Global classification of the planar Lotka-Volterra differential systems according to the configurations of invariant straight lines

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Abstract

In this article we consider the Lotka-Volterra planar differential systems, a class which is important in applications. We first give a history of studies of these systems and show that none of the four works (see Section 2) claiming to give a "complete" classification of these systems succeeded in this task. We then consider the class ***LV*** of quadratic differential systems which could be brought by affine transformations and time homotheties to Lotka-Volterra systems. We classify this family according to the configurations of invariant straight lines which these systems possess. This is a more powerful global classification tool than those used in the four mentioned works. We obtain a total of 65 such configurations which are distinguished, roughly speaking, by the multiplicity of their invariant lines and by the multiplicities of the singularities of the systems located on the configurations. We determine an algebraic subvariety of $\mathbb{R}^{12}$ which contains all these systems and we find the bifurcation diagram of the configurations within this algebraic subvariety of ***LV***-systems, in terms of polynomial invariants with respect to the group action of affine transformations and time homotheties. This classification will serve as a basis for the full topological classification of ***LV***-systems.

Résumé

Dans cet article nous considérons les systèmes différentiels Lotka-Volterra dans le plan qui forment une classe importante de systèmes pour les applications. Nous présentons d'abord une étude historique de ces systèmes (Section 2) et nous montrons qu'aucun des quatre travaux prétendant de donner une classification "complète" de ces systèmes n'accompli pas cette tâche. Nous considérons ensuite la classe ***LV*** de tous les systèmes différentiels qui peuvent être ramenés à des systèmes Lotka-Volterra par des transformations affines du plan et des homothéties de l'axe du temps. Nous classifions cette famille d'après les configurations de lignes droites invariantes que ces systèmes possèdent. Cet outil global de classification est plus puissants que tous ceux utilisés dans ces quatre travaux. Nous obtenons un total de 65 configurations, grosso modo distinguées par les multiplicités des droites invariantes et par les multiplicités des singularités des systèmes situées sur ces configurations. Nous déterminons une sous-varieté algebrique de $\mathbb{R}^{12}$ contenant tous ces systèmes et nous trouvons le diagramme de bifurcation de ces configurations à l'intérieur de cette sous-varieté en termes de polynômes invariants par rapport à l'action du groupe des transformations affines et des homotheties sur l'axe du temps. Cette classification servira ensuite pour réaliser la classification topologique des systèmes ***LV***.

1 Brief survey of the history of the classification problem of the Lotka-Volterra systems

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In this article we consider real autonomous polynomial differential systems of the form:

$$\begin{aligned} \dot{x} &= x(a_0 + a_1 x + a_2 y) \equiv p(x, y), \\ \dot{y} &= y(b_0 + b_1 x + b_2 y) \equiv q(x, y), \end{aligned} \tag{1.1}$$

which are quadratic, i.e. $\max(\deg(p), \deg(q)) = 2$ with p, q polynomials in x, y with real coefficients. The systems (1.1) are known as Lotka-Volterra Systems, as they were proposed independently by Alfred J. Lotka in 1925 [50] and Vito Volterra in 1926 [84]. The scientific literature on these systems has been steadily growing due to their many applications and also their theoretical interest and today a large literature devoted to their study exists. We sketch below the beginning history of these equations.

Alfred James Lotka (1880-1949) was an American scientist born in Lemberg, Austria, presently Lviv in Ukraine. He was a mathematician, a chemist and a statistician. He obtained his BSc in 1901 and became Doctor of Science in 1912. Both degrees were awarded by Birmingham University. He spent a year in Leipzig (1901-1902) where he studied chemistry and where he also became interested in the theory of evolution. He worked as a chemist from 1914 to 1919. In 1920 he published in a chemistry journal, his first work on these equations [49], modeling a chemical reaction. In [49] he actually only considered the particular case of (1.1) with two of the six coefficients of equations (1.1) being zero.

From 1922 to 1924, Lotka held a research appointment in Raymond Pearl's Human Biology group at Johns Hopkins University. The result of his work was the publication of the book Elements of Physical Biology (1925), also published as Elements of Mathematical Biology. Regarding the origin of this work Lotka says in the Preface "the first plan of the work was laid about 1902, in the author's student days at Leipzig. The development of the topic is recorded, in outline, in various publications, of which the first appeared in 1907 in the American Journal of Science." However the last stage of the work and "filling in the flesh about the skeleton framework elaborated in the journal literature, was carried out at the John Hopkins University in the Department of Biometrics and Vital Statistics". Lotka explains that "Physical Biology has been employed to denote the broad application of physical principles and methods in the contemplation of biological systems." This work has been largely one of systematization and of development of method. Lotka summarizes in this book his previous work and organizes his ideas of unity and universality of physical laws, making his works accessible to other scientists. In Chapter VII of this work (Fundamental Equations of Kinetics), on page 64 there appears the equation expressing the law of population growth $dX/dt = aX + bX^2$. Lotka cites Verhulst in relation to this equations.

Pierre François Verhulst (1804-1849) was a Belgian mathematician who proposed a model for growth of a population with an equation in just one dependent variable $p(t)$, representing the number of individuals in the population at time t. Verhulst was inspired by the law of Malthus which appeared in an essay [52] of Malthus in 1798 (see [53]). Malthus' law says that without obstacles to its development, a population increases as a geometrical progression. The Belgian astronomer, mathematician, statistician and sociologist Adolphe Quetelet (1796-1874), influential in introducing statistical methods to the social sciences completes Malthus' law with the following principle: "The resistance, or the sum of obstacles to the development of a population is like the square of the velocity with which the population tends to grow." Neither Malthus, nor Quetelet wrote any equation. It was Verhulst who wrote for the first time this law on the evolution of populations, called the logistic equation by Verhulst, also the Verhulst law or the Verhulst-Pearl law. In 1920 the American biologist Raymond Pearl (1879-1940), rediscovered independendently the equation of Verhulst and showed its precision in the evolution of populations of Drosophila flies. In [53] Verhulst is named "the father of mathematical demography" as this first equation of population dynamics $dp/dt = p(r - kp)$ appeared for the first time in his note of 1838 [83].

Although Lotka's first book covered a large amount of topics, from energetics of evolution to the physical nature of consciousness, the author is primarily known today for the Lotka-Volterra equations of population dynamics. These equations appear in Chapter VIII of the book entitled Fundamental equations of Kinetics - Special cases: two and three dependent variables. In this chapter Lotka distinguishes among the various equations of population dynamics. The equations (1.1) appear in this chapter in the particular case $a_1 = b_2 = 0$ and in the context of parasitology, where x is the number of host population and y is the number of parasitic population. The model is analyzed and yields periodic solutions. The purely periodic solutions have been previously discussed by Lotka in 1920 in [51] and Lotka points out an error in that paper. The chapter also includes equations in three variables such as those with two populations, each prey for a third one. The various possibilities which could occur such as competition for common food supply, competition combined with mutual destruction, bodies of one species consumed by other species, etc. are discussed. An analogy of mixed populations of different species of organisms with populations of different species of molecules of chemical substances are considered.

In 1924, Lotka began his employment as a Statistician at the Metropolitan Life Insurance Company in New York City, where he would stay until he retired in 1947. His book Theorie Analytique des Associations Biologiques, (Analytical Theory of Biological Associations) published in French in two parts in 1934 and 1939 in Paris, summarized the essentials of his work on the mathematical theory of evolution and on the mathematics of population analysis. Lotka is primarily known today for the LotkaVolterra equation of population dynamics.

The Italian mathematician Vito Volterra (1860-1940) was born in Ancona, then part of the Papal States, into a poor family. He studied in Pisa where he obtained his doctorate in 1882 under the direction of Enrico Betti with a thesis in Hydrodynamics. He became Professor of Mechanics at Pisa in 1883. In 1892 he became professor of mechanics at the University of Turin and in 1900, professor of mathematical physics at the University of Rome La Sapienza. Volterra became one of the leading mathematicians of his time with work on integral equations, on partial differential equations, on functional analysis, and on mathematical biology. Volterra turned his attention to the application of his mathematical ideas to biology after World War I, developing the work of Verhulst and the most famous outcome of this period is the Lotka-Volterra equations. He studied the Verhulst equation and the logistic curve and worked on predator-prey equations motivated by questions concerning statistics about fishing posed by Umberto D'Ancona who was a marine biologist. In [58] Nowak and May traced the beginning of mathematical ecology to D'Ancona's work in which he analyzed statistics of fish markets. D'Ancona noted that during the first world war the proportion of predatory fish had increased. Why should war favor sharks? This was the question posed by D'Ancona to Vito Volterra who was at that time professor at the University of Rome. Volterra wrote a particular case of the equations (1.1) to model the dynamics of biological systems in which two species interact:

$dx/dt = rx - axy$, $dy/dt = bxy - dy$.

Here x is the number of individuals which are prey and y is the number of predators. Suppose prey grow at the rate rx and are killed by predators at the rate axy. Predators grow at a rate proportional to their own density times the density of prey population, that is bxy. Without prey, predators die at the rate dy. Volterra found the answer to the question asked by D'Ancona, now known as Volterra's principle. The equilibrium of these equations is at $x* = d/b$ and $y* = r/a$ so the ration is $\rho = y * /x* = rb/(ad)$. Thus fishing effectively reduces the growth rate of prey and so ρ declined during the periods of heavy fishing. Volterra studied his model in great depth and his work was extended by three Russian mathematicians Gause, Kostizin and Kolmogoroff. Give Reference! For more information on the legacy of Pierre-François Verhulst and Vito Volterra in population dynamics see [54].

It was only after the publication of his work [84] , as Volterra explains in [85] page 2, that he became aware that these equations were also considered by Ross for question of parasitology relative to malaria and also by Lotka in the volume [50]. Lotka considered the case of two species and this case was developed by Volterra in the first part of his work(in the second section). Volterra obtained general laws for n species under the hypothesis of conservative and dissipative associations. As he pointed out in [85] these laws were new, stated there for the first time. He also mentioned his regret that he did not cite the "interesting works of Dr Lotka" which contain applications other than mathematical on questions of chemistry and biology.

This ends the first chapter of the history of the Lotka-Volterra equations.

Since the introduction of Lotka-Volterra equations, the literature on these systems has been steadily growing and we now have an ample bibliography on them. There are papers which studied their theory, for example deciding under what conditions on the parameters a_i and b_j the systems are integrable ([21], [25], [22], [23], [24], [43], [48], [1], [2], [3], [4], [33]) or drawing phase portraits for the systems ([30], [13], [68], [69]). Local studies on normalizability, linearizability and integrability were done in [27], [28], [29], [39], [47]. Attempts to give complete classifications to the family of Lotka-Volterra systems were made in [68] [37],

[30], [88] but as we shall explain in this paper none of these works arrived at a complete classification and the three papers [68], [37], [30] are not in agreement while in the forth paper the equivalence relation is not even explicitly specified and no phase portrait is given.

Naturally, we want to have a *complete* and efficient classification of this class. For this, far better global geometric tools than those used in [68] [37], [30] are needed and one of the motivations for this work is to obtain a complete and efficient classification of this family by using such tools.

The Lotka-Volterra class of differential systems is important for applications which abound in the literature. We already mentioned applications to population dynamics and demography; to epidemiology, ecology, immunology, parasitology and also to chemistry. But we also have applications to economy and game theory [59], to neural networks [57], laser physics [45], plasma physics [46], hydrodynamics [17], [18], [19] [20], discretization for the Korteweg-de-Vries equation [15], [14].

There is a large amount of work on Lotka-Volterra differential systems of order n with $n \geq 2$ and several monographs contain material on planar or higher order Lotka-Volterra systems for example [42], [41], [55], [58], [56], [32].

2 A critical review of the four papers claiming to give "*complete classifications*" of planar Lotka-Volterra differential systems

In all four works [68], [88], [37] , [30] published respectively in 1988, 1993, 2007 and 2008 the authors claim to give "*complete classifications*" of Lotka-Volterra systems. Why do we need to have four papers, all claiming to do the same thing, namely to give *"complete classifications"* of Lotka-Volterra systems?

A quick check of the references given in the last three papers indicates that the authors do not mention anyone of the previously published papers. It seems that each author believed that he or she was the first to have given a solution to the problem and with the result that the problem was considered again and again. But did they obtain results which are in agreement?

In this section we give a critical review of these papers. Based on this review the above question is settled negatively. We mention further below errors which we encountered in these studies. In fact none of the four papers is error free. A fresh new look at this family of systems with a much more rigorous approach is thus necessary. Furthermore, as this class involves a large amount of phase portraits, this new study must be based on efficient classifying tools so as to obtain an easy, good grasp on these many phase portraits.

We begin by observing that in a classification problem it is extremely important to state precisely and right from the start the equivalence relation under which the classification is to be taken. We mention below several possible equivalence relation which we may consider:

- **a) the topological equivalence preserving the orientation of orbits:** two real planar differential systems are topologically and orbit preserving equivalent if and only if there exist a homeomorphism of the plane carrying orbits to orbits and preserving orientation
- **b) the topological equivalence preserving or reversing the orientation of orbits:** Two real planar differential systems are topologically and orbit preserving or reversing orientation equivalent if and only if there exist a homeomorphism of the plane carrying orbits to orbits and preserving or reversing orientation.
- **c) the equivalence under the action of the group of affine transformations and time rescaling:** Two real planar differential systems are equivalent under the action of the group of affine transformations and time rescaling if and only if there exist a transformation of the plane in this group transforming one system into another.
- **d) the C^∞ equivalence relation preserving or reversing the orientation:** Two real planar differential systems are C^∞ if and only if there exist a C^∞ diffeomorphism of the plane carrying orbits to orbits and preserving or reversing orientation.

Are the four *complete* classifications done using distinct equivalence relations?

We now examine the four papers [68], [88], [37] , [30] from the point of view of the equivalence relation which the authors considered, or might have considered in case this relation is not explicitly stated.

[68]: In this paper the only time when the author mention an equivalence relation is on page 108 where the author referring to [5] says:

"In the usual definition of topological equivalency of phase portraits, as for example in [5], p.106, two phase portraits have the same qualitative structure if there exists a homeomorphism mapping the paths of both phase portraits onto each other."

Reyn's statement above has shortcomings. Indeed, a homeomorphism can only map paths from one system into paths of another system and not "paths of both phase portraits onto each other". It is best to see the definition of Andronov et al. in [5] which Reyn cites and which presumably he uses. This definition is on page 106 and involves the notion of "path" which the authors define on page 7 as the full image in the (x,y)-plane of a solution curve $(x, y) = (\phi(t), \psi(t))$. They distinguish between "paths" and "whole paths" (page 16) which is the image in the phase space of the solution taken in its maximal interval of existence. Hence "path" and "orbit" are not interchangeable and only "whole path" and "orbit are synonymous.

Definition 2.1. *The phase portraits of two planar differential systems* (3.1) *are said to be topologically equivalent if and only if they are equivalent according to Andronov et al.'s definition e) or if there exists an homeomorphism carrying phase curves to phase curves and preserving or reversing the orientation (?).*

e) Andronov et al.'s definition of equivalence: We shall say that the phase portraits of systems (A_1) and (A_2) have the same or identical topological (or qualitative) structure [or are topologically equivalent] in regions G_1 and G_2, respectively, if there exists a mapping T of G_1 onto G_2 satisfying the following conditions:

1) T is a topological mapping;

2) if two points of G_1 lie on the same path of system (A_1), their images undet T lie on the same path of system (A_2);

3) if two points of G_2 lie on the same path of System (A_2), their images under T^{-1} lie on the same path of system (A_1).

By topological mapping the authors mean (page 105) a one-to-one and onto mapping $T : A \to B$ which is bi-continuous.

Remark We observe that in this definition the orientation of orbits plays no role and in fact this definition only refers to the foliations with singularities induced by the systems.

It is clear that if two systems are equivalent under the second one of the first four definitions then the systems are also equivalent under the above definition of Andronov et al. The reciprocal proof is less evident (?). ...

[88]: We now examine the paper [88]. At the beginning of this article on page 103, the author says that she "will give a complete classification" of Lotka-Volterra systems into "eleven qualitatively different cases up to inversion of time and permutations of the projective coordinates". This phrase is already puzzling because the compactification on the projective space $P_2(\mathbb{R})$ of a Lotka-Volterra system of equations is a foliation with singularities and not a vector field. It makes no sense to talk about orientation and "inversion of time" in this compactification. The author does not specify from the start an equivalence relation. She merely considers various possibilities on the coefficients of the equations, expressed in terms of inequalities on the coefficients involving only linear expressions. It is only in the last pages (from 112 to 115) of the paper that the eleven cases (Case 1 to 9a) are indicated. These cases are claimed to be "qualitatively" distinct because the seven singularities of the compactification in the projective space of the systems, behave differently. For example the author distinguishes Case 1 from Case 2 as in the first we have three saddles, two stable nodes, two unstable nodes while in Case 2 we have three saddles, three stable nodes and one unstable node. The difference between these two cases amounts to the orientation of the orbits but as explained above, in passing to the projective plane we only get a foliation with singularities and no! orientation. In addition, the cases from 1 and up to 9a involve only inequalities. So what happens for example when the coefficient a_0 is zero? It is clear that the word *complete* in the description on page 103 is misused. We add that no phase portrait is given in [88]. For all these reasons in our further examination of the papers we shall restrict ourselves to the three papers [68], [37] , [30].

[37]: We now consider the work [37]. This is a Ph.D. thesis in which the Lotka-Volterra equations are discussed in Chapter 4. In Section 4.1 (page 55) of this chapter Georgescu says that she gives there a complete classification of the Lotka-Volterra systems. However she does not mention under what equivalence relation is this classification given. It seems that the notion used in [37] is a) although she did not say so. In addition the author distinguishes between a node and a focus although topologically they cannot be distinguished. We see this by looking at the tables on pages 62 and 63 where nodes and foci are distinguished.

[30]: Finally we consider [30]. In the Abstract of this paper, referring to Lotka-Volterra systems the authors say: "There are 143 different phase portraits in the Poincaré disc up to reversal of sense of all orbits". This is the only place in the whole paper where an equivalence relation is mentioned. They did not specify however what they mean by "different". Presumably what they had in mind was the definition b) above.

When confronting the results of [68], [37] and [30] it is important to recall the kind of equivalence relation they used. Summing up our discussion above we have seen that [68] used Andronov et al.'s definition, [37] presumably used a) adding the distinction between nodes and foci and [30] used b).

Are the three classifications obtained in the three papers complete? Taking care of the respective equivalence relations used, did the three papers produce the same phase portraits?

A critical review of [68] and [37] was done in [74]. Neither one of these two papers succeeds in giving a complete classification of planar Lotka-Volterra systems. In fact the whole class of phase portraits of Lotka-Volterra systems with only one finite singular point is missing in [37]. We also found (see further below) phase portraits of Lotka-Volterra systems which are missing in [68]. The comparison results in [74] for the cases of total finite multiplicity of singularities equal to 3 and 4 are given in a succession of twelve Lemmas containing all the details. These results are summed up in the concluding section as follows:

i) There are seven phase portraits in [37] which are missing in [68]: No. (10), (58) of the figure 4.1 and No. (6), (13)-(15), (26) of figure 4.3.

ii) There are six phase portraits of [68] missing in [37]: 2.DV, 2.AVIII, 2.AIX-X, 2.BI-II, 2.DV-VI, 4.4.IV-V.

iii) There are at least four incorrect phase portraits of [37]: No.(2), (6), (27) and (53) of figure 4.1. For example the phase portrait No.(6) in [37] is incorrect as this portrait is indicated as having one saddle and two nodes at infinity, one of them in the second quadrant. However this point is indicated as having three arrows heading away from the singularity and one arrow towards the singularity.

To detect this lack of agreement between these two papers the authors of [74] used the divisors as defined in [73] and introduced a number of geometric invariants. These also helped to prove which exactly among the phase portraits are topologically distinct. In fact none of the works [68], [37], [30] which give phase portraits of the systems contain proofs that the phase portraits listed are distinct in other words non-equivalent according to the corresponding equivalence relation chosen. Such proofs are only given in [74].

We now turn our attention to the article [30]. As both [68] or [37] have flaws, we no longer wished to compared them with [30]. Instead we looked to scrutinized directly the results in [30]. *Are the 143 phase portraits listed in [30] correct?* We observed that the phase portrait (24-3) in Figure 1 on page 801 is wrong. Indeed, this phase portraits possesses four finite singularities and six singularities at infinity on the Poincaré disc which yield three singular points at infinity for the foliation on $P_2(\mathbb{R})$. In other words we have here a generic situation with a total of seven singularities on the projective plane. However the singularity at infinity corresponding to the horizontal line has three hyperbolic sectors and one parabolic one. So clearly this phase portrait is wrong. Furthermore, the phase portraits (31-1-1) and (31-1-2) in Figure 1 on page 802 are given as "different" up to reversal of sense of all orbits. However, these two phase portraits can be transformed one into the other by using a symmetry and reversing the orientation. (Explain! and perhaps this discussion must be continued with some other observations.) So not all phase portraits given in [30] are distinct according to the equivalence relation b) which seems to be the one used by the authors.

In conclusion we see that all three classifications [68], [37], [30] have disturbing flaws. The most significant flaw of these papers resides in the lack of adequate global classification tools. The class of Lotka-Volterra systems yields many phase portraits. *How could we manage to get a good grasp of this maze of portraits if we only use poor classifying tools?*

Reyn's global classification tool in [68] was the number of finite singularities of the systems and he considered first those with 4 singularities, then with 3, 2 and finally with one singular point. Thus the class of Lotka-Volterra systems with four distinct finite singularities yielded a large number of 65 phase portraits in [68]. In [74] the authors found disagreement between [68] and [37] regarding this case, for example the phase portrait No.(10) in figure 4.1 in [37] does generate any phase portrait in [68] if we remove the orientation of orbits. But this phase portrait actually occurs in the family of Lotka-Volterra systems. Give Proof!

To have an efficient control on the large number of phase portraits one needs a splitting of this class into subclasses organized according to good geometric properties, each one of them with not too many phase portraits and this is not what these authors have done.

Let us now look at the approach in [37]. The classifying tool here was the total multiplicity of finite singularities. This classifying tool is not much better than Reyn's tool in [68]. We point out that Georgescu gets only 30 phase portraits with four finite singularities while Reyn gets 65! Georgescu also uses invariant polynomials in the classification but judging from their treatment in the thesis, it seems that the author is not entirely sure of their role. Tables are given where in the first column are listed algebraic or semi-algebraic subsets in the parameter space expressed in terms of the coefficients of the differential equations. Clearly these tables give a non-intrinsic classification. In [37] algebraic invariants are also used and the author gives Tables in terms of polynomial invariants. The author does not explain why she chose to do both types of classifications. Clearly the one in terms of invariants is intrinsic, i.e. it can be applied to any presentation of the systems. Judging from her introductory phrase at the beginning of Chapter 4 on page 55 it seems she does not realize the significance of invariants.

Finally *what are the classifying tools* used by the authors of [30]? This is the most recent paper on the Lotka-Volterra systems. In contrast with [68] or [37], in [30] we have no global geometric invariants such as the total number of finite singularities used in [68] or the total multiplicity of finite singularities used in [37] to help in the classification. It appears that with the exception of the separation of the family of Lotka-Volterra systems in those possessing a center and those without centers, there are no other organizing tools to help in the classification. The authors end up in Section 6 with 5 pages of the various global possibilities distinguished by algebraic equalities and inequalities in terms of the coefficients. This is a maze of various situations not grouped together according to global geometric concepts as it was done in the previous papers of [68] or [37]. We also point out that the authors did not know that all the phase portraits of the systems with center for the quadratic class as well as their bifurcation diagram were known ([86], [70], [61], [90]). Indeed, these references are all missing and as they struggle in the first six pages to obtain the Lotka-Volterra systems with center, it is quite clear that they ignored their existence.

Here are our motivations for this new study on the Lotka-Volterra class of differential systems:

- This class is important in applications.
- We are also interested in this class for theoretical reasons. The topological classification of general planar quadratic differential systems is not known and this problem is very hard. It is estimated that this class will yield over two thousand phase portraits. The work on this far more restrictive family of Lotka-Volterra systems yields approximately one hundred and fifty phase portraits. However, as indicated above the four papers claiming to give *complete classification* of the Lotka-Volterra family produced conflicting and incomplete results. If we fail to obtain a sound classification of the Lotka-Volterra class then surely the task of classifying larger families of polynomial systems looks hopeless.
- Modulo the action of the group of affine transformations and time homotheties, the planar Lotka-Volterra class is 3-dimensional while the class of quadratic differential systems modulo the same action is 5-dimensional. Due to the global result saying that any system in the Lotka-Volterra class has no limit cycles, it should be possible to draw the bifurcation diagram of this class. This was not attempted in anyone of the papers [68], [37], [30]. We know that the bifurcation set of singularities is algebraic. *Is the whole bifurcation set of all planar Lotka-Volterra systems an algebraic set? Or do we have bifurcation surfaces which are not algebraic?* The above questions were not raised in anyone of the papers [68], [37], [30].
- We have seen that the global tools for the classification in the above papers were either poor or non-existent. While the presence of the two real invariant lines was used in drawing the phase portraits

in [68], [37], [30], this important geometric feature of the Lotka-Volterra systems, was not adequately used in classifying the systems in anyone of these papers. In either [68] or [30] no algebraic invariants are used. We would like to obtain a classification which is easy to grasp. For this to occur we would like to introduce geometric concepts which will help to make this classification conceptually clear. Furthermore we want to be able to settle the question *Could we introduce polynomial invariants so as to be able to characterize each phase portraits in terms of these invariants?*

In conclusion the goal of the work we begin in this article is to obtain a much more comprehensive study at the family of planar Lotka-Volterra systems in which we shall answer the questions indicated above.

Definition 2.2. *We shall denote by **LV** the class of all planar differential systems which could be brought by affine transformations and time rescaling to the form (1.1) above and the systems in this class will be called **LV**-systems.*

In this paper we are interested in the global classification of the class ***LV*** according to their global geometric properties and in particular according to the configuration of invariant straight lines which the systems possess.

3 Algebro-geometric structures on planar polynomial differential systems

Planar polynomial vector fields are objects of a mixed nature: algebraic as they are defined by polynomials and their singularities are intersection points of algebraic curves; analytic, as their solutions are analytic curves; geometric as their phase curves determine a phase portrait with specific geometric properties such as for example presence or absence of limit cycles or graphics. Sometimes we could have planar algebraic curves which are unions of phase curves. As such curves are invariant under the flow, we refer to them as algebraic invariant curves. The presence of such algebraic invariant curves is an important information about the system. For example if we have sufficiently many such curves, the system is integrable, i.e. it has a non-constant analytic first integral. This result was obtained by Darboux in [31]. The history of polynomial differential systems possessing invariant algebraic curves was launched by this work of Darboux [31] and was followed in the 1890's and the beginning of the XXth century by several interesting works of Poincaré, Painlevé and Autonne (see [8, 62–66]). Poincaré and Painlevé stated problems on such systems (see [72] for references and for these problems). Dulac also mentioned briefly the method of integration of Darboux in his article on the problem of the center which he solved for quadratic systems in 1908. There followed a period of time when only occasionally a paper appeared on polynomial differential systems with invariant algebraic curves. This situation changed towards the end of the XXth century when a new period of more intensive investigations on polynomial differential systems with algebraic invariant curves began began, following the work of Jouanolou [44], of Prelle and Singer [67], followed by the papers of Schlomiuk [70] and those on the center [71], [72]. Since then many papers appeared and the area has been growing steadily into a field of research lying on the interface between dynamical systems and algebraic-geometry.

We give below some formal definitions to render precise the terms we use in this paper.

Definition 3.1. *An affine algebraic invariant curve (or an algebraic particular integral) of a polynomial system*

$$(S) \qquad \frac{dx}{dt} = p(x,y), \quad \frac{dy}{dt} = q(x,y), \tag{3.1}$$

or of a vector field

$$\tilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$$

where $p,\ q \in \mathbb{R}[x,y]$*, i.e.* $p,\ q$ *are polynomials in* $x,\ y$ *over* $\mathbb{R}$*, is a curve* $f(x,y)=0$ *where* $f \in \mathbb{C}[x,y]$, $\deg(f) \geq 1$*, such that there exists* $k(x,y) \in \mathbb{C}[x,y]$ *satisfying* $\tilde{D}f = fk$ *in* $\mathbb{C}[x,y]$*. We call* k *the cofactor of* $f(x,y)$ *with respect to the system.*

In [31] Darboux gave a sufficient condition of integrability for complex systems (S): *if a system* (S) *has s invariant algebraic curves* $f_i(x,y)=0$, $i=1,2,\ldots,s$ *such that* $s\geq m(m+1)/2$ *where* $m=\deg(S)$, *then either we have a first integral of the form* $F=\prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$ *for some* $\lambda=(\lambda_1,\ldots,\lambda_s)\in\mathbb{C}^s\setminus\{0\}$, *or the system has an integrating factor of the same form.*

We note that (S) is integrable at least on the open set U which is the complement in $\mathbb{C}^2$ of the union of the curves $f_i(x,y)=0$, $i=1,2,\ldots,s$.

In [44] Jouanolou proved a theorem saying that *if a system* (S) *has s invariant algebraic curves* $f_i(x,y)=0$, $i=1,2,\ldots,s$ *such that* $s\geq(m(m+1)/2)+2$ *where* $m=\deg(S)$, *then we have a first integral which is a rational function over* $\mathbb{C}$ *i.e. is a function of the form* $F=^s f_i(x,y)^{\lambda_i}$ *for some* $\lambda=(\lambda_1,\ldots,\lambda_s)\in\mathbb{Z}^s\setminus\{0\}$ *with* $f_i(x,y)$ *irreducible over* $\mathbb{C}$.

An important point is that even in the case when we are only interested in integrability of real systems (3.1) the method of Darboux of integrating complex systems (3.1) can be used successfully for obtaining real first integrals of the real systems (3.1) (see for example, [72] for explanations).

If a polynomial differential system has invariant algebraic curves then we shall say that ***the system is endowed with an algebro-geometric structure***. This structure involves the specific properties of the individual invariant curves such as their singularities and the specific way they intersect. It also involves the singularities of the system with their respective multiplicities. The way these two types of features interact will determine much of the properties of the systems. We either have an infinite number of invariant algebraic curves in which case due to the theorem of Jouanolou the system has a first integral which is rational over $\mathbb{C}$ (see [44]) or it only has a finite number of algebraic invariant curves in which case it may or it may not be integrable. In both cases the degrees of the invariant algebraic curves are bounded.

If a planar polynomial differential system has invariant algebraic curves then these curves could have ***multiplicities***. Just as a singularity of a system could be a multiple singularity, meaning that in perturbations this singularity splits into two or more singularities, so also algebraic invariant curves could have multiplicities, meaning that in neighboring systems this curve splits into two or more invariant algebraic curves. There are several notions of multiplicity for invariant algebraic curves. In [26] the authors define the notions of: ***infinitesimal multiplicity; integrable multiplicity; algebraic multiplicity; geometric multiplicity; holonomic multiplicity***. The authors show that for irreducible invariant algebraic curves these definitions are all equivalent. They also introduce the notion of ***strong geometric multiplicity***.

Assuming that a polynomial differential system is endowed with an algebro-geometric structure, in case the system has only a finite number of invariant curves, then let $f_i(x,y)=0$, $i=1,2,...,k$ be the irreducible invariant curves of this system and let m_i be their corresponding multiplicities. Then we can form the formal sum $\sum_i m_i f_i$ which is a way of encoding this global information regarding the invariant curves. Such a formal sum will be called ***the multiplicity divisor of invariant algebraic curves of the systems***. This is a useful global tool in classifying systems with algebro-geometric structure and will be useful in this work. We point out that Darboux theory of integrability was extended to include multiplicity of affine invariant curves in [89].

