Moduli spaces of planar quadratic vector fields with invariant lines of total multiplicity at least four and three distinct infinite singularities

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Abstract

In this article we consider the class \( QSL_4 \) of all real quadratic differential systems \[
\frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y)
\] with \( \gcd(p, q) = 1 \), having invariant lines of total multiplicity four and three distinct infinite singularities (real or complex). Firstly we enlarge the canonical forms for the class \( QSL_4 \) constructed in [26] so as to include limit points in the 12-dimensional parameter space of the set \( QSL_4 \). We next construct bifurcation diagrams for the canonical forms thus obtained. Finally we construct the moduli spaces under the action of the group of affine transformations and time homotheties, for all these families of systems.

Résumé

Dans cet article nous considérons la classe \( QSL_4 \) de tous les systèmes différentiels quadratiques \[
\frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y)
\] avec \( \gcd(p, q) = 1 \), ayant des droites invariantes de multiplicité totale quatre et trois points singuliers distincts à l’infini. Nous elargissons d’abord les formes canoniques de la classe \( QSL_4 \) construites en [26] afin d’inclure des points limites dans l’espace de dimension 12 des paramètres de la classe \( QSL_4 \). Ensuite nous construisons les diagrammes de bifurcation pour les formes canoniques ainsi obtenues. Finalement nous construisons les espaces de modules sous l’action du groupe des transformations affines et des hohotheties du temps.
1 Introduction

We consider here real planar differential systems of the form
\[ (S) \quad \frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y), \]
where \( p, q \in \mathbb{R}[x, y] \), i.e. \( p, q \) are polynomials in \( x, y \) over \( \mathbb{R} \), their associated vector fields
\[ \tilde{D} = p(x, y) \frac{\partial}{\partial x} + q(x, y) \frac{\partial}{\partial y} \]
and differential equations
\[ q(x, y)dx - p(x, y)dy = 0. \]
We call degree of a system (1.1) (or of a vector field (1.2) or of a differential equation (1.3)) the integer \( n = \deg(S) = \max(\deg p, \deg q) \). In particular we call quadratic a differential system (1.1) with \( n = 2 \).

A system (1.1) is said to be integrable on an open set \( U \) of \( \mathbb{R}^2 \) if there exists a \( C^1 \) function \( F(x, y) \) defined on \( U \) which is a first integral of the system, i.e. such that \( \tilde{D}F(x, y) = 0 \) on \( U \) and which is nonconstant on any open subset of \( U \). The cases of integrable systems are rare but as Arnold said in [1, p. 405] “...these integrable cases allow us to collect a large amount of information about the motion in more important systems...”.

In [7], Darboux gave a method of integration of planar polynomial differential equations in terms of invariant algebraic curves. Roughly speaking, an invariant algebraic curve of system (1.1) is a curve \( f(x, y) = 0, f(x, y) \in \mathbb{R}[x, y] \) which is invariant under the flow. For a number of reasons it is not convenient however to consider curves over the reals. Firstly \( \mathbb{R} \) is not an algebraically closed field. In particular the curve \( x^2 + y^2 + 1 = 0 \) is empty over the reals. As all systems (1.1) over the reals generate systems over \( \mathbb{C} \), it is better to talk about invariant algebraic curves over \( \mathbb{C} \) of differential systems (1.1) over \( \mathbb{C} \). The theory of Darboux was done over \( \mathbb{C} \) and it turns out that it can also be applied for real systems. Darboux formulated his theory for differential equations over the complex plane.

Definition 1.1 (Darboux [7]). An affine algebraic curve \( f(x, y) = 0, f \in \mathbb{C}[x, y], \deg f \geq 1 \) is invariant for a system (1.1) if and only if \( f | \tilde{D}f \) in \( \mathbb{C}[x, y] \), i.e. \( k = \frac{\tilde{D}f}{f} \in \mathbb{C}[x, y] \). In this case \( k \) is called the cofactor of \( f \).

Definition 1.2 (Darboux [7]). An algebraic solution of an equation (1.3) (respectively (1.1), (1.2)) is an invariant algebraic curve \( f(x, y) = 0, f \in \mathbb{C}[x, y], \deg f \geq 1 \) with \( f \) an irreducible polynomial over \( \mathbb{C} \).

Darboux showed that if an equation (1.3) or (1.1) or (1.2) possesses a sufficient number of such invariant algebraic solutions \( f_i(x, y) = 0, f_i \in \mathbb{C}[x, y], i = 1, 2, \ldots, s \) then the equation has a first integral of the form \( F = f_1(x, y)^{\lambda_1} \cdots f_s(x, y)^{\lambda_s}, \lambda_i \in \mathbb{C} \).

Definition 1.3. An expression of the form \( F = e^{G(x, y)}, G(x, y) \in \mathbb{C}(x, y) \), i.e. \( G \) is a rational function over \( \mathbb{C} \), is an exponential factor for a system (1.1) or an equation (1.3) if and only if \( k = \frac{\tilde{D}F}{F} \in \mathbb{C}[x, y] \). In this case \( k \) is called the cofactor of the exponential factor \( F \).

\[^5\text{Under the name degenerate invariant algebraic curve this notion was introduced by Christopher in [6].}\]
Proposition 1.1 (Christopher [6]). If an equation (1.3) admits an exponential factor $e^{G(x,y)}$ where $G(x,y) = \frac{G_1(x,y)}{G_2(x,y)}$, $G_1, G_2 \in \mathbb{C}[x,y]$ then $G_2(x,y) = 0$ is an invariant algebraic curve of (1.3).

Definition 1.4. We say that a system (1.1) or an equation (1.3) has a Darboux first integral (respectively Darboux integrating factor) if it admits a first integral (respectively integrating factor) of the form $e^{G(x,y)} \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$, where $G(x,y) \in \mathbb{C}(x,y)$ and $f_i \in \mathbb{C}[x,y]$, deg $f_i \geq 1$, $i = 1, 2, \ldots, s$, $f_i$ irreducible over $\mathbb{C}$ and $\lambda_i \in \mathbb{C}$. A system (1.1) or an equation (1.3) has a Liouvillian first integral (respectively a Liouvillian integrating factor) if it admits a first integral (respectively integrating factor) which is a Liouvillian function, i.e. a function which is built up from rational functions over $\mathbb{C}$ using exponentiation, integration and algebraic functions.

Proposition 1.2 (Darboux [7]). If an equation (1.3) (or (1.1), or (1.2)) has an integrating factor (or first integral) of the form $F = \prod_{i=1}^{s} f_i^{\lambda_i}$ then $\forall i \in \{1, \ldots, s\}$, $f_i = 0$ is an algebraic invariant curve of (1.3) (1.1), (1.2)).

In [7] Darboux proved the following theorem of integrability using invariant algebraic solutions of differential equation (1.3):

Theorem 1.1 (Darboux [7]). Consider a differential equation (1.3) with $p,q \in \mathbb{C}[x,y]$. Let us assume that $m = \max(\deg p, \deg q)$ and that the equation admits $s$ algebraic solutions $f_i(x,y) = 0$, $i = 1, 2, \ldots, s$ (deg $f_i \geq 1$). Then we have:

I. If $s = m(m+1)/2$ then there exists $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s \setminus \{0\}$ such that $R = \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$ is an integrating factor of (1.3).

II. If $s \geq m(m+1)/2 + 1$ then there exists $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s \setminus \{0\}$ such that $F = \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$ is a first integral of (1.3).

The simplest class of integrable quadratic systems due to the presence of invariant algebraic curves is the class of integrable quadratic systems due to the presence of invariant lines. The study of this class was initiated in articles [21,23–26]

An important ingredient in the classification of quadratic systems possessing invariant lines is the notion of configuration of invariant lines of a polynomial differential system.

Definition 1.5. We call configuration of invariant lines of a system (1.1) the set of all its (complex) invariant lines (which may have real coefficients), each endowed with its own multiplicity [21] and together with all the real singular points of this system located on these lines, each one endowed with its own multiplicity.

Notation 1.1. We denote by $\text{QSL}_4$ the class of all real quadratic differential systems (1.1) with $p, q$ relatively prime ($(p,q) = 1$), with only a finite number of singularities at infinity, and possessing a configuration of invariant straight lines of total multiplicity $M_{IL} = 4$ including the line at infinity and including possible multiplicities of the lines.

The study of $\text{QSL}_4$ was initiated in [23] where we proved a theorem of classification for this class. This classification, which is taken modulo the action of the group of real affine transformations and time rescaling, is given in terms of algebraic invariants and comitants and also geometrically, using cycles on the complex projective plane and on the line at infinity.

The following two results were proved in [26].
Theorem 1.2. Consider a quadratic system (1.1) in $QSL_4$. Then this system has either a polynomial inverse integrating factor which splits into linear factors over $C$ or an integrating factor which is Darboux generating in the usual way a Liouvillean first integral. Furthermore the quotient set of $QSL_4$ under the action of the affine group and time rescaling is formed by: a set of 20 orbits; a set of twenty-three one-parameter families of orbits and a set of ten two-parameter families of orbits. A system of representatives of the quotient space is also given.

Theorem 1.3. i) The total number of topologically distinct phase portraits in the class $QSL_4$ is 69.

ii) We give necessary and sufficient conditions, invariant with respect to the action of the affine group and time rescaling, for the realization of each one of the phase portraits corresponding to all possible configuration of invariant lines.

In this article we continue the work begun in [26]. The problems we are concerned with here, and which we solve in this article, are the following:

- We want to show how the phase portraits for $QSL_4$ found in [26] vary when we vary the parameters. As the systems in $QSL_4$ modulo the group action split into a set of isolated points, a set of two-parameter families of systems and a set of one parameter families of systems, we want to construct bifurcation diagrams for the two or one-parameter families of systems. The parameter spaces are here the affine plane or the affine line. We actually want to do more, namely to complete the affine plane to the projective plane (respectively the affine line to the projective line) including on the line at infinity (respectively on the point at infinity), systems which belong to the border set of $QSL_4$ in the 12-parameter space.

- We want to construct moduli spaces with respect to the group action for all the completed families of systems mentioned above.

This article is organized as follows:

In Section 2 we give the preliminary definitions and results needed in this article.

In Section 3 we construct canonical forms for all families of quadratic systems $(S) \in QS$ with invariant lines of total multiplicity at least four, completed with systems occurring at border points of $QSL_4$ in the parameter space.

In Section 4 we construct bifurcation diagrams for all the canonical forms obtained in Section 3. We also construct the moduli spaces obtained from these canonical forms when we identify points via the action of the group of affine transformations and time rescaling.

\section{Preliminary statements and definitions}

Consider real differential systems of the form:

\begin{equation}
(S) \left\{ \begin{array}{l}
\frac{dx}{dt} = p_0(a) + p_1(a, x, y) + p_2(a, x, y) \equiv p(a, x, y), \\
\frac{dy}{dt} = q_0(a) + q_1(a, x, y) + q_2(a, x, y) \equiv q(x, y),
\end{array} \right. \tag{2.1}
\end{equation}

where $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ and

$p_0 = a_{00}, \quad p_1(x, y) = a_{10}x + a_{01}y, \quad p_2(x, y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\
q_0 = b_{00}, \quad q_1(x, y) = b_{10}x + b_{01}y, \quad q_2(x, y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2.$

Notation 2.1. $\mathbb{R}[a, x, y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$. Let us denote by $a = (a_{00}, a_{10}, \ldots, b_{02})$ a point in $\mathbb{R}^{12}$. Each particular system (2.1) yields an ordered 12-tuple $a$ formed by its coefficients.
2.1 Divisors and zero-cycles associated to invariant lines configurations

Notation 2.2. Let

\[ P(X, Y, Z) = p_0(a)Z^2 + p_1(a, X, Y)Z + p_2(a, X, Y) = 0, \]
\[ Q(X, Y, Z) = q_0(a)Z^2 + q_1(a, X, Y)Z + q_2(a, X, Y) = 0. \]

We denote \( \sigma(P, Q) = \{ w \in \mathbb{P}_2(\mathbb{C}) \mid P(w) = Q(w) = 0 \}. \)

Definition 2.1. A formal expression of the form \( D = \sum n(w)w \) where \( w \in \mathbb{P}_2(\mathbb{C}) \), \( n(w) \) is an integer and only a finite number of the numbers \( n(w) \) are not zero, will be called a zero-cycle of \( \mathbb{P}_2(\mathbb{C}) \) and if \( w \) only belongs to the line \( Z = 0 \) will be called a divisor of this line. We call degree of the expression \( D \) the integer \( \deg(D) = \sum n(w) \). We call support of \( D \) the set \( \text{Supp}(D) \) of points \( w \) such that \( n(w) \neq 0 \).

