Integrability conditions for two-component Bose-Einstein condensates in periodic potentials

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Abstract. We consider a set of coupled Gross-Pitaevskii equations as a model for a two-component Bose-Einstein condensate. No assumption is made on the signs or magnitudes of the relevant parameters like the scattering lengths and the coupling coefficients. The formalism is therefore valid for asymmetric as well as symmetric coupled condensate wave states. The question of integrability is then addressed. We explicitly show that Hirota’s bilinear method is one of the simplest analyses to obtain the integrability conditions of the coupled Gross-Pitaevskii equations. In the obtained region of integrability, we generate analytical soliton solutions and, in a special case, we decouple the Gross-Pitaevskii equations under consideration. This allows us to identify a large class of solutions in terms of Jacobian elliptic functions.

Résumé. On considère un ensemble d’équations de Gross-Pitaevskii couplées comme modèle d’un condensat de Bose-Einstein à deux composantes sans aucune hypothèse sur les signes ou grandeurs des paramètres, telles la longueur des scatterings et les coefficients du couplage. Le formalisme est donc valide pour les états ondulaires d’un condensat asymétrique ou symétrique. On montre explicitement que la méthode d’Hirota bilinéaire est l’une des méthodes les plus simples pour obtenir les
conditions d’intégrabilité des équations de Gross-Pitaevskii couplées. On obtient les solutions-solitons analytiques dans la région d’intégrabilité et, dans un cas spécial, on découple l’équation de Gross-Pitaevskii, ce qui permet d’identifier une grande classe de solutions exprimées au moyen des fonctions de Jacobi elliptiques.

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I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation,

\[ i\partial_t \psi + \nabla^2 \psi + \gamma |\psi|^2 \psi = 0, \]

is a canonical and universal equation which is of major importance in continuum mechanics, plasma physics, nonlinear optics, nonlinear transmission lines, and condensed matter, where it describes the behavior of a weakly interacting Bose gas and is known as the Gross-Pitaevskii (GP) equation [1–8]. The reason for its importance and ubiquity is that it describes the evolution of the envelope of an almost monochromatic wave in a conservative system of weakly nonlinear dispersive waves. Similarly, systems of coupled nonlinear Schrödinger equations have been used to describe motions and interactions of more than one wave envelopes in cases where more than one order parameter is needed to specify the system. The coupled NLS equations have received a lot of attention with recent experimental advances in multi-component Bose-Einstein condensates (BECs) [9–16]. Multi-component BECs in the mean field approximation can be described by the coupled nonlinear Schrödinger equations, also called coupled Gross-Pitaevskii (CGP) equations [17]. BECs can excite various exotic topological defects and provide a perfect testing ground to investigate their physics, because almost all parameters of the system can be controlled experimentally. Topological defects in two-component BECs have been predicted theoretically, but there is still no understanding of what the complete families of these defects and solitary waves are, nor of their properties, formation mechanisms and dynamics.

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In theoretical and experimental investigations, the evolution of coupled BEC wavepackets was recently considered [18–20]. Pairs of nonlinearly coupled BECs are thus modelled via CGP equations, involving extra coupling terms whose sign and/or magnitude are \textit{a priori} not prescribed. Although theoretical modelling, quite naturally, first involved symmetric pairs of (identical) BECs, for simplicity, evidence from experiments suggests that asymmetric boson pairs deserve attention [21]. In this work, we investigate nonlinearly coupled BEC pair and present a soliton solution. Both BECs are assumed to lie in the ground state, for simplicity, although no other assumption is made on the sign and/or magnitude of the relevant physical parameters.

The two-component BEC is described by CGP equations

\[
i\hbar \frac{\partial \psi_j}{\partial t} = \left( -\frac{\hbar^2}{2m_j} \frac{\partial^2}{\partial x^2} + V_j(x) + \frac{4\pi\hbar^2 a'_{jj}}{m_j} |\psi_j|^2 + \frac{2\pi\hbar^2 a'_{j,3-j}}{m_{j,3-j}} |\psi_{3-j}| + \mu_j \right) \psi_j,
\]

\( j = 1, 2 \). Here, \( \psi_j(x,t) \) is the wave function for the \( j \)-th condensate and \( V_j(x) \) is the external potential experienced by the \( j \)-th condensate. In most situations \( V_j(x) = V(x) \) for all \( j = 1, 2 \). However, because of the different properties of the constituting atoms of the different condensates, it is possible for different condensates in the same physical trap to experience different external potentials. Also, \( m_j \) is the atom mass of the atom species of the \( j \)-th condensate; \( m_{j,3-j} = m_j m_{3-j}/(m_j + m_{3-j}) \) is the reduced mass corresponding to atom species of the \( j \)-th and \( l \)-th condensate; finally, \( a'_{jj} \) and \( a'_{12} = a'_{21} \) are, respectively, the scattering lengths of intra- and inter-species atomic collisions (\( j = 1, 2 \)). The (linear) last terms in each equation involve the chemical potential \( \mu_j \), which corresponds to a ground state of the condensate in a simplified model. These terms can be readily eliminated via a simple phase-shift transformation, viz. \( \psi_j = \tilde{\psi}_j \exp (i\mu_j t) \).