Example 3.1. *A family of systems endowed with algebro-geometric structure is the set of all planar quadratic vector fields which possess a singular point which is a center. In [71], [72] it was shown that in the generic case this class splits into four classes according to the type of invariant algebraic curves which these systems possess: i) the Hamiltonian systems which possess irreducible cubic invariant curves; ii) the systems with an invariant conic and an invariant straight line; iii) the systems with invariant straight lines of total multiplicity four (these are called the Lotka-Volterra systems); iv) the systems possessing an invariant parabola and an invariant nodal cubic tangent at a point at infinity.*

Example 3.2. *Another example is the family of all planar quadratic differential systems with the line at infinity filled up with singularities. All these systems have affine invariant straight lines of total multiplicity three. This family was first studied without using the algebro-geometric structure of the four invariant lines in [36] and without proving the Darboux integrability of these systems. The family was classified in [81] using the algebro-geometric structure which yielded also the Darboux integrability of the systems.*

Example 3.3. *Let* $\mathbf{QSL_i}$ *be the family of all non-degenerate quadratic differential systems possessing invariant straight lines (including the line at infinity) of total multiplicity i with* $i\in\{2,3,4,5,6\}$. *Each class* $\mathbf{QSL_i}$ *is an example of a family of systems with algebro-geometric structure. The families* $\mathbf{QSL_i}$

with $i \in \{4,5,6\}$ were classified using their algebro-geometric structures in [76], [79] for $i = 5,6$ and in classifying the classes $\mathbf{QSL_i}$ for $i = 5,6$ in [78] and [80] for $i = 4$.

Example 3.4. *The Lotka-Volterra systems which we consider in this work form yet another example of a whole class of planar differential systems which are endowed with algebro-geometric structure. Indeed, all planar Lotka-Volterra systems possess at least two distinct affine lines as well as the in the at infinity as invariant lines and these could also have multiplicities (see further below) other than one. This structure will be used along with some global algebro-geometric tools (zero-cycles or divisors on the projective plane) to be introduced below, for the purpose of classification of this class.*

None of the works [68], [37], [30] and [88] exploited the algebro-geometric structure of the Lotka-Volterra systems with the result that, contrary to what the authors claimed, all four classifications are incomplete and furthermore the results are not in agreement.

We use here this structure of the $\boldsymbol{LV}$-systems for the purpose of classifying this class. We propose to do a full study of this class of systems based on this algebro-geometric structure of this class. At the same time we want this study to be intrinsic, independent of the normal form given to the systems. For this purpose we shall use polynomial invariants and geometric invariants and this will help in obtaining a geometrically clear picture of this class. Also the invariants will help in proving that the phase portraits obtained are indeed topologically distinct. Such a proof was not given in anyone of the three papers [68], [37], [30] which gave phase portraits and in fact sometimes topologically equivalent phase portraits were claimed to be distinct.

In this article we only need to consider invariant straight lines of quadratic differential systems. A quadratic vector field determines a point in the 11-dimensional projective space $P_{11}(\mathbb{R})$. Indeed, if we multiply all coefficients of a quadratic vector field by a non-zero constant, the resulting vector field can be brought to the first one by a rescaling of time and clearly induces the same foliation with singularities in $P_2(\mathbb{R})$. An invariant straight line $\mathcal{L}(x,y) = ux + vy + w = 0$ of a vector field χ determines a point $[u : v : w]$ in $P_2(\mathbb{R})$. If we have a sequence of invariant straight lines $\mathcal{L}(x,y) = u_i x + v_i y + w_i = 0$, we say that this sequence converges towards a line $\mathcal{L}(x,y) = ux + vy + w = 0$, $(u,v) \neq 0$, if and only if the sequence $[u_i : v_i : w_i]$ converges towards $[u_i v_i : w_i]$ in $P_2(\mathbb{R})$.

For $\boldsymbol{LV}$ we shall only need the definition of multiplicity of straight lines given in [76]. This notion is the restriction to the quadratic class of the notion of strong geometric multiplicity used in [26].

Definition 3.2. *We say that an invariant straight line $\mathcal{L}(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$, $(u,v,w) \in \mathbb{C}^3$ for a quadratic vector field $\tilde{D}$ has multiplicity m if there exists a sequence of real quadratic vector fields $\tilde{D}_k$ converging to $\tilde{D}$, such that each $\tilde{D}_k$ has m distinct (complex) invariant straight lines $\mathcal{L}_k^1 = 0, \ldots, \mathcal{L}_k^m = 0$, converging to $\mathcal{L} = 0$ as $k \to \infty$, and this does not occur for $m+1$. We say that the line at infinity $Z = 0$ has multiplicity m for $\tilde{D}$ if and only there exists a sequence of vector fields $\tilde{D}_i$, $i = 1,2,\ldots$ within the quadratic class such that for each i the vector field $\tilde{D}_i$ has $m-1$ distinct invariant affine lines $\mathcal{L}^{i,j}(x,y) = u_{i,j}x + v_{i,j}y + w_{i,j} = 0$ such all these lines that tend to the line at infinity $Z = 0$ and $\tilde{D}_i$ tends to $\tilde{D}$ as i tends to infinity and this is not possible for $m+1$.*

Proposition 3.1. *[6] The maximum number of invariant lines (including the line at infinity and including multiplicities) which a quadratic system in $\mathbf{QS}$ could have is six.*

As all $\boldsymbol{LV}$-systems possess two distinct real invariant lines and also the line at infinity, the $\boldsymbol{LV}$-systems have invariant lines of total multiplicity at least 3.

We consider the invariant straight lines of $\boldsymbol{LV}$-systems marked by the real singular points of the systems lying on the two lines. The key global concept which will be used for classifying the $\boldsymbol{LV}$-systems is the concept of configuration of invariant algebraic curves of a planar polynomial system which is defined as follows:

Definition 3.3. *Consider a planar polynomial system of degree n. We call configuration of invariant algebraic curves of this system, the set of (complex) invariant algebraic curves (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant curves, each one endowed with its own multiplicity.*

This is a more powerful global classifying concept than anyone used in [68], [37], [30].

Remark 3.1. *When a planar polynomial system has a rational first integral, we have an infinite number of invariant algebraic curves and so the configuration is formed by all these curves, each having infinite multiplicity.*

Remark 3.2. *The concept of configuration of invariant algebraic curves of a polynomial vector field is a mixed one. It involves curves taken over $\mathbb{C}$, and the real singularities of the systems located on these algebraic curves.*

We may also consider sub-configurations by limiting the degrees of these curves to less than or equal to k or to curves of degree k. In this case we use the term *configuration of invariant curves of maximum degree k* or *of degree k*. In particular we shall be interested here in the case of straight lines, $k = 1$.

For ***LV***-systems we shall only need to work with invariant straight lines and the concept of configuration of invariant straight lines. Some ***LV***-systems have a rational first integral and hence an infinite number of invariant algebraic curves. In particular they may have have only an infinite number of invariant lines, a situation occurring in degenerate ***LV***-systems.

Suppose a planar polynomial differential systems possesses a finite number of invariant algebraic curves $f_i = 0$, $i = 1, 2, ..., k$ which we may assume to be irreducible. Suppose that they have the corresponding multiplicities m_i. In this case we can associate to the system ***the divisor of the projective plane*** $\sum_i m_i f_i$. These divisors play an important role in the theory of integrability. Indeed, in [89] the authors improve Darboux' theory if integrability by taking into account the multiplicities of the curves. In what way do these divisors relate to the global scheme of singularities of the systems, finite or infinite? This question suggests that to us that apart from the multiplicity divisors defined above we need to construct others encoding the multiplicities of singularities and these concepts taken together will form a framework for the classification of ***LV***-systems. Accumulation of data such as the data included in [76], [79], [78] and [80] and the present work, will help settle the above as well as similar questions.

4 Other Global Geometric Classification Tools

We consider a real polynomial differential system

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y), \tag{4.1}$$

with f and g relatively prime polynomials in $\mathbb{R}[x, y]$ with $max(deg(f), deg(g)) = m$.

Each such system generates a complex system when the variables range over $\mathbb{C}$.

4.1 Zero-cycles and divisors attached to the singularities of systems

To a planar polynomial system we can attach a number of cycles on the projective plane introduced in [75] and used in several papers among them [73], [77]. These cycles involve the notion of intersection numbers.

The intersection number of two affine algebraic curves [34] $C : f(x, y) = 0$ and $C' : g(x, y) = 0$ over $\mathbb{C}$ at a point in $\mathbb{C}^2$ is the number

$$I_a(f, g) = dim_{\mathbb{C}} O_a/(f, g),$$

where O_a is the local ring of the affine complex plane $A^2(\mathbb{C}) = \mathbb{C}^2$ at a; i.e., O_a is the ring of rational functions $r(x, y)/s(x, y)$ which are defined at a, i.e. $s(a) \neq 0$.

The intersection numbers $I_a(p, q)$ for p, q as in (3.1), at the singular point a in $\mathbb{C}^2$, can be computed easily by using the axioms [34]. For two projective curves in $F(X, Y, Z) = 0$ and $G(X, Y, Z) = 0$, where curves in?
F and G are homogeneous polynomials in the variables X, Y and Z over $\mathbb{C}$, assuming for instance that $W = [X_0 : Y_0 : Z_0]$ with $Z_0 \neq 0$, we consider the chart $Z \neq 0$ and we define $I_W(F, G) = I_w(f, g)$ where $f = F(X/Z, Y/Z, 1)$, $g = G(X/Z, Y/Z, 1)$ and $w = (X_0/Z_0, Y_0/Z_0)$. It is known that $I_W(F, G)$ is independent of the choice of a local chart, and of a projective change of variables, see again [34].

If we start with a planar polynomial system (4.1), then we can consider the projective completions of the curves $f = 0$ and $g = 0$. Let us denotes these by $F(X, Y, Z) = 0$, $G(X, Y, Z) = 0$. We may then consider the multiplicity of intersection at a point W at infinity of $f = 0$ with $g = 0$ as being $I_W(F, G)$

where $W \in \{Z=0\}$. We need a way of encoding globally these multiplicities and for this we recall here briefly the general definitions of zero-cycle and divisor.

Let V be an irreducible algebraic variety over a field K. A *cycle of dimension r or r-cycle on V* is a formal sum $\sum_W = n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of n_W's are non-zero. An $(n-1)$-cycle is called *a divisor* [40]. The *degree of a divisor* is the sum $\sum_W = n_W$.

To a system (4.1) we can associate several multiplicity zero-cycles and divisors:

i) A zero-cycle which encodes the information regarding the intersection numbers of the curves $F=0$ and $G=0$ in $\mathbb{C}P^2$:

$$D_{\mathbb{C}}(F,G) = \sum_{W \in \mathbb{C}P^2} I_W(F,G)W,$$

where $F(X,Y,Z) = Z^m f(X/Z, Y/Z)$, $G(X,Y,Z) = Z^m g(X/Z, Y/Z)$.

ii) A divisor on the line at infinity $Z=0$ which encodes the multiplicities of the intersection points of $F=0$ with $G=0$, which lie on $Z=0$:

$$D_{\mathbb{C}}(F,G;Z) = \sum_{W \in \{Z=0\} \cap \mathbb{C}P^2} I_W(F,G)W.$$

iii) A zero-cycle which encodes the information regarding the intersection numbers of the affine algebraic curves $f=0$ and $g=0$ in $\mathbb{C}^2$:

$$D_{\mathbb{C}}(f,g) = \sum_{W \in \mathbb{C}^2} I_W(f,g)W = D_{\mathbb{C}}(F,G) - D_{\mathbb{C}}(F,G;Z).$$

iv) The divisor which encodes the multiplicities of the complex singular points at infinity which count how many real or complex singular points at infinity will bifurcate at infinity in a complex perturbation of the system:

$$D_{\mathbb{C}}(C,Z) = \sum_{W \in \{Z=0\} \cap \mathbb{C}P^2} I_W(C,Z)W,$$

where $C(X,Y,Z) = YF(X,Y,Z) - XG(X,Y,Z)$ and Z does not divide C.

v) A zero-cycle which encodes the information regarding the intersection numbers of all the real or complex singularities W of the completion in $\mathbb{C}P^2$ of the foliation $g(x,y)dx - f(x,y)dy = 0$ in $\mathbb{C}^2$, associated to a system (4.1):

$$D_{\mathbb{C}} = D_{\mathbb{C}}(f,g) + D_{\mathbb{C}}(F,G;Z) + D_{\mathbb{C}}(C,Z),$$

where Z does not divide C.

The *support* of a cycle C is the set $Supp(C) = \{W | n_W \neq 0\}$. We denote by $Max(C)$ the maximum value of the coefficients n_W in C. For every $m \leq Max(C)$ let $s(m)$ be the number of the coefficients n_W in C which are equal to m. We call *type* of the cycle C the set of ordered couples $(s(m), m)$ where $1 \leq m \leq Max(C)$. This concept was introduced in [73] and it was also used in [77].

Definition 4.1. *A point w of the projective plane $\mathbb{P}^2(\mathbb{C})$ is said to be of multiplicity (r,s) for a system (S) if*

$$(r,s) = (I_W(F,G), I_W(C,Z)).$$

We fuse the divisors $D_{\mathbb{C}}(F,G;Z)$ and $D_{\mathbb{C}}(C,Z)$ on the line at infinity into just one but with values in the ring $\mathbb{Z}^2$:

Definition 4.2.

$$D_S = \sum_W (I_W(F,G), I_W(C,Z))W$$

where W belongs to the line $Z=0$ of the complex projective plane.

The above defined divisor describes the number of singularities which could arise in a perturbation of (S) from singularities at infinity of (S) in both the finite plane and at infinity.

4.2 Divisors attached to the configurations of invariant algebraic curves

We have two kinds of divisors: a) those of the projective plane defined by the multiplicities of the curves and b) those of the configuration, defined by using the multiplicities of the singularities.

The first kind was defined in Section 3 for a planar polynomial vector field with a finite number of invariant algebraic curves. This is the multiplicity divisor of the plane attached to the configuration formed by these curves. The type of this divisor is invariant under the group action.

Apart from this divisor of the plane we can also attach several divisors of the configuration. Assume the planar polynomial vector field has irreducible distinct invariant algebraic curves $f_i(x, y) = 0$, $i = 1, 2, ..., m$. We consider their projective completions $F_i(X, Y, Z) = 0$ as well as the line $Z = 0$ and we form the curve: G: $Z \prod_i F(X, Y, Z) = 0$. This curve will have two kinds of singularities: i) those which are intersection points of two of the curves F_i or intersection of one curve F_i with the line at infinity and ii) those which are singularities of the curve without being of the type i). Each one of these points is also a singular point of the system.

The singularities of the systems are of four kinds: i) those which are singularities of G without being singular points of any of the curves $F_i = 0$; ii) those which are singular points of some of the curves $F_i = 0$ without being singular points of G; iii) those which are singular points of some curves $F_i = 0$ and also singular points of G; iv) those which lie on the configuration but which are not singularities G; v) those which lie of outside of the configuration.

We consider four divisors $\sum_W n(W)W$ attached to the curve G by taking the summation index W to be running into singularities of kinds i), ii) iii), iv), v).

The types of these divisors are invariant under the action of the affine group and time homotheties.

4.3 Zero-cycles, divisors and geometrical properties of *LV*-systems

In this section we define global geometrical tools to be used in the classification of the Lotka-Volterra systems (1.1). For this we apply the notions of the preceding section to this special class of systems.

Notations. We denote by N the number of distinct singularities, finite (s_i) or infinite (r_i), of the Lotka-Volterra systems (1.1).

Proposition 4.1. *For any Lotka-Volterra system* (1.1) *all the singularities are real and there are at least two distinct singular points at infinity.*

Proof. The finite singularities of the Lotka-Volterra systems (1.1) are the intersection points of the lines $x = 0$ and $a_0 + a_1x + a_2y = 0$ with the lines $y = 0$ and $b_0 + b_1x + b_2y = 0$. As these lines are all real, all solutions are real. Clearly the lines $x = 0$ and $y = 0$ are invariant and they determine two distinct singular points at infinity $r_1 = [0 : 1 : 0]$ and $r_2 = [1 : 0 : 0]$ in $P_2(\mathbb{R})$. The singularities at infinity are the zeroes of the polynomial $yx(a_0 - b_0 + (a_1 - b_1)x + (a_2 - b_2)y) = 0$ and hence apart from r_1, r_2 we have a third real singular point $[a_2 - b_2 : b_1 - a_1 : 0]$ in case $(a_2 - b_2, b_1 - a_1) \neq 0$. ∎

As all singularities are real we shall drop the index in $D_{\mathbb{C}}$ and just write D.

Proposition 4.2. *A necessary and sufficient condition for the Lotka-Volterra systems* (1.1) *to have a line of singularities at infinity is* $a_1 = b_1$ *and* $a_2 = b_2$.

Proof. In order to have a line of singularities at infinity it is necessary and sufficient that $xy[(a_1 - b_1)x + (a_2 - b_2)y] \equiv 0$, i.e. $a_1 = b_1$ and $a_2 = b_2$. ∎

The zero-cycle D of multiplicities of singularities was used in [74] to confront results in the papers [68] and [37] and to reveal the lack of agreement of the results in these two papers for the Lotka-Volterra systems with finite total multiplicity of singularities equal to 3 or 4. The following proposition was instrumental in this work and it is an example of how to compute the total multiplicity divisor D with elementary methods.

Proposition 4.3. *Consider a non-degenerate Lotka-Volterra systems* (1.1) *with* $(a_1-b_1)^2+(a_2-b_2)^2 \neq 0$ *and with total multiplicity of finite singularities 3 or 4. Then the zero-cycle D is well defined and according to decreasing values of the number N of all singularities (all real) of the systems D has the following form:*

$$\begin{array}{ll}
N=7 & D_1 = s_1+s_2+s_3+s_4+q_1+q_2+q_3;\\
N=6 & D_2 = s_1+s_2+s_3+s_4+2q_1+q_2;\\
 & D_3 = s_1+s_2+s_3+2q_1+q_2+q_3;\\
 & D_4 = 2s_1+s_2+s_3+q_1+q_2+q_3;\\
N=5 & D_5 = s_1+s_2+s_3+2q_1+2q_2;\\
 & D_6 = 2s_1+s_2+s_3+2q_1+q_2;\\
 & D_7 = 2s_1+s_2+2q_1+q_2+q_3;\\
 & D_8 = 2s_1+2s_2+q_1+q_2+q_3;\\
N=4 & D_9 = 2s_1+s_2+2q_1+2q_2;\\
 & D_{10} = 2s_1+2s_2+2q_1+q_2;\\
 & D_{11} = 4s_1+q_1+q_2+q_3;\\
N=3 & D_{12} = 4s_1+2q_1+q_2.
\end{array}$$

Proof. Due to the Proposition 4.1 for $N=7$ and $N=6$ there is no singularity with higher multiplicity than 2 and hence the zero-cycle D can be constructed unambiguously as above. It remains to prove that for $N=5,4,3$, then D is as indicated above. We need to show that i) there is no finite singular point with multiplicity 3 and ii) there is no infinite singular point with finite multiplicity 3. Apart from $(0,0)$ the singularities are solutions of the following equations:

$$\begin{cases} x=0,\\ b_0+b_1x+b_2y=0; \end{cases} \tag{4.2}$$

$$\begin{cases} a_0+a_1x+a_2y=0,\\ y=0; \end{cases} \tag{4.3}$$

$$\begin{cases} a_0+a_1x+a_2y=0,\\ b_0+b_1x+b_2y=0. \end{cases} \tag{4.4}$$

We first prove that $(0,0)$ has multiplicity $m\leq 2$ or $m=4$. $I_{(0,0)}(p,q)=I_1+I_2+I_3+I_{(0,0)}(x,y)$ where $I_1=I_{(0,0)}(x,b_0+b_1x+b_2y)$, $I_2=I_{(0,0)}(y,a_0+a_1x+a_2y)$ and $I_3=I_{(0,0)}(a_0+a_1x+a_2y,b_0+b_1x+b_2y)$. As $I_{(0,0)}(x,y)=1$, we need to show that $I_1+I_2+I_3\leq 1$ or $I_1+I_2+I=3$. Each one of I_1, I_2 and I_3 could only be zero or one as we intersect lines and by hypothesis the systems we consider are non-degenerate. Suppose $I_3=0$, then necessarily $a_0\neq 0$ or $b_0\neq 0$. Assume $a_0\neq 0$. In this case $I_2=0$ and hence $m\leq 1$. Similarly if we assume $b_0\neq 0$. Suppose now $I_3=1$ then $a_0=0=b_0$ and as the system is non-degenerate we must have $a_1b_2\neq a_2b_1$. In this case $I_1=I_2=1$ and hence $I_1+I_2+I=3$.

It remains to show that if (α,β) is a finite singular point other than $(0,0)$ then $I_{(\alpha,\beta)}(p,q)$ is at most 2. If $\alpha\beta\neq 0$ then clearly $I_{(\alpha,\beta)}(p,q)=1$. If $\alpha=0\neq\beta$, then clearly (α,β) cannot be a solution of (4.3) and hence the multiplicity of (α,β) is $m\leq 2$. Analogously for $\alpha=0\neq 0=\beta$.

It remains to show that for singular points at infinity we cannot have the multiplicity 3. As $q_1=[0:1:0]$ and $q_2=[1:0:0]$) are always distinct infinite singularities, if we have an infinite singular point of multiplicity three, this implies that necessarily a singular point of the finite plane escaped to infinity and hence the total multiplicity of the finite singular points cannot be four. So let us assume that the total multiplicity of finite singularities is equal to three and that q_1 has multiplicity three. As we have three singular points in the finite plane, distinct or some with multiplicity, then the infinite singular point $q_3=[a_2-b_2:b_1-a_1:0]$ must coincide with q_1 which yields $b_2=a_2$ and $b_1\neq a_1$ and q_1 must be a solution at least of one of the homogenized systems of (4.2)-(4.4). It is clear that q_1 cannot be a solution of the homogenized equations of (4.3). So it must be a solution the homogenized equations of either (4.2) or (4.4). If it is a solution of the first equations then necessarily $b_2=a_2=0$. In this case it is also a solution of the second equations and viceversa. In both cases the multiplicity of q_1 is four.

In the same way one can verify that q_2 cannot have the multiplicity 3. If the point $[a_2-b_2:b_1-a_1:0]$ is distinct from q_1 or q_2 then it cannot have multiplicity three as this would mean that we would only have a total finite multiplicity of two.

Now we shall show that in view of the above, D is as indicated in the statement of the Proposition for $N = 5$. It is clear that for the total finite multiplicity three we have only two possibilities for $D(p, q)$, $s_1 + s_2 + s_3$, $2s_1 + s_2$ which yield D_5 and D_7. For the total finite multiplicity four we have the following possibilities for $D(p, q)$: $2s_1 + s_2 + s_3$, $2s_1 + 2s_2$. These yield respectively D_6 and D_8. The cases for $N = 4, 3$ are proved analogously.

According to the above proof we conclude that there are only 12 possibilities for zero-cycle D of the systems (1.1) as listed in the Proposition 4.3. ■

We now attach to every ***LV***-system possessing a configuration C of invariant straight lines a divisor and a zero-cycle on the projective plane.

Definition 4.3. *To every **LV**-system possessing a finite configuration of invariant lines C we attach a multiplicity divisor of the projective plane corresponding to the configuration C of invariant straight lines L:*

$$D_{\mathbb{C}}(C) = \sum_{L \in C} M(L)L.$$

We also attach a zero-cycle of the projective plane counting the multiplicities of the isolated singularities of the system which are located on the configuration C:

$$D^*(C) = \sum_{r \in C} m(r)r.$$

Notation 4.1. *We shall denote by $deg(D_{\mathbb{C}}(C)$ (respectively $deg(D^*(C))$ the degree of the divisor $D_{\mathbb{C}}(C)$ (respectively of $D^*(C)$).*

Notation 4.2. *We shall denote by $Max(D_{\mathbb{C}}(C))$ the maximum multiplicity of an invariant line of an **LV**-system and by $Max(D^*(C))$ the maximum multiplicity of a real singularity finite and infinite of an **LV**-system.*

The type of the configuration is an important invariant which we define as follows:

Definition 4.4. *To every configuration of invariant lines we consider for each multiplicity m of an invariant line, the number $l(m)$ of lines of the configuration which have multiplicity m. We then form the set S of ordered couples $\{(m, l(m)) | m = 1, ...MaxD(C)\}$.*

For ***LV***-system we consider only their configurations of invariant straight lines. As these are non-singular curves, we only have three types of singularities attached to a specific configuration: i) those which are located on the curve (G) and are also singularities of (G); those which belong to (G) but which are not singularities of (G); those not on (G).

Definition 4.5. *We also consider the types T_0, T_1, T_2 of the multiplicity divisors of G of the singularities of types i), ii), iii) above.*

The results obtained can be generalized as follows:

Theorem 4.1. *I. Any non-degenerate **LV**-system, of configuration (C) of invariant lines has all its singularities real and at least six of the singularities of the foliation with singularities in the projective plane, counted with multiplicity, are located on (C) ($deg(D^*(C) \geq 6$). The maximum multiplicity which an isolated singular point of such a system could have is either two or four. The maximum $m_{Sing(L)}$ total multiplicity of a singularity located on an invariant straight line finite or infinite satisfies $3 \leq m_{Sing(L)} \leq 6$.*

*II. All invariant straight lines of any non-degenerate **LV**-system are real and we could have at most six such lines. The maximum multiplicity which an invariant line could have in such a system is two and we could have at most two invariant lines with multiplicity two. In such cases the intersection point of the two lines is of multiplicity four.*

This theorem will result from work done in Section 6.

5 Group actions on polynomial systems

Consider real planar polynomial differential systems

$$\frac{dx}{dt} = p(x,y), \quad \frac{dx}{dt} = q(x,y). \tag{5.1}$$

We call degree of such a system $n = \max(\deg(p(x,y), \deg(q(x,y)))$. We shall denote by $\mathbf{PS}$ the set of all systems (5.1) of degree n. On the set of all polynomial differential systems $\mathbf{PS}$ acts the group $\mathrm{Aff}\,(2,\mathbb{R})$ of affine transformations on the plane. This action

$$\begin{array}{rcl} \mathrm{Aff}\,(2,\mathbb{R}) \times \mathbf{PS} & \longrightarrow & \mathbf{PS} \\ (g, \quad S) & \longrightarrow & \tilde{S} = gS \end{array}$$

is defined as follows:

Consider an affine transformation $g \in \mathrm{Aff}\,(2,\mathbb{R})$, $g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. For this transformation we have:

$$g: \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + B; \qquad g^{-1}: \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} - M^{-1}B.$$

where $M = ||M_{ij}||$ is a 2×2 nonsingular matrix and B is a 2×1 matrix over $\mathbb{R}$. For every $S \in \mathbf{PS}$ we can form its induced transformed system $\tilde{S} = gS$:

$$\frac{d\tilde{x}}{dt} = \tilde{p}(\tilde{x},\tilde{y}), \qquad \frac{d\tilde{y}}{dt} = \tilde{q}(\tilde{x},\tilde{y}), \tag{$\tilde{S}$}$$

where

$$\begin{pmatrix} \tilde{p}(\tilde{x},\tilde{y}) \\ \tilde{q}(\tilde{x},\tilde{y}) \end{pmatrix} = M \begin{pmatrix} (p \circ g^{-1})(\tilde{x},\tilde{y}) \\ (q \circ g^{-1})(\tilde{x},\tilde{y}) \end{pmatrix}.$$

The map

$$\begin{array}{rcl} \mathrm{Aff}\,(2,\mathbb{R}) \times \mathbf{PS} & \longrightarrow & \mathbf{PS} \\ (g, \quad S) & \longrightarrow & \tilde{S} = gS \end{array}$$

verifies the axioms for a left group action. For every subgroup $G \subseteq \mathrm{Aff}\,(2,\mathbb{R})$ we have an induced action of G on $\mathbf{PS}$.

Definition 5.1. *Consider a subset $\mathcal{A}$ of $\mathbf{PS}$ and a subgroup $\mathcal{G}$ of $\mathrm{Aff}\,(2,\mathbb{R})$. We say that the subset $\mathcal{A}$ is invariant with respect to the group $\mathcal{G}$ if for every g in $\mathcal{G}$ and for every system (S) in $\mathcal{A}$ the transformed system (gS) is also in $\mathcal{A}$.*

We can identify the set of systems in $\mathbf{PS}$ with a subset of $\mathbb{R}^m$ via the embedding $\mathbf{PS} \hookrightarrow \mathbb{R}^m$ which associates to each system (S) in $\mathbf{PS}$ the m-tuple $(\boldsymbol{a}_{00}, \ldots, \boldsymbol{b}_{0n})$ of its coefficients, where $m = (n+1)(n+2)$. We denote by $\mathbb{R}^m_{\mathcal{A}}$ the image of set $\mathcal{A}$ of $\mathbf{PS}$ under the embedding $\mathbf{PS} \hookrightarrow \mathbb{R}^m$.

For every $g \in \mathrm{Aff}\,(m,\mathbb{R})$ let $r_g: \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be the map which corresponds to g via this action.

We know (cf. [82]) that r_g is linear and that the map $r: \mathrm{Aff}\,(m,\mathbb{R}) \longrightarrow GL(m,\mathbb{R})$ thus obtained is a group homomorphism. For every subgroup G of $\mathrm{Aff}\,(m,\mathbb{R})$, r induces a representation of G onto a subgroup $\mathcal{G}$ of $GL(m,\mathbb{R})$.

We denote by $\mathbf{QS}$ the set of all quadratic differential systems, i.e. the set of all systems such that $\max(\deg(p), \deg(q)) \leq 2$. The group $\mathrm{Aff}\,(2,\mathbb{R})$ acts on $\mathbf{QS}$ and this action yields an action of this group on $\mathbb{R}^{12}$. For every subgroup G of $\mathrm{Aff}\,(2,\mathbb{R})$, r induces a representation of G onto a subgroup $\mathcal{G}$ of $GL(12,\mathbb{R})$.

5.1 Definitions of invariant polynomials

Definition 5.2. *A polynomial $U(a, x, y) \in \mathbb{R}[a,x,y]$ is called a comitant with respect to $(\mathcal{A}, G)$, where $\mathcal{A}$ is an affine invariant subset of $\mathbf{PS}$ and G is a subgroup of $\mathrm{Aff}\,(2,\mathbb{R})$, if there exists $\chi \in \mathbb{Z}$ such that for every $(g, \boldsymbol{a}) \in G \times \mathbb{R}^m_{\mathcal{A}}$ and the following identity holds in $\mathbb{R}[x,y]$:*

$$U(r_g(\boldsymbol{a}),\ g(x,y)) \equiv\ (\det g)^{-\chi}\, U(\boldsymbol{a}, x, y),$$

where $\det g = \det M$*. If the polynomial* U *does not explicitly depend on* x *and* y *then it is called invariant. The number* $\chi \in \mathbb{Z}$ *is called the weight of the comitant* $U(a, x, y)$*. If* $G = GL(2, \mathbb{R})$ *(or* $G = Aff(2, \mathbb{R})$ *) and* $\mathcal{A} = \mathbf{PS}$ *then the comitant* $U(a, x, y)$ *is called GL-comitant (respectively, affine comitant).*

Definition 5.3. *A subset* $X \subset \mathbb{R}^m$ *will be called G-invariant, if for every* $g \in G$ *we have* $r_g(X) \subseteq X$*.*

Let $T(2, \mathbb{R})$ be the subgroup of $\mathrm{Aff}\,(2, \mathbb{R})$ formed by translations. Consider the linear representation of $T(2, \mathbb{R})$ into its corresponding subgroup $\mathcal{T} \subset GL(m, \mathbb{R})$, i.e. for every $\tau \in T(2, \mathbb{R})$, $\tau: \ x = \tilde{x} + \alpha, y = \tilde{y} + \beta$ we consider as above $r_\tau : \ \mathbb{R}^m \longrightarrow \mathbb{R}^m$.