We define below the geometrical objects (divisors or zero-cycles) which play an important role in constructing the invariants of the systems.

Definition 2.2.

\[ D_s(P, Q) = \sum_{w \in \sigma(P, Q)} I_w(P, Q)w; \]
\[ D_s(P, Q; Z) = \sum_{w \in \{Z=0\}} I_w(P, Q)w; \]
\[ \hat{D}_s(P, Q, Z) = \sum_{w \in \{Z=0\}} \left( I_w(C, Z), I_w(P, Q) \right)w; \]
\[ D_s(C, Z) = \sum_{w \in \{Z=0\}} I_w(C, Z)w \quad \text{if} \quad Z \nmid C(X, Y, Z), \]

where \( C(X, Y, Z) = YP(X, Y, Z) - XP(X, Y, Z) \), \( I_w(F, G) \) is the intersection number (see, [9]) of the curves defined by homogeneous polynomials \( F, G \in \mathbb{C}[X, Y, Z] \), \( \deg(F), \deg(G) \geq 1 \) and \( \{Z = 0\} = \{[X : Y : 0] \mid (X, Y) \in \mathbb{C}^2 \setminus (0, 0)\} \).

Notation 2.3. The following integers are invariant with respect to the group of real affine transformations and real time rescaling:

\[ n^\infty = \#\{ w \in \text{Supp} D_s(C, Z) \mid w \in \mathbb{P}_2(\mathbb{R}) \}; \]
\[ d^\infty = \deg D_s(P, Q; Z). \] (2.2)

A complex projective line \( uX + vY + wZ = 0 \) is invariant for the system \((S)\) if either it coincides with \( Z = 0 \) or is the projective completion of an invariant affine line \( ux + vy + w = 0 \).

Notation 2.4. Let \((S) \in \text{QSL}\). Let us denote

\[ \text{IL}(S) = \left\{ l \mid \begin{array}{l} \text{l is a line in } \mathbb{P}_2(\mathbb{C}) \text{ such that } l \text{ is invariant for } (S) \end{array} \right\}; \]
\[ M(l) = \text{the multiplicity of the invariant line } l \text{ of } (S). \]

Remark 2.5. We note that the line \( l^\infty : Z = 0 \) is included in \( \text{IL}(S) \) for any \((S) \in \text{QSL}\).
Let \( l_i : f_i(x, y) = 0, \ i = 1, \ldots, k, \) be all the distinct invariant affine lines (real or complex) of a system \((S) \in \text{QSL}\). Let \( l'_i : \mathcal{F}_i(X, Y, Z) = 0 \) be the complex projective completion of \( l_i \).

**Notation 2.6.** We denote

\[
\mathcal{G} : \prod_i \mathcal{F}_i(X, Y, Z) Z = 0; \quad \text{Sing} \mathcal{G} = \{ w \in \mathcal{G} \mid w \text{ is a singular point of } \mathcal{G} \};
\]

\[
\nu(w) = \text{the multiplicity of the point } w, \text{ as a point of } \mathcal{G}.
\]

**Definition 2.3.**

\[
\mathbf{D}_{il}(S) = \sum_{l \in \mathbf{IL}(S)} M(l) l, \quad (S) \in \text{QSL};
\]

\[
\text{Supp} \mathbf{D}_{il}(S) = \{ l \mid l \in \mathbf{IL}(S) \}.
\]

**Notation 2.7.** The following integers are invariants of systems in \( \text{QSL} \) with respect to the group of real affine transformations and real time rescaling:

\[
M_{il} = \deg \mathbf{D}_{il}(S);
\]

\[
N_c = \# \text{Supp} \mathbf{D}_{il};
\]

\[
N_x = \# \{ l \in \text{Supp} \mathbf{D}_{il} \mid l : ux + vy + w = 0, [u : v : w] \in \mathbb{P}_2(\mathbb{R}) \};
\]

\[
n_{Γ,σ} = \# \{ ω \in \text{Supp} \mathbf{D}_s(P, Q) \mid ω \in \mathcal{G} \}_{\mathbb{R}^2};
\]

\[
d_{Γ,σ} = \sum_{ω \in \mathcal{G} \}_{\mathbb{R}^2} I_ω(P, Q);
\]

\[
m_\mathcal{G} = \max \{ ν(ω) \mid ω \in \text{Sing} \mathcal{G} \};
\]

\[
m_\infty = \max \{ ν(ω) \mid ω \in (\text{Sing} \mathcal{G}) \cap \{ Z = 0 \} \}.
\]

### 2.2 The main \( T \)-comitants associated to configurations of invariant lines

It is known that on the set \( \text{QS} \) of all quadratic differential systems \((2.1)\) acts the group \( \text{Aff}(2, \mathbb{R}) \) of affine transformation on the plane (cf. [21]). For every subgroup \( G \subseteq \text{Aff}(2, \mathbb{R}) \) we have an induced action of \( G \) on \( \text{QS} \). We can identify the set \( \text{QS} \) of systems \((2.1)\) with a subset of \( \mathbb{R}^{12} \) via the map \( \text{QS} \rightarrow \mathbb{R}^{12} \) which associates to each system \((2.1)\) the 12-tuple \( \mathbf{a} = (a_{00}, \ldots, b_{02}) \) of its coefficients.

For the definitions of an affine or \( \text{GL} \)-comitant and invariant as well as for the definition of a \( T \)-comitant and \( \text{CT} \)-comitant we refer the reader to [22], [5]. Here we shall only construct the necessary \( T \)-comitants associated to configurations of invariant lines for the class of quadratic systems with exactly four invariant lines including the line at infinity and including multiplicities.

Let us consider the polynomials

\[
C_i(a, x, y) = yp_i(a, x, y) - xq_i(a, x, y) \in \mathbb{R}[a, x, y], \ i = 0, 1, 2,
\]

\[
D_i(a, x, y) = \frac{∂}{∂x} p_i(a, x, y) + \frac{∂}{∂y} q_i(a, x, y) \in \mathbb{R}[a, x, y], \ i = 1, 2.
\]

As it was shown in [27] the polynomials

\[
\{ C_0(a, x, y), \ C_1(a, x, y), \ C_2(a, x, y), \ D_1(a), \ D_2(a, x, y) \}
\]

of degree one in the coefficients of systems \((2.1)\) are \( \text{GL} \)-comitants of these systems.
Notation 2.8. Let \( f, g \in \mathbb{R}[a, x, y] \) and
\[
(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^{k} \binom{k}{h} \frac{\partial^{k} f}{\partial x^{k-h} \partial y^{h}} - \frac{\partial^{k} g}{\partial x^{k-h} \partial y^{h}}.
\] (2.5)

\((f, g)^{(k)} \in \mathbb{R}[a, x, y]\) is called the transcendent of index \( k \) of \((f, g)\) (cf. [11], [16]).

Theorem 2.1. [28] Any GL-comitant of systems (2.1) can be constructed from the elements of the set (2.4) by using the operations: +, −, ×, and by applying the differential operation \((f, g)^{(k)}\).

Notation 2.9. Consider the polynomial \( \Phi_{\alpha, \beta} = \alpha P + \beta Q \in \mathbb{R}[a, X, Y, Z, \alpha, \beta] \) where \( P = Z^2p(X/Z, Y/Z), Q = Z^2q(X/Z, Y/Z), p, q \in \mathbb{R}[a, x, y] \) and \( \max(\deg_{(x,y)} p, \deg_{(x,y)} q) = 2 \). Then
\[
\Phi_{\alpha, \beta} = c_{11}(\alpha, \beta)X^2 + 2c_{12}(\alpha, \beta)XY + c_{22}(\alpha, \beta)Y^2 + 2c_{13}(\alpha, \beta)XZ + 2c_{23}(\alpha, \beta)YZ
\]
\[
+ c_{33}(\alpha, \beta)Z^2, \quad \Delta(\alpha, \beta) = \det |c_{ij}(\alpha, \beta)|_{i,j \in \{1,2,3\}},
\]
\[
D(a, \alpha, \beta) = 4\Delta(a, -\beta, \alpha), \quad H(a, \alpha, \beta) = 4 \left[ \det |c_{ij}(\beta, \alpha)|_{i,j \in \{1,2\}} \right].
\]

Lemma 2.1. [21] Consider two parallel invariant affine lines \( \mathcal{L}_i(x, y) \equiv ux + vy + w_i = 0, \mathcal{L}_i(x, y) \in \mathbb{C}[x, y], (i = 1, 2) \) of a quadratic system (S) of coefficients \( a \). Then \( H(a, -v, u) = 0 \), i.e. the T-comitant \( H(a, x, y) \) captures the directions of parallel invariant lines of systems (2.1).

We construct the following T-comitants:

Notation 2.10.
\[
B_3(a, x, y) = (C_2, D)^{(1)} = \text{Jacobi} (C_2, D),
\]
\[
B_2(a, x, y) = (B_3, B_3)^{(2)} - 6B_2(C_2, D)^{(3)},
\]
\[
B_1(a) = \text{Res}_x (C_2, D) / y^9 = -2^{-9}3^{-8} (B_2, B_3)^{(4)}.
\] (2.6)

Lemma 2.2. [21] A necessary condition for the existence of one (respectively 2; 3) invariant straight line(s) in one (respectively 2; 3 distinct) directions in the affine plane is \( B_1 = 0 \) (respectively \( B_2 = 0; B_3 = 0 \)).

Let us apply a translation \( x = x' + x_0, y = y' + y_0 \) to the polynomials \( p(a, x, y) \) and \( q(a, x, y) \). We obtain \( \tilde{p}(\tilde{a}(a, x_0, y_0), x', y') = p(a, x' + x_0, y' + y_0), \quad \tilde{q}(\tilde{a}(a, x_0, y_0), x', y') = q(a, x' + x_0, y' + y_0). \) Let us construct the following polynomials
\[
\Gamma_i(a, x_0, y_0) \equiv \text{Res}_x \left( C_i (\tilde{a}(a, x_0, y_0), x', y') \right) / (y')^{i+1},
\]
\[
\Gamma_i(a, x_0, y_0) \in \mathbb{R}[a, x_0, y_0], \quad (i = 1, 2).
\]

Notation 2.11.
\[
\tilde{E}_i(a, x, y) = \Gamma_i(a, x_0, y_0)|_{(x_0 = x, y_0 = y)} \in \mathbb{R}[a, x, y] \quad (i = 1, 2).
\] (2.7)

Observation 2.12. We note that the constructed polynomials \( \tilde{E}_1(a, x, y) \) and \( \tilde{E}_2(a, x, y) \) are affine comitants of systems (2.1) and are homogeneous polynomials in the coefficients \( a_{00}, \ldots, b_{02} \) and non-homogeneous in \( x, y \) and \( \deg_{x} \tilde{E}_1 = 3, \deg_{(x,y)} \tilde{E}_1 = 5, \deg_{a} \tilde{E}_2 = 4, \deg_{(x,y)} \tilde{E}_2 = 6 \).