In this paper, we first apply the Hirota bilinear transformation to build the integrability conditions for the coupled GP equations (1) and then, under these integrability conditions, we find exact solutions that generate soliton solutions of Eqs. (1) with null potential. Secondly, in a special case of the integrability conditions to be obtained, we decouple the coupled GP Eqs. (1) and present some other exact solutions.
II. INTEGRABILITY CONDITIONS FOR COUPLED BOSE-EINSTEIN CONDENSATES

The coupled Gross-Pitaevskii equations (1) can be made non-dimensional by re-scaling the dependent and independent variables. After this re-scaling and after a simple phase-shift transformation, the equations have the form

\[ i \frac{\partial \psi_j}{\partial t} = \left(-\frac{1}{2c_j} \frac{\partial^2}{\partial x^2} + V_j(x) + a_{jj} |\psi_j|^2 + a_{j,3-j} |\psi_{3-j}|^2 \right) \psi_j. \] (2)

We have not introduced new symbols for the dimensionless quantities, so as not to overburden the notation. The interaction matrix \( A = (a_{jl})_{j,l=1,2} \) contains all information about the nature of the interatomic interactions. Its entries are referred to as interaction coefficients. The parameters \( c_j, j = 1, 2 \), play the role of effective masses. Current experiments [10, 11, 22] with multiple condensates use different spin states or distinct isotopes of one atomic species, so that all \( c_j, j = 1, 2 \), are equal. Moreover, the parameters \( c_1 \) and \( c_2 \) can be rescaled by another scaling transformation, so that effectively \( c_1 = c_2 = 1 \).

In the quasi-1D limit, a condensate has three independent interaction coefficients,

\[ \gamma_{j,3-j} = 2\beta_{j,3-j} / (A_j + A_{3-j}), \]

where \( \beta_{j,3-j} = \beta_{3-jj} \) is the 3D s-wave scattering length for collisions between atoms in states \( j \) and \( 3-j \), and \( A_j \) is the cross-sectional area of the trap confining species \((j = 1, 2)\). We then re-scale \( \psi_{1,2} \) to set the diagonal coefficients to unity \((\psi_j \rightarrow \psi_j / \sqrt{\gamma_{jj}})\), thus obtaining the two off-diagonal coefficients \( a_{j,3-j} \) which appear in Eqs. (2). One can therefore show that

\[ a_{12} = \frac{\beta_{12}}{\beta_{11}} \left(1 + \frac{A_1 - A_2}{A_1 + A_2} \right), \quad a_{21} = \frac{\beta_{12}}{\beta_{22}} \left(1 - \frac{A_1 - A_2}{A_1 + A_2} \right). \]

This proves that one can have \( a_{12} \neq a_{21} \) in Eqs. (2) (even if \( c_1 = c_2 \)).

A large class of periodic potentials is given by \( V_j(x) = -V_{j0} \text{sn}^2(mx,k) \) with \(-V_{j0}\) as amplitude parameter. Here \( \text{sn}(mx,k) \) is the Jacobian elliptic sine function with elliptic modulus \( 0 \leq k \leq 1 [17] \). In the limit as \( k \rightarrow 1 - 0 \), each of the \( V_j(x) \) becomes an array of well-separated hyperbolic secant potential barriers or wells, while in the limit as \( k \rightarrow +0 \) it becomes purely sinusoidal. Because \( \text{sn}(mx,k) \) is a periodic function with period

\[ \frac{4K(k)}{m} = 4 \int_0^{\pi/2} d\zeta / \sqrt{1 - k^2 \sin^2 \zeta}, \]
each potential \( V_j(x) \) is a periodic function with period \( 2K(k)/m \). Moreover, the period \( 4K(k)/m \) approaches infinity as \( k \to 1 - 0 \). Thus, as \( k \to 1 - 0 \), the potential \( V_j(x) \) is a periodic lattice of separated peaks or troughs, depending on the sign of \( V_j^0 \). The reason for considering potentials that are more general than the standing light wave potential is that, by changing the parameter \( k \), various interesting regimes of the BECs are considered.