Definition 5.4. *A comitant* $U(a, x, y)$ *with respect to* $(\mathcal{A}, G)$ *is called a T-comitant if for every* $(\tau, \boldsymbol{a}) \in T(2, \mathbb{R}) \times \mathbb{R}^m_{\mathcal{A}}$ *the identity* $U(r_\tau \cdot \boldsymbol{a}, \ \tilde{x}, \ \tilde{y}) = U(\boldsymbol{a}, \ \tilde{x}, \ \tilde{y})$ *holds in* $\mathbb{R}[\tilde{x}, \tilde{y}]$*.*

Definition 5.5. *The polynomial* $U(a, x, y) \in \mathbb{R}[a, x, y]$ *has well determined sign on* $V \subset \mathbb{R}^m$ *with respect to* x*,* y *if for every fixed* $\boldsymbol{a} \in V$*, the polynomial function* $U(\boldsymbol{a}, x, y)$ *is not identically zero on* V *and has constant sign outside its set of zeroes on* V*.*

Observation 5.1. *We draw the attention to the fact, that if a T-comitant* $U(a, x, y)$ *with respect to* $(\mathcal{A}, \mathcal{G})$ *of even weight is a binary form in* x*,* y*, of even degree in the coefficients of* (5.1) *and has well determined sign on the affine invariant algebraic subset* $\mathbb{R}^m_{\mathcal{A}}$ *then this property is conserved by any affine transformation and the sign is conserved.*

5.2 The main invariant polynomials associated to *LV*-systems

Consider real quadratic systems, i.e. systems of the form:

$$(S) \qquad \begin{cases} \dfrac{dx}{dt} &= p_0 + p_1(x, y) + \ p_2(x, y) \equiv p(x, y), \\ \dfrac{dy}{dt} &= q_0 + q_1(x, y) + \ q_2(x, y) \equiv q(x, y) \end{cases} \tag{5.2}$$

with $\max(\deg(p), \deg(q)) = 2$ and

$$\begin{aligned} p_0 &= a_{00}, \quad p_1(x, y) = a_{10}x + a_{01}y, \quad p_2(x, y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, \quad q_1(x, y) = b_{10}x + b_{01}y, \quad q_2(x, y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of system (5.2) and denote $\mathbb{R}[a, x, y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$.

Notation 5.2. *Let us denote by* $\boldsymbol{a} = (\boldsymbol{a}_{00}, \boldsymbol{a}_{10} \ldots, \boldsymbol{b}_{02})$ *a point in* $\mathbb{R}^{12}$*. Each particular system* (5.2) *yields an ordered 12-tuple* $\boldsymbol{a}$ *of its coefficients.*

Let us consider the polynomials

$$\begin{aligned} C_i(a, x, y) &= yp_i(a, x, y) - xq_i(a, x, y) \in \mathbb{R}[a, x, y], \ i = 0, 1, 2, \\ D_i(a, x, y) &= \frac{\partial}{\partial x}p_i(a, x, y) + \frac{\partial}{\partial y}q_i(a, x, y) \in \mathbb{R}[a, x, y], \ i = 1, 2. \end{aligned} \tag{5.3}$$

As it was shown in [82] the polynomials

$$\{ \ C_0(a, x, y), \quad C_1(a, x, y), \quad C_2(a, x, y), \quad D_1(a), \quad D_2(a, x, y) \ \} \tag{5.4}$$

of degree one in the coefficients of systems (5.2) are GL-comitants of these systems.

Notation 5.3. *Let* f*,* $g \in \mathbb{R}[a, x, y]$ *and*

$$(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}. \tag{5.5}$$

$(f, g)^{(k)} \in \mathbb{R}[a, x, y]$ *is called the transvectant of index* k *of* (f, g) *(cf.* [38], [60]*)*

Theorem 5.1 (see [87]). *Any GL-comitant of systems* (5.2) *can be constructed from the elements of the set* (5.4) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(f,g)^{(k)}$.*

Remark 5.4. *We point out that the elements of the set* (5.4) *generate the whole set of GL-comitants and hence also the set of affine comitants as well as of set of T-comitants.*

Notation 5.5. *Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P + \beta Q \in \mathbb{R}[a, X, Y, Z, \alpha, \beta]$ where $P = Z^2 p(X/Z, Y/Z)$, $Q = Z^2 q(X/Z, Y/Z)$, $p,\ q \in \mathbb{R}[a, x, y]$ and $\max(\deg_{(x,y)} p, \deg_{(x,y)} q) = 2$. Then*

$$\begin{aligned}\Phi_{\alpha,\beta} =\ & c_{11}(a,\alpha,\beta)X^2 + 2c_{12}(a,\alpha,\beta)XY + c_{22}(a,\alpha,\beta)Y^2 + 2c_{13}(a,\alpha,\beta)XZ + 2c_{23}(a,\alpha,\beta)YZ \\ & + c_{33}(a,\alpha,\beta)Z^2, \qquad \Delta(a,\alpha,\beta) = \ \det||c_{ij}(a,\alpha,\beta)||_{i,j\in\{1,2,3\}}\end{aligned}$$

and we denote

$$D(a,x,y) = 4\Delta(a,y,-x), \qquad H(a,x,y) = 4\big[\det||c_{ij}(a,y,-x)||_{i,j\in\{1,2\}}\big]. \tag{5.6}$$

Lemma 5.1 (see [76]). *Consider two invariant affine lines $\mathcal{L}_i(x,y) \equiv ux + vy + w_i = 0$, $\mathcal{L}_i(x,y) \in \mathbb{C}[x,y]$, $(i = 1,2)$ of a quadratic system (S) of coefficients $\boldsymbol{a}$ with $w_1 \neq w_2$. In this case the lines are parallel. Then $H(\boldsymbol{a},-v,u)=0$, i.e. the T-comitant $H(a,x,y)$ captures the directions of parallel invariant lines of systems* (5.2).

We construct the following T-comitants:

Notation 5.6.

$$\begin{aligned} B_3(a,x,y) &= (C_2, D)^{(1)} = Jacob\,(C_2, D)\,, \\ B_2(a,x,y) &= (B_3, B_3)^{(2)} - 6B_3(C_2, D)^{(3)}, \\ B_1(a) &= \operatorname{Res}_x(C_2, D)\,/y^9 = -2^{-9}3^{-8}\,(B_2, B_3)^{(4)}\,. \end{aligned} \tag{5.7}$$

Lemma 5.2 (see [76]). *For the existence of an invariant straight line in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that $B_1 = 0$ (respectively $B_2 = 0$; $B_3 = 0$).*

At the moment we do not have necessary and sufficient conditions for the existence of an invariant straight line or for invariant lines in two or three directions.

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $p(a,x,y)$ and $q(a,x,y)$. We obtain $\tilde{p}(\tilde{a}(a,x_0,y_0),x',y') = p(a,x'+x_0,y'+y_0)$, $\quad \tilde{q}(\tilde{a}(a,x_0,y_0),x',y') = q(a,x'+x_0,y'+y_0)$. Let us construct the following polynomials

$$\begin{aligned}\Gamma_i(a,x_0,y_0) \equiv \operatorname{Res}_{x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big), C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1}, \\ \Gamma_i(a,x_0,y_0) \in \mathbb{R}[a,x_0,y_0],\ (i=1,2).\end{aligned}$$

Notation 5.7.

$$\tilde{\mathcal{E}}_i(a,x,y) = \Gamma_i(a,x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[a,x,y] \quad (i=1,2).$$

Observation 5.8. *We note that the constructed polynomials $\tilde{\mathcal{E}}_1(a,x,y)$ and $\tilde{\mathcal{E}}_2(a,x,y)$ are affine comitants of systems* (5.2) *and are homogeneous polynomials in the coefficients $a_{00},\ldots,b_{02}$ and non-homogeneous in x,y and $\deg_a \tilde{\mathcal{E}}_1 = 3$, $\deg_{(x,y)} \tilde{\mathcal{E}}_1 = 5$, $\deg_a \tilde{\mathcal{E}}_2 = 4$, $\deg_{(x,y)} \tilde{\mathcal{E}}_2 = 6$.*

Notation 5.9. *Let $\mathcal{E}_i(a,X,Y,Z)$ $(i=1,2)$ be the homogenization of $\tilde{\mathcal{E}}_i(a,x,y)$, i.e.*

$$\mathcal{E}_1(a,X,Y,Z) = Z^5\tilde{\mathcal{E}}_1(a,X/Z,Y/Z), \qquad \mathcal{E}_2(a,X,Y,Z) = Z^6\tilde{\mathcal{E}}_1(a,X/Z,Y/Z)$$

and $\qquad \mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{E}_1(a,X,Y,Z),\ \mathcal{E}_2(a,X,Y,Z)\Big)$ *in* $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [76]):

Lemma 5.3. *The straight line $\mathcal{L}(x,y) \equiv ux + vy + w = 0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant line for a quadratic system* (5.2) *if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\tilde{\mathcal{E}}_1(\boldsymbol{a},x,y)$ and $\tilde{\mathcal{E}}_2(\boldsymbol{a},x,y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a},x,y) = (ux+vy+w)\widetilde{W}_i(x,y) \quad (i=1,2),$$

where $\widetilde{W}_i(x,y) \in \mathbb{C}[x,y]$.

Lemma 5.4. *1) If* $\mathcal{L}(x,y) \equiv ux + vy + w = 0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ *is an invariant straight line of multiplicity* k *for a quadratic system* (5.2) *then* $[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$ *in* $\mathbb{C}[x,y]$*, i.e. there exist* $W_i(\boldsymbol{a},x,y) \in \mathbb{C}[x,y]$ $(i=1,2)$ *such that*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a},x,y) = (ux+vy+w)^k W_i(\boldsymbol{a},x,y), \quad i=1,2. \tag{5.8}$$

2) If the line $l_\infty : Z = 0$ *is of multiplicity* $k > 1$ *then* $Z^{k-1} \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$.

Let us consider the following GL-comitants of systems (5.2):

Notation 5.10.

$$\begin{array}{ll}
M(a,x,y) = 2\,\mathrm{Hessian}\big(C_2(x,y)\big), & \eta(a) = \mathrm{Discrim}\big(C_2(x,y)\big),\\
K(a,x,y) = \mathrm{Jacobian}\big(p_2(x,y), q_2(x,y)\big), & \mu_0(a) = \mathrm{Discrim}\big(K(a,x,y)\big)/16,\\
N(a,x,y) = K(a,x,y) + H(a,x,y), & \theta(a) = \mathrm{Discrim}\big(N(a,x,y)\big).
\end{array} \tag{5.9}$$

Remark 5.11. *We note that by the discriminant of the cubic form* $C_2(a,x,y)$ *we mean the expression given in Maple via the function "*discrim$(C_2,x)/y^6$*".*

The geometrical meaning of these invariant polynomials is revealed by the next 3 lemmas (see [76]).

Lemma 5.5. *Let* $S \in \mathbf{QS}$ *and let* $\boldsymbol{a} \in \mathbb{R}^{12}$ *be its 12-tuple of coefficients. The common points of* $P=0$ *and* $Q=0$ *on the line* $Z=0$ *are given by the common linear factors over* $\mathbb{C}$ *of* p_2 *and* q_2*. This yields the geometrical meaning of the comitants* μ_0*,* K *and* H*:*

$$\gcd(p_2(x,y), q_2(x,y)) = \begin{cases}
\text{constant} & \Leftrightarrow \quad \mu_0(\boldsymbol{a}) \neq 0;\\
bx+cy & \Leftrightarrow \quad \mu_0 = 0 \text{ and } K(\boldsymbol{a},x,y) \neq 0;\\
(bx+cy)(dx+ey) & \Leftrightarrow \quad \mu_0(\boldsymbol{a}) = 0,\ K(\boldsymbol{a},x,y) = 0 \text{ and } H(\boldsymbol{a},x,y) \neq 0;\\
(bx+cy)^2 & \Leftrightarrow \quad \mu_0(\boldsymbol{a}) = 0,\ K(\boldsymbol{a},x,y) = 0 \text{ and } H(\boldsymbol{a},x,y) = 0,
\end{cases}$$

where $bx+cy,\ dx+ey \in \mathbb{C}[x,y]$ *are some linear forms and* $be - cd \neq 0$.

Lemma 5.6. *A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a systems* (5.2) *corresponding to* $\boldsymbol{a} \in \mathbb{R}^{12}$ *is the condition* $\theta(\boldsymbol{a}) = 0$ *(respectively,* $N(\boldsymbol{a},x,y) = 0$*).*

Lemma 5.7. *The type (see Section 4.1) of the divisor* $D_S(C,Z)$ *for systems* (5.2) *is determined by the corresponding conditions indicated in Table 1, where we write* $\omega_1^c + \omega_2^c + \omega_3$ *if two of the points, i.e.* ω_1^c, ω_2^c*, are complex but not real. Moreover, for each type of the divisor* $D_S(C,Z)$ *given in Table 1 the quadratic systems* (5.2) *can be brought via a linear transformation to one of the following canonical systems* $(\mathbf{S}_I) - (\mathbf{S}_V)$ *corresponding to their behavior at infinity.*

Table 1

Case	Type of $D_S(C,Z)$	Necessary and sufficient conditions on the comitants
1	$\omega_1 + \omega_2 + \omega_3$	$\eta > 0$
2	$\omega_1^c + \omega_2^c + \omega_3$	$\eta < 0$
3	$2\omega_1 + \omega_2$	$\eta = 0, \quad M \neq 0$
4	3ω	$M = 0, \quad C_2 \neq 0$
5	$D_S(C,Z)$ undefined	$C_2 = 0$

$$\begin{cases} \dot{x} &= k + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} &= l + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{I})$$

$$\begin{cases} \dot{x} &= k + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} &= l + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} &= k + cx + dy + gx^2 + hxy, \\ \dot{y} &= l + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} &= k + cx + dy + gx^2 + hxy, \\ \dot{y} &= l + ex + fy - x^2 + gxy + hy^2, \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} &= k + cx + dy + x^2, \\ \dot{y} &= l + ex + fy + xy. \end{cases} \qquad (\mathbf{S}_{V})$$

In order to construct other necessary invariant polynomials let us consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ acting on $\mathbb{R}[a, x, y]$ constructed in [10] (see also [11]), where

$$\mathbf{L}_1 = 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}};$$
$$\mathbf{L}_2 = 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}$$

In [10] it is shown that if a polynomial $U \in \mathbb{R}[a, x, y]$ is a comitant of system (5.2) with respect to the group $GL(2, \mathbb{R})$ then $\mathcal{L}(U)$ is also a GL-comitant.

So, by using this operator and the GL-comitant $\mu_0(a) = \operatorname{Res}_x(p_2(x, y), q_2(x, y))/y^4$ we construct the following polynomials:

$$\mu_i(a, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i = 1, .., 4, \quad \text{where} \quad \mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)). \tag{5.10}$$

These polynomials are in fact comitants of systems (5.2) with respect to the group $GL(2, \mathbb{R})$.

To reveal the geometrical meaning of the comitants $\mu_i(a, x, y),\ i = 0, 1, \ldots, 4$ we use the following resultants whose calculation yield:

$$Res_{X}(P, Q) = \mu_0 Y^4 + \mu_{10} Y^3 Z + \mu_{20} Y^2 Z^2 + \mu_{30} Y Z^3 + \mu_{40} Z^4; \tag{5.11}$$
$$Res_{Y}(P, Q) = \mu_0 X^4 + \mu_{01} X^3 Z + \mu_{02} X^2 Z^2 + \mu_{03} X Z^3 + \mu_{04} Z^4, \tag{5.12}$$

where $\mu_{ij} = \mu_{ij}(a) \in \mathbb{R}[a_{00}, \ldots, b_{02}]$.

On the other hand for μ_i, $i = 0, 1, \ldots, 4$ from (5.10) we have

$$\begin{aligned} \mu_0(a) &= \mu_0; \\ \mu_1(a, x, y) &= \mu_{10}x + \mu_{01}y; \\ \mu_2(a, x, y) &= \mu_{20}x^2 + \mu_{11}xy + \mu_{02}y^2; \\ \mu_3(a, x, y) &= \mu_{30}x^3 + \mu_{21}x^2y + \mu_{12}xy^2 + \mu_{03}y^3; \\ \mu_4(a, x, y) &= \mu_{40}x^4 + \mu_{31}x^3y + \mu_{22}x^2y^2 + \mu_{13}xy^3 + \mu_{04}y^4. \end{aligned}$$

We observe that the leading coefficients of the comitants μ_i, $i = 0, 1, \ldots, 4$ with respect to x (respectively y) are the corresponding coefficients in (5.11) (respectively (5.12)).

We draw the attention to the fact, that if the comitant $\mu_i(a, x, y)$ $(i = 0, 1, \ldots, 4)$ is not equal to zero then we may assume that its leading coefficients are both non zero, as this can be obtained by applying a rotation of the phase plane of systems (5.2). From here and (5.11), (5.12) and the above values of μ_i, $i = 0, 1, \ldots, 4$ we have:

Lemma 5.8. *The system $P(X, Y, Z) = Q(X, Y, Z) = 0$ possesses m $(1 \leq m \leq 4)$ solutions $[X_i : Y_i : Z_i]$ with $Z_i = 0$ $(i = 1, \ldots, m)$ (considered with multiplicities) if and only if for every $i \in \{0, 1, \ldots, m-1\}$ we have $\mu_i(a, x, y) = 0$ in $\mathbb{R}[a, x, y]$ and $\mu_m(a, x, y) \neq 0$.*

Remark 5.12. *It can easily be checked that the following identity holds*

$$\mu_4(a,X,Y) \;=\; Res_Z\left(P(X,Y,Z),Q(X,Y,Z)\right).$$

Hence, clearly for any solution $[X_0 : Y_0 : Z_0]$ *(including those with* $Z_0 = 0$*) of the system of equations* $P(X,Y,Z) = Q(X,Y,Z) = 0$*, the following relation is satisfied:* $\mu_4(a,X_0,Y_0) = 0$.

Lemma 5.9. *A quadratic system* (5.2) *is degenerate (i.e.* $\gcd(p,q) \neq$ constant *) if and only if* $\mu_i(a,x,y) = 0$ *in* $\mathbb{R}[x,y]$ *for every* $i = 0,1,2,3,4$.

Using the transvectant differential operator (5.5) and the constructed earlier invariant polynomials (5.3), (5.6) and (5.9) we shall construct the following needed invariant polynomials:

$$\begin{aligned}
H_1(a) &= -\big((C_2,C_2)^{(2)},C_2)^{(1)},D\big)^{(3)};\\
H_2(a,x,y) &= (C_1,\ 2H-N)^{(1)} - 2D_1N;\\
H_3(a,x,y) &= (C_2,D)^{(2)};\\
H_4(a) &= \big((C_2,D)^{(2)},(C_2,D_2)^{(1)}\big)^{(2)};\\
H_5(a) &= \big((C_2,C_2)^{(2)},(D,D)^{(2)}\big)^{(2)} + 8\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)};\\
H_6(a,x,y) &= 16N^2(C_2,D)^{(2)} + H_2^2(C_2,C_2)^{(2)};\\
H_7(a) &= (N,C_1)^{(2)};\\
H_8(a) &= 9\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)} + 2\Big[(C_2,D)^{(3)}\Big]^2;\\
H_9(a) &= -\Big(\big((D,D)^{(2)},D,\big)^{(1)}D\Big)^{(3)};\\
H_{10}(a) &= \big((N,D)^{(2)},\ D_2\big)^{(1)};\\
H_{11}(a,x,y) &= 8H\big[(C_2,D)^{(2)} + 8(D,D_2)^{(1)}\big] + 3H_2^2;\\
H_{12}(a,x,y) &= (D,D)^{(2)} \equiv \text{Hessian}\,(D);\\
H_{13}(a,x,y) &= A_1A_2 - A_{14} - A_{15};\\
H_{14}(a,x,y) &= 156A_2A_5 - 20A_2A_3 - 33A_2A_4 + 396A_1A_6 - 20A_{22} + 168A_{23} + 84A_{24};\\
N_1(a,x,y) &= C_1(C_2,C_2)^{(2)} - 2C_2(C_1,C_2)^{(2)},\\
N_2(a,x,y) &= D_1(C_1,C_2)^{(2)} - \big((C_2,C_2)^{(2)},C_0\big)^{(1)},\\
N_3(a,x,y) &= (C_2,C_1)^{(1)},\\
N_4(a,x,y) &= 4\,(C_2,C_0)^{(1)} - 3C_1D_1,\\
N_5(a,x,y) &= \big[(D_2,C_1)^{(1)} + D_1D_2\big]^2 - 4\big(C_2,C_2\big)^{(2)}\big(C_0,D_2\big)^{(1)},\\
N_6(a,x,y) &= 8D + C_2\Big[8(C_0,D_2)^{(1)} - 3(C_1,C_1)^{(2)} + 2D_1^2\Big].
\end{aligned}$$

We observe here that the polynomials A_i $i \in \{1,2,3,4,5,6,14,15,22,23,24\}$ used above are the elements of minimal polynomial basis of affine invariants (see Appendix), constructed in [16].

6 Classification of *LV*-systems according to their configurations of invariant lines

In classifying ***LV***-systems we shall rely on their algebro-geometric structure encoded in the divisor $D_{\mathbb{C}}(\mathcal{C})$ of the configuration $\mathcal{C}$ of invariant straight lines as well as in the other multiplicity divisors of singularities attached to the configuration (see Section 4). The study of quadratic systems possessing invariant straight lines was begun in [76] and continued in [79], [78], and [80]. The four works jointly taken cover the study of quadratic differential systems possessing invariant lines of at least four total multiplicity. Among these systems some but not all, belong to the class ***LV***. One of our goals is to give a topological classification of the class $\boldsymbol{LV}$ which is transparent and easy to understand. The algebro-geometric features of the $\boldsymbol{LV}$-systems is helpful and this geometric structure is also important for questions regarding integrability of systems in the class $\boldsymbol{LV}$.

The following is a corollary of Lemma 5.2.

Corrolary 6.1. *A necessary condition for a quadratic system* (5.2) *to be in the class* **LV** *(i.e. to possess two intersecting real invariant lines) is that the condition* $B_2(\boldsymbol{a},x,y)=0$ *hold in* $\mathbb{R}[x,y]$.

According to [76] and [78] we have:

Lemma 6.1. *If a quadratic system* (S) *corresponding to a point* $\boldsymbol{a}\in\mathbb{R}^{12}$ *belongs to the class* $\mathbf{QSL_4}\cup$ $\mathbf{QSL_5}\cup$ $\mathbf{QSL_6}$*, then for this system one of the following sets of conditions are satisfied in* $\mathbb{R}[x,y]$*, respectively:*

$$
\begin{array}{lll}
(S)\in\mathbf{QSL_4} & \Rightarrow & \textit{either } \theta(\boldsymbol{a})\neq 0,\ B_3(\boldsymbol{a},x,y)=0, \quad \textit{or} \quad \theta(\boldsymbol{a})=0=B_2(\boldsymbol{a},x,y);\\
(S)\in\mathbf{QSL_5} & \Rightarrow & \textit{either } \theta(\boldsymbol{a})=0=B_3(\boldsymbol{a},x,y), \quad \textit{or} \quad N(\boldsymbol{a},x,y)=0=B_2(\boldsymbol{a},x,y);\\
(S)\in\mathbf{QSL_6} & \Rightarrow & N(\boldsymbol{a},x,y)=0=B_3(\boldsymbol{a},x,y).
\end{array}
$$

We associate to each system in $\boldsymbol{LV}$ its configuration of invariant lines. We view the configurations of real lines on the disc representing the projective plane when two opposite points on the circumference are identified. In general to imagine the full configurations, we complete the picture by drawing dashed lines to indicate complex lines. However for $\boldsymbol{LV}$-systems this is not needed as all invariant lines of the systems are real.

Remark 6.2. As an $\boldsymbol{LV}$-system possesses two real intersecting invariant straight lines, the condition $B_2=0$ must be fulfilled according to Corollary 6.1. Moreover, from proposition 4.1 such a system must have at least two pairs of opposite real distinct infinite singularities. Hence, we can only have two possibilities: $\eta>0$ or $\eta=0$. This also follows from Lemma 5.7 saying that either the conditions $\eta>0$ or $\eta=0$ and $M\neq 0$ or $C_2=0$ (then $\eta=0$) have to be satisfied.

In what follows we shall construct and classify all possible configurations for a system in the family $\boldsymbol{LV}$ and determine the necessary and sufficient conditions for the realization of each one of them. For this purpose and considering Remark 6.2, we need to investigate only three canonical systems: $(\mathbf{S}_I)$, $(\mathbf{S}_{III})$ and $(\mathbf{S}_V)$. Considering Lemma 5.7 we obtain that two systems belonging to two distinct canonical forms will possess distinct configurations of the invariant lines.

Remark 6.3. Assume that for a quadratic system the condition $C_2=0$ holds. Then for this system we have $\eta=\theta=\mu_0=0=B_3$ and $KN\neq 0$.

This could be checked directly calculating for systems $(\mathbf{S}_V)$ the values of the invariant polynomials θ and μ_0.

6.1 Systems from the family $(\mathbf{S}_I)$

Theorem 6.1. *The planar* **LV** *quadratic differential systems with* $\eta>0$ *yield a 3-dimensional topological quotient space under the action of the affine group and time homotheties. There is a total of 29 distinct configurations of invariant lines of these systems indicated in Figure 1. Four out of these configurations have at least one and at most two lines of singularities and correspond to degenerate systems. Twenty configurations contain all singularities of the systems and nine have exactly one finite singularity outside their configurations. The points at infinity of the systems are either simple or of multiplicity 2 expressing a collision of only one finite with one infinite singularity and in this case we have at most two such points at infinity. All finite singularities have multiplicities different from 3. The maximum multiplicity of an invariant line is 2 and we have a maximum of two such invariant lines.*

These systems are split into five distinct classes according to the multiplicity of their invariant lines (including the line at infinity) as follows:

(i) The **LV** *systems with exactly three simple invariant straight lines (S=(1,3). These have 8 configurations Config. 3.j,* $j=1,2,\ldots,8$ *indicated in Fig.1, four with all singular points of the systems on the configurations and four with one singular point outside the configuration. All singularities of the systems*

are either simple or double. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as its respective representative are indicated in Table 2.

(ii) The ***LV*** *systems with four invariant straight lines counted with multiplicity. These have 10 configurations Config. 4.j with* $j \in \{1,3,4,5,9,10,16,17,18,22\}$*, seven of them with only simple invariant lines and three of them with exactly one double line located at infinity. The maximum multiplicity of a singularity finite or infinite is 2. The invariant necessary and sufficient conditions for the realization of each one of these configurations are:*

$$
\begin{array}{lll}
\textit{Config. 4.1} & \Leftrightarrow & \theta \neq 0,\ B_3 = 0,\ H_7 \neq 0; \\
\textit{Config. 4.3} & \Leftrightarrow & \theta \neq 0,\ B_3 = H_7 = 0,\ H_1 \neq 0,\ \mu_0 \neq 0; \\
\textit{Config. 4.4} & \Leftrightarrow & \theta \neq 0,\ B_3 = H_7 = 0,\ H_1 \neq 0,\ \mu_0 = 0; \\
\textit{Config. 4.5} & \Leftrightarrow & \theta \neq 0,\ B_3 = H_7 = H_1 = 0; \\
\textit{Config. 4.9} & \Leftrightarrow & B_3\mu_0 H_9 \neq 0,\ \theta = 0 \ \ \textit{and} \begin{cases} N \neq 0,\ H_7 = 0,\ H_{10}N > 0, \textit{or} \\ N = 0,\ H_4 \neq 0,\ H_8 > 0; \end{cases} \\
\textit{Config. 4.10} & \Leftrightarrow & B_3\mu_0 \neq 0,\ \theta = H_9 = 0 \ \ \textit{and} \begin{cases} N \neq 0,\ H_7 = 0,\ H_{10}N > 0, \textit{or} \\ N = 0,\ H_4 \neq 0,\ H_8 > 0; \end{cases} \\
\textit{Config. 4.16} & \Leftrightarrow & B_3 N \neq 0,\ \theta = \mu_0 = H_7 = 0,\ H_9 \neq 0; \\
\textit{Config. 4.17} & \Leftrightarrow & B_3 N \neq 0,\ \theta = \mu_0 = H_7 = H_9 = 0,\ H_{10} \neq 0; \\
\textit{Config. 4.18} & \Leftrightarrow & N \neq 0,\ B_3 = \theta = \mu_0 = 0,\ H_7 \neq 0; \\
\textit{Config. 4.22} & \Leftrightarrow & B_3\mu_0 \neq 0,\ \theta = 0 \ \ \textit{and} \begin{cases} N \neq 0,\ H_7 = H_{10} = 0, \textit{or} \\ N = H_8 = 0,\ H_4 \neq 0. \end{cases}
\end{array}
$$

(iii) The ***LV*** *systems with five invariant straight lines counted with multiplicity. These have 5 configurations Config. 5.j with* $j \in \{1,3,7,8,12\}$*, two of them have only simple invariant lines (*$j \in \{1,3\}$*) and 3 with exactly one double line. Two configurations have only simple singularities (*$j \in \{1,3\}$*, two have each two singularities of multiplicity 2 (*$j \in \{7,12\}$*) and one configuration (*$j = 8$*) has a singularity of multiplicity 4. The invariant necessary and sufficient conditions for the realization of each one of these configurations are:*

$$
\begin{array}{lll}
\textit{Config. 5.1} & \Leftrightarrow & \theta = 0,\ N \neq 0,\ B_3 = H_7 = 0,\ \mu_0 \neq 0,\ H_9 \neq 0; \\
\textit{Config. 5.3} & \Leftrightarrow & \theta = 0,\ N = 0,\ B_3 \neq 0,\ H_4 = 0,\ H_1 > 0,\ H_5 > 0; \\
\textit{Config. 5.7} & \Leftrightarrow & \theta = 0,\ N \neq 0,\ B_3 = H_7 = 0,\ \mu_0 = 0; \\
\textit{Config. 5.8} & \Leftrightarrow & \theta = 0,\ N \neq 0,\ B_3 = H_7 = 0,\ \mu_0 \neq 0,\ H_9 = 0; \\
\textit{Config. 5.12} & \Leftrightarrow & \theta = 0,\ N = 0,\ B_3 \neq 0,\ H_4 = 0,\ H_1 > 0,\ H_5 = 0.
\end{array}
$$

(iv) The ***LV*** *systems with six invariant straight lines counted with multiplicity have only two possible configurations Config. 6.j with* $j \in \{1,5\}$*, one with simple lines (*$j = 1$*) and the other with two double lines intersecting at a singular point of multiplicity 4 (*$j = 5$*). The invariant necessary and sufficient conditions for the realization of each one of these configurations are:*

$$
\begin{array}{lll}
\textit{Config. 6.1} & \Leftrightarrow & \theta = N = B_3 = 0,\ H_1 > 0; \\
\textit{Config. 6.5} & \Leftrightarrow & \theta = N = B_3 = 0,\ H_1 = 0.
\end{array}
$$

(v) The degenerate ***LV*** *systems defined by the conditions* $\mu_i = 0,\ i = 0,1,\dots,4$ *and possessing at least one and at most two affine lines filled with singularities. These have 4 configurations Config.* $LV_d.j$ *with* $j \in \{1,2,3,4\}$*, only one with exactly two lines of singularities (*$j = 4$*). The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as its respective representative are indicated in Table 3.*

Proof: In this case $\eta > 0$ and according to Lemma 5.7 we shall examine the systems of the form

$$
\begin{aligned}
\dot{x} &= a + cx + dy + gx^2 + (h-1)xy, \\
\dot{y} &= b + ex + fy + (g-1)xy + hy^2,
\end{aligned} \tag{6.1}
$$

for which $C(x,y) = xy(x-y),\ \theta = -8(g-1)(h-1)(g+h)$.