Notation 2.13. Let \( \mathcal{E}_i(a, X, Y, Z) \) \( (i = 1, 2) \) be the homogenization of \( \tilde{E}_i(a, x, y) \), i.e.
\[
\mathcal{E}_1(a, X, Y, Z) = Z^5 \tilde{E}_1(a, X/Z, Y/Z), \quad \mathcal{E}_2(a, X, Y, Z) = Z^6 \tilde{E}_1(a, X/Z, Y/Z)
\]
and \( H(a, X, Y, Z) = \gcd \left( \mathcal{E}_1(a, X, Y, Z), \mathcal{E}_2(a, X, Y, Z) \right) \) in \( \mathbb{R}[a, X, Y, Z] \).
The geometrical meaning of these affine comitants is given by the two following lemmas (see [21]):

**Lemma 2.3.** The straight line $L(x, y) \equiv ux + vy + w = 0, u, v, w \in \mathbb{C}, (u, v) \neq (0, 0)$ is an invariant line for a quadratic system (2.1) if and only if the polynomial $L(x, y)$ is a common factor of the polynomials $\tilde{E}_1(a, x, y)$ and $\tilde{E}_2(a, x, y)$ over $\mathbb{C}$, i.e.

$$\tilde{E}_i(a, x, y) = (ux + vy + w)\tilde{W}_i(x, y) \quad (i = 1, 2),$$

where $\tilde{W}_i(x, y) \in \mathbb{C}[x, y]$.

**Lemma 2.4.** Let $(S) \in \text{QS}$ and let $a \in \mathbb{R}^{12}$ be its 12-tuple of coefficients. 1) If $L(x, y) \equiv ux + vy + w = 0, u, v, w \in \mathbb{C}, (u, v) \neq (0, 0)$ is an invariant straight line of multiplicity $k$ for a quadratic system (2.1) then $[L(x, y)]^k \mid \gcd(\tilde{E}_1, \tilde{E}_2)$ in $\mathbb{C}[x, y]$, i.e. there exist $W_i(a, x, y) \in \mathbb{C}[x, y]$ $(i = 1, 2)$ such that

$$\tilde{E}_i(a, x, y) = (ux + vy + w)^kW_i(a, x, y), \quad i = 1, 2. \quad (2.8)$$

2) If the line $l_\infty : Z = 0$ is of multiplicity $k > 1$ then $Z^{k-1} \mid \gcd(\tilde{E}_1, \tilde{E}_2)$.

Let us consider the following $GL$-comitants of systems (2.1):

**Notation 2.14.**

$$
\begin{align*}
M(a, x, y) &= 2\text{Hessian } (C_2(x, y)), \\
K(a, x, y) &= \text{Jacobian } (p_2(x, y), q_2(x, y)), \\
N(a, x, y) &= K(a, x, y) + H(a, x, y),
\end{align*}
$$

$$
\begin{align*}
\eta(a) &= \text{Discriminant } (C_2(x, y)), \\
\mu(a) &= \text{Discriminant } (K(a, x, y)), \\
\theta(a) &= \text{Discriminant } (N(a, x, y)).
\end{align*}
$$

The geometrical meaning of these invariant polynomials is revealed by the next 3 lemmas (see [21]).

**Lemma 2.5.** Let $(S) \in \text{QS}$ and let $a \in \mathbb{R}^{12}$ be its 12-tuple of coefficients. The common points of $P = 0$ and $Q = 0$ on the line $Z = 0$ are given by the common linear factors over $\mathbb{C}$ of $p_2(x, y)$ and $q_2(x, y)$. Moreover,

$$
\deg \gcd(p_2(x, y), q_2(x, y)) = \begin{cases} 
0 & \text{iff } \mu(a) \neq 0; \\
1 & \text{iff } \mu(a) = 0, \ K(a, x, y) \neq 0; \\
2 & \text{iff } K(a, x, y) = 0.
\end{cases}
$$

**Lemma 2.6.** A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a systems (2.1) corresponding to $a \in \mathbb{R}^{12}$ is the condition $\theta(a) = 0$ (respectively, $N(a, x, y) = 0$).

From [22] it easily follows

**Lemma 2.7.** The type (as defined in [21]) of the divisor $D_S(C, Z)$ for systems (1.1) is determined by the corresponding conditions indicated in Table 1, where we write $\omega_1^2 + \omega_2^2 + \omega_3$ if two of the points, i.e. $\omega_1, \omega_2$, are complex but not real. Moreover, for each type of the divisor $D_S(C, Z)$ given in Table 1 the quadratic systems (1.1) can be brought via a linear transformation to one of the following canonical systems $(S_i) - (S_V)$ corresponding to their behavior at infinity.
Table 1

<table>
<thead>
<tr>
<th>Case</th>
<th>Type of $D_S(C, Z)$</th>
<th>Necessary and sufficient conditions on the comitants</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\omega_1 + \omega_2 + \omega_3$</td>
<td>$\eta &gt; 0$</td>
</tr>
<tr>
<td>2</td>
<td>$\omega_1^c + \omega_2^c + \omega_3$</td>
<td>$\eta &lt; 0$</td>
</tr>
<tr>
<td>3</td>
<td>$2\omega_1 + \omega_2$</td>
<td>$\eta = 0, \ M \neq 0$</td>
</tr>
<tr>
<td>4</td>
<td>$3\omega$</td>
<td>$M = 0, \ C_2 \neq 0$</td>
</tr>
<tr>
<td>5</td>
<td>$D_S(C, Z)$ undefined</td>
<td>$C_2 = 0$</td>
</tr>
</tbody>
</table>

$$
\begin{align*}
\frac{dx}{dt} &= k + cx + dy + gx^2 + (h - b)xy, \\
\frac{dy}{dt} &= l + ex + fy + (g - b)xy + hy^2; \\
\end{align*}
(S_I)
$$

$$
\begin{align*}
\frac{dx}{dt} &= k + cx + dy + gx^2 + (h + b)xy, \\
\frac{dy}{dt} &= l + ex + fy - bx^2 + gxy + hy^2; \\
\end{align*}
(S_{II})
$$

$$
\begin{align*}
\frac{dx}{dt} &= k + cx + dy + gx^2 + hxy, \\
\frac{dy}{dt} &= l + ex + fy + (g - b)xy + hy^2; \\
\end{align*}
(S_{III})
$$

$$
\begin{align*}
\frac{dx}{dt} &= k + cx + dy + gx^2 + hxy, \\
\frac{dy}{dt} &= l + ex + fy - bx^2 + gxy + hy^2; \\
\end{align*}
(S_{IV})
$$

$$
\begin{align*}
\frac{dx}{dt} &= k + cx + dy + gx^2, \\
\frac{dy}{dt} &= l + ex + fy + gxy. \\
\end{align*}
(S_{V})
$$

In this paper we shall use also the following invariant polynomials, constructed in papers [21]–
$H_1(a) = -((C_2, C_2)_{(2)}, C_2)_{(1)}, D)_{(3)}$;
$H_2(a, x, y) = (C_1, 2H - N)_{(1)} - 2D_1N$;
$H_3(a, x, y) = (C_2, D)_{(2)}$;
$H_4(a) = ((C_2, D)_{(2)}, (C_2, D_2)_{(1)})_{(2)}$;
$H_5(a) = ((C_2, C_2)_{(2)}, (D, D)_{(2)})_{(2)} + 8((C_2, D)_{(2)}, (D, D_2)_{(1)})_{(2)}$;
$H_6(a, x, y) = 16N^2(C_2, D)_{(2)} + H_2^2(C_2, C_2)_{(2)}$;
$H_7(a) = (N, C_1)_{(2)}$;
$H_8(a) = 9((C_2, D)_{(2)}, (D, D_2)_{(1)})_{(2)} + 2[(C_2, D)_{(2)}]_3$;
$H_9(a) = -(((D, D)_{(2)}, D_2)_{(1)}D)_{(3)}$;
$H_{10}(a) = (N, D)_{(2)}, D_2)_{(1)}$;
$H_{11}(a, x, y) = 8H[(C_2, D)_{(2)} + 8(D, D_2)_{(1)}] + 3H_2^2$;
$N_1(a, x, y) = C_1(C_2, C_2)_{(2)} - 2C_2(C_1, C_2)_{(2)}$;
$N_2(a, x, y) = D_1(C_1, C_2)_{(2)} - ((C_2, C_2)_{(2)}, C_0)_{(1)}$;
$N_3(a, x, y) = (C_2, C_1)_{(1)}$;
$N_4(a, x, y) = 4(C_2, C_0)_{(1)} - 3C_1D_1$;
$N_5(a, x, y) = [(D_2, C_1)_{(1)} + D_1D_2]_{(2)} - 4(C_2, C_2)_{(2)}(C_1, D_1)_{(1)}$;
$N_6(a, x, y) = 8D + C_2[8(C_0, D_2)_{(1)} - 3(C_1, C_1)_{(2)} + 2D_1]$,\[26]:
$\mathcal{G}_1(a) = ((C_2, \tilde{E})_{(2)}, D_2)_{(1)}$;
$\mathcal{G}_2 = 8H_8 - 9H_5$;
$\mathcal{G}_3 = (\mu_0 - \eta)H_1 - 6\eta(H_4 + 12H_{10})$.

where

\[\tilde{E}(a, x, y) = \left[D_1(2\omega_1 - \omega_2) - 3(C_1, \omega_1)_{(1)} - D_2(3\omega_3 + D_1D_2)\right]/72,\]
\[\omega_1(a, x, y) = (C_2, D_2)_{(1)}; \quad \omega_2(a, x, y) = (C_2, C_2)_{(2)}; \quad \omega_3(a, x, y) = (C_1, D_2)_{(1)}.\]

To construct some more needed invariant polynomials we use the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ acting on $\mathbb{R}[a, x, y]$ (see [4]), where

\[
\mathbf{L}_1 = 2a_{00} \frac{\partial}{\partial a_{10}} + a_{10} \frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01} \frac{\partial}{\partial a_{11}} + 2b_{00} \frac{\partial}{\partial b_{10}} + b_{10} \frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01} \frac{\partial}{\partial b_{11}},
\]
\[
\mathbf{L}_2 = 2a_{00} \frac{\partial}{\partial a_{01}} + a_{01} \frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10} \frac{\partial}{\partial a_{11}} + 2b_{00} \frac{\partial}{\partial b_{01}} + b_{01} \frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10} \frac{\partial}{\partial b_{11}}.
\]

Then setting $\mu_0(a) = \mu(a) = \text{Res}_{x}(p_2, q_2)/y^4$ we construct the following polynomials:

\[\mu_i(a, x, y) = \frac{1}{i!} \mathcal{L}^{(i)}(\mu_0), \quad i = 1, \ldots, 4;\]
\[\kappa(a) = (M, K)_{(2)}/4; \quad \kappa_1(a) = (M, C_1)_{(2)};\]
\[ L(a, x, y) = 4K(a, x, y) + 8H(a, x, y) - M(a, x, y); \]
\[ R(a, x, y) = L(a, x, y) + 8K(a, x, y); \]
\[ K_1(a, x, y) = p_1(x, y)q_2(x, y) - p_2(x, y)q_1(x, y); \]
\[ K_2(a, x, y) = 4 \text{Jacob}(J_2, \xi) + 3 \text{Jacob}(C_1, \xi)D_1 - \xi(16J_1 + 3J_3 + 3D_2^2); \]
\[ K_3(a, x, y) = 2C_2^2(2J_1 - 3J_3) + C_2(3C_0K - 2C_1J_4) + 2K_1(3K_1 - C_1D_2), \]

where \( \mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)) \) and
\[ J_1 = \text{Jacob}(C_0, D_2), \quad J_2 = \text{Jacob}(C_0, C_2), \quad J_3 = \text{Discrim}(C_1), \quad J_4 = \text{Jacob}(C_1, D_2), \quad \xi = M - 2K. \]

### 3 Construction of canonical forms for systems \((S) \in \text{QS}\) with invariant lines of total multiplicity at least four.