Generally, the coupled Gross-Pitaevskii equations (2) are not integrable. However, in the case \( a_{jj} = a_{j,3-j} \) \((a_{11} = a_{22} = a_{12} = a_{21})\) and \( c_1 = c_2 \), Eqs. (2) with zero potential amount to an integrable Manakov system [23, 24]. In this case, some exact solutions are generated in Ref. [12] by means of the Darboux transformation [25, 26]. Exact solutions with nonzero potential in the present case can be found in Ref. [27, 28]. In this section, we apply the Hirota method to build a new integrability conditions for Eqs. (2) which contain the already known integrability conditions. Then, under these conditions, we construct some exact solutions of Eqs. (2) that generate soliton solutions. In a special case where \( c_1 = c_2 \), we reduce the coupled GP Eqs. (2) to a single GP equation and find some exact solutions.

### A. Integrability conditions and soliton solutions

To obtain new integrability conditions and some exact solutions for the coupled GP Eqs. (2), we follow Hirota [29] and introduce the following transformation

\[
\psi_j = \frac{\Psi_j}{\Phi}, \quad j = 1, 2, 3
\]

where \( \Psi_j \) are complex functions and \( \Phi \) is a real function. If we insert ansatz (3) into Eqs. (2), we obtain

\[
\Phi \left[ i \left( \Phi \frac{\partial \Psi_j}{\partial t} - \Psi_j \frac{\partial \Phi}{\partial t} \right) + \frac{1}{2c_j} \left( \Phi \frac{\partial^2 \Psi_j}{\partial x^2} - 2 \frac{\partial \Phi \partial \Psi_j}{\partial x} + \Psi_j \frac{\partial^2 \Phi}{\partial x^2} \right) \right]
- \left[ \frac{1}{c_j} \left( \Phi \frac{\partial^2 \Phi}{\partial x^2} - \left( \frac{\partial \Phi}{\partial x} \right)^2 \right) + V_j \Phi + a_{jj} |\Psi_j|^2 + a_{j,3-j} |\Psi_{3-j}|^2 \right] \Psi_j = 0.
\]

A particular solution of this equation is obtained if each of the two terms vanishes:

\[
\begin{cases}
2c_j i \left( \Phi \frac{\partial \Psi_j}{\partial t} - \Psi_j \frac{\partial \Phi}{\partial t} \right) + \Phi \frac{\partial^2 \Psi_j}{\partial x^2} - 2 \frac{\partial \Phi \partial \Psi_j}{\partial x} + \Psi_j \frac{\partial^2 \Phi}{\partial x^2} = 0, \\
\frac{1}{c_j} \left( \Phi \frac{\partial^2 \Phi}{\partial x^2} - \left( \frac{\partial \Phi}{\partial x} \right)^2 \right) + V_j \Phi + a_{jj} |\Psi_j|^2 + a_{j,3-j} |\Psi_{3-j}|^2 = 0.
\end{cases}
\]

The second equation in system (4) will always be verified as soon as

\[
\frac{c_1}{c_2} = \frac{a_{21}}{a_{11}} = \frac{a_{22}}{a_{12}} = \frac{V_{20}}{V_{10}}.
\]
In the case of zero potential \((V_j(x) = 0)\), the last term in Eq. (5) is omitted. These conditions are the integrability conditions of system (2) by means of Hirota’s method. It is evident that Eqs. (5) admit the already known integrability conditions \(a_{11} = a_{12} = a_{21} = a_{22}, c_1 = c_2\) when \(V_j(x) = 0\). Under these last integrability conditions Eqs. (2) amount to a Manakov system [23, 24] and, in this case, an exact vectorial solution, which accounts for a soliton interacting with a plane wave, is found in Ref. [12] by means of the Darboux transformation. Because conditions (5) are in terms of almost all parameters, we believe that these conditions may be very useful for the experimental generation of exact solutions of the coupled GP equations (2).