A) Non-degenerate systems

Table 2

Orbit representative	Necessary and sufficient conditions	Configuration
$(III.1)\ \begin{cases} \dot{x} = x[1+gx+(h-1)y], \\ \dot{y} = y[f+(g-1)x+hy], \\ f,g,h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_1) \end{cases}$	$\eta > 0,\ \mu_0 B_3 H_9 \neq 0$ and either $\theta \neq 0$ or $(\theta = 0\ \&\ NH_7 \neq 0)$	Config. 3.1
$(III.2)\ \begin{cases} \dot{x} = x[1+gx+(h-1)y], \\ \dot{y} = y[(g-1)x+hy], \\ g,h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_2) \end{cases}$	$\eta > 0,\ \mu_0 B_3 \neq 0,\ H_9 = H_{13} = 0$ and either $\theta \neq 0$ or $(\theta = 0\ \&\ NH_7 \neq 0)$	Config. 3.2
$(III.3)\ \begin{cases} \dot{x} = x[g+gx+(h-1)y], \\ \dot{y} = y[g-1+(g-1)x+hy], \\ g,h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_2) \end{cases}$	$\eta > 0,\ \mu_0 B_3 H_{13} \neq 0,\ H_9 = 0$ and either $\theta \neq 0$ or $(\theta = 0\ \&\ NH_7 \neq 0)$	Config. 3.3
$(III.4)\ \begin{cases} \dot{x} = x[1+(h-1)y], \\ \dot{y} = y(f-x+hy), \\ f,h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_3) \end{cases}$	$\eta > 0,\ \theta B_3 H_9 \neq 0,\ \mu_0 = H_{14} = 0$	Config. 3.4
$(III.5)\ \begin{cases} \dot{x} = x[1+(1-h)(x-y)], \\ \dot{y} = y(f-hx+hy), \\ f,h \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_3) \end{cases}$	$\eta > 0,\ \theta B_3 H_9 H_{14} \neq 0,\ \mu_0 = 0$	Config. 3.5
$(III.6)\ \begin{cases} \dot{x} = x[1+(h-1)y], \\ \dot{y} = y(-x+hy), \\ h \in \mathbb{R},\ h(h-1) \neq 0 \end{cases}$	$\eta > 0,\ \theta B_3 \neq 0,\ \mu_0 = H_9 = 0,$ $H_{13} = H_{14} = 0$	Config. 3.6
$(III.7)\ \begin{cases} \dot{x} = x[h-1+(h-1)y], \\ \dot{y} = y(h-x+hy), \\ h \in \mathbb{R},\ h(h-1) \neq 0 \end{cases}$	$\eta > 0,\ \theta B_3 H_{13} \neq 0,\ \mu_0 = H_9 = H_{14} = 0$	Config. 3.7
$(III.8)\ \begin{cases} \dot{x} = x[1+(1-h)(x-y)], \\ \dot{y} = hy(y-x), \\ h \in \mathbb{R},\ h(h-1) \neq 0 \end{cases}$	$\eta > 0,\ \theta B_3 H_{14} \neq 0,\ \mu_0 = H_9 = 0$	Config. 3.8
$(III.9)\ \begin{cases} \dot{x} = x(1+gx+y), \\ \dot{y} = y(f-x+gx+y), \\ f,g \in \mathbb{R}, \quad \text{cond. } (\mathcal{A}_4) \end{cases}$	$\eta = 0,\ \theta H_4 B_3 \mu_0 H_9 \neq 0$	Config. 3.9
$(III.10)\ \begin{cases} \dot{x} = x(g+gx+y), \\ \dot{y} = y[g-1+(g-1)x+y], \\ g \in \mathbb{R}, \quad g(g-1) \neq 0 \end{cases}$	$\eta = 0,\ \theta H_4 B_3 \mu_0 H_{13} \neq 0,\ H_9 = 0$	Config. 3.10
$(III.11)\ \begin{cases} \dot{x} = x(1+gx+y), \\ \dot{y} = y(-x+gx+y), \\ g \in \mathbb{R}, \quad g(g-1) \neq 0 \end{cases}$	$\eta = 0,\ \theta H_4 B_3 \mu_0 \neq 0,\ H_9 = H_{13} = 0$	Config. 3.11
$(III.12)\ \begin{cases} \dot{x} = x(1+y), \\ \dot{y} = y(f+x+y), \\ f \in \mathbb{R},\ f(f-1) \neq 0 \end{cases}$	$\eta = 0,\ \theta H_4 B_3 H_9 \neq 0,\ \mu_0 = 0$	Config. 3.12
$(III.13)\ \begin{cases} \dot{x} = x(1+y), \\ \dot{y} = y(x+y), \end{cases}$	$\eta = 0,\ \theta H_4 B_3 \neq 0,\ \mu_0 = H_9 = 0$	Config. 3.13
$gh(g+h-1)(g-1)(h-1)f(f-1)(fg+h)(1-g+fg)(f+h-fh) \neq 0;$		$(\mathcal{A}_1)$
$gh(g+h-1)(g-1)(h-1) \neq 0;$		$(\mathcal{A}_2)$
$h(h-1)f(f-1)(f+h-fh) \neq 0.$		$(\mathcal{A}_3)$
$g(g-1)f(f-1)(1-g+fg) \neq 0.$		$(\mathcal{A}_4)$

6.1.1 Case $\theta \neq 0$

The condition $\theta \neq 0$ yields $(g-1)(h-1) \neq 0$ and without lost of generality we may assume $d = e = 0$ due to a translation. Thus, we get the canonical systems

$$\begin{aligned} \dot{x} &= a + cx + 2gx^2 + 2(h-1)xy, \\ \dot{y} &= b + fy + 2(g-1)xy + 2hy^2, \end{aligned} \tag{6.2}$$

Table 3

Orbit representative	Necessary and sufficient conditions	Configuration
$(LV_d.1)\ \begin{cases}\dot x = x(1+gy-y),\\ \dot y=(g-1)xy,\\ g(g-1)\neq 0\end{cases}$	$\eta>0,\ \mu_{0,1,2,3,4}=0,\ \theta\neq 0,\ H_7\neq 0$	Config. $\mathrm{LV}_d.1$
$(LV_d.2)\ \begin{cases}\dot x = x(gx-y),\\ \dot y=(g-1)xy,\\ g(g-1)\neq 0\end{cases}$	$\eta>0,\ \mu_{0,1,2,3,4}=0,\ \theta\neq 0,\ H_7=0$	Config. $\mathrm{LV}_d.2$
$(LV_d.3)\ \begin{cases}\dot x = x(1+y),\\ \dot y=xy,\end{cases}$	$\eta>0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,\ H_4=0,\ H_7\neq 0$	Config. $\mathrm{LV}_d.3$
$(LV_d.4)\ \begin{cases}\dot x = xy,\\ \dot y=xy,\end{cases}$	$\eta>0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,\ H_4=0,\ H_7=0$	Config. $\mathrm{LV}_d.4$
$(LV_d.5)\ \begin{cases}\dot x = xy,\\ \dot y=y(1-x+y),\end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta\neq 0,\ H_7\neq 0$	Config. $\mathrm{LV}_d.5$
$(LV_d.6)\ \begin{cases}\dot x = xy,\\ \dot y=y(-x+y),\end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta\neq 0,\ H_7=0$	Config. $\mathrm{LV}_d.6$
$(LV_d.7)\ \begin{cases}\dot x = x(1+gx),\ g\neq 0\\ \dot y=(g-1)xy,\ g\neq 1\end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,\ K\neq 0,\ H_2\neq 0$	Config. $\mathrm{LV}_d.7$
$(LV_d.8)\ \begin{cases}\dot x = gx^2,\ g(g-1)\neq 0\\ \dot y=(g-1)xy,\end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,\ K\neq 0,\ H_2=0$	Config. $\mathrm{LV}_d.8$
$(LV_d.9)\ \begin{cases}\dot x = x,\\ \dot y=xy,\end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $K=0,\ N\neq 0,\ H_7=0,\ H_2\neq 0$	Config. $\mathrm{LV}_d.9$
$(LV_d.10)\ \begin{cases}\dot x = 0,\\ \dot y=xy,\end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $K=0,\ N\neq 0,\ H_7=0,\ H_2=0,\ D=0$	Config. $\mathrm{LV}_d.10$
$(LV_d.11)\ \begin{cases}\dot x = x^2-1,\\ \dot y=0,\end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $K=0,\ N=0,\ D=N_1=0,\ N_5>0$	Config. $\mathrm{LV}_d.11$
$(LV_d.12)\ \begin{cases}\dot x = x^2,\\ \dot y=0,\end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $K=0,\ N=0,\ D=N_1=0,\ N_5=0$	Config. $\mathrm{LV}_d.12$
$(LV_d.13)\ \begin{cases}\dot x = x(1+x),\\ \dot y=xy,\end{cases}$	$C_2=0,\ \mu_{1,2,3,4}=0,\ H_2\neq 0$	Config. $\mathrm{LV}_d.13$
$(LV_d.14)\ \begin{cases}\dot x = x^2,\\ \dot y=xy,\end{cases}$	$C_2=0,\ \mu_{1,2,3,4}=0,\ H_2=0$	Config. $\mathrm{LV}_d.14$

for which we calculate: Coefficient$[B_2(\boldsymbol{a},x,y),x^3y]=2592ab(g-1)^2(h-1)^2$. Therefore, the condition $B_2=0$ (see Corollary 6.1) implies $ab=0$ and we can assume $a=0$ due to the change $(x,y,a,b,c,f,g,h)\mapsto(y,x,b,a,f,c,h,g)$.

So, $a=0$ and for systems (6.2) we have

$$B_2=-648b(g-1)^2[b(g+h)^2+(c-f)(fg+ch)]x^4.$$

Hence, since $\theta\neq 0$ (i.e. $(g-1)(h-1)(g+h)\neq 0$) from $B_2=0$ we get either $b=0$ or $b(g+h)^2+(c-f)(fg+ch)=0$. Due to an affine transformation without loss of generality we may consider $b=0$. Indeed, assuming $b\neq 0$ the second relation yields $b=(f-c)(fg+ch)/(g+h)^2$ and then via the transformation $x_1=x$, $y_1=y-x+(f-c)/(g+h)$ we obtain the new system of the same form (6.2) but with $a=b=0$.

Thus we obtain the family of Lotka-Volterra systems

$$\dot x=x\big[c+gx+(h-1)y\big],\quad \dot y=y\big[f+(g-1)x+hy\big],\tag{6.3}$$

for which we shall determine the possible invariant line configurations. For these systems we have

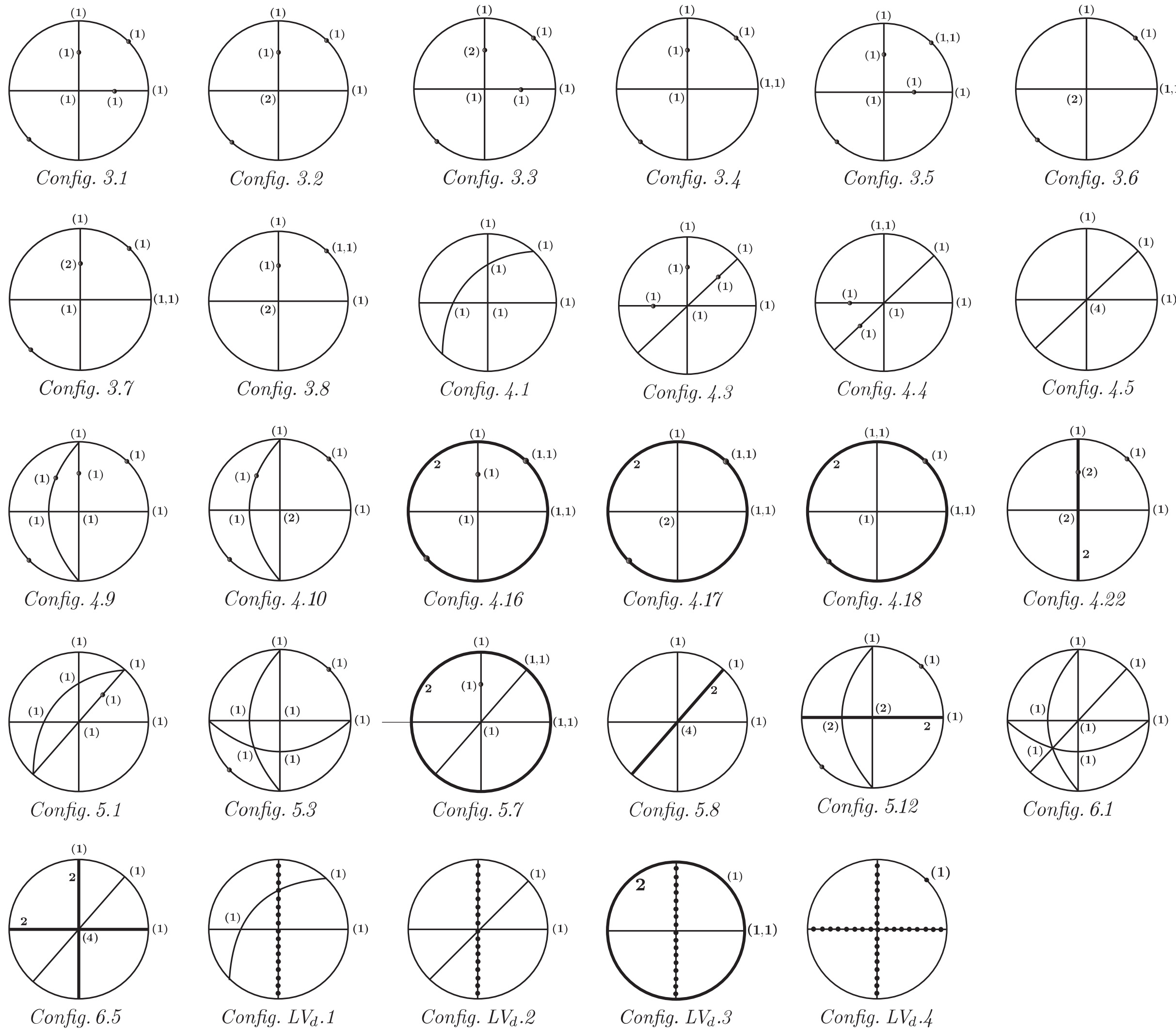


Figure 1: **The case $\boldsymbol{\eta \neq 0}$**

$B_3(x,y) = 3(c-f)(fg+ch)x^2y^2$ and according to [78] if $B_3 = 0$ we arrive to one of the configurations *Config. 4.j*, $j = 1,3,4,5$ as it is indicated in the statement of Theorem 6.1.

In what follows we assume $B_3 \neq 0$. As $\theta \neq 0$ by Lemmas 5.2 and 5.6 systems (6.3) possesses exactly three simple lines (considering the line at infinite). In order to determine the respective configurations of invariant lines we shall consider the finite singularities, which for these systems are as follows:

$$M_1(0,0), \quad M_2(-c/g,0), \quad M_3(0,-f/h); \quad M_4\left(\frac{fh-f-ch}{g+h-1}, \frac{cg-c-fg}{g+h-1}\right). \tag{6.4}$$

On the other hand for systems (6.3) calculations yield:

$$\begin{gathered}\mu_0 = gh(g+h-1), \quad H_9 = -576c^2f^2(fh-f-ch)^2(cg-c-fg)^2,\\ B_3 = 3(c-f)(fg+ch)x^2y^2, \quad \theta = -8(g-1)(h-1)(g+h),\\ H_{13} = -cf(ch+fg)(g-1)(h-1)/2, \quad H_{14} = 30gh(c-f)(fg+ch).\end{gathered} \tag{6.5}$$

Herein considering (6.4) we observe, that μ_0 governs the situation when a finite point goes to the infinity. Moreover, according to Lemma 5.12 the number of such points are defined by the number of the vanishing polynomials μ_i $(i = 0,1,2,3)$.

On the other hand the polynomial H_9 ($\equiv 576\mathbf{D}$, see [7]) is responsible for the collapsing of the finite singularities. So, we use these geometric facts to split the proof in several subcases.

6.1.1.1 Subcase $\mu_0 \neq 0$ Then all three infinite singular points remain simple ones and we shall examine two possibilities: $H_9 \neq 0$ and $H_9 = 0$.

6.1.1.1.1 Possibility $H_9 \neq 0$. In this case clearly all singularities are simple and we get a configuration of invariant line given by *Config. 3.1*. We note that $H_9 \neq 0$ implies $c \neq 0$ and we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. So, we get the family of systems (*III.1*) (see Table 2) for which the conditions $B_3 H_9 \mu_0 \neq 0$ and either $\theta \neq 0$ or $\theta = 0$ and $NH_7 \neq 0$ (considering Remark 6.5) implies the condition $(\mathcal{A}_1)$.

6.1.1.1.2 Possibility $H_9 = 0$. Then $cf(fh - f - ch)(cg - c - fg) = 0$ and taking into account (6.5) we consider two cases: $H_{13} = 0$ and $H_{13} \neq 0$.

1) Assume first $H_{13} = 0$. As $\theta B_3 \neq 0$ from (6.5) we get $cf = 0$ and without loss of generality we can assume $f = 0$ due to the change $(x, y, c, f, g, h) \to (y, x, f, c, h, g)$. Therefore by (6.4) the point M_3 coincides with $M_1(0, 0)$ and since in this case we have $B_3 = 3c^2hx^2y^2 \neq 0$ this leads to *Config. 3.2*. Moreover applying the same rescaling as above we may consider $c = 1$. Thus we get the family of systems (*III.2*) from Table 2), for which the condition $B_3\mu_0 \neq 0$ and either $\theta \neq 0$ or $\theta = 0$ and $NH_7 \neq 0$ (considering Remark 6.6) implies the condition $(\mathcal{A}_2)$.

2) Admit now $H_{13} \neq 0$. Then $cf \neq 0$ and the condition $H_9 = 0$ gives $(fh - f - ch)(cg - c - fg) = 0$. And again without loss of generality we can assume $cg - c - fg = 0$ due to the change $(x, y, c, f, g, h) \to (y, x, f, c, h, g)$. In this case according to (6.4) the point M_4 coincides with $M_3(0, -f/h)$. As $g \neq 0$ we may consider $c = ug$ (u is a new parameter) and then $f = u(g - 1)$. Then we calculate $B_3 = 3u^2g(g + h - 1)x^2y^2 \neq 0$ and we may assume $u = 1$ due to the change $(x, y, t) \mapsto (ux, uy, t/u)$. This leads to *Config. 3.3* and to the respective representative (*III.3*) (see Table 2), for which the condition $B_3\mu_0 H_{13} \neq 0$ and either $\theta \neq 0$ or $\theta = 0$ and $NH_7 \neq 0$ (considering Remark 6.7) implies the condition $(\mathcal{A}_2)$.

6.1.1.2 Subcase $\mu_0 = 0$ In this case $gh(g + h - 1) = 0$ and at least one of the finite singular points (6.4) is gone to infinity. On the other hand due to the condition $B_3 \neq 0$ and (6.5) the alternative: either $gh = 0$ (when one singularity of the configuration becomes double) or $g + h - 1 = 0$ (when all three its singularities remain simple), is governed by the invariant H_{14}.

6.1.1.2.1 *Possibility* $H_{14} = 0$. In this case $gh = 0$ and without loss of generality we can assume $g = 0$ due to the change $(x, y, c, f, g, h) \to (y, x, f, c, h, g)$. In this case calculations yield:

$$B_3 = 3c(c - f)hx^2y^2, \quad \theta = 8h(h - 1), \quad H_9 = -576c^4f^2(f + ch - fh)^2, \quad H_{13} = c^2fh(h - 1)/2. \tag{6.6}$$

Considering the finite singularities

$$M_1(0, 0), \qquad M_3(0, -f/h); \quad M_4\left(\frac{fh - f - ch}{h - 1}, -\frac{c}{h - 1}\right) \tag{6.7}$$

we observe that due to the condition $B_3\theta \neq 0$ all three points remain in the finite part of the phase plane when the parameters c, f, and h vary. Moreover, the coincidence of two of them is governed by the polynomial H_9.

1) Assume first $H_9 \neq 0$. In these cases all three finite points are simple and we arrive to *Config. 3.4*. Moreover, as $c \neq 0$ we may assume $c = 1$ due to a rescaling. So we get the family of systems (*III.4*) (see Table 2) for which the condition $\theta B_3 H_9 \neq 0$ implies the condition $(\mathcal{A}_3)$.

2) Suppose now $H_9 = 0$. Considering (6.6) and $B_3 \neq 0$ the condition $H_9 = 0$ implies $f(f + ch - fh) = 0$. Moreover due to $\theta B_3 \neq 0$ the condition $f = 0$ is equivalent to $H_{13} = 0$.

a) If $H_{13} = 0$ then $f = 0$ and from (6.7) we observe, that the point M_3 coincides with $M_1(0, 0)$. This leads to the configuration given by *Config. 3.6* and we considering $c = 1$ (due to a rescaling) we get the family of systems (*III.6*) (see Table 2), for which the condition $B_3\theta \neq 0$ implies $h(h - 1) \neq 0$.

b) Assume now $H_{13} \neq 0$. In this case the condition $H_9 = 0$ yields $f + ch - fh = 0$ and as $h \neq 0$, setting $f = hu$ we obtain $c = u(h - 1)$). We observe that the point M_4 coincides with M_3 (see (6.7)) and this leads to *Config. 3.7*. To determine the respective representative we make $u = 1$ via the rescaling $(x, y, t) \mapsto (ux, uy, t/u)$ and we obtain the family of systems (*III.7*) from Table 2, for which the condition $\theta B_3 H_{13} \neq 0$ implies $h(h-1) \neq 0$.

6.1.1.2.2 *Possibility* $H_{14} \neq 0$. Then $gh \neq 0$ and the condition $\mu_0 = 0$ gives $g = 1 - h$. In this case for systems (6.3) we obtain

$$B_3 = 3(c-f)(f+ch-fh)x^2y^2, \ \theta = 8h(h-1), \ H_9 = -576c^2f^2(f+ch-fh)^4. \tag{6.8}$$

As $\theta \neq 0$ then all three finite singularities

$$M_1(0,0), \quad M_2(c/(h-1), 0), \quad M_3(0, -f/h) \tag{6.9}$$

of systems (6.3) in this case remain in the finite part of the phase plane when the parameters c, f, and h vary.

1) Assume first $H_9 \neq 0$. In this case singular points (6.9) are simple and we arrive to *Config. 3.5*. As $c \neq 0$ we may assume $c = 1$ due to a rescaling and then we get the family of systems (*III.5*) from Table 2. For this family the condition $\theta B_3 H_{14} H_9 \neq 0$ implies the condition $(\mathcal{A}_3)$.

2) Admit now $H_9 = 0$. As $B_3 \neq 0$ from (6.8) we obtain $cf = 0$ and we may consider $f = 0$ due to the change $(x, y, c, f, h) \to (y, x, f, c, 1-h)$, which keeps systems (6.3) with $g = 1-h$. Then the point M_3 coincides with $M_1(0,0)$ and we get *Config. 3.8*. To determine the respective representative we observe, that if $f = 0$ then $c \neq 0$ (due to $B_3 \neq 0$) and via a rescaling we set $c = 1$. So we obtain the family of systems (*III.8*) from Table 2, for which the condition $\theta B_3 H_{14} \neq 0$ implies $h(h-1) \neq 0$.

6.1.2 Case $\theta = 0, \ N \neq 0$

Remark 6.4. *According to [78], (see Table 2) and [79], (see Table 5) if for a quadratic systems either the conditions $\eta > 0, \ \theta = 0, \ N \neq 0$ and $B_2 = 0$ or $\eta = 0, \ M \neq 0$ and $B_2 = 0$ are verified, then this system could belong to the family* $\mathbf{QSL_4} \bigcup \mathbf{QSL_5}$ *only if the condition $B_3 H_7 = 0$ holds.*

The condition $\theta = 0$ yields $(g-1)(h-1)(g+h) = 0$ and without lost of generality we may assume $g+h = 0$ due to the respective linear transformation, which replace the line $x = 0$ with $y = x$ (if $h-1 = 0$) or the line $y = 0$ with $y = x$ (if $g - 1 = 0$).

So, $g = -h$ and then $N = (h^2-1)(x-y)^2$. Hence, the conditions $N \neq 0$ implies $h^2 - 1 \neq 0$ and we may assume $d = e = 0$ due to a translation. Therefore we get the family of systems

$$\begin{aligned} \dot{x} &= a + cx - hx^2 + (h-1)xy, \\ \dot{y} &= b + fy - (h+1)xy + hy^2, \end{aligned} \tag{6.10}$$

for which we calculate: $\text{Coefficient}[B_2(\boldsymbol{a}, x, y), x^3y] = 2592ab(h+1)^2(h-1)^2$. So the condition $B_2 = 0$ implies $ab = 0$ and we can assume $a = 0$ due to the change $(x, y, a, b, c, f, h) \to (y, x, b, a, f, c, -h)$. So, $a = 0$ and for systems (6.10) we have

$$\begin{aligned} B_2 = -648bh(h+1)^2(c-f)^2x^4, \quad \mu_0 = h^2, \\ N = (h^2-1)(x-y)^2, \quad H_7 = 4(c-f)(h^2-1). \end{aligned} \tag{6.11}$$

6.1.2.1 Subcase $\mu_0 \neq 0$ Then we have $h \neq 0$ and since $N \neq 0$ (i.e. $(h+1) \neq 0$) the condition $B_2 = 0$ yields $b(c-f) = 0$. We shall consider two possibilities: $H_7 \neq 0$ and $H_7 = 0$.

6.1.2.1.1 *Possibility* $H_7 \neq 0$. In this case considering (6.11) we obtain $(c-f) \neq 0$ and hence the condition $B_2 = 0$ yields $b = 0$. This leads to the family of systems

$$\dot{x} = cx - hx^2 + (h-1)xy, \quad \dot{y} = fy - (h+1)xy + hy^2, \tag{6.12}$$

possessing the invariant lines $x = 0$ and $y = 0$ and the following finite singularities:

$$M_1(0,0), \quad M_2(c/h, 0), \quad M_3(0, -f/h); \quad M_4(ch+f-fh, ch+c-fh). \tag{6.13}$$

For these systems calculations yield:

$$\begin{gathered}B_3=3h(c-f)^2x^2y^2, \quad \mu_0=h^2, \quad H_7=4(c-f)(h^2-1),\\ H_9=-576c^2f^2(ch+f-fh)^2(ch+c-fh)^2, \quad H_{13}=cf(c-f)h(h^2-1)/2.\end{gathered} \tag{6.14}$$

We observe that the condition $\mu_0 H_7\neq 0$ implies $B_3\neq 0$ and considering Remark 6.4 we conclude that besides the invariant lines $x=0$ and $y=0$ systems (6.12) could not possess other affine invariant lines. Moreover, these lines are simple as well as the infinite line is.

Thus, to construct the configurations of invariant lines for these systems it remains to examine when two of singularities (6.13) coincide. And since this geometrical property is governed by the invariant polynomial H_9 we consider two cases: $H_9\neq 0$ and $H_9=0$.

1) Assume first $H_9\neq 0$. Then all four singularities are simple and we obtain *Config. 3.1*. In order to determine the respective representative we observe that $c\neq 0$ (due to $H_9\neq 0$) and via a rescaling we can set $c=1$.

Remark 6.5. *It is obvious to observe that the obtained family of systems coud be included in the family of systems (III.1) from Table 2 allowing $g=-h$. We note also that in this case the condition $(\mathcal{A}_1)$ implies $NH_7\neq 0$.*

2) Suppose now $H_9=0$. In this case we have either $cf=0$ or $(ch+f-fh)(ch+c-fh)=0$ and we have to examine two cases: $H_{13}=0$ and $H_{13}\neq 0$.

a) Assume $H_{13}=0$. Then $cf=0$ and without loss of generality we can assume $f=0$ due to the change $(x,y,c,f,h)\to(y,x,f,c,-h)$. Therefore the point M_3 coincides with $M_1(0,0)$ (see (6.13)) and this leads to *Config. 3.2*. As in the previous case we may assume $c=1$ and we could make a similar remark.

Remark 6.6. *The obtained family of systems in this case could be included in the family of systems (III.2) from Table 2 allowing $g=-h$. We note that in this case the condition $(\mathcal{A}_2)$ implies $NH_7\neq 0$.*

b) Suppose now $H_{13}\neq 0$. Then the condition $H_9=0$ gives $(ch+f-fh)(ch+c-fh)=0$ and again due to the same change above we can assume $ch+c-fh=0$. In this case according to (6.13) the point M_4 coincides with $M_3(0,-f/h)$ we get to *Config. 3.3*. As $h\neq 0$ we can set $c=-uh$ (u is a new parameter) and then $f=-u(1+h)$. Then we calculate $B_3=3u^2hx^2y^2\neq 0$ and we may assume $u=1$ due to the change $(x,y,t)\mapsto(ux,uy,t/u)$. This leads to *Config. 3.3* and to the respective representative for which the following remark is valid:

Remark 6.7. *The obtained family of systems corresponding to Config. 3.3 in this case could be included in the family of systems (III.3) from Table 2 allowing $g=-h$. We note that in this case the condition $(\mathcal{A}_2)$ implies $NH_7\neq 0$*

6.1.2.1.2 *Possibility $H_7=0$.* In this case considering (6.11) and the condition $N\neq 0$ we obtain $c-f=0$ (then $B_2=0$) and this leads to the family of systems

$$\dot{x}=cx-hx^2+(h-1)xy, \quad \dot{y}=b+cy-(h+1)xy+hy^2. \tag{6.15}$$

So, besides the invariant line $x=0$ these systems possess two invariant parallel lines

$$h(x-y)^2-c(x-y)+b=0,$$

which could be real distinct, complex or coinciding. So the systems from this family possess at least three affine invariant line (considered with their multiplicities) and we could detect the respective configurations applying the results obtained in [78] and [79]. To do this we need the values of the respective invariant polynomials.

For systems (6.15) calculations yield:

$$\begin{gathered}\eta=1>0, \quad \theta=B_2=H_7=0, \quad B_3=-3b(h+1)^2x^2(x-y)^2, \quad \mu_0=h^2,\\ N=(h^2-1)(x-y)^2 \quad H_{10}=8(h^2-1)(c^2-4bh), \quad H_4=48b(h-1)(h+1)^2,\\ H_9=-576(c^2-4bh)^2\big[c^2+b(h-1)^2\big]^2, \quad H_1=288\big[2c^2+b(h^2-6h+1)\big].\end{gathered} \tag{6.16}$$

We observe that the condition $B_3N \neq 0$ implies $H_4 \neq 0$ and $\operatorname{sign}(c^2-4bh) = \operatorname{sign}(H_{10}N)$ if $H_{10} \neq 0$ and $c^2-4bh=0$ if $H_{10}=0$.

Then considering the conditions $\mu_0 N \neq 0$ and $H_7=0$, according to [78, Theorem 4.1] and [79, Theorem 5.1] we obtain the following configurations and the respective invariant criteria:

- *Config. 4.9* $\Leftrightarrow$ $B_3 \neq 0$, $H_{10}N>0$, $H_9 \neq 0$;
- *Config. 4.10* $\Leftrightarrow$ $B_3 \neq 0$, $H_{10}N>0$, $H_9 = 0$;
- *Config. 4.22* $\Leftrightarrow$ $B_3 \neq 0$, $H_{10}=0$;
- *Config. 5.1* $\Leftrightarrow$ $B_3=0$, $H_1 \neq 0$;
- *Config. 5.8* $\Leftrightarrow$ $B_3=0$, $H_1=0$.

Remark 6.8. *If for systems* (6.15) *the condition $B_3=0$ (i.e. $b=0$) holds, then the condition $H_1=0$ is equivalent to $H_9=0$.*

Herein we obtain the respective conditions indicated in the statement of Theorem 6.1.

6.1.2.2 Subcase $\mu_0=0$

Then $h=0$ and as $a=0$ systems (6.10) become

$$\dot{x} = x(c-y), \quad \dot{y} = b+fy-xy, \tag{6.17}$$

for which we calculate:

$$\begin{aligned}\mathcal{E}_1 =& \big[fy^3-cx^2y+(b+2cf)xy+(b-2cf)y^2+bcx+\\ &(c^2f-cf^2-2bc-bf)y-b(b-c^2+cf)\big]\,\mathcal{H},\\ \mathcal{E}_2 =&(y-c)(cx-fy-b)(xy-fy-b)\,\mathcal{H},\end{aligned} \tag{6.18}$$

where $\mathcal{H}=x$. We observe, that the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ evaluated for systems (6.17) are of degree 4 and 5, respectively, and as according to Observation 5.8 in generic case they are polynomials of degree 5 and 6, respectively, we conclude (see [79], Lemmas 3.3 and 3.4) that the infinite line for these systems is of multiplicity 2. So, systems (6.17) belong to the family $\mathbf{QSL_3} \bigcup \mathbf{QSL_4} \bigcup \mathbf{QSL_5}$, however there are not two real intersection lines.

Clearly, to have an invariant line intersecting $x=0$, according to Lemma 5.3 this line must be a common factor of the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$. In other words the following condition must hold:

$$\operatorname{Res}_y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = bf(c-f)x^2(cx-b-cf)^6=0.$$

We observe, that f is the coefficient in front of y^3 in $\mathcal{E}_1$, and therefore the condition $f=0$ does not lead to the appearance of a common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$, depending on y. Therefore for systems (6.17) to belong to the class ***LV*** the condition $b(c-f)=0$ must be fulfilled.