In [23] we constructed the canonical forms, depending on at most two parameters, for systems in \(\text{QSL}_4\). To obtain these canonical forms we imposed the consecutive restrictions: that the systems be non-degenerate, that \(C_2 \neq 0\), and that they have not more than four invariant lines. These conditions reduced the number of parameters.

Now we want to include in our discussion some of the eliminated systems which will be lying on the border set of \(\text{QSL}_4\) in the parameter space. As for \(\text{QSL}_4\) the maximum number of parameters in the canonical forms is two [23], giving bifurcation diagrams in the two-dimensional real affine space, we would like to complete the affine plane to the projective plane so as to include in our treatment border points of the class \(\text{QSL}_4\) corresponding to the systems we discarded. Ideally we would like to construct canonical forms for the union of all these systems.

We indicate below the invariant polynomials (with respect to the group \(\text{Aff}(2, \mathbb{R}) \times \mathbb{R}^*\)) which govern the types of configurations of invariant lines for the canonical forms \((S_i) - (S_f)\) above. More precisely these are

\[ \eta, M, B_2, B_3, \theta, N, K, K_1, K_2, K_3, G_1, G_2, G_3, L, R, \mu_0, \ldots, \mu_4, N_1, \ldots, N_6, H_1, \ldots, H_{11}, \]

as they appear in Tables 4 and 5 of [24], in Table 1 of [25] and in Table 2 of [26].

#### 3.1 Systems with the divisor \(D_S(C, Z) = \omega_1 + \omega_2 + \omega_3\)

**Theorem 3.1.** (i) Any system \((S) \in \text{QSL}_4\) with divisor \(D_S(C, Z) = \omega_1 + \omega_2 + \omega_3\) can be brought via an affine transformation and time rescaling to one of the following eight canonical forms:

\[
\begin{align*}
\dot{x} &= gx + gx^2 + (h-b)xy, & \dot{y} &= -hy + (g-b)xy + hy^2; \\
\dot{x} &= x + gx^2 + (h-b)xy, & \dot{y} &= y + (g-b)xy + hy^2; \\
\dot{x} &= gx^2 + (h-b)xy, & \dot{y} &= (g-b)xy + hy^2; \\
\dot{x} &= g(x^2 - g^2), & \dot{y} &= y[(g-b)x + by - 2bh]; \\
\dot{x} &= g(x^2 + g^2), & \dot{y} &= y[(g-b)x + by - 2bh]; \\
\dot{x} &= gx^2, & \dot{y} &= y[(g-b)x + by - 2bh]; \\
\dot{x} &= h + gx, & \dot{y} &= -bxy + by^2; \\
\dot{x} &= c(c + d) + cx + dy, & \dot{y} &= -xy + y^2.
\end{align*}
\]
These canonical forms depend on the parameter $[b : g : h] \in \mathbb{P}_2[\mathbb{R}]$ for (3.1)-(3.7) and on $[c : d] \in \mathbb{P}_1[\mathbb{R}]$ for (3.8).

(ii) All systems $(S) \in \text{QLS}_4$ included in the family (3.1) (respectively (3.2); (3.3); (3.4); (3.5); (3.6); (3.7); (3.8)) have the configuration of invariant lines Config. 4.1 (respectively Config. 4.3 or 4.4; Config. 4.5; Config. 4.9 or 4.10; Config. 4.13; Config. 4.22; Config. 4.16 or 4.17 or 4.34; Config. 4.18).

Proof: According to Lemma 2.7 the systems with this type of divisor can be brought by linear transformations to the canonical form $(S_I)$ for which we calculate

$$\theta = 8(b - h)(g - b)(g + h)b.$$ (3.9)

### 3.1.1 The case $\theta \neq 0, B_3 = 0$

The condition $\theta \neq 0$ yields $b(g - b)(b - h) \neq 0$ and in $(S_I)$ we may assume $d = e = 0$ via the translation $x \rightarrow x + d/(b - h)$ and $y \rightarrow y + e/(b - g)$. Thus we obtain the systems

$$\dot{x} = k + cx + gx^2 + (h - b)xy, \quad \dot{y} = l + fy + (g - b)xy + hy^2,$$ (3.10)

for which we calculate

$$B_3 = 3bl(g - b)^2x^3(2y - x) + 3bk(h - b)^2y^3(y - 2x) +$$

$$3bl[(c - f)(fg + ch) - k(g + b)(g + 2h - b) + l(h + b)(2g + h - b)]x^2y^2,$$

$$H_7 = 4(f - c)(b - g)(b - h).$$

#### 3.1.1.1 Subcase $H_7 \neq 0$ Then $c - f \neq 0$ and hence, the condition $B_3 = 0$ yields $k = l = 0$ and $fg + ch = 0$. Since the condition $\theta \neq 0$ implies $g^2 + h^2 \neq 0$ by introducing a new parameter $u$ we can set $c = gu$ and $f = -hu$. Then the condition $H_7 \neq 0$ implies $u \neq 0$ and we may assume $u = 1$ due to the Remark A ($\gamma = u, s = 1$). This leads to the systems:

$$\dot{x} = gx + gx^2 + (h - b)xy, \quad \dot{y} = -hy + (g - b)xy + hy^2,$$ (3.11)

i.e. the family (3.1), which includes the systems with Configuration 4.1. Due to the transformation $(x, y, t) \mapsto (x, y, t/\lambda)$ the parameter here could be consider $\Lambda = [b : g : h] \in \mathbb{P}_2[\mathbb{R}]$.

#### 3.1.1.2 Subcase $H_7 = 0$. Then $f = c$ and we obtain the systems

$$\dot{x} = cx + gx^2 + (h - b)xy, \quad \dot{y} = cy + (g - b)xy + hy^2,$$ (3.12)

for which $H_1 = 576c^2b^4$.

#### 3.1.1.2.1 Subcase $H_1 \neq 0$. Then $c = 1$ (due to the Remark A) and we get the family

$$\dot{x} = x + gx^2 + (h - b)xy, \quad \dot{y} = y + (g - b)xy + hy^2,$$ (3.13)

which coincides with (3.2) and which includes the systems with Configurations 4.3 and 4.4. Due to the transformation $(x, y, t) \mapsto (x/\lambda, y/\lambda, t)$ the parameter here could be consider $\Lambda = [b : g : h] \in \mathbb{P}_2[\mathbb{R}]$. 11
3.1.1.2 Subcase \( H_1 = 0 \). Then \( c = 0 \) (since \( \theta \neq 0 \)) and we get the family
\[
\dot{x} = gx^2 + (h - b)xy, \quad \dot{y} = (g - b)xy + hy^2,
\] (3.14)
which coincides with (3.3) and which includes the systems with Configuration 4.5. Due to the transformation \((x, y, t) \mapsto (x, y, t/\lambda)\) the parameter here could be consider \( \Lambda = [b : g : h] \in \mathbb{P}_2(\mathbb{R}) \).

3.1.2 The case \( \theta = B_2 = 0 \)
According to (3.9) the condition \( \theta = 0 \) yields \((g - b)(h - b)(g + h) = 0\) and without loss of generality we can consider \( h = b \). Indeed, if \( g = b \) (respectively, \( g + h = 0 \)) we can apply the linear transformation which will replace the straight line \( x = 0 \) with \( y = 0 \) (respectively, \( x = 0 \) with \( y = x \)) reducing this case to \( h = b \). Assuming \( h = b \) for the systems \((S_I)\) we calculate \( N = (g^2 - b^2)x^2 \).

3.1.2.1 Subcase \( N \neq 0 \). Then \((g - b)(g + b) \neq 0\). For systems \((S_I)\) with \( h = b \) we have \( \mu = b^2g^2 \) and we shall consider two subcases: \( \mu \neq 0 \) and \( \mu = 0 \).

3.1.2.1.1 Subcase \( \mu \neq 0 \) Then \( g \neq 0 \) and we may assume \( c = f = 0 \) via the translation \( x \rightarrow x - c/(2g), \ y \rightarrow y + [c(g - b) - 2fg]/(4bg) \). Thus we obtain the systems
\[
\dot{x} = k + dy + gx^2, \quad \dot{y} = l + ex + (g - b)xy + by^2,
\] (3.15)
for which \( H_7 = 4d(g^2 - b^2) \). According to [23] for a given system \( S(a) \) of the form (3.15) to belong to the class \( \text{QSL}_4 \) the condition \( H_7(a) = 0 \) (i.e. \( d = 0 \)) is necessary. Then for systems (3.15) we calculate:
\[
B_2 = -648b^2[be^2 + l(b - g)^2][be^2 + (l - k)(b + g)^2] x^4
\]
and the condition \( B_2 = 0 \) yields either \((I)\) \( be^2 + l(b - g)^2 = 0 \) or \((II)\) \( be^2 + (l - k)(b + g)^2 = 0 \). We claim that the case \((I)\) can be reduced by a linear transformation and time rescalling to the case \((II)\) and viceversa. Indeed, via the transformation \( x_1 = x, \ y_1 = x - y \) and \( t_1 = -t \) systems (3.15) with \( d = 0 \) keep the same form
\[
\dot{x}_1 = \tilde{k} + \tilde{g}x_1^2, \quad \dot{y}_1 = \tilde{l} + \tilde{e}x_1 + (\tilde{g} - \tilde{b})x_1y_1 + \tilde{b}y_1^2
\]
but with new parameters: \( \tilde{k} = -k, \ \tilde{g} = -g, \ \tilde{l} = l - k, \ \tilde{e} = e \) and \( \tilde{b} = b \). Then obviously we have:
\[
\tilde{b}\dot{e}^2 + \tilde{l}(b - \tilde{g})^2 = be^2 + (l - k)(b + g)^2, \quad \tilde{b}\dot{e}^2 + (\tilde{l} - \tilde{k})(\tilde{b} + \tilde{g})^2 = be^2 + l(b - g)^2
\]
and this proves our claim.

In what follows we assume that the condition \((I)\) holds. Since \( g - b \neq 0 \) we may set \( e = u(g - b) \) (where \( u \) is a new parameter) and then we obtain \( l = -bu^2 \). So, applying the translation \((x, y) \mapsto (x, y - u)\) we get the systems
\[
\dot{x} = k + gx^2, \quad \dot{y} = y[(g - b)x + by - 2bu],
\] (3.16)
for which we have \( H_{10}N = -32gk(g^2 - b^2)x^2 \).

1) Assume first \( H_{10}N > 0 \). Then \( gk < 0 \) and since \( \mu \neq 0 \) (i.e. \( g \neq 0 \)) we may set \( k = -g^3v^2 \), where \( v \neq 0 \) is a new parameter. We may assume \( v = 1 \) via Remark A \((\gamma = v, s = 1)\) and setting \( u = h \) systems (3.16) become
\[
\dot{x} = g(x^2 - g^2), \quad \dot{y} = y[(g - b)x + by - 2bh],
\] (3.17)
i.e. the family (3.4), which includes the systems with Configurations 4.9 and 4.10. Due to the transformation \((x, y, t) \mapsto (\lambda x, \lambda y, \lambda^{-2} t)\) the parameter here could be consider \(\Lambda = [b : g : h] \in \mathbb{P}_2(\mathbb{R})\).

2) Suppose now \(H_{10} N < 0\). Then \(gk > 0\) and we set \(k = g^2 v^2\). We may assume \(v = 1\) via Remark A (\(\gamma = v, s = 1\)) and setting \(u = h\) systems (3.16) become

\[
\dot{x} = g(x^2 + g^2), \quad \dot{y} = y[(g - b)x + by - 2bh], \quad (3.18)
\]

coinciding with the family (3.5), which includes the systems with Configuration 4.22. Due to the transformation \((x, y, t) \mapsto (\lambda x, \lambda y, \lambda^{-2} t)\) the parameter here could be consider \(\Lambda = [b : g : h] \in \mathbb{P}_2(\mathbb{R})\).