Now we construct exact solutions of system (2) under conditions (5), in the case of zero potential \((V_j(x) = 0, j = 1, 2)\). For this purpose, we expand the dependent variables \(\Psi_j\) and \(\Phi\) in powers of real \(\epsilon\): \(\Psi_j = \Psi_{j1}\epsilon\) and \(\Phi = 1 + \Phi_2\epsilon^2\). Inserting this ansatz in the first Eq. (4) yields

\[
i \frac{\partial \Psi_{j1}}{\partial t} + \frac{1}{2} \frac{\partial^2 \Psi_{j1}}{\partial x^2} = 0,
\]

and

\[
\frac{1}{2c_j} \left( \Phi_2 \frac{\partial^2 \Psi_{j1}}{\partial x^2} - 2 \frac{\partial \Psi_{j1}}{\partial x} \frac{\partial \Phi_2}{\partial x} + \Psi_{j1} \frac{\partial^2 \Phi_2}{\partial x^2} \right) + i \left( \Phi_2 \frac{\partial \Psi_{j1}}{\partial t} - \Psi_{j1} \frac{\partial \Phi_2}{\partial t} \right) = 0.
\]

The linear Schrödinger equation (6) gives

\[
\Psi_{11}(x,t) = \exp (\alpha_1) \quad \text{and} \quad \Psi_{21}(x,t) = \exp (\alpha_2 + \xi),
\]

where

\[
\alpha_j = \alpha_j(x,t) = k_1 x + i \frac{k_1^2}{2c_j} t + k_{10}
\]

and \(k_1, \xi,\) and \(k_{10}\) are two complex constants. If we insert \(\Psi_{j1}(x,t)\) into Eq. (7) and use the second Eq. (4), we obtain

\[
\Phi_2 = \frac{c_j}{(k_1 + k_1^*)^2} \left[ a_{jj} \exp (\alpha_j + \alpha_j^*) + a_{j,3-j} \exp (\alpha_j + \alpha_j^* + \xi + \xi^*) \right].
\]

Thus, under the integrability conditions (5), we have generated the following exact vectorial (two-component) soliton solutions of the coupled GP equations (2):

\[
\psi_j(x,t) = \frac{\epsilon \Psi_{j1}(x,t)}{1 + \epsilon^2 \Phi_2(x,t)}.
\]
\[ \Psi_{11}(x,t) = \exp(\alpha_1), \]
\[ \Psi_{21}(x,t) = \exp(\alpha_2 + \xi), \]
\[ \Phi_2(x,t) = \frac{c_j}{(k_1 + k_1^*)^2} \left[ a_{jj} \exp(\alpha_j + \alpha_j^*) + a_{j,3-j} \exp(\alpha_{3-j} + \alpha_{3-j}^* + \xi + \xi^*) \right], \quad (9) \]
\[ \alpha_j = \alpha_j(x,t) = k_1 x + i \frac{k_1^2}{2c_j} t + k_{10}, \]

Solutions (8) with data (9) contain four parameters, namely, the real parameter \( \epsilon \) and the three complex parameters \( k_{10}, k_1, \) and \( \xi, \) with \( k_1 \notin i\mathbb{R}. \) These parameters have to be chosen from the condition that \( 1 + \epsilon^2 \Phi_2(x,t) \neq 0. \) In the case where either \( \epsilon \to 0 \) or \( \epsilon \to \infty, \) solutions (8) amount to \( \psi_j(x,t) = 0. \)

From a detailed investigation, we find that the coupled GP equations (2) admit a higher order soliton solutions for the integrability conditions \( a_{12} = a_{21} = a_{11} = a_{22}, \) \( c_1 = c_2, \) and the resultant solutions are not in agreement with the results reported earlier [12]. So the Darboux transformation method and the Hirota bilinear method in the case of the integrability conditions \( a_{12} = a_{21} = a_{11} = a_{22}, \) \( c_1 = c_2 \) do not give the same solutions of Eqs. (2). Hence the solutions obtained in this paper in the case when \( a_{12} = a_{21} = a_{11} = a_{22}, \) \( c_1 = c_2 \) are new and increase the family of exact solutions of Eqs. (2) under the mentioned integrability conditions. However, the results obtained here for these integrability conditions are consistent with those reported in Refs. [23, 30], which were obtained by means of Painlevé’s method [23, 30].