On the other hand for systems systems (6.17) calculations yield:

$$\begin{gathered}\theta=B_2=\mu_0=0, \quad B_3=-3bx^2(x-y)^2,\\ H_9=-576c^2f^2(b+cf)^2 \quad H_{10}=-8cf, \quad H_7=4(f-c)\end{gathered} \tag{6.19}$$

and evidently the condition above is equivalent to $B_3H_7=0$.

However in this case we are in the family $\mathbf{QSL_4} \bigcup \mathbf{QSL_5}$ (see Remark 6.4) and according to [78], (Theorem 4.1) and [79], (Theorem 5.1) for non-degenerate systems (6.17) we obtain the following configurations of invariant lines and the respective invariant criteria:

- *Config. 4.16* $\Leftrightarrow$ $B_3 \neq 0$, $H_7=0$, $H_9 \neq 0$;
- *Config. 4.17* $\Leftrightarrow$ $B_3 \neq 0$, $H_7=0$, $H_9=0$, $H_{10} \neq 0$;
- *Config. 4.18* $\Leftrightarrow$ $B_3=0$, $H_7 \neq 0$;
- *Config. 5.7* $\Leftrightarrow$ $B_3=0$, $H_6=0$;

As in the case $b=0$ (i.e.$B_3=0$) for systems (6.17) we have $H_6=128(f-c)(x-y)^2(cx^2-fy^2)(x^2-xy+y^2)$ we conclude that for $B_3=0$ the conditions $H_7=0$ and $H_6=0$ are equivalent. Then, considering Remark 6.4 we arrive to the respective conditions of the theorem.

6.1.3 Case $\theta = 0 = N$

For the systems (6.1) we calculate

$$N(\boldsymbol{a}, x, y) \quad = \quad (g^2-1)x^2 + 2(g-1)(h-1)xy + (h^2-1)y^2.$$

Hence the condition $N = 0$ yields $(g-1)(h-1) = g^2-1 = h^2-1 = 0$ and we obtain 3 possibilities: (a) $g = 1 = h$; (b) $g = 1 = -h$; (c) $g = -1 = -h$. The cases (b) and (c) can be brought by linear transformations to the case (b). Hence the resulting polynomials are: $p_2(x, y) = x^2$ and $q_2(x, y) = y^2$. Then the term in x of the first equation and the term in y in the second equation can be eliminated via a translation. Thus, we obtain the systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + y^2 \tag{6.20}$$

for which we have Coefficient$[B_2, x^3y] = 2592d^2e^2$ and hence the condition $B_2 = 0$ yields $de = 0$. Without loss of generality we may consider $d = 0$ due to the change $x \leftrightarrow y$, $d \leftrightarrow e$ and $a \leftrightarrow b$. So, $d = 0$ and then

$$\mu_0 = 1, \quad B_2 = 648e^2(4a-4b-e^2)x^4, \quad H_4 = 96e^2, \quad H_8 = -3456ae^2.$$

6.1.3.1 Subcase $H_4 \neq 0$ Then $e \neq 0$ and the condition $B_2 = 0$ yields $b = (4a-e^2)/4$ and we arrive to the family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = (4a-e^2)/4 + ex + y^2 \tag{6.21}$$

which possess three invariant lines: $x^2 + a = 0$ and $2(x-y) - e = 0$. So, in this case we are in the family $\mathbf{QSL_4} \bigcup \mathbf{QSL_5} \bigcup \mathbf{QSL_6}$ and for these systems we have:

$$\theta = N = B_2 = H_7 = 0, \quad B_3 = -3e^2x^2(x-y)^2, \quad H_9 = -9216a^2\big(4a+e^2\big)^2. \tag{6.22}$$

As the condition $H_4 \neq 0$ (i.e. $e \neq 0$) implies $B_3 \neq 0$ then according to [78], (Theorem 4.1) for non-degenerate systems (6.21) in **LV** we obtain: *Config. 4.9* if $H_8 > 0$ and $H_9 \neq 0$; *Config. 4.10* if $H_8 > 0$ and $H_9 = 0$ and *Config. 4.22* if $H_8 = 0$. This leads to the respective conditions of the theorem.

6.1.3.2 Subcase $H_4 = 0$ Then $e = 0$ and systems (6.20) become as systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + y^2 \tag{6.23}$$

for which $B_2 = 0$, $B_3 = 12(b-a)x^2y^2$, $H_1 = -1552(a+b)$, $H_5 = 6144ab$. We observe that systems (6.23) possess two couples of parallel lines $x^2 + a = 0$ and $y^2 + b = 0$ and hence we are in the class $\mathbf{QSL_5} \bigcup \mathbf{QSL_6}$.

So, according to [79], Theorem 5.1 for non-degenerate systems (6.23) in ***LV*** we obtain the following configurations of invariant lines and the respective invariant criteria:

- *Config. 5.3* $\Leftrightarrow$ $B_3 \neq 0$, $H_1 > 0$, $H_5 > 0$;
- *Config. 5.12* $\Leftrightarrow$ $B_3 \neq 0$, $H_1 > 0$, $H_5 = 0$;
- *Config. 6.1* $\Leftrightarrow$ $B_3 = 0$, $H_1 > 0$;
- *Config. 6.5* $\Leftrightarrow$ $B_3 = 0$, $H_1 = 0$;

This leads to the respective conditions of Theorem 6.1.

B) Degenerate systems

According to Lemma 5.9 a quadratic system is degenerate if and only if the condition $\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$ holds. We shall apply this condition to the family of systems (6.1) considering two cases: $\theta \neq 0$ and $\theta = 0$.

6.1.4 Case $\theta \neq 0$

As for systems (6.1) $\mu_0 = gh(g+h-1) = 0$ and without lost of generality we may assume $h = 0$ due to the respective linear transformation, which replace the line $x = 0$ with $y = 0$ (if $g = 0$) or the line $x = 0$ with $y = x$ (if $g + h - 1 = 0$).

So $h = 0$ and then $\theta = 8g(g-1)$ and as $\theta \neq 0$ we may assume $d = e = 0$ due to a translation. Therefore we get the family of systems

$$\dot{x} = a + cx + gx^2 - xy, \quad \dot{y} = b + fy + (g-1)xy, \tag{6.24}$$

for which we have $\mu_1 = fg(g-1)x = 0$. As $\theta \neq 0$ we obtain $f = 0$ and then we calculate: $\mu_2 = g(g-1)(b - a + ag)x^2$ and again since $\theta \neq 0$ the condition $\mu_2 = 0$ gives $b = a(1-g)$. Herein for systems (6.24) we obtain

$$\mu_3 = acg(g-1)^2x^3, \quad \mu_4 = a(g-1)^2x^3(c^2y - ag^2x)$$

and as $g(g-1) \neq 0$ the condition $\mu_3 = \mu_4 = 0$ implies $a = 0$.

Thus we obtain the family of degenerate systems

$$\dot{x} = x(c + gx - y), \quad \dot{y} = (g-1)xy, \tag{6.25}$$

for which we have $H_7 = 4c(g-1)$ and $\theta = 8g(g-1) \neq 0$. Taking into consideration the existence of the invariant line $g(x-y) + c = 0$ of systems above we obtain *Config.* LV_d*.1* if $c \neq 0$ (i.e. $H_7 \neq 0$) and *Config.* LV_d*.2* if $c = 0$ (i.e. $H_7 = 0$). It remains to note that in the case $c \neq 0$ we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. This leads to the respective representatives from Table 3.

6.1.5 Case $\theta = 0$

It was shown earlier (see Subsection 6.1.2) that if for systems (6.1) the condition $\theta = (g-1)(h-1)(g+h) = 0$ holds then without lost of generality we may assume $g + h = 0$ (i.e. $g = -h$). In this case $\mu_0 = h^2 = 0$ and hence we obtain $g = h = 0$. Moreover due to a translation we may assume $d = e = 0$. Therefore we get the family of systems

$$\dot{x} = a + cx - xy, \quad \dot{y} = b + fy - xy, \tag{6.26}$$

for which $\mu_0 = \mu_1 = 0$ and $\mu_2 = cfxy$. So the condition $\mu_2 = 0$ gives $cf = 0$ and we may assume $f = 0$ due to the change $(x, y, c, f) \mapsto (y, x, f, c)$. In this case for systems (6.26) we calculate

$$\mu_3 = -(a-b)cx^2y, \quad \mu_4 = -bc^2x^3y + (a-b)^2x^2y^2.$$

Therefore the condition $\mu_3 = \mu_4 = 0$ implies $b = a$ and we get the systems

$$\dot{x} = a + cx - xy, \quad \dot{y} = a - xy,$$

for which the condition $ac = 0$ holds (as $\mu_4 = 0$). For these systems calculation yields: $H_4 = -96a$. Clearly to be in the class $\boldsymbol{LV}$-systems above have to possess an invariant line either of the form $x + \alpha = 0$ or $y + \beta = 0$. Hence the condition $a = 0$ (i.e. $H_4 = 0$) must hold. Then changing $y \to -y$ we get the systems

$$\dot{x} = x(c + y), \quad \dot{y} = xy, \tag{6.27}$$

for which $H_7 = -4c$.

If $H_7 = 0$ (i.e. $c = 0$) both the lines $x = 0$ and $y = 0$ of system above are filled up with singularities and evidently we get the configuration of invariant lines given by *Config.* LV_d*.4*.

Assume now $H_7 \neq 0$. Then $c \neq 0$ and we may consider $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. We observe that in this case system (6.27) possesses the invariant line $y = 0$ and the line $x = 0$ which is filled up with singularities. We claim that infinite line for this system is of multiplicity 2. To prove this claim, following [81] we associate to system (6.27) with $c = 1$ its projective differential equation:

$$-XYZ\,dX + XZ(Y-Z)\,dY + XY(X - Y + Z)dZ = 0.$$

Then applying the real projective transformation of $\mathbb{P}_2(\mathbb{R})$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

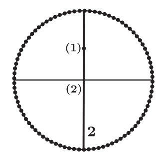
we obtain the projective differential equation

$$Y_1 Z_1 (X_1 - Y_1 + Z_1)\, dX_1 - X_1 Z_1 (X_1 - Y_1)\, dY_1 - X_1 Y_1 Z_1 dZ_1 = 0.$$

This equation is associated in the chart $Z_1 \neq 0$ ($Z_1 = 1$, $dZ_1 = 0$) to the following system

$$\dot{x} = x(x - y), \quad \dot{y} = y(1 + x - y). \tag{6.28}$$

For this system we have $C_2(a, x, y) \equiv 0$, i.e. it belongs to the class $\mathbf{QS}_{\mathbf{C_2 \equiv 0}}$ of quadratic differential systems with the line at infinity filled up with singularities. According to [81, Theorem 3.1] for system (6.28) we have $H_{10} = 36 \neq 0$, $H_9 = 0$ and $H_{12} = -8x^2 \neq 0$ and hence, the invariant line configuration corresponds to *Config.* C_2.3. As the the double invariant line $x = 0$ of system (6.28) corresponds to the line at infinity for system (6.27) we conclude that our claim is proved.



Config. C2.3

6.2 Systems from the family ($\mathbf{S}_{III}$)

Theorem 6.2. *There are 30 distinct configurations of planar **LV**-systems quadratic differential systems with $\eta = 0$. The maximum multiplicity of an invariant line is 2, indicated in Figure 2. Seven configurations have only simple invariant lines; eleven configurations have exactly one double line; four configurations have exactly two double lines; eight have each exactly one line of singularities and correspond to degenerate systems. Seventeen of the configurations have all the singularities on the configuration, each with finite multiplicity; five configurations have only one singularity of the system outside the configuration. The systems are split into four distinct classes according to the multiplicity of their invariant lines (including the line at infinity) as follows:*

*(i) The **LV**-systems with exactly three simple invariant straight lines. These have 5 configurations Config. 3.j, $j = 9, 10, \ldots, 13$, all with singularities of multiplicity at most 2. For $j = 9$ we have exactly one double singularity; for $j = 13$ we have three double singularities and all others have two double singularities. For $j \neq 10$ the systems have exactly one singularity not on the configuration and all singularities of the system are on the configuration for $j = 10$. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as its respective representative are indicated in Table 2.*

*(ii) The **LV**-systems with four invariant straight lines counted with multiplicity, which could be at most two. These have 9 configurations Config. 4.j with $j \in \{11, 12, 19, 20, 21, 23, 24, 25, 26\}$ and have each exactly one double invariant line. Three of them have a point of multiplicity 4, three others have exactly two points of multiplicity 2 and one has three points of multiplicity 2. The invariant necessary and sufficient conditions for the realization of each one of these configurations are:*

Config. 4.11 $\Leftrightarrow$ $\theta = H_7 = 0,\ B_3\mu_0 \neq 0,\ H_{10} > 0;$
Config. 4.12 $\Leftrightarrow$ $\theta = B_3 = H_7 = \mu_0 = 0,\ H_6 \neq 0,\ K \neq 0,\ H_{11} > 0;$
Config. 4.19 $\Leftrightarrow$ $\theta = B_3 = H_7 = \mu_0 = 0,\ NH_6 \neq 0,\ K = 0,\ H_{11} \neq 0;$
Config. 4.20 $\Leftrightarrow$ $\theta \neq 0,\ B_3 = H_7 = 0,\ D = 0;$
Config. 4.21 $\Leftrightarrow$ $\theta \neq 0,\ B_3 = H_7 = 0,\ D \neq 0,\ \mu_0 \neq 0;$
Config. 4.23 $\Leftrightarrow$ $\theta = H_7 = 0,\ B_3\mu_0 \neq 0,\ H_{10} = 0;$
Config. 4.24 $\Leftrightarrow$ $\theta = B_3 = H_7 = \mu_0 = 0,\ H_6 \neq 0,\ K \neq 0,\ H_{11} = 0;$
Config. 4.25 $\Leftrightarrow$ $\theta \neq 0,\ B_3 = 0,\ H_7 \neq 0;$
Config. 4.26 $\Leftrightarrow$ $\theta \neq 0,\ B_3 = H_7 = 0,\ D \neq 0,\ \mu_0 = 0.$

*(iii) The **LV**-systems with five invariant straight lines counted with multiplicity, which could be at most two. These have 6 configurations Config. 5.j with $j \in \{11, 13, 14, 17, 18, 19\}$, 4 of them with exactly one double line and 2 with exactly two double invariant lines. The invariant necessary and sufficient conditions for the realization of each one of these configurations are:*

Config. 5.11 $\Leftrightarrow$ $\theta = B_3 = 0,\ \mu_0 \neq 0,\ D \neq 0;$
Config. 5.13 $\Leftrightarrow$ $\theta = B_3 = \mu_0 = N = H = N_1 = 0,\ DN_2 \neq 0,\ H_5 > 0;$
Config. 5.14 $\Leftrightarrow$ $\theta = B_3 = \mu_0 = H_7 = H_6 = 0,\ N \neq 0,\ K \neq 0;$
Config. 5.17 $\Leftrightarrow$ $\theta = B_3 = \mu_0 = N = H = N_1 = 0,\ DN_2 \neq 0,\ H_5 = 0;$
Config. 5.18 $\Leftrightarrow$ $\theta = B_3 = \mu_0 = H_7 = H_6 = 0,\ N \neq 0,\ K = 0;$
Config. 5.19 $\Leftrightarrow$ $\theta = B_3 = 0,\ \mu_0 \neq 0,\ D = 0;$

(iv) *The **LV**-systems with six invariant straight lines counted with multiplicity, which could be at most two. These have 2 configurations Config. 6.j with $j \in \{7, 8\}$, each with exactly two double invariant lines. The invariant necessary and sufficient conditions for the realization of each one of these configurations are:*

Config. 6.7 $\Leftrightarrow$ $\theta = B_3 = \mu_0 = N = H = N_1 = N_2 = 0,\ D \neq 0;$
Config. 6.8 $\Leftrightarrow$ $\theta = B_3 = \mu_0 = N = H_6 = 0,\ H \neq 0,\ H_{11} > 0;$

(v) *The degenerate **LV**-systems defined by the conditions $\mu_i = 0$, $i = 0, 1, \ldots, 4$ and possessing at least one affine line filled with singularities. These have 8 configurations Config. $LV_d.j$ with $j \in \{5, 6, \ldots, 12\}$, two of them with two lines of singularities. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as its respective representative are indicated in Table 3.*

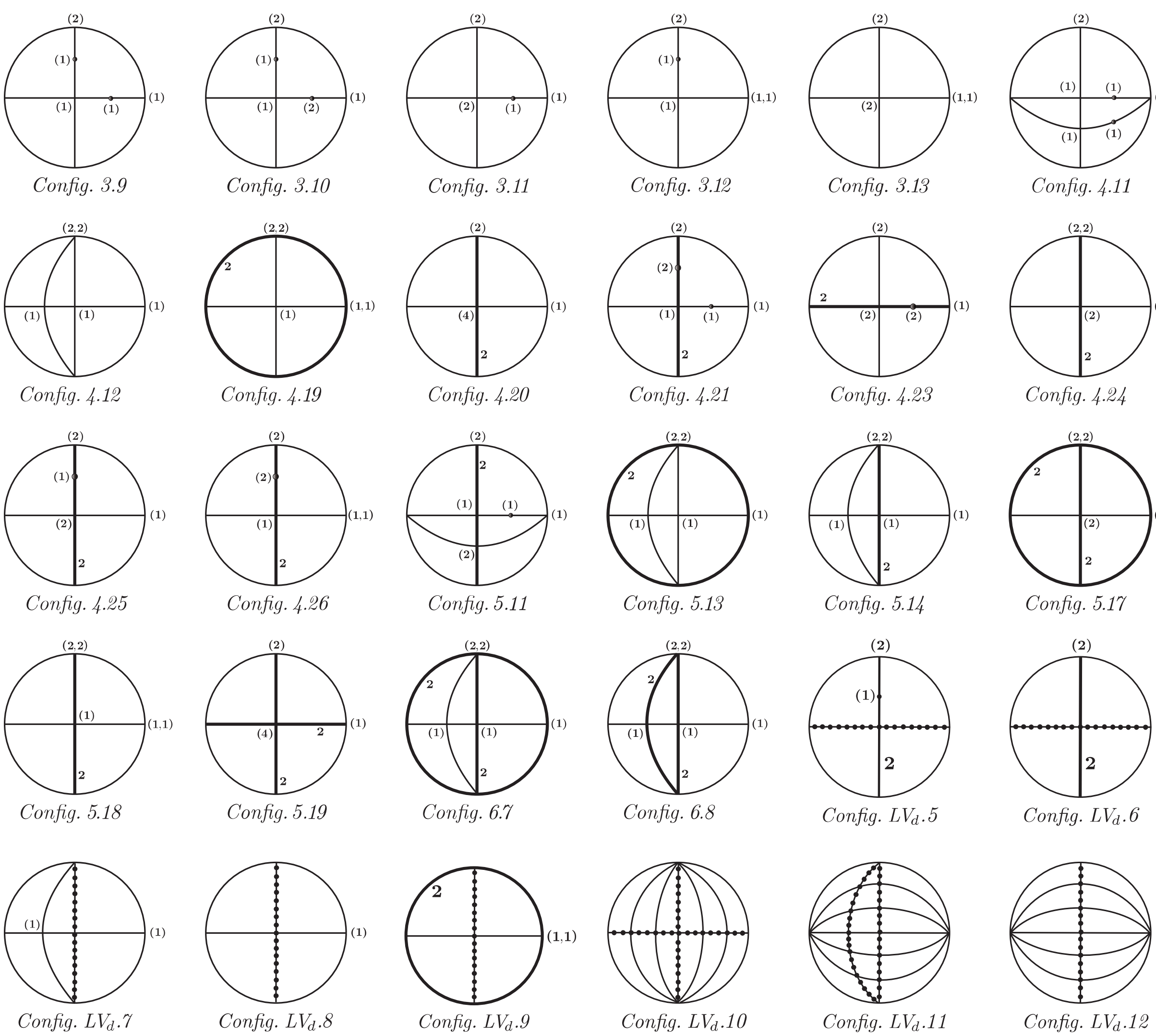


Figure 2: **The case $\boldsymbol{\eta = 0, C_2 \neq 0}$**

Proof: In the case of systems $(\mathbf{S}_{III})$ the conditions $\eta = 0$, $M \neq 0$ hold and we shall examine respective canonical form

$$\begin{aligned}\dot{x} &= a + cx + dy + gx^2 + hxy,\\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2,\end{aligned} \tag{6.29}$$

for which

$$\theta = -8(g-1)h^2,\ \mu_0 = gh^2,\ H = -(g-1)^2x^2 - 2(g+1)hxy - h^2y^2 = 0.$$

First of all we observe (since $C_2(x, y) = x^2y$) that systems (6.29) have two real infinite points $N_1(0, 1, 0)$ and $N_2(1, 0, 0)$. This means, that these systems could have invariant lines only of the type either $x+\alpha = 0$ or $y + \beta = 0$. We need to construct the conditions when both such possibilities occur simultaneously.

A) Non-degenerate systems

We shall examine step by step all the possibilities given by polynomials $\theta(a)$, $\mu_0(a)$ and $H(a, x, y)$.

6.2.1 Subcase $\theta \neq 0$

The condition $\theta \neq 0$ yields $h(g-1) \neq 0$ and we may consider $d = e = 0$ due to a translation. Moreover, since $h \neq 0$ we may assume $h = 1$ due to the rescaling $y \to y/h$. Thus we obtain the family of systems

$$\dot{x} = a + cx + gx^2 + xy, \quad \dot{y} = b + fy + (g-1)xy + y^2, \tag{6.30}$$

for which we calculate Coefficient$[B_2(\boldsymbol{a}, x, y), y^4] = -648a^2$. Hence the necessary condition $B_2 = 0$ yields $a = 0$ and then

$$\begin{gathered} B_2 = -648b(b + c^2 - cf)(g-1)^2x^4, \quad H_4 = 48(b + c^2 - cf), \quad \theta = 8(1-g)\\ B_3 = -3\big[b(g-1)^2x^2 - (b + c^2 - cf)y^2\big]x^2 \end{gathered} \tag{6.31}$$

We shall consider two possibilities: $H_4 \neq 0$ and $H_4 = 0$.

6.2.1.1 Possibility $H_4 \neq 0$. In this case the condition $B_2 = 0$ yields $b = 0$ and we arrive to the family of systems

$$\dot{x} = x(c + gx + y), \quad \dot{y} = y[f + (g-1)x + y], \tag{6.32}$$

possessing the invariant lines $x = 0$ and $y = 0$ and the following singularities:

$$M_1(0, 0), \quad M_2(-c/g, 0), \quad M_3(0, -f); \quad M_4\big(f - c,\ cg - c - fg\big). \tag{6.33}$$

For these systems calculations yield:

$$\begin{gathered} B_3 = 3c(c-f)x^2y^2, \quad \mu_0 = g, \quad H_4 = 48c(c-f), \quad H_7 = 4(f-c)(g-1),\\ H_9 = -576c^2f^2(c-f)^2(cg - c - fg)^2, \quad H_{13} = c^2f(1-g)/2. \end{gathered} \tag{6.34}$$

We observe that the condition $\theta H_4 \neq 0$ implies $B_3H_7 \neq 0$ and considering Remark 6.4 we conclude that besides the invariant lines $x = 0$ and $y = 0$ systems (6.12) could not possess other affine invariant lines. Moreover, these lines are simple as well as the infinite line is.

Thus, to construct the configurations of invariant lines for these systems it remains to examine their singularities (6.33) and we shall consider two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

6.2.1.1.1 *Assume first $\mu_0 \neq 0$.* Then all singularities (6.33) remain in the finite part of the phase plane and only could coincide. So, we shall distinguish the cases $H_9 \neq 0$ and $H_9 = 0$.

1) If $H_9 \neq 0$ then all singular points (6.33) are distinct and this leads to *Config. 3.9*. On the other hand as $c \neq 0$ we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. So, we get the family of systems (*III.9*) (see Table 2) for which the condition $\theta H_4 H_9 \mu_0 \neq 0$ implies the condition $(\mathcal{A}_4)$.

2) Suppose now $H_9 = 0$. Since $H_4 \neq 0$ from (6.34) we have either $f = 0$ or $cg - c - fg = 0$ and these two possibilities are governed by invariant polynomial H_{13}.

a) If $H_{13} \neq 0$ then $f \neq 0$ and we get $cg - c - fg = 0$. As $g \neq 0$ ($\mu_0 \neq 0$) setting $c = gu$ we obtain $f = u(g-1)$). We observe that the point M_4 coincides with M_2 (see (6.33)) and this leads to *Config. 3.10*. To determine the respective representative we make $u = 1$ via the rescaling $(x, y, t) \mapsto (ux, uy, t/u)$ and we obtain the family of systems (*III.10*) from Table 2, for which the condition $\theta H_4 B_3 H_{13} \neq 0$ implies $g(g-1) \neq 0$.

b) Suppose now $H_{13} = 0$. Then $f = 0$ and the point M_3 coincides with $M_1(0, 0)$. Hence in this case we arrive to *Config. 3.11*. We again assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$ and we obtain the family of systems (*III.11*) (see Table 2) for which the condition $\theta H_4 \mu_0 \neq 0$ implies $g(g-1) \neq 0$.

6.2.1.1.2 *Assume now* $\mu_0 = 0$. Then $g = 0$ and as due to $B_3 \neq 0$ it holds $c \neq 0$, then we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. So, we get the family of systems

$$\dot{x} = x(1+y), \quad \dot{y} = y(f - x + y), \tag{6.35}$$

possessing the finite singularities: $M_1(0, 0)$, $M_3(0, -f)$ and $M_4(f-1, -1)$. On the other hand for these systems we have $H_9 = -576f^2(f-1)^2$.

1) If $H_9 \neq 0$ (i.e. $f \neq 0$) then the three singular points are distinct and this leads to *Config. 3.12*. The respective representative is given by the family of systems (*III.12*) (see Table 2) for which the condition $\theta H_4 B_3 H_9 \neq 0$ implies $f(f-1) \neq 0$.

2) Assuming $H_9 = 0$ due to $B_3 \neq 0$ we obtain $f = 0$. So, obviously we get *Config. 3.13* with its representative given by a single system (*III.13*) from Table 2.

6.2.1.2 Possibility $H_4 = 0$.

In this case $b = c(f - c)$ and this leads to the family of systems

$$\dot{x} = x(c + gx + y), \quad \dot{y} = c(f-c) + fy + (g-1)xy + y^2, \tag{6.36}$$

possessing the invariant line $x = 0$ which is double because $\mathcal{H} = x^2$ (see Notation 5.9). So, these systems possess invariant lines of total multiplicity at least 3. Thus, a system (6.36) will be a Lotka-Volterra system only if it possesses an invariant line in the second direction (i.e. the line of the form $y + \alpha = 0$). However in this case we are in the family $\mathbf{QSL_4} \bigcup \mathbf{QSL_5} \bigcup \mathbf{QSL_6}$. Moreover, since $\theta \neq 0$ by Lemma 6.1 a system (6.36) could belong only to the family $\mathbf{QSL_4}$ and in this case the condition $B_3 = 0$ has to be fulfilled. For these systems we have:

$$\begin{gathered} B_3 = 3c(c-f)(g-1)^2x^4, \quad \theta = 8(1-g), \\ H_7 = 4(c-f)(1-g), \quad \mu_0 = g, \quad D(a, x, y)|_{c=f} = -f^2x^2y. \end{gathered} \tag{6.37}$$

So, according to [78, Theorem 4.1] for non-degenerate systems (6.36) in the class ***LV*** we obtain the following configurations of invariant lines and the respective invariant criteria:

- *Config. 4.20* $\Leftrightarrow$ $B_3 = 0$, $H_7 = 0$, $D = 0$;
- *Config. 4.21* $\Leftrightarrow$ $B_3 = 0$, $H_7 = 0$, $D \neq 0$, $\mu_0 \neq 0$;
- *Config. 4.25* $\Leftrightarrow$ $B_3 = 0$, $H_7 \neq 0$;
- *Config. 4.26* $\Leftrightarrow$ $B_3 = 0$, $H_7 = 0$, $D \neq 0$, $\mu_0 = 0$.

It remains to observe considering (6.31), that the condition $B_3 = 0$ implies $H_4 = 0$.

6.2.2 Subcase $\theta = 0$, $\mu_0 \neq 0$

The conditions $\theta = -8(g-1)h^2$ and $\mu_0 = gh^2 \neq 0$ yield $g = 1$ and $h \neq 0$. Hence we may consider $h = 1$ due to the rescaling $y \to y/h$ and we may assume $d = f = 0$ due to a translation. Thus we obtain the family of systems

$$\dot{x} = a + cx + x^2 + xy, \quad \dot{y} = b + ex + y^2, \tag{6.38}$$

for which we have Coefficient$[B_2, y^4] = -648a^2$ and therefore the condition $B_2 = 0$ implies $a = 0$. Then calculations yield:

$$B_2 = -648e^2(b + c^2)x^4, \quad H_7 = -4e.$$

Since $a = 0$ systems (6.38) posses invariant line $x = 0$. It is obvious to observe, that these systems could possess invariant line of the type $y + \alpha = 0$ if and only if $e = 0$. Hence, a system (6.38) belongs to the class $\boldsymbol{LV}$ if and only if the condition $H_7 = 0$ holds.

Thus, for $B_2 = 0$ and $H_7 = 0$ we obtain the family of systems

$$\dot{x} = x(c + x + y), \quad \dot{y} = b + y^2, \tag{6.39}$$

which possess invariant lines $x = 0$ and $y^2 + b = 0$ and hence, belong to the class $\mathbf{QSL_4} \bigcup \mathbf{QSL_5} \bigcup \mathbf{QSL_6}$. For these systems we have:

$$\begin{gathered} B_3 = 3(b + c^2)x^2y^2, \quad \mu_0 = 1, \quad N = y^2, \quad H_7 = 0, \\ H_{10} = -32b, \quad D = xy\big[4bx + (b + c^2)y\big], \end{gathered} \tag{6.40}$$

and since $N \neq 0$ according to Lemma 6.1 a system (6.39) could not belong to $\mathbf{QSL_6}$.

Considering the conditions $B_2 = \theta = 0$ and $\mu_0 \neq 0$, according to [78, Theorem 4.1] and [79, Theorem 5.1] for non-degenerate systems (6.39) in $\boldsymbol{LV}$ we obtain the following configurations of invariant lines and the respective invariant criteria:

- *Config. 4.11* $\Leftrightarrow$ $B_3 \neq 0$, $H_7 = 0$, $H_{10} > 0$;
- *Config. 4.23* $\Leftrightarrow$ $B_3 \neq 0$, $H_7 = 0$, $H_{10} = 0$;
- *Config. 5.11* $\Leftrightarrow$ $B_3 = 0$, $H_7 = 0$, $D \neq 0$;
- *Config. 5.19* $\Leftrightarrow$ $B_3 = 0$, $H_7 = 0$, $D = 0$.

We observe, that for systems (6.38) the condition $B_3 = 0$ yields $a = e = b + c^2 = 0$ and then $H_7 = 0$. So, the last condition can be removed when $B_3 = 0$ and we arrive to the conditions of the theorem.

6.2.3 Subcase $\theta = \mu_0 = 0$, $H \neq 0$

For the systems (6.29) the conditions $\theta = -8(g-1)h^2 = 0$ and $\mu_0 = gh^2 = 0$ imply $h = 0$ and then $H = -(g-1)^2x^2$. Therefore the condition $H \neq 0$ implies $g \neq 1$ and we may assume $e = f = 0$ via a translation. Thus, we get the family of systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + (g-1)xy, \tag{6.41}$$

for which we calculate

$$\begin{gathered} B_2 = 648d\Big[bc(g-1)^3x^4 + 4bd(g-1)^2x^3y - d^3g^2y^4\Big], \\ B_3 = -3\Big[b(g-1)^2x^4 + cd^2(g-1)x^2y^2 + 2d^2gxy^3\Big], \quad H_7 = 4d(g^2-1). \end{gathered} \tag{6.42}$$

It obviously can be observed, that systems (6.41) could possess invariant lines of the type $x + \alpha = 0$ (respectively $y + \beta = 0$) if and only if $d = 0$ (respectively $b = 0$). And in order to be in the class $\boldsymbol{LV}$ the conditions $b = d = 0$ must be verified.

