for higher order delta operators, nonlinear lattices could be useful. A procedure to compute the basic polynomials has been proposed in [36], [9]. The problem of the determination of their zeroes in a closed form is essentially open.

The approach developed here can be easily adapted to the study of other physically relevant models, especially nonlinear lattice field theories, for instance the Liouville theory. In general, the higher–dimensional theory can be developed on the same footing. Work is in progress on these aspects.

An interesting open problem is to extend the proposed technique to the discretization of *isochronous dynamical systems*, in the spirit of the general framework proposed in [37].

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References

[1] K. Wilson, Confinement of quarks, Phys. Rev. D **10**, 2445 (1974).
[2] B. Grammaticos Y. Kosmann–Schwarzbach and T. Tamizhmani (Eds.), Discrete integrable systems, Lect. Notes in Physics **644**, 2004.
[3] I. Montvay and G. Müunster, *Quantum Fields on a lattice*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1994.
[4] T. D. Lee, Can time be a discrete dynamical variable? Phys. Lett. **122B**, 217 (1983).
[5] A. Ashtekar, in *General Relativity and Gravitation*, Durban, 2001, edited by N. T. Bishop and S. D. Maharaj (World Scientific, New York, 2002).
[6] S. Kauffman and L. Smolin, Combinatorial Dynamics and Time in Quantum Gravity, Lect. Notes Phys. **541**, 101 (2000).
[7] T. Thiemann, Lectures on Loop Quantum Gravity, Lect. Notes in Phys. **631**, 41 (2003).
[8] R. Friedberg and T. D. Lee, Discrete quantum mechanics, Nucl. Phys. B **225**, 1 (1983).
[9] D. Levi, P. Tempesta and P. Winternitz, Umbral Calculus, Difference Equations and the Discrete Schrödinger Equation, J. Math. Phys. **45**, 4077 (2004).
[10] D. Levi, P. Tempesta and P. Winternitz, Lorentz and Galilei invariance on lattices, Phys. Rev. D **69**, 105011 (2004).
[11] C. Zachos, Umbral deformations on discrete space–time, Intern. Journal Mod. Phys. A **23**, 2005 (2008).
[12] Yu. B. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach*, Progress in Mathematics, Vol. 219. Basel: Birkhuser, 2003.
[13] M. J. Ablowitz and J. F. Ladik, Nonlinear differential–difference equations, J. Math. Phys. **16**, 598 (1975).
[14] M. J. Ablowitz and J. F. Ladik, Nonlinear differential–difference equations and Fourier analysis, J. Math. Phys. **17**, 1011 (1976).
[15] A. I. Bobenko and Y. Suris, *Discrete differential geometry. Integrable structure*, Graduate Studies in Mathematics, **98**. American Mathematical Society, Providence, RI, xxiv+404 pp. (2008).
[16] I. Dynnikov and S. P. Novikov, Geometry of the triangle equation on two-manifolds, Mosc. Math. J. **3**, no. 2, 419 (2003).
[17] B. Konopelchenko, Discrete integrable systems and deformations of associative algebras, J. Phys. A **42**, 454003 (2009).
[18] B. A. Kuperschmidt, Discrete Lax equations and differential–difference calculus, Astrisque No. **123**, 212 pp. (1985).
[19] J. Moser and A. P. Veselov, Discrete versions of some classical integrable systems and factorization of matrix polynomials, Commun. Math. Phys. **139**, 217 (1991).
[20] D. Levi, J. Negro, M. A. del Olmo, Discrete derivatives and symmetries of difference equations. J. Phys. A, 2023 (2001).
[21] R. Quispel, F. Nijhoff, H. Capel and J. Van der Linden, Linear integral equations and non–linear differential-difference equations, Physica A, **125**, 344 (1984).
[22] O. Ragnisco, A simple method to generate integrable symplectic maps, pp. 227-231 in: F.J. Lambert, I. Antoniou (Eds.), Solitons and Chaos, Springer, 1991,
[23] D. Levi and P. Winternitz, Continuous symmetries of difference equations, J. Phys. A **39** R1–R63 (2006).
[24] C. Rovelli and L. Smolin, Nucl. Phys. B **442**, 593 (1995).
[25] G. 't Hooft, Class. Quantum Grav. **13**, 1023 (1996).
[26] R. Gambini and J. Pullin, Canonical Quantization of General Relativity in Discrete Space–Times Phys. Rev. Lett. **90**, 021301 (2003).

[27] P. Tempesta, Discretization of nonlinear evolution equations over associative function algebras, Nonlinear Analysis, **72**, 3237 (2010).
[28] R. S. Ward, Discretization of integrable systems, Phys. Lett. A **165**, 325 (1992).
[29] Y. Bouguenaya and D. B. Fairlie, A finite difference scheme with a Leibnitz rule, J. Phys. A **19**, 1049 (1986).
[30] M. H. Ismail *Classical and quantum orthogonal polynomials in one variable*, Cambridge University Press, 2004.
[31] S. Roman, *The Umbral Calculus*, Academic Press, New York, 1984.
[32] S. Roman and G. C. Rota, The umbral calculus, Adv. Math. **27**, 95 (1978).
[33] G. C. Rota, *Finite Operator Calculus*, Academic Press, New York, 1975.
[34] P. M. Santini, A. Doliwa, M. Nieszporski, Integrable dynamics of Toda type on square and triangular lattices, Phys. Rev. E (3) **77**, no. 5, 056601, (2008).
[35] P. Tempesta, On Appell sequences of polynomials of Bernoulli and Euler type, J. Math. Anal. Appl. **341** 1295 (2008).
[36] A. Dimakis, F. Müller–Hoissen and T. Striker, Umbral Calculus, Discretization and Quantum Mechanics on a Lattice, J. Phys. A **29**, 6861 (1996).
[37] F. Calogero, *Isochronous systems*, Oxford University Press, Oxford, 2008.

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