In the special case where \( c_1 = c_2 \) (as it has been done above, we can then take \( c_j = 1 \) and \( V_{10} = V_{20} = V_0, \) the coupled GP equations (2) admit, under the integrability conditions (5), the particular vector solutions \( (\psi_1, \psi_2) = (\psi, \alpha \psi), \) where \( \alpha \) is any (real or complex) constant, and \( \psi \) is a condensate wave function which, in scaled form, is described by the one-dimensional single GP equation

\[ i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V_0 \text{sn}^2(mx,k)\psi + (a_{11} + |\alpha|^2 a_{22}) |\psi|^2 \psi. \quad (10) \]

Some exact solutions of Eq. (10) and their stability are discussed in detail in Refs. [7, 31] for \( V_0 \neq 0. \) These solutions are given by

\[ \psi(x,t) = r(x) \exp(i\theta(x) - i\omega t), \]
with \( \theta'(x) = c/r^2 \). Relation \( \theta'(x) = c/r^2 \) is an expression of conservation of angular momentum. Null angular momentum solutions, which constitute an important special case, satisfy \( c = 0 \). As an example of these solutions, we can write (see Refs. [7, 31])

\[
r^2(x) = A \sin^2(mx) + B,
\]

with

\[
A = \frac{V_0 + mk^2}{a_{11} + |\alpha|^2 a_{22}},
\]

\[
\omega = \frac{1}{2} m^2 \left( 1 + k^2 \right) + \frac{(a_{11} + |\alpha|^2 a_{22})(3m^2k^2 + 2V_0)}{2(V_0 + mk^2)} B,
\]

\[
c^2 = m^2 B \left[ 1 + \frac{B(a_{11} + |\alpha|^2 a_{22})}{V_0 + mk^2} \right] \left[ \frac{V_0 + mk^2}{a_{11} + |\alpha|^2 a_{22}} + k^2 B \right].
\]

For a given potential, this solution has two free parameters, \( B \) and \(|\alpha|\), that play the role of constant background level offset. The freedom in choosing the potential gives a total of five parameters, \( V_0, k, m, |\alpha|, \) and \( B \). The requirements that both \( r^2(x) \) and \( c^2 \) be positive imposes conditions on the range of these parameters, either

\[
\frac{V_0 + mk^2}{a_{11} + |\alpha|^2 a_{22}} > 0 \quad \text{and} \quad B \geq 0,
\]

or

\[
-\frac{V_0 + mk^2}{k^2(a_{11} + |\alpha|^2 a_{22})} \geq B \geq -\frac{V_0 + mk^2}{a_{11} + |\alpha|^2 a_{22}} > 0.
\]

These conditions are in terms of \( B \) and \(|\alpha|\).

Let us examine the case where \( V_0 = 0 \). Solutions (8) with data (9) in the special case where \( c_1 = c_2 = 1 \) satisfy \( \psi_2 = \alpha \psi_1 \) only when \( a_{11} = a_{22} = a_{12} = a_{21} \). Here we find \( \alpha = \exp(\xi) \). One can seek other solutions of Eqs. (2) in this special case by solving the second order ordinary differential equation

\[
r'' = -2\omega r + 2 \left( a_{11} + |\alpha|^2 a_{22} \right) r^3 + \frac{C^2}{r^3},
\]

where the constant \( C \) is defined by \( \theta'(x) = C/r^2 \). Then

\[
\psi(x, t) = r(x) \exp(i\theta(x) - i\omega t)
\]

will be a solution of Eq. (10) with zero potential \( (V_0 = 0) \). For the integration of this equation, one may first find a first integral and put it in the form

\[
\zeta^2 = -2C^2 + K \zeta - 8\omega \zeta^2 + 4 \left( a_{11} + |\alpha|^2 a_{22} \right) \zeta^3,
\]
which is an elliptic ordinary differential equation. Depending on the constants $C$ and $K$, it can admit nonnegative soliton solutions and bounded nonnegative and periodic solutions in terms of Jacobian elliptic functions, where $\zeta = r^2$ and $K$ is a constant. For example, when $K = 0$ and $C = 0$, we find the soliton solution

$$r^2(x) = \zeta(x) = \frac{-8\omega}{(a_{11} + |\alpha|^2a_{22})} \frac{1}{-2 \pm \cos \left(2\sqrt{-2\omega}x\right)},$$

if $a_{11} + |\alpha|^2a_{22} < 0$ and for every $\omega < 0$.

### III. CONCLUSION

We have shown that Hirota’s bilinear method is one of the simplest methods to identify the integrability of a binary Bose-Einstein condensates (BEC) described by a coupled nonlinear Schrödinger equations with many parameters. The bilinear method used in this work is very simple when compared with the Painlevé analysis. In the integrability region, we have found that the coupled Gross-Pitaevskii (CGP) equations under consideration with zero potential admit the higher order soliton solutions and we have generated these soliton solutions. In a special case we decouple the coupled GP equations under consideration, which allows us to identify some exact non-stationary solutions of these CGP equations. The general bilinear integrability conditions obtained through a bilinear form may be useful for the experimental generation of solitons in a binary BEC with zero potential. Our results can be tested, and can hopefully be confirmed by designed experiments.

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