We claim that the last conditions are equivalent to $B_3 = H_7 = 0$ (in this case $B_2 = 0$). Indeed, if $b = d = 0$ from (6.42) we get $B_3 = H_7 = 0$. Conversely, suppose that the conditions $B_3 = H_7 = 0$ hold. Since $H \neq 0$ (i.e. $g-1 \neq 0$) from $B_3 = 0$ we obtain $b = dg = 0$, and with additional condition $d(g^2-1) = 0$ (i.e. $H_7 = 0$) we get $b = d = 0$.

So, assuming $B_3 = H_7 = 0$ we obtain the systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = (g-1)xy, \tag{6.43}$$

possessing invariant lines $y=0$ and $a+cx+gx^2=0$. Therefore these systems belong to the class $\mathbf{QSL_4}\bigcup\mathbf{QSL_5}\bigcup\mathbf{QSL_6}$. For these systems we have:

$$\begin{gathered}K=2g(g-1)x^2,\quad N=(g^2-1)x^2,\quad H=-(g-1)^2x^2,\quad H_2=4c(g-1)^2x^2,\\ H_3=8a(g-1)^2x^2,\ \ H_6=-128(g-1)^4\big[c^2-a(g+1)^2\big]x^6,\ \ H_{11}=48(c^2-4ag)(g-1)^4x^4.\end{gathered}\tag{6.44}$$

Clearly, in order to have two real intersecting invariant lines it is necessary and sufficient either $H_{11}>0$ or $H_{11}=0$ and $K\neq 0$.

Considering the conditions $\theta=\mu_0=B_3=H_7=0$ and $H\neq 0$, according to [78, Theorem 4.1] and [79, Theorem 5.1] for non-degenerate systems (6.43) in ***LV*** we obtain the following configurations of invariant lines and the respective invariant criteria:

- *Config. 4.12* $\Leftrightarrow$ $H_6\neq 0,\ K\neq 0,\ H_{11}>0$;
- *Config. 4.19* $\Leftrightarrow$ $H_6\neq 0,\ K=0,\ H_{11}\neq 0$;
- *Config. 4.24* $\Leftrightarrow$ $H_6\neq 0,\ K\neq 0,\ H_{11}=0$;
- *Config. 5.14* $\Leftrightarrow$ $H_6=0,\ N\neq 0,\ K\neq 0$;
- *Config. 5.18* $\Leftrightarrow$ $H_6=0,\ N\neq 0,\ K=0$;
- *Config. 6.8* $\Leftrightarrow$ $N=0,\ H_2=0,\ H\neq 0,\ H_3>0$;

Observation 6.9. *Considering* (6.44) *we obtain the following relations:*

1. *The condition $N\neq 0$ implies $H\neq 0$;*
2. *If $N=0$ and $H\neq 0$ then $g=-1$ and the conditions $H_2=0$ is equivalent to $H_6=0$. Moreover, if $H_2=0$ (i.e. $c=0$) then* $\operatorname{sign}(H_3)=\operatorname{sign}(H_{11})=\operatorname{sign}(a)$.

So, taking into account these observations we arrive to the conditions from the statement of the theorem.

6.2.4 Subcase $\theta=\mu_0=H=0$

For the systems (6.29) the condition $H=-(g-1)^2x^2-2(g+1)hxy-h^2y^2=0$ in $\mathbb{R}[x,y]$ yields $h=0$ and $g=1$. Then $\theta=\mu_0=0$ and the condition $B_2=-648d^4y^4=0$ implies $d=0$. Moreover, we can assume $c=0$ due to a translation and hence we get the systems

$$\dot{x}=a+x^2,\quad \dot{y}=b+ex+fy,\tag{6.45}$$

possessing two parallel invariant lines $x^2+a=0$. Clearly, to have an invariant line of the form $y+\alpha=0$ it is necessary $e=0$ which is equivalent to $N_1=8ex^4=0$. Then for systems (6.45) we calculate

$$B_3=N=H=N_1=0,\quad D=-f^2x^2y,\quad N_2=4(4a+f^2)x,\quad N_5=-64ax^2,$$

and according to [79, Theorem 5.1] for non-degenerate systems (6.45) in ***LV*** we obtain the following configurations of invariant lines and the respective invariant criteria:

- *Config. 5.13* $\Leftrightarrow$ $D\neq 0,\ N_2\neq 0,\ N_5>0$;
- *Config. 5.17* $\Leftrightarrow$ $D\neq 0,\ N_2\neq 0,\ N_5=0$;
- *Config. 6.7* $\Leftrightarrow$ $D\neq 0,\ N_2=0$.

Thus, taking into consideration the conditions $B_3=\mu_0=N=H=N_1=0$ we arrive to the conditions given by the theorem.

B) Degenerate systems

According to Lemma 5.9 a quadratic system is degenerate if and only if the condition $\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=0$ holds. We shall apply this condition to the family of systems (6.29) considering two cases: $\theta\neq 0$ and $\theta=0$.

6.2.5 Case $\theta \neq 0$

As for systems (6.1) $\theta = 8h^2(1-g)$ and $\mu_0 = gh^2$, the conditions $\mu_0 = 0$ and $\theta \neq 0$ yield $g = 0$ and $h \neq 0$. Then without loss of generality we may assume $d = e = 0$ (due to a translation) and $h = 1$ (due to the rescaling $y \to y/h$) and we get the family of systems

$$\dot{x} = a + cx + xy, \quad \dot{y} = b + fy - xy + y^2. \tag{6.46}$$

For these systems we have $\mu_1 = -cy = 0$ and hence, $c = 0$. Then we calculate: $\mu_2 = (a+b)y^2$ and the condition $\mu_2 = 0$ gives $b = -a$. Herein for systems (6.46) we obtain

$$\mu_3 = afy^3, \quad \mu_4 = ay^3(f^2x + ay)$$

and therefore the condition $\mu_3 = \mu_4 = 0$ implies $a = 0$.

Thus we obtain the family of degenerate systems

$$\dot{x} = xy, \quad \dot{y} = y(f - x + y), \tag{6.47}$$

for which the line $y = 0$ is filled up with singularities and the invariant line is double one. Indeed, following [81] we associate to systems (6.47) the projective differential equation:

$$YZ(-X + Y + fZ) - XYZ\,dY + XY(X - fZ)dZ = 0.$$

Then applying the real projective transformation of $\mathbb{P}_2(\mathbb{R})$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

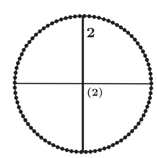
we obtain the projective differential equation

$$Y_1Z_1(fY_1 - X_1 + Z_1)\,dX_1 - X_1Z_1(X_1 - Y_1)\,dY - X_1Y_1Z_1dZ_1 = 0.$$

This equation is associated in the chart $Z_1 \neq 0$ ($Z_1 = 1$, $dZ_1 = 0$) to the following system

$$\dot{x} = x(-x + fy), \quad \dot{y} = y(1 - x + fy). \tag{6.48}$$

For this system we have $C_2(a, x, y) \equiv 0$, i.e. it belongs to the class $\mathbf{QS}_{\mathbf{C_2 \equiv 0}}$ of quadratic differential systems with the line at infinity filled up with singularities. According to [81] for systems (6.48) we calculate: $H_{10} = 36f^2$, $H_9 = 0$ and $H_{12} = -8x^2 \neq 0$. Therefore by [81, Theorem 3.1] the invariant line configuration corresponds to *Config.* $C_2.3$ (see page 33) if $f \neq 0$. If $f = 0$ then $H_{11} = 0$ and then we get *Config.* $C_2.7$.



Config. C2.7

As the the double invariant line $x = 0$ of systems (6.48) corresponds to the invariant line $x = 0$ of systems (6.47) we obtain *Config.* $LV_d.5$ if $f \neq 0$ and *Config.* $LV_d.6$ if $f = 0$. It remains to note that for systems (6.47) we have $H_7 = -4f$ and in the case $f \neq 0$ we may assume $f = 1$ due to the rescaling $(x, y, t) \mapsto (fx, fy, t/f)$. This leads to the respective representatives from Table 3.

6.2.6 Subcase $\theta = 0$

For systems (6.29) the conditions $\theta = -8(g-1)h^2 = 0$ and $\mu_0 = gh^2 = 0$ imply $h = 0$ and then $K = 2g(g-1)x^2$. We shall consider the possibilities $K \neq 0$ and $K = 0$.

6.2.6.1 Possibility $K \neq 0$ Then $g(g-1) \neq 0$ and we may consider $e = f = 0$ due to a translation. Thus we get the family of systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + (g-1)xy, \tag{6.49}$$

for which the condition $\mu_1 = dg(g-1)^2x = 0$ by $K \neq 0$ implies $d = 0$. Then $\mu_2 = ag(g-1)^2x^2$ and again due to $g(g-1) \neq 0$ we get $a = 0$. In this case calculations yield:

$$\mu_3 = bc(1-g)gx^3, \quad \mu_4 = bx^3(bg^2x - c^2y + c^2gy).$$

So the conditions $\mu_3 = \mu_4 = 0$ implies $b = 0$ and we get the family of systems

$$\dot{x} = x(c+gx), \quad \dot{y} = (g-1)xy, \tag{6.50}$$

for which we have $H_2 = 4c(g-1)^2x^2$ ($H_{11} = 48c^2(-1+g)^4x^4$) and $K = 2g(g-1)x^2 \neq 0$. Taking into consideration the existence of the invariant lines $y = 0$ and $gx + c = 0$ we obtain *Config.* LV_d*.7* if $c \neq 0$ (i.e. $H_2 \neq 0$) and *Config.* LV_d*.8* if $c = 0$ (i.e. $H_2 = 0$). It remains to note that in the case $c \neq 0$ we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. This leads to the respective representatives from Table 3.

6.2.6.2 Possibility $K = 0$.

In this case $g(g-1) = 0$ and as for systems (6.29) with $h = 0$ we have $N = (g^2-1)x^2$, hence the condition $N = 0$ is equivalent to $g = 1$.

6.2.6.2.1 *Assume first* $N \neq 0$. Then $g = 0$ and due to a translation we may consider $e = f = 0$. So we obtain the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b - xy, \tag{6.51}$$

for which we have

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = -cdxy, \quad H_7 = -4d$$

and the condition $\mu_2 = 0$ gives $cd = 0$.

a) If $H_7 \neq 0$ then $d \neq 0$ (we assume $d = 1$ due to a rescaling) and this implies $c = 0$. Then we calculate $\mu_3 = axy^2$ and $\mu_4 = xy^2(a^2x - by)$ and therefore the condition $\mu_3 = \mu_4 = 0$ gives $a = b = 0$. So we get the degenerate system $\dot{x} = y, \ \ \dot{y} = -xy$ which evidently is not in the class $\boldsymbol{LV}$.

b) Suppose now $H_7 = 0$, i.e. $d = 0$. Then we have

$$\mu_0 = \mu_1 = \mu_2 = 0, \ \mu_3 = -acx^2y, \ \mu_4 = x^2y(-bc^2x + a^2y), \ H_2 = 4cx^2$$

and the condition $\mu_3 = \mu_4 = 0$ yields $a = cb = 0$.

b_1) If $H_2 \neq 0$ then $c \neq 0$ and we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. Then for systems (6.51) we obtain $a = b = 0$ and after changing $y \to -y$ we obtain the system

$$\dot{x} = x, \quad \dot{y} = xy, \tag{6.52}$$

having the line $x = 0$ filled up with singularities. We claim that infinite line for this system is of multiplicity 2. Indeed following [81] we associate to this system its projective differential equation:

$$-XYZ\,dX - xZ^2\,dY + XY(X+Z)dZ = 0.$$

Then applying the real projective transformation of $\mathbb{P}_2(\mathbb{R})$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

we obtain the projective differential equation

$$Y_1Z_1(X_1+Z_1)\,dX_1 - X_1^2Z_1\,dY_1 - X_1Y_1Z_1\,dZ_1 = 0.$$

This equation is associated in the chart $Z_1 \neq 0$ ($Z_1 = 1$, $dZ_1 = 0$) to the following system

$$\dot{x} = x^2, \quad \dot{y} = y(1+x) \tag{6.53}$$

For this system we have $C_2(a, x, y) \equiv 0$, i.e. it belongs to the class $\mathbf{QS}_{\mathbf{C_2 \equiv 0}}$ of quadratic differential systems with the line at infinity filled up with singularities. According to [81, Theorem 3.1] for system (6.53) we calculate $H_{10} = H_{11} = 0$ and $H_{12} = -8x^2 \neq 0$ and hence, the invariant line configuration corresponds to *Config.* $C_2.7$ (see page 39). As the double invariant line $x = 0$ of system (6.53) corresponds to the line at infinity for system (6.52) we conclude that our claim is proved. Moreover, taking into consideration *Config.* $C_2.7$ we obtain for system (6.52) *Config.* $LV_d.9$.

b_2) In the case $H_2 = 0$ we get $c = 0$ and this leads to the family of degenerate systems

$$\dot{x} = 0, \quad \dot{y} = b - xy, \tag{6.54}$$

for which we may assume $b \in \{0, 1\}$ due to the rescaling $y \to by$ if $b \neq 0$. As for these systems we have $D = bx^3$ hence we conclude that if $D \neq 0$ (i.e. $b = 1$) system (6.54) does not belong to the class ***LV***.

In the case $D = 0$ (i.e. $b = 0$ we get the configuration given by *Config.* $LV_d.10$ and vis the change $x \to -x$ we obtain the orbit representative indicated in Table 3.

6.2.6.2.2 *Assume now* $N = 0$. This implies $g = 1$ and due to a translation we may assume $c = 0$. So considering the condition $h = 0$ systems (6.29) become as systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + fy. \tag{6.55}$$

For these systems we have $\mu_0 = \mu_1 = 0$ and $\mu_2 = f^2x^2$. So the condition $\mu_2 = 0$ gives $f = 0$ and then calculation yields:

$$\mu_0 = \mu_1 = \mu_2 = 0, \ \mu_3 = de^2x^3, \ \mu_4 = (b^2 + ae^2)x^4 - bdex^3y, \ D = -d^2y^3.$$

a) If $D \neq 0$ then $d \neq 0$ (we assume $d = 1$ due to a rescaling) and the condition $\mu_3 = \mu_4 = 0$ gives $b = e = 0$. So applying an additional translation (to exclude parameter a) we get the system

$$\dot{x} = y + x^2, \quad \dot{y} = 0$$

which clearly does not belong to the class ***LV***.

b) Assume $D = 0$, i.e. $d = 0$. So we obtain the family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + ex, \tag{6.56}$$

for which we have

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \ \mu_4 = (b^2 + ae^2)x^4, \ N_1 = 8ex^4, \ N_5 = -64ax^2.$$

As the condition $\mu_4 = 0$ implies $b^2 + ae^2 = 0$, we shall consider two subcases: $N_1 \neq 0$ and $N_1 = 0$.

b_1) If $N_1 \neq 0$ then we may assume $e = 1$ due to a rescaling and we obtain $a = -b^2$. This leads to the family of degenerate systems

$$\dot{x} = (x - b)(x + b), \quad \dot{y} = b + x,$$

for which we may assume $b \in \{0, 1\}$ due to the rescaling $(x, y, t) \mapsto (bx, y, t/b)$ if $b \neq 0$. It can be easy to convince that both systems do not not belong to the class ***LV***.

b_1) Admit now $N_1 = 0$, i.e. $e = 0$. Then the condition $\mu_4 = 0$ gives $b = 0$ and we obtain the degenerate systems

$$\dot{x} = a + x^2, \quad \dot{y} = 0.$$

Clearly these systems belongs to the class ***LV*** if and only if $a \leq 0$ that is equivalent to $N_5 \geq 0$. Moreover we get the configuration given by *Config.* $LV_d.11$ if $N_5 > 0$ and *Config.* $LV_d.12$ if $N_5 = 0$. It remains to observe that in the first case we may assume $a = -1$ due to the rescaling $(x, y, t) \mapsto (\sqrt{-b}x, y, t/\sqrt{-b})$. So we obtain the respective orbit representatives, indicated in Table 3.

As all the cases are examined Theorem 6.2 is proved. ∎

6.3 Systems with a line of singularities at infinity(case $C_2 = 0$)

Theorem 6.3. *The planar **LV** quadratic differential systems with $C_2 = 0$ have six distinct configurations of invariant straight lines, indicated in Figure 3. Four of these are non-degenerate (Config C2.j with $j \in \{1,3,5,7\}$) and two are degenerate (Config. $LV_d.j$ with $j \in \{13,14\}$. These two distinct classes are split according to the degree of degeneracy as follows:*

*(i) The 4 non-degenerate **LV**-systems $C_2.j$ with $j \in \{1,3,5,7\}$ have either three distinct invariant affine straight lines ($C_2.j$ with $j \in \{1,5\}$), or exactly two affine invariant lines, only one of them double ($C_2.j$ with $j \in \{3,7\}$). The invariant necessary and sufficient conditions for the realization of each one of these configurations are:*

Config. $C_2.1$ $\Leftrightarrow$ $H_{10} \neq 0,\ H_9 < 0$;
Config. $C_2.3$ $\Leftrightarrow$ $H_{10} \neq 0,\ H_9 = 0,\ H_{12} \neq 0$;
Config. $C_2.5$ $\Leftrightarrow$ $H_{10} = 0,\ H_{12} \neq 0,\ H_{11} > 0$;
Config. $C_2.7$ $\Leftrightarrow$ $H_{10} = 0,\ H_{12} \neq 0,\ H_{11} = 0$;

*(ii) The degenerate **LV** systems defined by the conditions $\mu_i = 0$, $i = 1, \ldots, 4$ and possessing at least one affine line filled with singularities. These have 2 configurations Config. $LV_d.j$ with $j \in \{13,14\}$, which have either two distinct affine invariant lines for $j = 14$ or three such lines for $j = 13$. The invariant necessary and sufficient conditions for the realization of each one of these configurations as well as its respective representative are indicated in Table 3.*

Proof: In the case of non-degenerate quadratic systems the proof follows immediately from [81], Theorem 3.1. It remains only to note that for a degenerate system the conditions $\mu_i = 0$ ($i = 0, 1, \ldots, 4$) (see Lemma 5.9) implies $H_9 = H_{10} = H_{12} = 0$.

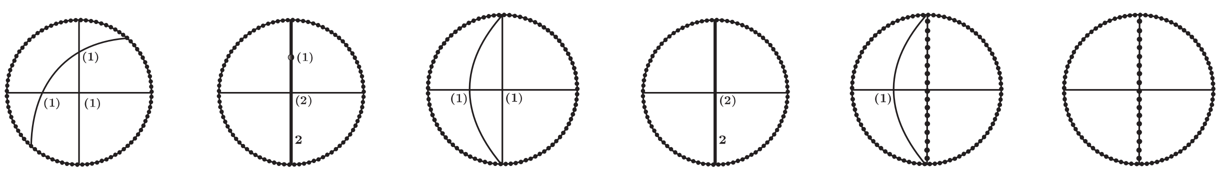
Assume now that for a system with $C_2 = 0$ the conditions $\mu_i = 0$ ($i = 0, 1, \ldots, 4$) are verified, i.e. by Lemma 5.9 this system is degenerate. According to [81, see page 10] a system with $C_2 = 0$ via an affine transformation will be brought to the form:

$$\dot{x} = a + cx + dy + x^2, \quad \dot{y} = b + xy. \tag{6.57}$$

For these systems we have $\mu_0 = 0$ and $\mu_1 = dx$ and hence, the condition $\mu_1 = 0$ gives $d = 0$. Then we calculate $\mu_2 = ax^2 = 0$ that implies $a = 0$ and considering these relations for systems (6.57) calculation yields: $\mu_3 = -bcx^3$ and $\mu_4 = bx^3(bx + c^2y)$. Therefore the conditions $\mu_3 = \mu_4 = 0$ implies $b = 0$ and we get the systems

$$\dot{x} = x(c + x), \quad \dot{y} = xy. \tag{6.58}$$

It is not too hard to detect that the invariant line configuration correspond to *Config. LV_d.10* if $c \neq 0$ and to *Config. LV_d.11* if $c = 0$. It remains to note that for systems (6.58) we have $H_2 = 4cx^2$ and the condition $c = 0$ is given by $H_2 = 0$. We observe also that in the case $c \neq 0$ we may assume $c = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$. This leads to the respective representatives from Table 3. ∎



Config. C2.1 *Config. C2.3* *Config. C2.5* *Config. C2.7* *Config. LV_d.13* *Config. LV_d.14*

Figure 3: **The case $C_2 = 0$**

7 The global bifurcation diagram in an algebraic variety of the space $\mathbb{R}^{12}$, of configurations of invariant lines of *LV*-systems

The main geometric feature of the ***LV***-systems is that they possess two invariant affine lines in distinct directions and a necessary condition for this to occur is that they satisfy the condition $B_2 = 0$. This requirement determines an algebraic variety $\mathbb{R}^{12}_{B_2} \subset \mathbb{R}^{12}$ in the parameter space $\mathbb{R}^{12}$. In this Section we construct the bifurcation diagram of $\boldsymbol{LV}$-systems within the variety $\mathbb{R}^{12}_{B_2}$. The splitting of $\boldsymbol{LV}$-systems into subclasses will be done in terms of affine invariant polynomials.

We begin by recalling two facts: i) Apart from the condition $B_2 = 0$, for a quadratic differential system to be an $\boldsymbol{LV}$-system it is necessary to have either $\eta > 0$ or $\eta = 0$ and in this second case $M \neq 0$ or $C_2 = 0$ (see Corollary 6.1 and Remark 6.2). ii) Apart from $B_2 = 0$, by Lemma 5.6 a necessary condition for a quadratic system (5.2) to have one couple (respectively, two couples) of parallel invariant straight affine lines is $\theta = 0$ (respectively $N = 0$).

Theorem 7.1. *Consider the family of $\boldsymbol{LV}$-systems. This subclass of $\mathbb{R}^{12}_{B_2}$ splits into the following three disjoint classes defined according to the values of the polynomial invariants θ and N which govern when the systems possess or not affine parallel invariant lines and how many such couples. For each one of these subclasses we give the bifurcation diagram of their configurations.*

(i) The set of $\boldsymbol{LV}$-systems satisfying the condition $\theta \neq 0$. The systems in this subclass do not have of a couple of parallel affine invariant lines and in Diagram 1 we give the bifurcation diagram of their configurations.

(ii) The set of $\boldsymbol{LV}$-systems satisfying the condition $\theta = 0$ and $N \neq 0$. The system in this subclass possess at most one couple of parallel lines. We give in Diagram 2 the bifurcation diagram of invariant line configurations for this subfamily of systems within the algebraic variety $\mathbb{R}^{12}_{B_2} \subset \mathbb{R}^{12}$.

(iii) The set of $\boldsymbol{LV}$-systems satisfying the condition $\theta = 0 = N$. The systems in this class could possess two couples of invariant affine straight lines in two distinct direction. We give in Diagram 3 the bifurcation diagram of configurations of invariant lines for this subfamily within the algebraic variety $\mathbb{R}^{12}_{B_2} \subset \mathbb{R}^{12}$.

In these diagrams the notations 4.i, 5.j, 6.s and C_2.k correspond to Config. 4.i, Config. 5.j, Config. 6.s and Config. C_2.k from the articles [78], [76] and [81].

Proof: We shall examine step by step the conditions given by each of the Diagrams 1, 2 and 3, using in accordance with Lemma 5.7 the families of systems $(\mathbf{S}_I)$ (respectively $(\mathbf{S}_{III})$; $(\mathbf{S}_V)$) defined by the conditions $\eta > 0$ (respectively $\eta = 0$ and $M \neq 0$; $C_2 = 0$).

7.1 $\boldsymbol{LV}$-systems without a couple of parallel invariant straight lines

Due to $\theta \neq 0$, by Remark 6.3 we deduce that to this class of systems could belong only systems from the families $(\mathbf{S}_I)$ and $(\mathbf{S}_{III})$.

7.1.1 The case $B_3 \neq 0$: systems possessing invariant straight lines in at most two directions

According to Theorem 6.1 (see Table 2) if $B_2 = 0$ and $\theta B_3 \neq 0$ then in the case $\eta > 0$ (i.e. when systems possess three real infinite singularities) for the systems in this class we have the configurations of invariant lines defined by the following conditions, respectively:

- $\mu_0 \neq 0$, $H_9 \neq 0$ $\Rightarrow$ *Config. 3.1*;
- $\mu_0 \neq 0$, $H_9 = 0$, $H_{13} \neq 0$ $\Rightarrow$ *Config. 3.3*;
- $\mu_0 \neq 0$, $H_9 = 0$, $H_{13} = 0$ $\Rightarrow$ *Config. 3.2*;
- $\mu_0 = 0$, $H_9 \neq 0$, $H_{14} \neq 0$ $\Rightarrow$ *Config. 3.5*;
- $\mu_0 = 0$, $H_9 \neq 0$, $H_{14} = 0$ $\Rightarrow$ *Config. 3.4*;
- $\mu_0 = 0$, $H_9 = 0$, $H_{14} \neq 0$ $\Rightarrow$ *Config. 3.8*;
- $\mu_0 = 0$, $H_9 = 0$, $H_{14} = 0$, $H_{13} \neq 0$ $\Rightarrow$ *Config. 3.7*;
- $\mu_0 = 0$, $H_9 = 0$, $H_{14} = 0$, $H_{13} = 0$ $\Rightarrow$ *Config. 3.6*.

Similarly in the case $\eta = 0$ (and $M \neq 0$) by Table 2 we have:

- $H_4 \neq 0$, $\mu_0 \neq 0$, $H_9 \neq 0$ $\Rightarrow$ *Config. 3.9*;

Diagram 1 ($B_2 = 0$, $\eta \geq 0$, $\theta \neq 0$)

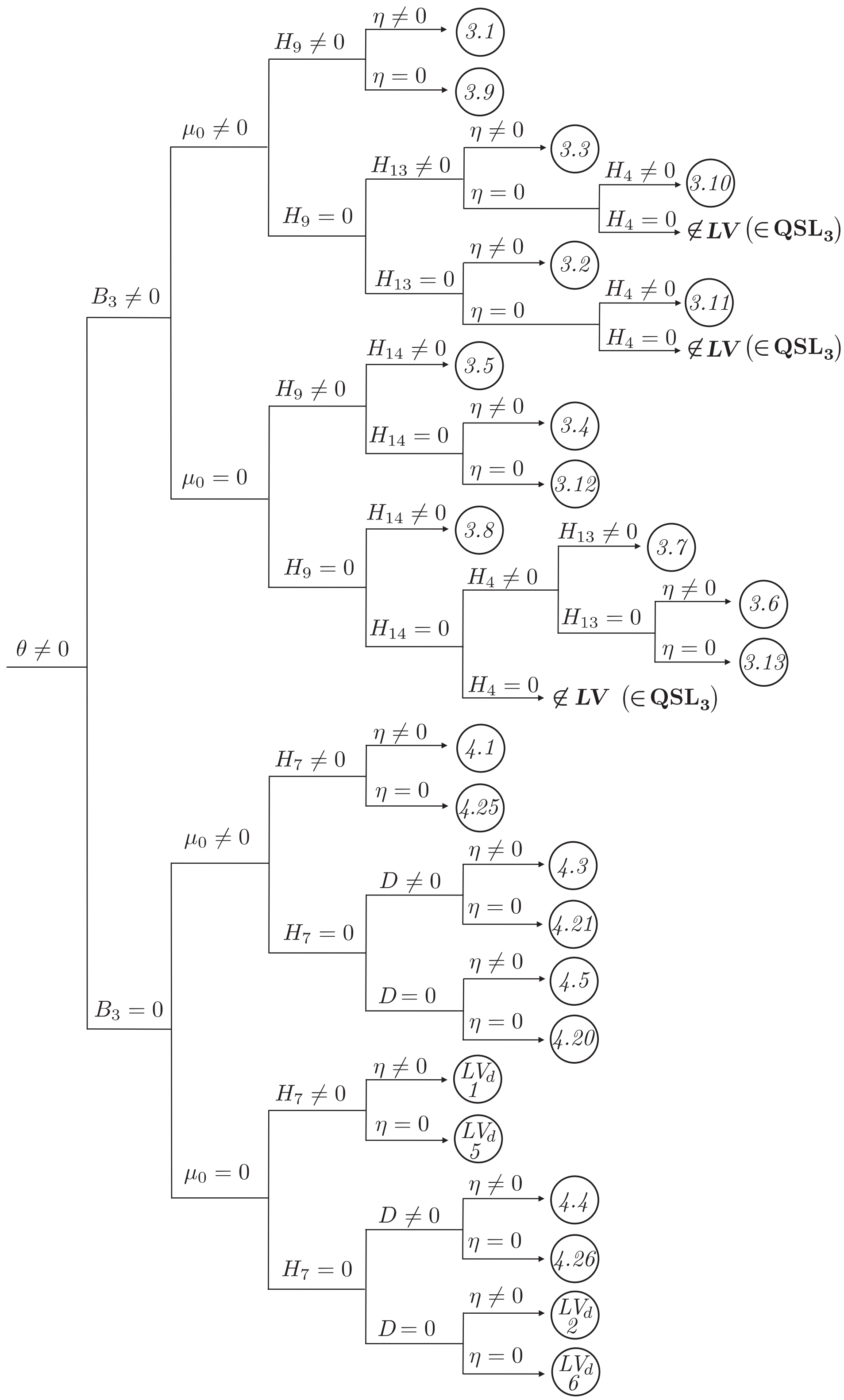


Diagram 2 ($B_2 = 0$, $\eta \geq 0$, $\theta = 0$, $N \neq 0$)