3) Assume finally, \(H_{10} = 0\). Since \(g \neq 0\) we obtain \(k = 0\) and setting \(u = h\) we get the family of systems

\[
\dot{x} = gx^2, \quad \dot{y} = y[(g - b)x + by - 2bh], \quad (3.19)
\]

i.e. the family (3.6), which includes the systems with Configuration 4.22. Due to the transformation \((x, y, t) \mapsto (\lambda x, \lambda y, \lambda^{-2} t)\) the parameter here could be consider \(\Lambda = [b : g : h] \in \mathbb{P}_2(\mathbb{R})\).

### 3.1.2.1.2 Subcase \(\mu = 0\)

Then \(g = 0\) and we may consider \(e = f = 0\) via the translation \(x \mapsto x + (2e + f)/b, y \mapsto y + e/b\). So the systems \((S_1)\) with \(\theta = \mu = 0\) (i.e. \(h = b, g = 0\)) become

\[
\dot{x} = k + cx + dy, \quad \dot{y} = l - bxy + by^2 \quad (3.20)
\]

for which \(B_2 = -648b^5l(c^2 + cd - bk + bl)x^4 = 0\). Hence the condition \(B_2 = 0\) yields either (I) \(l = 0\) or (II) \(c^2 + cd - bk + bl = 0\). We claim that the case (II) can be reduced by an affine transformation and time rescalling to the case (I) and viceversa. Indeed, via the transformation \(x_1 = x + 2c + d, y_1 = x - y + c + d, t_1 = -t\) the systems (3.20) keep the same form

\[
\dot{x}_1 = \bar{k} + \bar{c}x_1 + \bar{d}y_1, \quad \dot{y}_1 = \bar{l} - \bar{b}x_1y_1 + \bar{b}y_1^2,
\]

but with new parameters: \(\bar{k} = 2c(c + d)/b - k, \quad \bar{c} = -(c + d), \quad \bar{l} = l - k + c(c + d)/b, \quad \bar{b} = b, \quad \bar{d} = d\). Then obviously we have: \(\bar{l} = (c^2 + cd - bk + bl)/b, \quad \bar{c}^2 + \bar{c}d - \bar{b}k + \bar{b}l = bl\) and this proves our claim.

Hence we only need to consider \(l = 0\) and thus we arrive to the following 4-parameter family of systems:

\[
\dot{x} = k + cx + dy, \quad \dot{y} = -bxy + by^2. \quad (3.21)
\]

According to [23] in order to be in the class \(\text{QSL}_4\) it is necessary \(H_7 B_3 = 0\), where

\[H_7 = -4b^2d, \quad B_3 = 3b^2(c^2 + cd - bk)x^2y^2.\]

This leads to the condition

\[d(c^2 + cd - bk) = 0.\]

1) Assume \(H_7 = 0\). Then \(d = 0\) and setting \(c = g\) and \(k = h\) we get the family of systems

\[
\dot{x} = h + gx, \quad \dot{y} = -bxy + by^2, \quad (3.22)
\]

i.e. the family (3.7), which includes the systems with Configurations 4.16, 4.17 and 4.34. Due to the transformation \((x, y, t) \mapsto (x, y, t/\lambda)\) the parameter here could be consider \(\Lambda = [b : g : h] \in \mathbb{P}_2(\mathbb{R})\).

2) Assume \(H_7 \neq 0\). As \(b \neq 0\) due to time rescalling we may set \(b = 1\) and replacing then the condition \(3 = 0\) yields \(k = c(c + d)\). Therefore we get the family of systems:

\[
\dot{x} = c(c + d) + cx + dy, \quad \dot{y} = -xy + y^2, \quad (3.23)
\]

i.e. the family (3.8), which includes the systems with Configurations 4.18. Due to the transformation \((x, y, t) \mapsto (\lambda x, \lambda y, t/\lambda)\) the parameter here could be consider \(\Lambda = [c : d] \in \mathbb{P}_1(\mathbb{R})\).
3.2 Systems with the divisor \( D_S(C, Z) = \omega_1^c + \omega_2^c + \omega_3 \)

**Theorem 3.2.** (i) Any system \((S) \in \mathbb{QSL}_4\) with divisor \( D_S(C, Z) = \omega_1^c + \omega_2^c + \omega_3 \) can be brought via an affine transformation and time rescaling to one of the following four canonical forms:

\[
\begin{align*}
\dot{x} &= gx^2 + (h + b)xy, \\
\dot{y} &= h[g^2 + (h + b)^2] + (g^2 + b^2 - h^2)x + 2ghy - bx^2 + gxy + hy^2; \\
\dot{x} &= gx^2 + (h + b)xy, \\
\dot{y} &= -b + gx + (h - b)y - bx^2 + gxy + hy^2; \\
\dot{x} &= gx^2 + (h + b)xy, \\
\dot{y} &= -bx^2 + gxy + hy^2; \\
\dot{x} &= 2cx + 2dy, \\
\dot{y} &= c^2 + d^2 - x^2 - y^2.
\end{align*}
\]

These canonical forms depend on the parameter \([b : g : h] \in \mathbb{P}_2[\mathbb{R}]\) for (3.24)–(3.26) and on \([c : d] \in \mathbb{P}_1[\mathbb{R}]\) for (3.27).

(ii) All systems \((S) \in \mathbb{QSL}_4\) included in the family (3.24) (respectively (3.25); (3.26); (3.27)) have the configuration of invariant lines Config. 4.2 (respectively Config. 4.6 or 4.7; Config. 4.8; Config. 4.27) and lie in the affine chart corresponding to \(b \neq 0\) for (3.24)–(3.26) and to \(d \neq 0\) for (3.27).

**Proof:** According to Lemma 2.7 the systems with this type of divisor can be brought by linear transformations to the canonical form \((S_H)\) for which we calculate the main classifying invariant polynomials (see Table 2 [26]):

\[
\begin{align*}
\theta = & 8b(h + b)((h - b)^2 + g^2), \\
C_2 = & bx(x^2 + y^2), \\
N = & (g^2 - 2bh + 2b^2)x^2 + 2g(h + b)xy + (h^2 - b^2)y^2.
\end{align*}
\]

**Remark 3.1.** We observe that the condition \(C_2 = 0\) is equivalent to \(b = 0\) and this leads to systems with the line at infinity filled up with singularities. This class was studied in [25].

**Remark 3.2.** We note that two of the infinite points of the systems \((S_H)\) are not real. Therefore according to [23] a system of this class could belong to \(\mathbb{QSL}_4\) only if for this system the condition \(B_3 = 0\) holds.

In what follows we shall assume that for a system \((S_H)\) the condition \(B_3 = 0\) is fulfilled.

We shall view \(\mathbb{P}_2[\mathbb{R}]\) as a disk with opposite points on the circumference identified. We shall use the homogeneous coordinates \([b : g : h] \in \mathbb{P}_2[\mathbb{R}]\) placing the line \(b = 0\) on the circumference of the disk.

### 3.2.1 The case \(\theta \neq 0\)

The condition \(\theta \neq 0\) yields \((h + b) \neq 0\) and we may assume \(c = d = 0\) in \((S_H)\) via the translation \(x \rightarrow x - d/(h + b)\) and \(y \rightarrow y + (2dg - c(h + b))/(h + b)^2\). Thus we obtain the systems

\[
\begin{align*}
\dot{x} = & k + gx^2 + (h + b)xy, \\
\dot{y} = & l + ex + fy - bx^2 + gxy + hy^2,
\end{align*}
\]

for which we have: Coefficient \([B_3, y^4] = -3bk(h + b)^2\). Hence the condition \(B_3 = 0\) implies \(k = 0\) and we have

\[
B_3 = 3b[ef(h + b) + 2gl(h - b) - f^2g]x^2(x^2 - y^2) + 6b[bf^2 + ef g - e^2h + l(h - b)^2 - g^2l]x^3y.
\]

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So, the condition $B_3 = 0$ yields the following system of equations

$$
\begin{align*}
    Eq_1 &\equiv e(f(h + b) + 2g(h - b)) - f^2g = 0, \\
    Eq_2 &\equiv b(f^2 + ef - e^2h + [(h - b)^2 - g^2])l = 0.
\end{align*}
$$

(3.30)

Both equations are linear in $l$. We first note that we cannot have both coefficients of $l$ zero. Indeed, if we suppose $g(h - b) = (h - b)^2 - g^2 = 0$ then this contradicts $\theta \neq 0$.

Hence, at least one of the coefficients in front of $l$ is not zero. We consider two cases: $g(h - b)[(h - b)^2 - g^2] \neq 0$ and $g(h - b)[(h - b)^2 - g^2] = 0$.

3.2.1.1 Case $g(h - b)[(h - b)^2 - g^2] \neq 0$. In this case we calculate $\text{Res}_1(Eq_1, Eq_2) = (be - eh + fg)[2egh + f(h^2 - b^2 - g^2)]$. We observe that for systems (3.29) with $k = 0$ we have: $H_7 = 4(h + b)[be - eh + fg]$ and we shall consider two subcases: $H_7 \neq 0$ and $H_7 = 0$.

3.2.1.1.1 Subcase $H_7 \neq 0$ Then the equality $\text{Res}_1(Eq_1, Eq_2) = 0$ yields $2egh = f(g^2 + b^2 - h^2)$. Since $\theta \neq 0$ from (3.28) we have $(gh)^2 + (g^2 + b^2 - h^2)^2 \neq 0$ then without loss of generality we may set: $e = (g^2 + b^2 - h^2)u$ and $f = 2ghu$ where $u$ is a new parameter. Therefore from (3.30) we obtain

$$
    g(h - b)[l - hu^2(g^2 + (h + b)^2)] = 0 = [(h - b)^2 - g^2][l - hu^2(g^2 + (h + b)^2)]
$$

and hence, $l = hu^2[g^2 + (h + b)^2]$. In this case $H_7 = 4u(h + b)^2[g^2 + (h - b)^2] \neq 0$ and we may assume $u = 1$ via Remark A ($\gamma = u, s = 1$). This leads to the systems:

$$
\begin{align*}
    \dot{x} &= gx^2 + (h + b)xy, \\
    \dot{y} &= h[g^2 + (h + b)^2] + (g^2 + b^2 - h^2)x + 2ghy - bx^2 + gxy + hy^2.
\end{align*}
$$

(3.31)

Hence the conditions

$$
\eta < 0, \quad B_3 = 0, \quad \theta \neq 0, \quad H_7 \neq 0
$$

(3.32)

necessarily lead us to the canonical form (3.31), which coincides with (3.24) and which according to Table 2 ([26]) due to (3.32) includes the systems with Config. 4.2.

We now observe that the change of parameter $(b, g, h) \mapsto (\lambda b, \lambda g, \lambda h)$ leads to a system $(S_\lambda)$ equivalent to (3.31) under the group action. Indeed the change $(x, y, t) \mapsto (\lambda x, \lambda y, \lambda^{-2}t)$ applied to $(S_\lambda)$ yields the system (3.31).

Remark 3.3. We point out that whenever we have the conditions (3.32), they lead us to the canonical form (3.31). However not every system of the form (3.31) satisfies these conditions. Hence if not all three parameters are zero, the parameter space is $\mathbb{P}_2(\mathbb{R})$ for the form (3.31) and it contains points in the border of the set of the systems possessing the Config. 4.2.

3.2.1.1.2 Subcase $H_7 = 0$ Then $e(h - b) = fg$ and since the condition $\theta \neq 0$ yields $g^2 + (h - b)^2 \neq 0$, we may assume $e = gu$ and $f = (h - b)u$, where $u$ is a new parameter. Then from (3.30) we have $g(h - b)(l + bu^2) = 0 = [(h - b)^2 - g^2](l + bu^2)$. Due to $g^2 + (h - b)^2 \neq 0$ we obtain $l = -bu^2$ and this leads to the systems:

$$
\begin{align*}
    \dot{x} &= gx^2 + (h + b)xy, \\
    \dot{y} &= -bu^2 + gu x + u(h - b)y - bx^2 + gxy + hy^2.
\end{align*}
$$

(3.33)

For these systems we have $H_9 = 2304b^4u^8(h + b)^8$. 