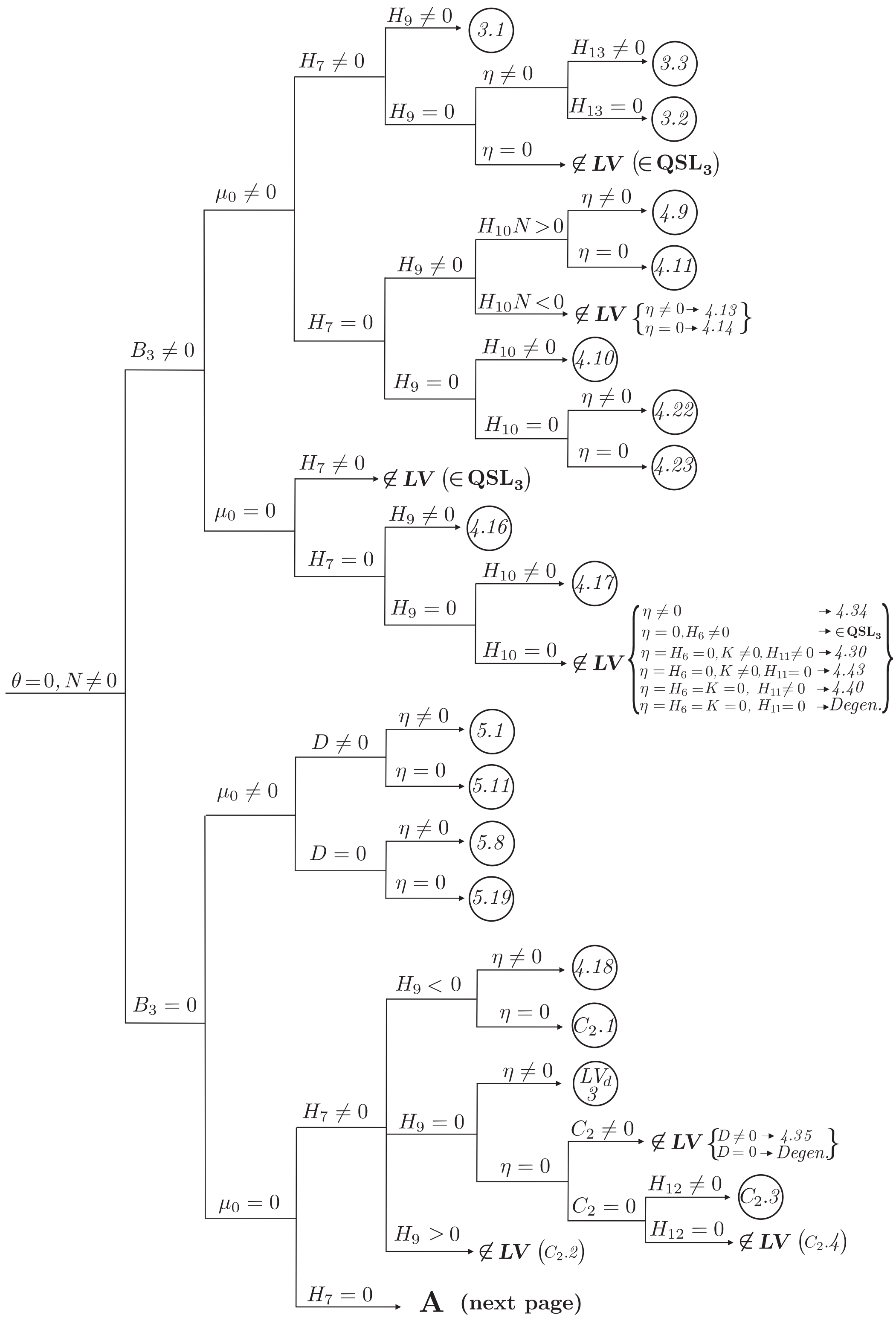
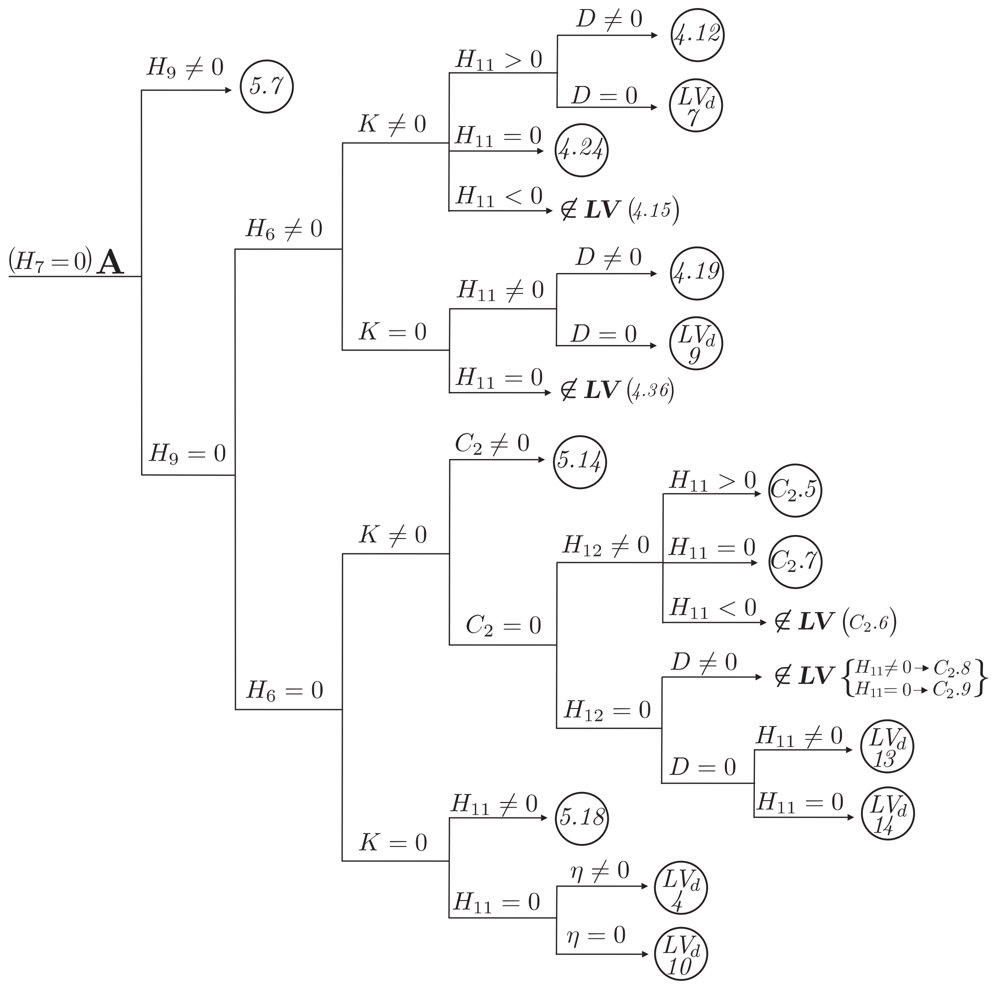


Diagram 2 (*continued*)



- $H_4 \neq 0,\ \mu_0 \neq 0,\ H_9 = 0,\ H_{13} \neq 0 \quad \Rightarrow \quad$ *Config. 3.10*;
- $H_4 \neq 0,\ \mu_0 \neq 0,\ H_9 = 0,\ H_{13} = 0 \quad \Rightarrow \quad$ *Config. 3.11*;
- $H_4 \neq 0,\ \mu_0 = 0,\ H_9 \neq 0 \quad \Rightarrow \quad$ *Config. 3.12*;
- $H_4 \neq 0,\ \mu_0 = 0,\ H_9 = 0 \quad \Rightarrow \quad$ *Config. 3.13*;

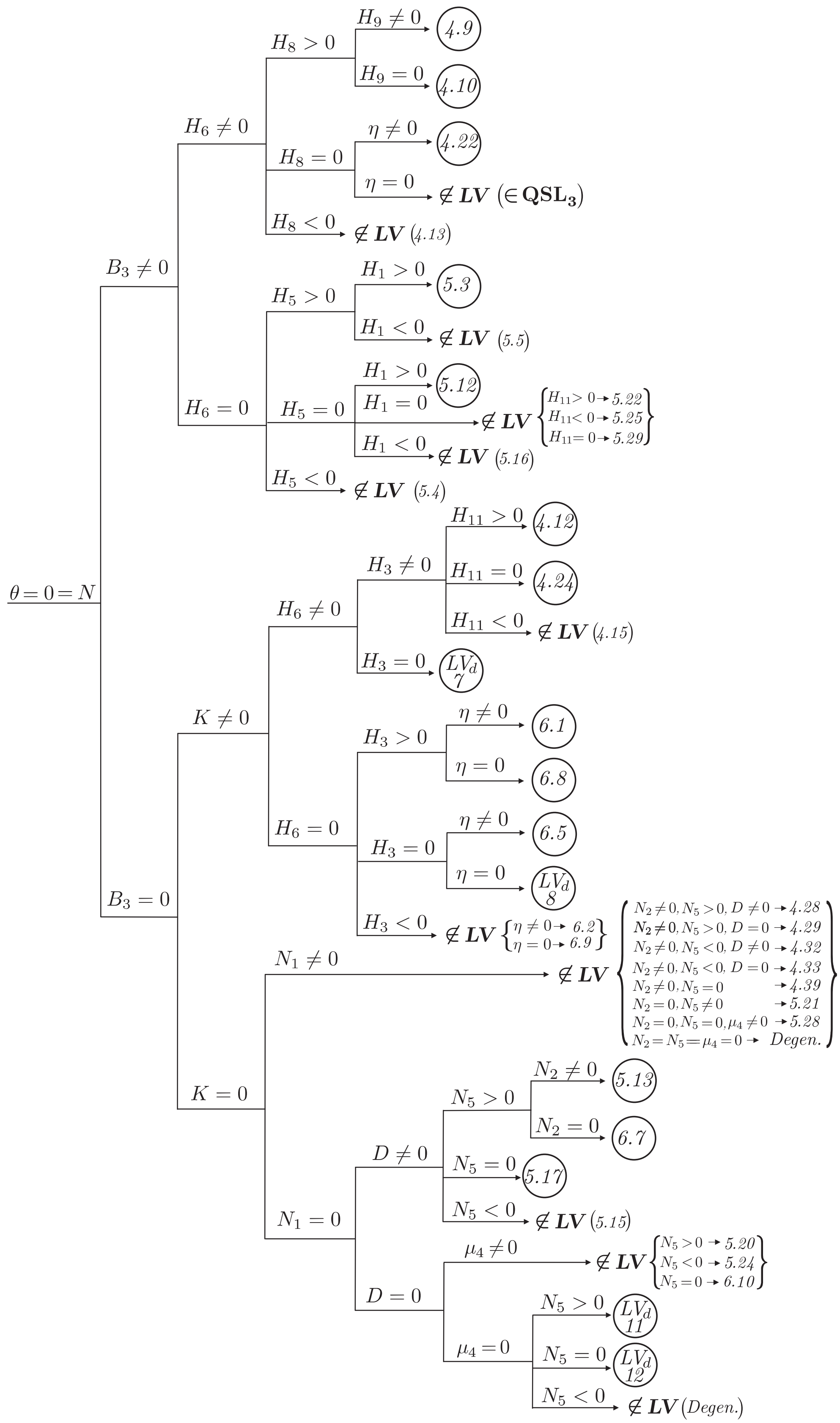
We observe that if for a system in the family $(\mathbf{S}_I)$ the conditions $\theta \neq 0$, $B_2 = 0$ and $B_3 \neq 0$ hold, then this system belongs to the class $\boldsymbol{LV}$. At the same time a system in the family $(\mathbf{S}_{III})$ subject of the same conditions above belongs to $\boldsymbol{LV}$ if and only if $H_4 \neq 0$.

We note that in the case $H_4 = 0$ systems $(\mathbf{S}_{III})$ could be brought via an affine transformation to the systems (6.36), for which the condition $H_9 = 0 = H_{14}$ holds. Furthermore if $H_4 \neq 0$ then due to an affine transformation systems $(\mathbf{S}_{III})$ could be brought to the systems (6.32) for which $H_{14} = 0$ and according to (6.34) the conditions $\mu_0 = H_9 = 0$ implies $H_{13} = 0$. Therefore we arrive to the Diagram 1 (case $B_3 \neq 0$). It remains to note, that systems (6.36) possess the invariant line $x = 0$, which is double and hence in the case $H_4 = 0$ we obtain systems in $\mathbf{QSL_3}$.

7.1.2 The case $B_3 = 0$: systems possessing invariant straight lines in a maximum possible of three directions

7.1.2.1 Systems with three distinct real infinite singularities. In this case the condition $\eta > 0$ holds. Then it was shown earlier (see page 25) that a quadratic system with the conditions $B_2 = 0$ and $\theta \neq 0$

Diagram 3 ($B_2 = 0$, $\eta \geq 0$, $\theta = 0 = N$)



due to an affine transformation could be brought to form (6.3), for which $\theta = -8(g-1)(h-1)(g+h) \neq 0$ and calculation yields:

$$B_3 = 3(c-f)(fg+ch)x^2y^2,\ H_7 = -4(c-f)(-1+g)(-1+h),\ \mu_0 = gh(-1+g+h).$$

So, the condition $B_3 = 0$ gives either $f = c$ (if $H_7 = 0$) or $fg + ch = 0$ if (if $H_7 \neq 0$).

7.1.2.1.1 *Assume first $H_7 \neq 0$.* If $\mu_0 \neq 0$ then by Lemma 5.9 systems (6.3) are non-degenerate and by Theorem 6.1 we obtain *Config. 4.1.*

Suppose now $\mu_0 = 0$, i.e. $gh(g+h-1) = 0$. If $gh = 0$ we can assume $h = 0$ due to the change $(x, y, c, f, g, h) \mapsto (y, x, f, c, h, g)$. Then the conditions $B_3 = 3(c-f)fgx^2y^2 = 0$, $H_7 = 4(c-f)(g-1) \neq 0$ and $\theta = 8g(g-1) \neq 0$ imply $f = 0$ and we obtain the systems (6.25) with the configuration *Config. LV_d.1* as $H_7 \neq 0$. Assuming $gh \neq 0$ we get $h = 1-g$ and then the conditions $B_3 = 3(c-f)(c-cg+fg)x^2y^2 = 0$ and $H_7 = 4(c-f)g(g-1) \neq 0$ yield $f = c(g-1)/g$ and this leads to degenerate systems

$$\dot{x} = x(c+gx-gy), \quad \dot{y} = ((-1+g)y(c+gx-gy))/g.$$

As for these systems the condition $H_7 \neq 0$ holds, according to Theorem 6.1 (see Table 3) the invariant line configurations again corresponds to *Config. LV_d.1.*

7.1.2.1.2 *Assume now $H_7 = 0$.* Then $f = c$ and this implies $B_3 = 0$. So we get the family of systems

$$\dot{x} = x(c+gx-y+hy), \quad \dot{y} = y(c-x+gx+hy),$$

for which we have

$$\mu_0 = gh(-1+g+h),\ H_1 = 576c^2,\ D = -c^2x(x-y)y.$$

So the condition $H_1 = 0$ is equivalent to $D = 0$. If $\mu_0 \neq 0$ then by Lemma 5.9 systems above are non-degenerate an according to Table 2 we obtain the respective configurations and conditions from Diagram 1 (case $B_3 = 0 = H_7$ and $\mu_0 \neq 0$).

Admit now $\mu_0 = 0$, i.e. $gh(-1+g+h) = 0$. We claim that without loss of generality we may assume $h = 0$ due to an affine transformation and a change of parameters. Indeed, if $g = 0$ we could reach this via the simple change $(x, y, c, g, h) \mapsto (y, x, c, h, g)$. Assume now $h \neq 0$ and $g = 1-h$. Then via the affine transformation $x_1 = x - y$, $y_1 = -y$ systems above with $g = 1-h$ will be transformed to the systems

$$\dot{x}_1 = x_1(c+g_1x_1-y_1), \quad \dot{y}_1 = y_1(c-x_1+g_1x_1)$$

with $g_1 = h-1$ and this proves our claim.

So $h = 0$ and it is clear that if $c \neq 0$ (i.e. $D \neq 0$ and that is equivalent with $H_1n \neq 0$) by Theorem 6.1 (see Table 2) we get *Config. 4.4*, whereas in the case $c = 0$ (i.e. $D = 0$) according to Table 3 we obtain *Config. LV_d.2.*

7.1.2.2 Systems with two distinct real infinite singularities. In this case the condition $\eta = 0$ holds and we consider the family ($\mathbf{S}_{III}$). It was mentioned above (see the case $B_3 \neq 0$) that these systems in the case $\theta H_4 \neq 0$ and $B_2 = 0$ could be brought to the systems (6.32) for which we have $B_3 = 3c(c-f)x^2y^2$ and $H_4 = 48c(c-f)$ and hence the condition $H_4 \neq 0$ implies $B_3 \neq 0$. Consequently in order to have the condition $B_3 = 0$ it is necessary $H_4 = 0$ and then due to an affine transformation systems ($\mathbf{S}_{III}$) could be brought to the systems (6.36), for which

$$\begin{aligned} B_3 &= 3c(c-f)(g-1)^2x^4, \quad \theta = 8(1-g), \\ H_7 &= 4(c-f)(1-g), \qquad \mu_0 = g \end{aligned} \tag{7.1}$$

7.1.2.2.1 *Assume first $H_7 \neq 0$.* In this case $B_3 = 0$ gives $c = 0$ and we get the family of systems

$$\dot{x} = x(gx+y), \quad \dot{y} = y[fy+(g-1)x+y]. \tag{7.2}$$

If $\mu_0 \neq 0$ then by Lemma 5.9 these systems are non-degenerate and by Theorem 6.2 we obtain *Config. 4.25.*

Assume $\mu_0 = 0$, i.e. $g = 0$. Then systems (7.2) become degenerate and as in this case we have $H_4 = 0$ and $H_7 \neq 0$, according to Theorem 6.2 (see Table 3) we obtain *Config. LV_d.5.*

7.1.2.2.2 *Suppose now* $H_7 = 0$, i.e. by (7.1) we get $f = c$ and this implies $B_3 = 0$. This leads to the systems

$$\dot{x} = x(c + gx + y), \quad \dot{y} = y[cy + (g-1)x + y], \tag{7.3}$$

which evidently belong to the class $\boldsymbol{LV}$. For these systems we have

$$B_3 = H_7 = 0, \quad \theta = 8(1-g), \quad \mu_0 = g, \; D = -c^2x^2y.$$

If $\mu_0 \neq 0$ then by Lemma 5.9 these systems are non-degenerate and according to Theorem 6.2 we obtain *Config. 4.21* if $D \neq 0$ and *Config. 4.20* if $D = 0$.

Assuming $\mu_0 = 0$ (i.e. $g = 0$) evidently systems (7.3) could be degenerate if and only if $c = 0$ (i.e. $D = 0$). So by Theorem 6.2 we get *Config. 4.26* if $D \neq 0$ whereas for $D = 0$ according to Table 3 we obtain *Config.* LV_d*.6*. This completes the proof of the theorem regarding the Diagram 1.

7.2 $\boldsymbol{LV}$-systems which could possess at most one couple of parallel invariant straight lines

In this case by Lemma 5.6 the condition $\theta = 0$ and $N \neq 0$ have to be satisfied. We again shall consider two cases: $B_3 \neq 0$ and $B_3 = 0$.

7.2.1 The case $B_3 \neq 0$: systems possessing invariant straight lines in at most two directions

7.2.1.1 Systems with three distinct real infinite singularities. In this case the condition $\eta > 0$ hold and our attention will be concentrated on Theorem 6.1. We shall consider two subcases: $\mu_0 \neq 0$ (when no finite point is gone to infinity) and $\mu_0 = 0$ (when by Lemma 5.12 at least one finite point is gone to infinity).

7.2.1.1.1 *Assume first* $\mu_0 \neq 0$. According to Theorem 6.1 (see also Table 2) if $B_2 = 0$, $\theta = 0$, $N \neq 0$ and $\mu_0 B_3 \neq 0$ then for the systems in this class we have the configurations of invariant lines defined by the following conditions, respectively:

$(\boldsymbol{a_1})$ $H_7 \neq 0$, $H_9 \neq 0$ $\Rightarrow$ *Config. 3.1*;

$(\boldsymbol{a_2})$ $H_7 \neq 0$, $H_9 = 0$, $H_{13} \neq 0$ $\Rightarrow$ *Config. 3.3*;

$(\boldsymbol{a_3})$ $H_7 \neq 0$, $H_9 = 0$, $H_{13} = 0$ $\Rightarrow$ *Config. 3.2*;

$(\boldsymbol{a_4})$ $H_7 = 0$, $H_9 \neq 0$, $H_{10}N > 0$ $\Rightarrow$ *Config. 4.9*;

$(\boldsymbol{a_5})$ $H_7 = 0$, $H_9 = 0$, $H_{10}N > 0$ $\Rightarrow$ *Config. 4.10*;

$(\boldsymbol{a_6})$ $H_7 = 0$, $H_{10} = 0$ $\Rightarrow$ *Config. 4.22*.

We observe that all the cases above correspond to the respective cases from Diagram 1 (case $B_3\mu_0\eta \neq 0$) except the last two cases: $(\boldsymbol{a_5})$ and $(\boldsymbol{a_6})$. More precisely we must prove the following two things:

α) *the conditions* $H_7 = 0$ *and* $H_{10} = 0$ *imply* $H_9 = 0$; β) *if* $H_7 = H_9 = 0$ *then the condition* $H_{10} \neq 0$ *is equivalent to* $H_{10}N > 0$. And it remains also to do the following step: γ) *to examine the case* $H_7 = 0$, $H_9 \neq 0$ *and* $H_{10}N < 0$.

α) As it was shown earlier in the proof of Theorem 6.1 (see page 29) in the case $B_2 = \theta = 0$, $N\mu_0 B_3 \neq 0$ and $H_7 = 0$ systems $(\mathbf{S}_I)$ could be brought to the systems (6.15) for which we have

$$N = (h^2-1)(x-y)^2, \; H_9 = -576(c^2-4bh)^2\big[c^2 + b(h-1)^2\big]^2, \; H_{10} = 8(h^2-1)(c^2-4bh),$$
$$B_3 = -3b(h+1)^2x^2(x-y)^2, \quad H_4 = 48b(h-1)(h+1)^2.$$

Clearly due to the condition $N \neq 0$ (i.e. $h^2 - 1 \neq 0$) the condition $H_{10} = 0$ imply $H_9 = 0$ and this proves our claim α).

β) Assume $H_9 = 0$ and $H_{10} \neq 0$. Then $c^2 + b(h-1)^2 = 0$ and due to $N \neq 0$ we could take $b = -c^2/(h-1)^2$. Therefore we calculate $H_{10} = 8c^2(1+h)^3)/(h-1) \neq 0$ and then $H_{10}N = 8c^2(1+h)^4(x-y)^2 > 0$. Our claim β) is proved.

γ) Assuming that the conditions $B_2 = \theta = 0$, $N\mu_0 B_3 \neq 0$, $H_7 = 0$ and $H_9 \neq 0$ are fulfilled by the formulas above we deduce $H_{10} \neq 0$. Moreover as $B_3 \neq 0$ we obtain $H_4 \neq 0$. Therefore if $H_{10}N < 0$ then according to [78, Theorem 4.1] (see Table 2) we obtain the configuration of invariant lines given by *Config. 4.13.*

7.2.1.1.2 *Suppose now* $\mu_0 = 0$. According to Theorem 6.1 for the systems in this class we have the configurations of invariant lines defined by the following conditions, respectively:

- $H_7 = 0$, $H_9 \neq 0$ $\quad\Rightarrow\quad$ *Config. 4.16*;
- $H_7 = 0$, $H_9 = 0$, $H_{10} \neq 0$ $\quad\Rightarrow\quad$ *Config. 4.17.*

So in order to obtain the whole open subset of an algebraic variety, defined by the conditions

$$B_2 = \theta = 0,\ N \neq 0,\ B_3 \neq 0,\ \eta > 0,\ \mu_0 = 0 \tag{7.4}$$

we have to add and examine the cases $H_7 \neq 0$ and $H_7 = H_9 = H_{10} = 0$.

It was shown earlier in the proof of Theorem 6.1 (see page 30) that systems $(\mathbf{S}_I)$ subject to the conditions (7.4) due to an affine transformation could be brought to systems (6.17), which belong to the class $\mathbf{QSL_3}$. However these systems are not ***LV***-systems and to be in this class the additional condition $B_3H_7 = 0$ has to be fulfilled. Hence as $B_3 \neq 0$ in the case $H_7 \neq 0$ we have not ***LV***-systems as it is indicated in Diagram 2 for this case.

Assume now that for systems (6.17) the conditions $H_7 = H_9 = H_{10} = 0$ hold. Considering the conditions (7.4) according to [78] we obtain *Config. 4.34*, i.e. in this case systems also do not belong to the class ***LV***.

7.2.1.2 Systems with two distinct real infinite singularities. We consider the family $(\mathbf{S}_{III})$, for which the conditions $\eta = 0$ and $M \neq 0$ hold.

7.2.1.2.1 *Assume first* $\mu_0 \neq 0$. According to Theorem 6.2 (see also Table 2) if $B_2 = 0$, $\theta = 0$, $N \neq 0$ and $\mu_0 B_3 \neq 0$ then for the systems in this class we have the configurations of invariant lines defined by the following conditions, respectively:

- $H_7 = 0$, $H_{10} > 0$ $\quad\Rightarrow\quad$ *Config. 4.11*;
- $H_7 = 0$, $H_{10} = 0$ $\quad\Rightarrow\quad$ *Config. 4.23.*

So it remains to examine the cases: *(i)* $H_7 \neq 0$ and *(ii)* $H_7 = 0$, $H_{10} < 0$.

It was shown in the proof of Theorem 6.2 (see page 36) that subject to the conditions $B_2 = \theta = 0$, $B_3\mu_0 N \neq 0$ systems $(\mathbf{S}_{III})$ could be brought via an affine transformation to the systems

$$\dot{x} = x(c + x + y), \quad \dot{y} = b + ex + y^2, \tag{7.5}$$

for which $B_2 = -648(b + c^2)e^2x^4 = 0$ implies $e(b + c^2) = 0$. We note that for these systems $H_7 = -4e$ and hence, if $H_7 \neq 0$ then $b = -c^2$ and for these systems we obtain (see notation 5.9) $\mathcal{H} = x^2$, i.e. the invariant line $x = 0$ is double. Therefore these systems belong to the class $\mathbf{QSL_3}$ as it is indicated in Diagram 2 for this case.

Assume now that for systems (7.5) the condition $H_7 = 0$ (i.e. $e = 0$) holds and then evidently they possess invariant lines $y^2 + b = 0$. On the other hand for these systems we calculate:

$$N = y^2,\ B_3 = 3(b + c^2)x^2y^2,\ H_9 = -9216b^2(b + c^2)^2,\ H_{10} = -32b.$$

Therefore as $B_3 \neq 0$ the condition $H_{10} \neq 0$ is equivalent to $H_9 \neq 0$. In the case $H_{10} < 0$ we have $b > 0$ and hence the invariant lines $y^2 + b = 0$ are imaginary. Moreover according to [78] (see Table 2) in this case we have the invariant line configuration given by *Config. 4.14*. As $\operatorname{sign}(H_{10}) = \operatorname{sign}(H_{10}N)$ we obtain the respective cases from Diagram 2.

7.2.1.2.2 *Suppose now* $\mu_0 = 0$. In this case for systems $(\mathbf{S}_{III})$ the conditions $\theta = 8(1-g)h^2 = 0$ and $\mu_0 = gh^2 = 0$ imply $h = 0$ and then $N = (g^2-1)x^2 \neq 0$. Therefore due to a translation we may assume $e = f = 0$, obtaining the family of systems (6.41). Due to the condition $N \neq 0$ (i.e. $g^2 - 1 \neq 0$ for these systems the condition condition $B_2 = 0$ implies $bd = dg = 0$. On the other hand for systems (6.41) we have $H_7 = 4d(g^2-1)$.

If $H_7 \neq 0$ we get $b = g = 0$, $d \neq 0$ and this leads to the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = (g-1)xy,$$

which could not be in the class ***LV*** because there does not exists an invariant line of the type $x + \alpha = 0$ due to $d \neq 0$. On the other hand we observe that these systems possess the invariant line $y = 0$ and the line at infinity is double, i.e. they belong to the class $\mathbf{QSL_3}$ as it is indicate in Diagram 2 in this case.

Assuming $H_7 = 0$ we get $d = 0$ and then $B_2 = 0$. Thus we obtain the family of systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + (g-1)xy,$$

for which calculations yield:

$$N = (g^2-1)x^2,\ B_3 = -3b(g-1)^2x^4,\ K = 2g(g-1)x^2,$$
$$H_6 = -128(g-1)^4[c^2 - a(g+1)^2]x^6,\ H_{11} = 48(g-1)^4(c^2 - 4ag)x^4.$$

Due to the condition $B_3 \neq 0$ (i.e. $b \neq 0$) these systems could not be in the class ***LV*** as there does not exists the line of the type $y + \beta = 0$, intersecting the two invariant lines $a + cx + gx^2 = 0$. These last line are simple ones if $H_6 \neq 0$ and hence in this case the systems above belong to $\mathbf{QSL_3}$.

Assuming $H_6 = 0$ and $K^2 + H_{11}^2 \neq 0$ (at the contrary systems become degenerate, see below), according to [78, Theorem 4.1] systems above possess the configuration of invariant line given by *Config. 4.30* (respectively *Config. 4.43*; *Config. 4.40*) if $KH_{11} \neq 0$ (respectively $K \neq 0$ and $H_{11} = 0$; $K = 0$ and $H_{11} \neq 0$).

If $K = H_6 = H_{11} = 0$ we obtain $g = c = a = 0$ and this leads to degenerate systems. Thus we have obtained the respective conditions of Diagram 2 in the case $B_3 \neq 0$ and $m\mu_0 = 0$.

7.2.2 The case $B_3 = 0$: systems possessing invariant straight lines in a maximum possible of three directions

As $\theta = B_3 = 0$ by Remark 6.3 we need to examine not only the families of systems $(\mathbf{S}_I)$ and $(\mathbf{S}_{III})$ but also the family $(\mathbf{S}_V)$ (in the case $\mu_0 = 0$), which corresponds to the condition $C_2 = 0$.

7.2.2.1 Systems with three distinct real infinite singularities. In this case the condition $\eta > 0$ holds and we shall consider two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

7.2.2.1.1 *Assume first* $\mu_0 \neq 0$. According to Theorem 6.1 if $B_2 = 0$, $\theta = 0$, $N \neq 0$, $B_3 = 0$ and $\mu_0 \neq 0$ then for the systems in this class we have the configurations of invariant lines defined by the following conditions, respectively:

- $H_7 = 0$, $H_9 \neq 0 \quad \Rightarrow \quad$ *Config. 5.1*;
- $H_7 = 0$, $H_9 = 0 \quad \Rightarrow \quad$ *Config. 5.8.*

We claim that in the case under consideration these conditions are equivalent to $D \neq 0$ and $D = 0$, respectively. Indeed, it was shown earlier in the proof of Theorem (see page 28) that in the case $\theta = 0$ and $N \neq 0$ systems $(\mathbf{S}_I)$ could be brought via an affine transformation to the systems (6.10), for which we have $N = (h^2-1)(x-y)^2$ and Coefficient$[B_3, x^4] = -3b(h+1)^2$, Coefficient$[B_3, y^4] = 3a(h-1)^2$. Therefore as $N \neq 0$ the condition $B_3 = 0$ gives $a = b = 0$ and then we obtain the family of systems

$$\dot{x} = x(c - hx - y + hy), \quad \dot{y} = y(f - x - hx + hy), \tag{7.6}$$

for which we have $B_3 = 3(c-f)^2hx^2y^2$ and $\mu_0 = h^2$. Due to $\mu_0 \neq 0$ the condition $B_3 = 0$ implies $f = c$ and for these obtain $D = c^2xy(y-x)y$ and $H_9 = -576c^8$. Evidently the condition $D_2 = 0$ is equivalent to $H_9 = 0$ and as for these systems we have $H_7 = 0$, our claim is proved.

7.2.2.1.2 *Suppose* $\mu_0 = 0$. In this case for systems (7.6) we have $h = 0$ and then $B_3 = 0$. Furthermore we obtain the family of systems

$$\dot{x} = x(c - y), \quad \dot{y} = y(f - x), \tag{7.7}$$

for which calculation yields:

$$H_7 = 4(f - c),\ H_9 = -576c^4 f^4,\ H_6 = 128(f - c)(x - y)^2(x^2 - xy + y^2)(cx^2 - fy^2),\ H_4 = 0.$$

(i) If $H_7 \neq 0$ according to Theorem 6.1 the configuration of invariant lines of systems (7.7) corresponds to *Config. 4.18* if $H_9 \neq 0$ (then $H_9 < 0$). In the case $H_9 = 0$ (i.e. $cf = 0$) we get degenerate systems and as $H_4 = 0$ by Table 3 we obtain *Config.* LV_d*.3*.

(ii) Assume now $H_7 = 0$, i.e. $f = c$. Then we calculate

$$H_9 = -576c^8,\ H_6 = 0,\ K = 0,\ H_{11} = 48c^2(x - y)^2(x^2 - 6xy + y^2),\ H_4 = 0$$

and if $H_9 \neq 0$ according to [76, Theorem 57] (see Table 4) we get *Config. 5.7*.

In the case $H_9 = 0$ (then we obtain the degenerate system (7.7) and by Table 3 we get *Config.* LV_d*.4*. On the other hand as in this case $H_6 K = H_{11} = 0$ we arrive to the respective conditions given by Diagram 2.

7.2.2.2 Systems with two distinct real infinite singularities. We consider the family $(\mathbf{S}_{III})$ and assume first $\mu_0 \neq 0$. Then as $\theta = 0 = B_3$ according to Theorem 6.2 we get the invariant line configuration given by *Config. 5.11* if $D \neq 0$ and by *Config. 5.19* if $D = 0$. This leads exactly to the respective conditions of Diagram 2.

Suppose now that for family $(\mathbf{S}_{III})$ the condition $\mu_0 = 0$ holds. Therefore the conditions $\theta = -8(g-1)h^2$ and $\mu_0 = gh^2 = 0$ imply $h = 0$ and then $N = (g^2 - 1)x^2$. Hence the condition $N \neq 0$ implies $g \neq 1$ and we may assume $e = f = 0$ via a translation. Thus, we get the family of systems (6.41) for which we have

$$B_3 = -3b(-1 + g)^2 x^4 - 3cd(-1 + g)x^2y^2 - 6d^2gxy^3,\ H_7 = 4d(g^2 - 1).$$

and we shall consider two subcases: $H_7 \neq 0$ and $H_7 = 0$.