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1) Assume first $H_9 \neq 0$. Then $u \neq 0$ and we may assume $u = 1$ via Remark A (?) $(\gamma = u, s = 1)$. So we get the family of systems
\[
\dot{x} = gx^2 + (h + b)xy, \quad \dot{y} = -b + gx + (h - b)y - bx^2 + gxy + hy^2
\]
which coincides with (3.25) and which includes the systems with the Configurations 4.6 and 4.7. Due to the transformation $(x, y, t) \mapsto (x, y, t/\lambda)$ the parameter here could be consider $\Lambda = [b : g : h] \in \mathbb{P}_2(\mathbb{R})$.

2) Suppose now $H_9 = 0$. Then $u = 0$ and we obtain the family of systems
\[
\dot{x} = gx^2 + (h + b)xy, \quad \dot{y} = -bx^2 + gxy + hy^2,
\]
i.e. the family (3.26), which includes the systems with Configuration 4.8. Due to the transformation $(x, y, t) \mapsto (x, y, t/\lambda)$ the parameter here could be consider $\Lambda = [b : g : h] \in \mathbb{P}_2(\mathbb{R})$.

3.2.2.1 Subcase $N \neq 0$. Then assuming $b \neq 0$ (see Remark 3.1) by (3.28) the condition $\theta = 0$ yields $h = -b$ and in addition we may assume $f = 0$ due to the translation: $x \to x$ and $y \to y + f/(2b)$. Hence, we obtain the systems
\[
\dot{x} = k + cx + dy + gx^2, \quad \dot{y} = l + ex - bx^2 + gxy - by^2,
\]
for which by Remark 3.2 the condition $B_3 = 0$ must be satisfied. Calculations yield: $H_7 = 4d(g^2 + 4b^2)$, Coefficient$[B_3, y^4] = -3bd^2g$. So the condition $B_3 = 0$ implies $dg = 0$. According to [23] for systems (3.36) to be in the class $\text{QLSL}_4$ the condition $d \neq 0$ (i.e. $H_7 \neq 0$) must be fulfilled. Then $g = 0$ and we may assume $e = 0$ via the translation: $x \to x + e/(2b)$, $y \to y$. After that for systems (3.36) calculations yield: $B_3 = 12b^3kx^2(x^2 - y^2) - 6(c^2 - 4bl + d^2)x^3y$. Therefore the condition $B_3 = 0$ yields $k = 0$ and $4bl = c^2 + d^2$. As $b \neq 0$ due to time rescalling we may set $b = 1$ and replacing $c$ with $2c$ and $d$ with $2d$ we get the systems:
\[
\dot{x} = 2cx + 2dy, \quad \dot{y} = c^2 + d^2 - x^2 - y^2,
\]
i.e. the family (3.27), which includes the systems with Configuration 4.27. Due to the transformation $(x, y, t) \mapsto (\lambda x, \lambda y, \lambda^{-1}t)$ the parameter here is $\Lambda = [c : d] \in \mathbb{P}_1(\mathbb{R})$.

3.2.2.2 Subcase $N = 0$. According to [23] in this case systems $(\text{S}_H)$ cannot belong to the class $\text{QLSL}_4$. 

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4 Construction of the bifurcation diagrams and of the moduli spaces.

4.1 Systems with the divisor $D_S(C, Z) = \omega_1 + \omega_2 + \omega_3$

There are the systems for which we have $\eta > 0$ (see Table 1, page 8) and each such system has the configuration of invariant lines corresponding to one of the configurations Config. 4.1, j=1,3-5,9,10,13,22,16,17,34. We shall examine these configurations one by one.

4.1.1 Configuration 4.1

According to Theorem 3.1 all the systems having the Configuration 4.1 are included in the family:

$$\dot{x} = gx + gx^2 + (h - b)xy, \quad \dot{y} = -hy + (g - b)xy + hy^2,$$  \hspace{1cm} (4.1)

where $[b : g : h] \in \mathbb{P}_2(\mathbb{R})$. We construct now the bifurcation diagram for this canonical form.

This diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line $b = 0$.

4.1.1.1 The case $b \neq 0$ In this case we may assume $b = 1$. Then for systems (4.1) calculations yield:

$$\theta = 8(1 - g)(h - 1)(g + h) = 2H_7, \quad \mu_0 = gh(g + h - 1) = H_1/576, \quad B_3 = 0,$$

$$N = (g^2 - 1)x^2 + 2(g - 1)(h - 1)xy + (h^2 - 1)y^2, \quad K = 2g(g - 1)x^2 + 4ghxy + 2(h - 1)y^2.$$

4.1.1.1.1 The subcase $\mu_0 \neq 0$ According to [26] if $\theta \neq 0$ (then $H_7 \neq 0$) the phase portrait of a system (4.1) corresponds to Picture 4.1(a) (respectively Picture 4.1(b); Picture 4.1(c)) if $\mu_0 > 0$ (respectively $\mu_0 < 0$, $K < 0$; $\mu_0 < 0$, $K > 0$). So, if $gh(g + h - 1)(1 - g)(h - 1)(g + h) \neq 0$ then we arrive at the situation indicated in Diagram 4.1.1 for these conditions.

Assume now $\theta = 0$, i.e. $(1 - g)(h - 1)(g + h) = 0$. If $N \neq 0$, (i.e. $(1 - g)^2 + (h - 1)^2 + (g + h)^2 \neq 0$) then from [24] it follows that the points on the lines $g = 1$, $h = 1$ and $h = -g$ excepting the points (1, 1), (-1, 1) and (1, -1), correspond to Picture 5.1. On the other hand the indicated last three points correspond to Picture 6.1 since for these values of parameters $g$ and $h$ we have for which $N = 0$ and $H_1 > 0$.

4.1.1.2 The subcase $\mu_0 = 0$ Then we have $gh(g + h - 1) = 0$ and systems (4.1) become degenerate.

Assume first $g = 0$. In this case systems we get the family of degenerate systems:

$$\dot{x} = (h - 1)xy, \quad \dot{y} = -hy - xy + hy^2$$  \hspace{1cm} (4.2)

having the singular invariant line $y = 0$. Examining the respective linear systems

$$\dot{x} = (h - 1)x, \quad \dot{y} = -h - x + hy,$$

and considering the singular line $y = 0$ we obtain, that the phase portrait of a system (4.2) corresponds to:

Picture D.1 if $h(h - 1) < 0$; Picture D.2 if $h(h - 1) > 0$; Picture D.3 if $h(h - 1) = 0$.

We observe, that for systems (4.1) the case $h = 0$ (respectively $g + h - b = 0$ could be obtained from $g = 0$ via the transformation $(x, y, t, b, h) \mapsto (-x_1, -y_1, -t_1, b_1, g_1)$ (respectively $(x, y, t, b, h) \mapsto (x_1, x_1 - y_1 + 1, t_1, -b_1, g_1 - b_1)$). Thus in the case $b \neq 0$ ($b = 1$) on the lines $g = 0$, $h = 0$ and $g + h - 1 = 0$ we get the bifurcations given by Diagram 4.1.1.
4.1.1.2 The case $b = 0$

Then we get the family of systems:

$$\dot{x} = gx + gx^2 + hxy, \quad \dot{y} = -hy + gxy + hy^2 \quad (4.3)$$

possessing the infinite line fulfilled with singularities. For these systems calculations yield:

$$C_2 = 0, \quad H_{10} = 36g^2h^2(g + h)^2, \quad H_9 = -576g^4h^4(g + h)^4$$

and hence, if $H_{10} \neq 0$ we have $H_9 < 0$. According to [22] in this case the phase portrait of systems (4.3) corresponds to $Picture C_2.1$.

Assume now $H_{10} = 0$, i.e. $gh(g + h) = 0$. We observe that in all three cases (i.e. $g = 0$ or $h = 0$ or $g + h = 0$) systems (4.3) become degenerate and the phase portrait corresponds to $Picture D.4$.

We now see that in the bifurcation Diagram 4.1.1 the portraits on the upper side of the first bisectrix $g = h$ coincide with those at symmetrical points with respect to this bisectrix and we wonder if they could be identified via the group action. Indeed, this is the case as we see by using the transformation $(x, y, t) \mapsto (-y - x, x, -t)$ inducing the map $(b, g, h) \mapsto (b, h, g)$. Therefore we can limit ourselves to the upper side of the first bisectrix removing the part of this Diagram which lies below the line $g - h = 0$. The transformation above is not the only one which identifies points in the parameter space via the group action. In fact we have a group of six transformations isomorphic to the symmetric group $S_3$ of permutations of the three real affine invariant lines of the systems (4.1). We list below all the six transformations yielding linear transformations of the
Diagram 4.1.1(\(\mathfrak{M}\))

affine parameter space \((b, g, h)\) and projective transformations in the parameter plane \(\mathbb{P}_2(\mathbb{R})\) with homogeneous coordinates \([b : g : h]\):

\[
\begin{align*}
(i) \quad (x, y, t) &\mapsto (-y, -x, -t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, h, g); \\
(ii) \quad (x, y, t) &\mapsto (y - x - 1, -x, t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, b - g - h, g); \\
(iii) \quad (x, y, t) &\mapsto (x, x - y + 1, -t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, b - g - h, h); \\
(iv) \quad (x, y, t) &\mapsto (y - x - 1, y, -t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, g, b - g - h); \\
v) \quad (x, y, t) &\mapsto (-y, x - y + 1, t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, h, b - g - h); \\
v) \quad (x, y, t) &\mapsto (x, y, t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, g, h). 
\end{align*}
\]

Consider now the chart \(b \neq 0\) and we can put \(b=1\). The transformations above yield corresponding ones on this affine plane of coordinates \((g, h)\) of the form: \(g_1 = b_{11}g + b_{12}h + b_1, h_1 = b_{21}g + b_{22}h + b_2\).

This affine plane can be viewed on the interior of the disk of radius 1.

Under the transformation \((ii)\) above, the third quadrant goes to the region delimited by the lines \(h = 0, h - 1 + g = 0\) for which we have \(h - 1 + g \geq 0\) and \(h \leq 0\) and this last region goes to the region delimited by the lines \(g = 0, h = 0\) and \(h - 1 + g = 0\) for which we have \(h - 1 + g \geq 0\), \(g \leq 0\). So for the construction of the moduli space, as well as for the bifurcation diagram, it suffices to limit ourselves to only one of these three regions removing the other two.

Analogously via the transformation \((ii)\) from (4.4), the region in the second quadrant delimited by the lines \(h = 0, g = 0\) and \(h - 1 + g = 0\) for which we have \(g \leq 0\) and \(h - 1 + g \leq 0\) goes to the region delimited by the lines \(g = 0, h = 0\) and \(h - 1 + g = 0\) and for which \(h \leq 0\) and \(h - 1 + g \leq 0\) and this last region goes to the region for which we have \(h \geq 0, g \geq 0\) and \(h - 1 + g \geq 0\). So for the construction of the moduli space, as well as for the bifurcation diagram, it suffices to limit ourselves to only one of these three regions and remove the other two.

We observe that under the transformation \((ii)\) the triangle determined by the three points \((0, 0), (1, 0)\) and \((0, 1)\) is invariant. Furthermore we see that this transformation fixes the point \((1/3, 1/3)\) and moves the triangle determined by the three points \((0, 0), (1/3, 1/3)\) and \((0, 1)\) into the triangle determined by the three points \((0, 0), (1/3, 1/3)\) and \((1, 0)\) and moves this last triangle into the
triangle determined by the three points (1,0), (1/3,1/3) and (0,1). For the construction of the moduli space we can thus just keep the last triangle and in view of the action of the transformation (i) from (4.4) and from this last triangle we just keep its part located above the first bisectrix.