7.2.2.2.1 *Assume first* $H_7 \neq 0$. Then $d \neq 0$ and due to $N \neq 0$ the condition $B_3 = 0$ implies $b = c = g = 0$. So we obtain the family of systems

$$\dot{x} = a + dy, \quad \dot{y} = -xy,$$

for which $D = -ax^2y$. Evidently these systems could not be in the class ***LV*** (as there could not exist an invariant line of the type $x + \alpha = 0$. We observe that these systems are non-degenerate if $D \neq 0$ and degenerate if $D = 0$. In the first case according to Theorem [78, Theorem 4.1] (see Table 2) we get *Config. 4.35*.

As for systems above $H_9 = 0$ and $C_2 = x^2y \neq 0$ we get the respective condition given by Diagram 2.

7.2.2.2.2 *Admit now* $H_7 = 0$, i.e. $d = 0$ and the condition $B_3 = 0$ gives $b = 0$. So we obtain the family of systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = (g - 1)xy, \tag{7.8}$$

for which we have

$$B_3 = 0 = \mu_0 = H_7 = H_9,\ K = 2g(g - 1)x^2,\ H_{11} = 48(g - 1)^4(c^2 - 4ag)x^4,$$
$$D = -a(g - 1)^2x^2y,\ H_6 = -128(g - 1)^4[c^2 - a(g + 1)^2]x^6,\ C_2 = x^2y \neq 0.$$

We observe that due to $N \neq 0$ (i.e. $g - 1 \neq 0$) systems above are degenerate if and only if $a = 0$ that is equivalent to $D = 0$.

(i) Assume $H_6 \neq 0$. Then in the case of non-degenerate systems (i.e. $D \neq 0$) according to [78, Theorem 4.1] we get *Config. 4.12* (respectively *4.15*; *4.24*; *4.19*; *4.36*) if $K \neq 0$ and $H_{11} > 0$ (respectively $K \neq 0$ and $H_{11} < 0$; $K \neq 0$ and $H_{11} = 0$; $K = 0$ and $H_{11} \neq 0$; $K = 0$ and $H_{11} = 0$).

If $D = 0$ (i.e. $a = 0$) systems (7.8) became degenerate and we have $H_6 = -128c^2(-1+g)^4x^6$ and $H_{11} = 48c^2(-1+g)^4x^4$. So, the condition $H_6 \neq 0$ implies $H_{11} > 0$ and $H_2 = 4c(-1+g)^2x^2 \neq 0$. According to Theorem 6.2 (see Table 3) we obtain *Config.* LV_d*.7* if $K \neq 0$ and *Config.* LV_d*.9* if $K = 0$. As for systems (7.8) we have $H_9 = 0$ in the case $H_6 \neq 0$ we get exactly the respective conditions given by Diagram 2.

(ii) Suppose now $H_6 = 0$. If $K \neq 0$ by Theorem 6.2 systems (7.8) possess the configuration of invariant lines given by *Config. 5.14*. If $K = 0$ then $g(g-1) = 0$ and due to $N \neq 0$ this implies $g = 0$. Then we obtain $H_6 = 128(a - c^2)x^6 = 0$ that gives $a = c^2$. So we get the systems

$$\dot{x} = c(c+x), \quad \dot{y} = -xy,$$

for which we have $H_{11} = 48c^2x^4$ and evidently these systems are degenerate if and only if $c = 0$ (i.e. $H_{11} = 0$). In in this case due to the change $x \to -x$ we obtain the orbit representative (LV_d*.10*) from Table 3 possessing the configuration given by *Config.* LV_d*.10*. If $H_{11} \neq 0$ systems above are non-degenerate and by Theorem 6.2 we get *Config. 5.18*.

7.2.2.3 Systems with a line of singularities at infinity.

We consider the family ($\mathbf{S}_V$) in which we may assume $e = f = 0$ due to a translation. So we get the family of systems (6.57) (see page 42) for which we have $H_7 = 4d$, $H_{10} = 36d^2$, $\mu_0 = 0$ and $\mu_1 = dx$.

7.2.2.3.1 *Assume first $H_7 \neq 0$.* Then $\mu_1 \neq 0$ and by Lemma 5.9 these systems are non-degenerate. As in this case we have also $H_{10} \neq 0$ then by [81, Theorem 3.1] (see Table 1) we obtain: *Config.* C_2*.1* if $H_9 < 0$; *Config.* C_2*.2* if $H_9 > 0$; *Config.* C_2*.3* if $H_9 = 0$ and $H_{12} \neq 0$; *Config.* C_2*.4* if $H_9 = H_{12} = 0$. Since $H_{10} \neq 0$ and the condition $C_2 = 0$ implies $\eta = 0$ (see Remark 6.3) the conditions above are equivalent to the respective conditions in Diagram 2.

7.2.2.3.2 *Suppose now $H_7 = 0$.* This gives $d = 0$ and we get the family of systems

$$\dot{x} = a + cx + x^2, \quad \dot{y} = b + xy,$$

for which calculations yield:

$$H_9 = H_6 = 0,\ K = 2x^2 \neq 0,\ H_{12} = -8a^2x^2,\ H_{11} = 48(c^2 - 4a)x^4,\ D = x^2(bx - ay),\ \mu_2 = ax^2.$$

(i) If $H_{12} \neq 0$ then $a \neq 0$ and this implies $\mu_2 \neq 0$. Therefore systems above are non-degenerate (see Lemma 5.9) and according to [81, Theorem 3.1] (see Table 1) we obtain: *Config.* C_2*.5* if $H_{11} > 0$; *Config.* C_2*.6* if $H_{11} < 0$ and *Config.* C_2*.7* if $H_{11} = 0$.

(ii) Assuming $H_{12} = 0$ (i.e. $a = 0$) we have $D = bx^3$ and $H_{11} = 48c^2x^4$. Clearly we have degenerate systems if and only if $D = 0$ (i.e. $b = 0$). If $D \neq 0$ by [81] we obtain *Config.* C_2*.8* if $H_{11} \neq 0$ and *Config.* C_2*.9* if $H_{11} = 0$.

In the case $D = 0$ we get degenerate systems (6.58) which were investigated earlier (see page 42):

$$\dot{x} = x(c+x), \quad \dot{y} = xy. \tag{7.9}$$

It is not too hard to detect that the invariant line configuration correspond to *Config.* LV_d*.10* if $c \neq 0$ and to *Config.* LV_d*.11* if $c = 0$. It remains to note that for systems (7.9) we have $H_2 = 4cx^2$ and $H_{11} = 48c^2x^4$. As the condition $H_{11} \neq 0$ is equivalent $H_2 \neq 0$ by Theorem 6.3 (see Table 3) we get *Config.* LV_d*.13* if $H_{11} \neq 0$ and *Config.* LV_d*.14* if $H_{11} = 0$. Taking into account that the condition $C_2 = 0$ implies $K \neq 0$ (see Remark 6.3) we arrive exactly to the respective conditions given in Diagram 2 for this class of systems.

7.3 *LV*-systems which could possess two couples of parallel invariant straight lines

In this case by Lemma 5.6 the condition $\theta = 0 = N$ have to be satisfied. Hence by Remark 6.3 systems from the family ($\mathbf{S}_V$) could not be in this class and it remain to consider only the families ($\mathbf{S}_I$) and ($\mathbf{S}_{III}$).

7.3.1 The case $B_3 \neq 0$: systems possessing invariant straight lines in at most two directions

7.3.1.1 Systems with three distinct real infinite singularities. In this case the condition $\eta > 0$ holds and we shall consider the family of systems $(\mathbf{S}_I)$. As it was shown earlier in the proof of Theorem 6.1 (see page 31) if $\theta = 0 = N$ via a translation systems $(\mathbf{S}_I)$ could be brought to the systems (6.20) for which we have Coefficient$[B_2, x^3y] = 2592d^2e^2$. Hence the condition $B_2 = 0$ yields $de = 0$ and without loss of generality we may consider $d = 0$ due to the change $(x, y, a, b, de) \mapsto (y, x, b, a, ed)$. So, $d = 0$ and then calculations yield:

$$B_2 = 648e^2(4a - 4b - e^2)x^4, \; H_6 = -2048e^2x^4(x^2 - xy + y^2), \; H_4 = 96e^2$$

and we shall consider two possibilities: $H_6 \neq 0$ and $H_6 = 0$.

7.3.1.1.1 *Assume first* $H_6 \neq 0$. Then $e \neq 0$ and due to the rescaling $(x, y, t) \mapsto (ex/2, ey/2, 2t/e)$ we may assume $e = 2$. Then the condition $B_2 = 0$ gives $b = a - 1$ and we get the family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = a - 1 + 2x + y^2,$$

for which we have

$$\mu_0 = 1, \; B_3 = -12x^2(x - y)^2, \; H_4 = 384, \; H_7 = 0, \; H_8 = -13824a, \; H_9 = -147456a^2(1 + a)^2.$$

We observe that for the systems above the conditions $H_4 \neq 0$ and $H_7 = 0$ hold. Moreover we have $\mu_0 \neq 0$ and by Lemma 5.9 these systems are non-degenerate. Therefore according to [78, Theorem 4.1] (see Table 2) we get *Config. 4.9* if $H_8 > 0$ and $H_9 \neq 0$; *Config. 4.10* if $H_8 > 0$ and $H_9 = 0$; *Config. 4.13* if $H_8 < 0$ and *Config. 4.22* if $H_8 = 0$. So we obtain exactly the respective conditions given in Diagram 3.

7.3.1.1.2 *Suppose now* $H_6 = 0$. Then $e = 0$ and we obtain the family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + y^2,$$

for which we calculate

$$B_2 = 0, \; \mu_0 = 1, \; B_3 = 12(b - a)x^2y^2, \; H_4 = 0, \; H_5 = 6144ab, \; H_1 = -1152(a + b).$$

As $\mu_0 \neq 0$ we obtain that these systems are non-degenerate (see Lemma 5.9). And since $H_4 = 0$ according to [76, Theorem 57] (see Table 4) for systems above we have the configurations of invariant lines defined by the following conditions, respectively:

(b_1) $H_5 > 0$, $H_1 > 0 \quad \Rightarrow \quad$ *Config. 5.3*;

(b_2) $H_5 > 0$, $H_1 < 0 \quad \Rightarrow \quad$ *Config. 5.5*;

(b_3) $H_5 = 0$, $H_1 > 0 \quad \Rightarrow \quad$ *Config. 5.12*;

(b_4) $H_5 = 0$, $H_1 < 0 \quad \Rightarrow \quad$ *Config. 5.16*;

(b_5) $H_5 < 0 \quad \Rightarrow \quad$ *Config. 5.4*.

Taking into account that due to $B_3 \neq 0$ the condition $H_5 \geq 0$ implies $H_1 \neq 0$ we arrive exactly to the respective conditions given in Diagram 3 in the case under consideration.

7.3.1.2 Systems with two distinct real infinite singularities. We consider the family $(\mathbf{S}_{III})$ for which we have $N = (g^2 - 1)x^2 + 2(g - 1)hxy + h^2y^2$ and hence, the condition $N = 0$ gives $h = 0$ and $g = \pm 1$. Then we may assume $c = 0$ due to a translation and we calculate: Coefficient$[B_2, y^4] = -648d^4g^2$. So the condition $B_2 = 0$ implies $d = 0$ and we get the family of systems

$$\dot{x} = a + gx^2, \quad \dot{y} = b + ex + fy + (g - 1)xy \quad (g = \pm 1), \tag{7.10}$$

for which we have $B_2 = 0$ and $B_3 = 3(g-1)(b+ef-bg)x^4$. Hence the condition $B_3 \neq 0$ implies $g \neq 1$ and therefore we obtain $g = -1$ that leads to the systems

$$\dot{x} = a - x^2, \quad \dot{y} = b + ex + fy - 2xy.$$

For these systems we calculate

$$\mu_0 = 0,\ B_3 = -6(2b+ef)x^4,\ H_6 = -2048f^2x^6,\ H_8 = H_5 = H_1 = 0,\ H_2 = -16fx^2,\ H = -4x^2.$$

We observe that in order to have an invariant line of the type $y + \beta = 0$ (i.e. to be in the class $\boldsymbol{LV}$) it is necessary either $b = e = 0$ or $b = f = 0$. However in both cases this implies a contradiction: $B_3 = 0$. So systems above could not be of Lotka-Volterra type. On the other hand as there exist two affine invariant lines $x^2 - a = 0$ we conclude that these systems belong to the the class $\mathbf{QSL_3} \cup \mathbf{QSL_4} \cup \mathbf{QSL_5} \cup \mathbf{QSL_6}$. We claim that if $H_6 \neq 0$ (i.e. $f \neq 0$) the system above belong to the class $\mathbf{QSL_3}$. Indeed according to [78, Theorem 4.1, Table 2] (respectively [76, Theorem 57, Table 4]; [76, Theorem 50, Table 2]) a system in $(\mathbf{S}_{III})$ with the conditions $N = B_2 = 0$ and $B_3 \neq 0$ could be in $\mathbf{QSL_4}$ (respectively $\mathbf{QSL_5}$; $\mathbf{QSL_6}$) only if $\mu_0 \neq 0$ (respectively $H_2 = 0$; $H_2H = 0$). However for systems above we have $\mu_0 = 0$ and $H_2H \neq 0$, as the condition $H_6 \neq 0$ implies $H_2 \neq 0$. Our claim is proved.

Assume now $H_6 = 0$, i.e. $e = 0$ and this implies $f = 0$. Then $H_2 = 0$, $H_{11} = 3072ax^4$ and $H_3 = 32ax^2$, i.e. if $H_{11} \neq 0$ then $\operatorname{sign}(H_3) = \operatorname{sign}(H_{11})$. Therefore according to [76, Theorem 57, Table 4] we get *Config. 5.22* if $H_{11} > 0$; *Config. 5.25* if $H_{11} < 0$ and *Config. 5.29* if $H_{11} = 0$. We note that in this case systems above could not be degenerate as $B_3 = -12bx^4 \neq 0$. So we obtain the respective conditions given in Diagram 3.

7.3.2 The case $B_3 = 0$: systems possessing invariant straight lines in a maximum possible of three directions

7.3.2.1 Systems with three distinct real infinite singularities. As it was shown earlier (see page 31) if $\theta = 0 = N$ via a translation systems $(\mathbf{S}_I)$ could be brought to the systems (6.20) for which we have

$$B_3 = 3e^2x^3(2y - x) - 12(a-b)x^2y^2 - 3d^2(2x - y)y^3.$$

Therefore the condition $B_3 = 0$ implies $d = e = a - b = 0$ and we get the family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = a + y^2,$$

for which calculations yield:

$$K = 4xy,\ H_6 = 0,\ H_3 = -32a(x^2 - xy + y^2),\ H_1 = -2304a.$$

We observe that if $H_3 \neq 0$ then $\operatorname{sign}(H_1) = \operatorname{sign}(H_3)$. Therefore according to [76, Theorem 50, Table 2] the configuration of invariant lines of systems above is given by *Config. 6.1* if $H_3 > 0$; *Config. 6.2* if $H_3 < 0$ and *Config. 6.5* if $H_3 = 0$. In other words considering that $K \neq 0$ and $H_6 = 0$ we obtain exactly the respective conditions given in Diagram 3.

7.3.2.2 Systems with two distinct real infinite singularities. As the condition $B_3 = 0$ implies $B_2 = 0$ it was shown earlier (see page 54) that systems in family $(\mathbf{S}_{III})$ for which $N = B_2 = 0$ via a translation could be brought to the family of systems (7.10) (with $g = \pm 1$), for which we have $B_3 = 3(g-1)(b+ef-bg)x^4$ and $K = 2g(g-1)x^2$. As $g = \pm 1$ the condition $K = 0$ is equivalent to $g = 1$ and we shall consider two cases: $K \neq 0$ and $K = 0$.

7.3.2.2.1 *Assume first $K \neq 0$.* Then $g = -1$ and in this case the condition $B_3 = -6(2b+ef)x^4 = 0$ gives $b = -ef/2$ and we get the family of systems

$$\dot{x} = a - x^2, \quad \dot{y} = (f - 2x)(2y - e)/2.$$

For these systems we have

$$\mu_0 = H_7 = 0,\ H_2 = -16fx^2,\ H_6 = -2048f^2x^6,\ H_3 = 8(4a - f^2)x^2,\ H_{11} = 3072ax^4,\ H = -4x^2.$$

We observe that the systems above are degenerate if and only if $f^2 = 4a$ and this condition is equivalent to $H_3 = 0$.

(i) Suppose $H_6 \neq 0$. As $\mu_0 = H_7 = 0$ according to [78, Theorem 4.1] (see Table 2) for a non-degenerate system (i.e. if $H_3 \neq 0$) we get *Config. 4.12* if $H_{11} > 0$; *Config. 4.15* if $H_{11} < 0$ and *Config. 4.24* if $H_{11} = 0$. In the case of degenerate systems we obtain $a = f^2/4$ (i.e. $H_3 = 0$) and as $K \neq 0$ and the condition $H_6 \neq 0$ implies $H_2 \neq 0$, according to Theorem 6.2 (see Table 3) we obtain *Config.* LV_d*.7*.

(ii) Assume now $H_6 = 0$, i.e. $f = 0$. Then $H_3 = 32ax^2$ and hence $\operatorname{sign}(H_3) = \operatorname{sign}(a)$ if $H_3 \neq 0$. As in this case the conditions $H_2 = 0$ and $H \neq 0$ hold, it follows from [76, Theorem 50, Table 2] that the configuration of invariant lines of the systems under consideration is given by *Config. 6.8* if $H_3 > 0$ and *Config. 6.9* if $H_3 < 0$. If $H_3 = 0$ (i.e. systems are degenerate) by Table 3 we get *Config.* LV_d*.8*.

7.3.2.2.2 *Admit finally* $K = 0$. In this case $g = 1$ and this implies $B_3 = 0$. So we get the systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + ex + fy, \tag{7.11}$$

for which calculations yield:

$$D = -f^2x^2y,\ N_1 = 8ex^4,\ N_2 = -4(4a + f^2)x,\ N_5 = -64ax^2,\ H = 0,$$
$$\mu_0 = \mu_1 = 0,\ \mu_2 = f^2x^2,\ \mu_3 = 2bfx^3,\ \mu_4 = (b^2 + ae^2)x^4 + afx^2y(2ex + fy).$$

We observe applying Lemma 5.9 that systems above become degenerate if and only if $D = \mu_4 = 0$.

(i) If $N_1 \neq 0$ then $e \neq 0$ and evidently the systems above could not be in the class ***LV*** (as there could not exist an invariant line of the type $y + \beta = 0$). On the other hand, as for these systems the condition $H = 0$ holds, according to [78, Theorem 4.1,Table 2] and [76, Theorem 57,Table 4] for a non-degenerate systems (i.e. if $D^2 + \mu_4^2 \neq 0$) we have the configurations of invariant lines defined by the following conditions, respectively:

- $N_2 \neq 0,\ N_5 > 0,\ D \neq 0 \quad \Rightarrow \quad$ *Config. 4.28*;
- $N_2 \neq 0,\ N_5 > 0,\ D = 0 \quad \Rightarrow \quad$ *Config. 4.29*;
- $N_2 \neq 0,\ N_5 < 0,\ D \neq 0 \quad \Rightarrow \quad$ *Config. 4.32*;
- $N_2 \neq 0,\ N_5 < 0,\ D = 0 \quad \Rightarrow \quad$ *Config. 4.33*;
- $N_2 \neq 0,\ N_5 = 0 \quad \Rightarrow \quad$ *Config. 4.39*;
- $N_2 = 0,\ D \neq 0 \quad \Rightarrow \quad$ *Config. 5.21*;
- $N_2 = 0,\ D = 0 \quad \Rightarrow \quad$ *Config. 5.28.*

Taking into considerations that in the case $D = \mu_4 = 0$ the systems become degenerate, we obtain exactly the respective conditions in Diagram 3 for this subcase.

(ii) Suppose now $N_1 = 0$, i.e. $e = 0$. If $D^2 + \mu_4^2 \neq 0$ (i.e. when systems are non-degenerate) considering $H = 0$, according to [76, Theorems 50 and 57, Tables 2 and 4] we get the configurations of invariant lines defined by the following conditions, respectively:

(c_1) $D \neq 0,\ N_5 > 0,\ N_2 \neq 0 \quad \Rightarrow \quad$ *Config. 5.13*;

(c_2) $D \neq 0,\ N_5 < 0,\ N_2 \neq 0 \quad \Rightarrow \quad$ *Config. 5.15*;

(c_3) $D \neq 0,\ N_5 = 0,\ N_2 \neq 0 \quad \Rightarrow \quad$ *Config. 5.17*;

(c_4) $D = 0,\ N_5 > 0,\ N_2 \neq 0 \quad \Rightarrow \quad$ *Config. 5.20*;

(c_5) $D = 0,\ N_5 < 0,\ N_2 \neq 0 \quad \Rightarrow \quad$ *Config. 5.24*;

(c_6) $D \neq 0,\ N_2 = 0 \quad \Rightarrow \quad$ *Config. 6.7*;

(c_7) $D = 0,\ N_2 = 0 \quad \Rightarrow \quad$ *Config. 6.7.*

Thus we observe that in the case $D^2 + \mu_4^2 \neq 0$ (when systems are non-degenerate) all the cases above correspond to the respective cases from Diagram 3 (case $B_3 = K = N_1 = 0$) except the cases $(\boldsymbol{c_2})$, $(\boldsymbol{c_3})$, $(\boldsymbol{c_6})$ and $(\boldsymbol{c_7})$. More precisely for having the equivalence of the respective conditions we must prove the following two things:

α) *the conditions* $D \neq 0$ *and* $N_5 \leq 0$ *imply* $N_2 \neq 0$; β) *the condition* $N_2 = 0$ *implies either* $N_5 > 0$ *if* $D \neq 0$ *or* $N_5 = 0$ *if* $D = 0$.

α) Assume that for systems (7.11) the condition $D \neq 0$ and $N_5 \leq 0$ are fulfilled. The we have $f \neq 0$ and $a \geq 0$ and hence $4a + f^2 > 0$, i.e. $N_2 \neq 0$.

β) Assume now that the condition $N_2 = 0$ holds for systems (7.11). Then $a = -f^2/4$ and we obtain $N_5 = 16f^2$. So $N_5 > 0$ if $D \neq 0$ and $N_5 = 0$ if $D = 0$, i.e. our claim is proved.

Assume now that the condition $D = \mu_4 = 0$ holds, i.e. for systems (7.11) in this case we have $e = f = b = 0$ and this leads to the degenerate systems

$$\dot{x} = a + x^2, \quad \dot{y} = 0,$$

for which $N_5 = -64a$. So these systems belong to the class $\boldsymbol{LV}$ if and only if $N_5 \geq 0$. According to Table 3 we get *Config.* LV_d*.11* if $N_5 > 0$ and *Config.* LV_d*.12* if $N_5 = 0$. Thus we obtain the respective conditions given by Diagram 3.

As all the cases are examined, Theorem 7.1 is completely proved. ∎

8 APPENDIX: Minimal polynomial basis of affine invariants of quadratic systems of degrees up to 12

Using the initial polynomials $C_i(\boldsymbol{a},x,y)$ $(i=0,1,2)$ and $D_j(\boldsymbol{a},x,y)$ $(i=1,2)$ from (5.4), we construct the following GL-comitants of the second degree with respect to coefficients of initial systems:

$$
\begin{array}{lll}
T_1=(C_0,C_1)^{(1)}, & T_2=(C_0,C_2)^{(1)}, & T_3=(C_0,D_2)^{(1)},\\
T_4=(C_1,C_1)^{(2)}, & T_5=(C_1,C_2)^{(1)}, & T_6=(C_1,C_2)^{(2)},\\
T_7=(C_1,D_2)^{(1)}, & T_8=(C_2,C_2)^{(2)}, & T_9=(C_2,D_2)^{(1)}.
\end{array}
$$

In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall use here a family of T-comitants expressed through C_i $(i=0,1,2)$ and D_j $(j=1,2)$:

$$
\begin{aligned}
\tilde{A}(a)=&\big(C_1,T_8-2T_9+D_2^2\big)^{(2)}/144,\\
\widetilde{B}(a,x,y)=&\big\{16D_1\big(D_2,T_8\big)^{(1)}\big(3C_1D_1-2C_0D_2+4T_2\big)\\
&+32C_0\big(D_2,T_9\big)^{(1)}\big(3D_1D_2-5T_6+9T_7\big)\\
&+2\big(D_2,T_9\big)^{(1)}\big(27C_1T_4-18C_1D_1^2-32D_1T_2+32\big(C_0,T_5\big)^{(1)}\big)\\
&+6\big(D_2,T_7\big)^{(1)}\big[8C_0(T_8-12T_9)-12C_1(D_1D_2+T_7)+D_1(26C_2D_1+32T_5)\\
&+C_2(9T_4+96T_3)\big]+6\big(D_2,T_6\big)^{(1)}\big[32C_0T_9-C_1(12T_7+52D_1D_2)-32C_2D_1^2\big]\\
&+48D_2\big(D_2,T_1\big)^{(1)}\big(2D_2^2-T_8\big)-32D_1T_8\big(D_2,T_2\big)^{(1)}+9D_2^2T_4\big(T_6-2T_7\big)\\
&+12D_1\big(C_1,T_8\big)^{(2)}\big(C_1D_2-2C_2D_1\big)+6D_1D_2T_4\big(T_8-7D_2^2-42T_9\big)\\
&-16D_1\big(C_2,T_8\big)^{(1)}\big(D_1^2+4T_3\big)+12D_1\big(C_1,T_8\big)^{(1)}\big(T_7+2D_1D_2\big)\\
&+96D_2^2\big[D_1\big(C_1,T_6\big)^{(1)}+D_2\big(C_0,T_6\big)^{(1)}\big]-16D_1D_2T_3\big(2D_2^2+3T_8\big)\\
&-4D_1^3D_2\big(D_2^2+3T_8+6T_9\big)+6D_1^2D_2^2\big(7T_6+2T_7\big)-252D_1D_2T_4T_9\big\}/(2^83^3),\\
\widetilde{D}(a,x,y)=&\Big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6)-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2\Big]/36,\\
\widetilde{E}(a,x,y)=&\big[D_1(2T_9-T_8)-3\big(C_1,T_9\big)^{(1)}-D_2(3T_7+D_1D_2)\big]/72,\\
\widetilde{F}(a,x,y)=&\big[6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)+48C_0\big(D_2,T_9\big)^{(1)}\\
&-9D_2^2T_4+288D_1\widetilde{E}-24\big(C_2,\widetilde{D}\big)^{(2)}+120\big(D_2,\widetilde{D}\big)^{(1)}\\
&-36C_1\big(D_2,T_7\big)^{(1)}+8D_1\big(D_2,T_5\big)^{(1)}\big]/144.\\
\widetilde{K}(a,x,y)=&(T_8+4T_9+4D_2^2)/72\equiv\Big(p_2(x,y),q_2(x,y)\Big)^{(1)}/4,\\
\widetilde{H}(a,x,y)=&(-T_8+8T_9+2D_2^2)/72.
\end{aligned}
$$

We note here that $\widetilde{K}(a,x,y)=K(a,x,y)/4$, $\widetilde{H}(a,x,y)=-H(a,x,y)/4$ and $\widetilde{D}(a,x,y)=D(a,x,y)$, where $D(a,x,y)$, $H(a,x,y)$ and $K(a,x,y)$ are defined in (5.6) and (5.9).

A minimal polynomial basis of T-comitants of systems (5.2) of degrees up to 12 was constructed in terms of the T-comitants above in [16]. The following are the elements of the polynomial basis of affine invariants:

$$
\begin{aligned}
A_1 &= \tilde{A}, \\
A_2 &= (C_2, D)^{(3)}/12, \\
A_3 &= \big[C_2, D_2)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/48, \\
A_4 &= (\widetilde{H}, \widetilde{H})^{(2)}, \\
A_5 &= (\widetilde{H}, \widetilde{K})^{(2)}/2, \\
A_6 &= (\widetilde{E}, \widetilde{H})^{(2)}/2, \\
A_7 &= \big[C_2, \widetilde{E})^{(2)}, D_2\big)^{(1)}/8, \\
A_8 &= \big[\widetilde{D}, \widetilde{H})^{(2)}, D_2\big)^{(1)}/8, \\
A_9 &= \big[\widetilde{D}, D_2)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/48, \\
A_{10} &= \big[\widetilde{D}, \widetilde{K})^{(2)}, D_2\big)^{(1)}/8, \\
A_{11} &= (\widetilde{F}, \widetilde{K})^{(2)}/4, \\
A_{12} &= (\widetilde{F}, \widetilde{H})^{(2)}/4, \\
A_{13} &= \big[C_2, \widetilde{H})^{(1)}, \widetilde{H}\big)^{(2)}, D_2\big)^{(1)}/24, \\
A_{14} &= (\widetilde{B}, C_2)^{(3)}/36, \\
A_{15} &= (\widetilde{E}, \widetilde{F})^{(2)}/4, \\
A_{16} &= \big[\widetilde{E}, D_2)^{(1)}, C_2\big)^{(1)}, \widetilde{K}\big)^{(2)}/16, \\
A_{17} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, D_2\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{18} &= \big[\widetilde{D}, \widetilde{F})^{(2)}, D_2\big)^{(1)}/16, \\
A_{19} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, \widetilde{H}\big)^{(2)}/16, \\
A_{20} &= \big[C_2, \widetilde{D})^{(2)}, \widetilde{F}\big)^{(2)}/16, \\
A_{21} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, \widetilde{K}\big)^{(2)}/16, \\
A_{22} &= \big[C_2, \widetilde{D})^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)} D_2\big)^{(1)}/1152, \\
A_{23} &= \big[\widetilde{F}, \widetilde{H})^{(1)}, \widetilde{K}\big)^{(2)}/8, \\
A_{24} &= \big[C_2, \widetilde{D})^{(2)}, \widetilde{K}\big)^{(1)}, \widetilde{H}\big)^{(2)}/32, \\
A_{25} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, \widetilde{E}\big)^{(2)}/16, \\
A_{26} &= (\widetilde{B}, \widetilde{D})^{(3)}/36, \\
A_{27} &= \big[\widetilde{B}, D_2)^{(1)}, \widetilde{H}\big)^{(2)}/24, \\
A_{28} &= \big[C_2, \widetilde{K})^{(2)}, \widetilde{D}\big)^{(1)}, \widetilde{E}\big)^{(2)}/16, \\
A_{29} &= \big[\widetilde{D}, \widetilde{F})^{(1)}, \widetilde{D}\big)^{(3)}/96, \\
A_{30} &= \big[C_2, \widetilde{D})^{(2)}, \widetilde{D}\big)^{(1)}, \widetilde{D}\big)^{(3)}/288, \\
A_{31} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, \widetilde{K}\big)^{(1)}, \widetilde{H}\big)^{(2)}/64, \\
A_{32} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, D_2\big)^{(1)}, \widetilde{H}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{33} &= \big[\widetilde{D}, D_2)^{(1)}, \widetilde{F}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128, \\
A_{34} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, D_2\big)^{(1)}, \widetilde{K}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{35} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, \widetilde{E}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128, \\
A_{36} &= \big[\widetilde{D}, \widetilde{E})^{(2)}, \widetilde{D}\big)^{(1)}, \widetilde{H}\big)^{(2)}/16, \\
A_{37} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, \widetilde{D}\big)^{(1)}, \widetilde{D}\big)^{(3)}/576, \\
A_{38} &= \big[C_2, \widetilde{D})^{(2)}, \widetilde{D}\big)^{(2)}, \widetilde{D}\big)^{(1)}, \widetilde{H}\big)^{(2)}/64, \\
A_{39} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, \widetilde{F}\big)^{(1)}, \widetilde{H}\big)^{(2)}/64, \\
A_{40} &= \big[\widetilde{D}, \widetilde{D})^{(2)}, \widetilde{F}\big)^{(1)}, \widetilde{K}\big)^{(2)}/64, \\
A_{41} &= \big[C_2, \widetilde{D})^{(2)}, \widetilde{D}\big)^{(2)}, \widetilde{F}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{42} &= \big[\widetilde{D}, \widetilde{F})^{(2)}, \widetilde{F}\big)^{(1)}, D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "[" is used in order to avoid placing the otherwise necessary up to five parenthesizes "(".

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