It can easily be verified that all the remaining transformations (4.4) cannot identify two distinct points inside the curvilinear quadrangle delimiting Diagram 4.1.1(\mathcal{M}).

Now it remains to find the action of the transformations (4.4) of $P_2(\mathbb{R})$ on the line $b = 0$ with homogeneous coordinates $[g:h]$. The transformations on this line $P_1(\mathbb{R})$ are of the form: $g_1 = b_{11}g + b_{12}h$, $h_1 = b_{21}g + b_{22}h$. We now consider the affine chart corresponding to $h \neq 0$ so we can put $h = 1$. On this line with coordinate $g$, the above transformations yield the change: $g_1 = (b_{11}g + b_{12})/(b_{21}g + b_{22})$.

We now consider the action of the first five transformations (4.4) on the segment of Diagram 4.1.1(\mathcal{M}) located on $b = 0$, so as to see if under the group action we can still identify points. The first transformations is the inversion $g_1 = 1/g$ with fixed points -1,1. This inversion sends points in the segment [-1,1] to points outside this segment and hence it cannot further identify points inside this segment. Consider now the second transformation of the affine line $h = 1$ with coordinate $g$. This transformation is: $g_1 = (-g - 1)/g$ and it sends the segment [-1/2,0] to [0,1], the segment [-1/2,0) onto $[1,\infty)$ which is outside [-1,1]. The same transformation sends the segment (0,1] onto $(-\infty,-2)$ which is also outside [-1,1]. Due to the inverse transformation $g_1 = 1/(1 + g)$ (which is actually the transformation (v) in (4.4)) we can identified the segment [0,1] for the construction of the moduli space with the segment [-1/2,0]. The other two segments on the circle in Diagram 4.1.1(\mathcal{M}) are sent outside of this segment of circle via transformation (v).

As we already know the behavior of the transformation (v) in (4.4), it remains to consider just the transformations (iii) and (iv) in (4.4) and see if these last two transformations could further identify points in the segment [-1,0]. So we now consider the third transformation from (4.4) and its induced transformation on the line $b = 0$, $h = 1$: $g_1 = -g - 1$. This transformation has -1/2 as a fixed point, it sends the segment [-1,-1/2] onto [-1/2,0] and the segment [-1/2,0] onto [-1,-1/2]. Hence we can identify via this transformation the points in the segment [-1/2,0] with those in the segment [-1,1/2] via this transformation and so we can discard [-1/2,0].

Finally we consider the transformation (iv) in (4.4) which written on the affine line $b = 0$, $h = 1$ is: $g_1 = -g/(g + 1)$. We observe that this transformation is of order 2 so it is equal to its inverse. It remains to see if this transformation further identifies points in the segment [-1,1]. We observe that it has no fixed points in this interval. We note that this transformation maps the interval $[-1,1]$ onto the interval $[1,\infty)$ so it cannot identify points in $[-1,1]$. We observe that we have the same phase portraits and see if we can identify via the group action points on these two segments. We observe that the only transformation which does this is the transformation (iii) from (4.4). Thus via this transformation we can identify these two segments finally obtaining the picture in Diagram 4.1.1(\mathcal{M}).

We conclude that the moduli set for the configuration 4.1 is the picture in the Diagram 4.1.1(\mathcal{M}) where 1) we only take from the segment on the circumference the segment corresponding to the interval $[-1,-1/2]$, the segment $[-1/2,1]$ being identified with $[-1,-1/2]$; 2) we only take one of the segments $g = 0$ ($h > 1$) and $h - g - 1 = 0$ ($h > 1$), via the arguments mentioned above.

### 4.1.2 Configurations 4.3 and 4.4

According to Theorem 3.1 all the systems having the Configurations 4.3 and 4.4 are included in the family:

\[ \ddot{x} = x + gx^2 + (h-b)xy, \quad \ddot{y} = y + (g-b)xy + hy^2, \]  

(4.5)
where \([b : g : h] \in \mathbb{P}_2(\mathbb{R})\). We construct now the bifurcation diagram for this canonical form. This diagram will again be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line \(b = 0\).

### 4.1.2.1 The case \(b \neq 0\)

In this case we may assume \(b = 1\) Then following the Table 2 of [26] for systems (4.5) we compute invariants necessary for each one of the Pictures 4.3(u), \(u \in \{a, b, c\}\):

\[
\begin{align*}
\theta &= 8(1 - g)(h - 1)(g + h), \mu_0 = gh(g + h - 1), B_3 = 0, H_7 = 0, H_1 = 576, \\
N &= (g^2 - 1)x^2 + 2(g - 1)(h - 1)xy + (h^2 - 1)y^2, K = 2g(g - 1)x^2 + 4ghxy + 2(h - 1)y^2.
\end{align*}
\]

#### 4.1.2.1.1 The subcase \(\mu_0 \neq 0\)

Since \(H_1 \neq 0\), according to [26] if \(\theta \neq 0\) the phase portrait of a system (4.5) corresponds to Picture 4.3(a) (respectively Picture 4.3(b); Picture 4.3(c)) if \(\mu_0 > 0\) (respectively \(\mu_0 < 0\), \(K < 0\); \(\mu_0 < 0\), \(K > 0\)). So, if \(gh(g + h - 1)(1 - g)(h - 1)(g + h) \neq 0\) then we arrive to the situation given by Diagram 4.1.2.

**Diagram 4.1.2**

Assume now \(\theta = 0\), i.e. \((1 - g)(h - 1)(g + h) = 0\). If \(N \neq 0\), (i.e. \((1 - g)^2 + (h - 1)^2 + (g + h)^2 \neq 0\)) then from [24] it follows that the points on the lines \(g = 1\), \(h = 1\) and \(h = -g\) except the points
(1, 1), (−1, 1) and (1, −1) correspond to Picture 5.1. On the other hand the indicated last three
triangles correspond to Picture 6.1 since for these values of parameters g and h we have N = 0 and
H₁ > 0.

4.1.2.1.2 The subcase μ₀ = 0 Then gh(g + h − 1) = 0 and we assume first g = 0. In this
case we get the family of systems
\[ \dot{x} = x + (h - 1)xy, \quad \dot{y} = y - xy + hy^2, \]
for which we have \( \theta = 8h(h - 1) \) and \( K = 2h(h - 1)y^2. \)

1) Assume \( \theta \neq 0. \) Then \( K \neq 0 \) and according to [26] the phase portrait of a system (4.6)
corresponds to Picture 4.4(a) if \( K < 0 \) (i.e. \( h(h - 1) < 0 \)) and it corresponds to Picture 4.4(b) if
\( K > 0 \) (i.e. \( h(h - 1) > 0 \)).

In a similar way if \( h = 0 \) (respectively if \( h = 1 - g \)) we have \( \theta = 8g(g - 1) \) and \( K = 2g(g - 1)x^2 \)
(respectively \( \theta = 8g(g - 1) \) and \( K = 2g(g - 1)(x - y)^2 \)). So, we get Picture 4.4(a) if \( g(g - 1) < 0 \)
and Picture 4.4(b) if \( g(g - 1) > 0 \) as it is given in Diagram 4.1.2.

Remark 4.1. We observe, that for systems (4.5) with \( b = 1 \) the case \( h = 0 \) (respectively \( h = 1 - g \)
could be obtained from \( g = 0 \) (i.e. from the systems (4.6)) via the transformation \( (x, y, t, h) \mapsto (x_1, y_1, t_1, g_1) \) (respectively \( (x, y, t, h) \mapsto (-x_1, y_1 - x_1, t_1, 1 - g_1) \)).

2) Assume now \( \theta = 0. \) Then for systems (4.6) we get \( h(h - 1) = 0. \) For these systems we have
\[ N = -x^2 - 2(h - 1)xy + (h^2 - 1)y^2 \neq 0, \quad H₀ = μ₀ = θ = 0 \]
and according to [24] the points \( (g, h) \in \{(0, 0), (0, 1)\} \) on Diagram 4.1.2 corresponds to systems
possessing the same phase portrait and namely, Picture 5.7.

Considering Remark 4.1 we conclude, that the points (1, 0) on Diagram 4.1.2 corresponds to the
same Picture 5.7.

4.1.2.2 The case \( b = 0 \) Then from (4.5) we get the following family of degenerate systems:
\[ \dot{x} = x(1 + gx + hy), \quad \dot{y} = y(1 + gx + hy), \]
possessing the affine singular line \( gx + hy + 1 = 0. \) Since for \( b = 0 \) we have \( y^2 + h^2 \neq 0 \) we arrive to
the phase portrait corresponding to Picture D.4.

We now see that the lines drawn on the bifurcation Diagram 4.1.2. are the same as those in
Diagram 4.1.1. Furthermore we observe that the bifurcation Diagram 4.1.2 is symmetrical with
respect to the first bisectrix \( g = h. \) So we suspect that analogous arguments could be applied in
this case.

Indeed, this can be seen by using the transformations preserving the canonical form (4.5), which
can be shown to be the following ones:

\[
\begin{align*}
(i) & \quad (x, y, t) \mapsto (y, x, t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, h, g); \\
(ii) & \quad (x, y, t) \mapsto (y - x, -x, t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, b - g - h, g); \\
(iii) & \quad (x, y, t) \mapsto (-x, y - x, t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, b - g - h, h); \\
(iv) & \quad (x, y, t) \mapsto (x - y, -y, t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, g, b - g - h); \\
(v) & \quad (x, y, t) \mapsto (-y, x - y, t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, h, b - g - h); \\
vii) & \quad (x, y, t) \mapsto (x, y, t) \quad \Rightarrow \quad (b, g, h) \mapsto (b, g, h). 
\end{align*}
\]
We now observe that the transformations of the affine plane \((x, y)\) and time homotheties above are different than those in (4.4) but the transformations of the parameters space with homogeneous coordinates \((b, g, h)\) are here exactly the same as in (4.4). As our discussion here depends only on these last transformations all the arguments used for Config. 4.1 apply also here.

Thus we obtain the moduli set and bifurcation diagram indicated in Diagram 4.1.2(\(\mathcal{M}\)), where on the arc of the circle we have similar identifications as in Diagram 4.1.1(\(\mathcal{M}\)).

### 4.1.3 Configuration 4.5

According to Theorem 3.1 all the systems having the Configuration 4.5 are included in the family of quadratic homogeneous systems:

\[
\dot{x} = gx^2 + (h-b)xy, \quad \dot{y} = (g-b)xy + hy^2,
\]

where \([b : g : h] \in \mathbb{P}_2(\mathbb{R})\). We construct now the bifurcation diagram for this canonical form. This diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line \(b = 0\).

#### 4.1.3.1 The case \(b = 1\)

Then for systems (4.9) calculations yield:

\[
\begin{align*}
\theta &= 8(1-g)(h-1)(g+h), \\
\mu_0 &= gh(g+h-1), \\
B_3 &= 0, \\
H_7 &= 0, \\
H_1 &= 0, \\
N &= (g^2-1)x^2 + 2(g-1)(h-1)xy + (h^2-1)y^2, \\
K &= 2g(g-1)x^2 + 4ghxy + 2(h-1)y^2.
\end{align*}
\]

#### 4.1.3.1.1 The subcase \(\mu_0 \neq 0\)

Since \(H_1 = 0 = H_7\), according to [26] if \(\theta \neq 0\) the phase portrait of a system (4.9) corresponds to Picture 4.5(a) (respectively Picture 4.5(b); Picture 4.5(c)) if \(\mu_0 > 0\) (respectively \(\mu_0 < 0\), \(K < 0\); \(\mu_0 < 0\), \(K > 0\)). So, if \(gh(g+h-1)(1-g)(h-1)(g+h) \neq 0\) then we arrive to the situation given by Diagram 4.1.3.
Assume now $\theta = 0$, i.e. $(1 - g)(h - 1)(g + h) = 0$. If $N \neq 0$, (i.e. $(1 - g)^2 + (h - 1)^2 + (g + h)^2 \neq 0$) then (as for systems (4.9) $H_1 = 0$) according to [24] the points on the lines $g = 1$, $h = 1$ and $h = -g$ except the points $(1, 1)$, $(-1, 1)$ and $(1, -1)$ correspond to Picture 5.8. On the other hand the indicated last three points correspond to Picture 6.5 since for these values of parameters $g$ and $h$ we have $N = 0 = H_1$.

4.1.3.1.2 The subcase $\mu_0 = 0$ Then we have $gh(g + h - 1) = 0$ and systems (4.9) become degenerate.

Assume first $g = 0$. In this case systems we get the family of systems:

$$
\dot{x} = (h - 1)xy, \quad \dot{y} = -xy + hy^2
$$

(4.10)

having the singular invariant line $y = 0$. Examining the respective linear systems

$$
\dot{x} = (h - 1)x, \quad \dot{y} = -x + hy,
$$

and considering the singular line $y = 0$ we obtain, that the phase portrait of a system (4.10) corresponds to:

- Picture D.5 if $h(h - 1) < 0$;
- Picture D.6 if $h(h - 1) > 0$;
- Picture D.3 if $h(h - 1) = 0$. 

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We observe, that for systems (4.9) with \( b = 1 \) the case \( h = 0 \) (respectively \( h = 1 - g \)) could be obtained from \( g = 0 \) (i.e. from the systems (4.10)) via the transformation \((x, y, t, h) \mapsto (x_1, y_1, t_1, g_1)\) (respectively \((x, y, t, h) \mapsto (-x_1, y_1 - x_1, t_1, 1 - g_1)\)). Thus in the case \( b = 1 \) on the lines \( g = 0, h = 0 \) and \( g + h - 1 = 0 \) we get the bifurcations given by Diagram 4.1.3.

4.1.3.2 The case \( b = 0 \) Then from (4.9) we get the following family of degenerate systems:
\[
\begin{align*}
\dot{x} &= x(gx + hy), \\
\dot{y} &= y(gx + hy),
\end{align*}
\] (4.11)
possessing the affine singular line \( gx + hy = 0 \). Since for \( b = 0 \) we have \( g^2 + h^2 \neq 0 \) we arrive to the phase portrait corresponding to Picture D.7.

We now observe that the lines drawn on the bifurcation Diagram 4.1.3. are the same as those in Diagram 4.1.2. Furthermore we observe that the bifurcation Diagram 4.1.3 is symmetrical with respect to the first bisectrix \( g = h \). So we suspect that analogous arguments could be applied in this case.

Indeed, this can be seen by using the transformations preserving the canonical form (4.9), which can be shown to be exactly the same (4.8).

Using the same arguments we used for Config. 4.1 we can conclude that the moduli set and the bifurcation diagram are as in Diagram 4.1.3(\( \mathfrak{M} \)) with the corresponding identifications.

4.1.4 Configurations 4.9 and 4.10

According to Theorem 3.1 all the systems having the Configurations 4.9 and 4.10 are included in the family:
\[
\begin{align*}
\dot{x} &= g(x^2 - g^2), \\
\dot{y} &= y[(g - b)x + by - 2bh],
\end{align*}
\] (4.12)
where \([b : g : h] \in \mathbb{P}_2(\mathbb{R})\). We construct now the bifurcation diagram for this canonical form. As before, this diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line \(b = 0\).

### 4.1.4.1 The case \(b = 1\)  
Then for systems (4.12) calculations yield:

\[
\theta = B_2 = H_7 = 0, \quad \mu_0 = g^2, \quad B_3 = 3g[g^2(g + 1)^2 - 4h^2]x^2y^2, \\
N = (g^2 - 1)x^2, \quad H_4 = 48g(1 - g)[g^2(g + 1)^2 - 4h^2], \quad H_{10} = 32g^4(g^2 - 1), \\
H_9 = -9216g^4[g^2(g - 1)^2 - 4h^2] , \quad G_2 = 2^93^3g^5[g^2(g - 1)^2 - 4h^2], \\
G_3 = 288g^2(1 - g)[g^2(g + 1)^2 - 4h^2].
\]  

\[(4.13)\]

\subsection{4.1.4.1.1 The subcase \(H_4 \neq 0\)}  
Then \(g(g - 1)[g^2(g + 1)^2 - 4h^2] \neq 0\) and this implies \(B_3G_3 \neq 0\). We shall consider two cases: \(H_9 \neq 0\) and \(H_9 = 0\).

1) Assume first \(H_9 \neq 0\). Then \(g^2(g - 1)^2 - 4h^2 \neq 0\) that yields \(G_2 \neq 0\).

a) If \(N \neq 0\) then \(H_{10}N > 0\) and according to [26] the phase portrait of a system (4.12) with \(b = 1\) corresponds to Picture 4.9(a) (respectively Picture 4.9(b); Picture 4.9(c)) if \(G_2 > 0, H_4 > 0, G_3 < 0\) (respectively if either \(G_2 < 0\) or \(G_2 > 0, H_4 < 0\); if \(G_2 > 0, H_4 > 0, G_3 > 0\)).

So, if \(g(g - 1)[g^2(g + 1)^2 - 4h^2][g^2(g - 1)^2 - 4h^2] \neq 0\) then we arrive to the situation given by Diagram 4.14.

b) Suppose now \(N = 0\). Since \(H_4 \neq 0\) we get \(g = -1\) and then calculation yields:

\[
\mu_0 = 1, \quad B_3 = 12h^2x^2y^2, \quad H_4 = 384h^2, \quad H_8 = 2^93^3h^2 \\
H_9 = -2^{14}3^2(h^2 - 1)^2, \quad G_2 = 2^{11}3^3(h^2 - 1), \quad G_3 = -2304h^2.
\]

Since \(H_4H_9 \neq 0\) we have \(h(h^2 - 1) \neq 0\) and this implies \(H_8 > 0, G_3 < 0\) and \(H_4 > 0\). So, according to [26] the phase portrait of a system (4.12) with \(b = 1\) and \(g = -1\) corresponds to Picture 4.9(a) if \(G_2 > 0\) and to Picture 4.9(b) if \(G_2 < 0\).

2) Assume now \(H_9 = 0\). Since \(H_4 \neq 0\) (i.e. \(g \neq 0\)) we get \(g(g - 1) = \pm 2h\).

\textbf{Remark 4.2.} We observe that the change \((x, y, t, g, h) \mapsto (-x, -y, -t, g, -h)\) keeps the family systems (4.12) with \(b = 1\).

For \(h = \pm g(g - 1)/2\) calculation yields

\[
\mu_0 = g^2, \quad \theta = B_2 = H_7 = H_9 = 0, \quad B_3 = 12g^4x^2y^2, \quad N = (g^2 - 1)x^2, \\
H_4 = 192g^4(1 - g), \quad H_{10} = 32g^4(g^2 - 1), \quad G_3 = 1152g^5(1 - g).
\]

a) If \(N \neq 0\) then \(g \neq -1\) and we get \(H_{10}N > 0\). According to [26] the phase portrait of a system (4.12) with \(b = 1\) and \(u = \pm g(g - 1)/2\) corresponds to Picture 4.10(a) (respectively Picture 4.10(b); Picture 4.10(c)) if \(H_4 > 0, G_3 > 0\) (respectively \(H_4 < 0; H_4 > 0, G_3 < 0\)).

b) Suppose now \(N = 0\). Due to \(H_4 \neq 0\) we obtain \(g = -1\) and then \(H_8 = 2^93^3 > 0\). Hence by [26] we obtain Picture 4.10(c).

### 4.1.4.1.2 The subcase \(H_4 = 0\)  
Then \(g(g - 1)[g^2(g + 1)^2 - 4h^2] = 0\) and we shall consider two cases: \(\mu_0 \neq 0\) and \(\mu_0 = 0\).

1) Assume first \(\mu_0 \neq 0\). Then \(g \neq 0\) and this implies \((g - 1)[g^2(g + 1)^2 - 4h^2] = 0\).

a) If \(B_3 \neq 0\) then from (4.13) we get \(g^2(g + 1)^2 - 4h^2 \neq 0\) and this implies \(g = 1\). Thus we obtain the family of systems

\[
\dot{x} = x^2 - 1, \quad \dot{y} = y(y - 2h).
\]  

\[(4.14)\]
for which calculations yield:

\[
B_2 = N = H_4 = 0, \quad B_3 = -12(h^2 - 1)x^2y^2, \quad H_1 = 1152(h^2 + 1), \quad H_5 = 6144h^2.
\]

Therefore according to [24] the phase portrait of a system (4.14) with \( B_3 \neq 0 \) (i.e. \( h \neq \pm 1 \)) corresponds to Picture 5.3 if \( H_5 \neq 0 \) (then \( H_5 > 0 \)) and to Picture 5.12 if \( H_5 = 0 \) (i.e. \( h = 0 \)).

b) Suppose now \( B_3 = 0 \). Then \( h = \pm g(g + 1)/2 \) and for systems (4.12) with \( b = 1 \) we calculate:

\[
\mu_0 = g^2, \quad \theta = B_3 = 0, \quad N = (g^2 - 1)x^2, \quad H_1 = 2304g^4.
\]

If \( N \neq 0 \) (i.e. \( g \neq \pm 1 \)) since \( H_1 \neq 0 \) (due to \( \mu_0 \neq 0 \)) according to [24] the phase portrait of a system (4.12) with \( b = 1 \) and \( h = \pm g(g + 1)/2 \) corresponds to Picture 5.1.

Assume now \( N = 0 \), i.e. \( g = \pm 1 \). Then we get the conditions \( \eta > 0, \quad B_3 = N = 0, \quad H_1 > 0 \) and by [24] we arrive to Picture 6.1.

2) Suppose now \( \mu_0 = 0 \). Then \( g = 0 \) and we get the family of degenerate systems

\[
\dot{x} = 0, \quad \dot{y} = y(-x + y - 2h).
\]

with the phase portrait given by Picture D.3.
4.1.4.2 The case $b = 0$ Then from (4.12) we get the following family of systems:

$$
\dot{x} = g(x^2 - g^2), \quad \dot{y} = gxy,
$$

(4.16)
possessing the infinite line filled with singularities. For these systems calculations yield:

\[ C_2 = 0, \ H_{10} = 0, \ H_{12} = -8g^{10}x^2, \ H_{11} = 192g^8x^2, \ \mu_2 = -g^6x^2 \]

and hence, if \( H_{12} \neq 0 \) we have \( H_{11} > 0 \) and \( \mu_2 < 0 \). According to [22] in this case the phase portrait of systems (4.3) corresponds to Picture C.2.5(a).

For \( H_{12} = 0 \) we get the trivial system \( \dot{x} = 0, \ \dot{y} = 0 \) the phase portrait of which is given (symbolically) by Picture D.0.

We now observe that the bifurcation Diagram 4.1.4 is symmetrical with respect to the line \( h = 0 \).

In fact the transformation \((x, y, t) \mapsto (-x, -y, -t)\) induces on the parameters the transformation \((b, g, h) \mapsto (b, g, -h)\) and hence the points on the lower side of the diagram are identified with their symmetrical ones on the the upper-half plane.

On the affine chart \( b \neq 0 \) (i.e. \( b = 1 \)) it can easily be checked that inside the upper-half disk and outside the line \( g = 0, \ g = 1 \) and outside the parabolas \( 2h = \pm(g^2 + g) \) we cannot identify two distinct points under the group action.

We now look at the curves we excluded and we consider the parabola \( 2h = (g^2 + g) \). We note that the transformation \((x, y, t) \mapsto (-x, y - x + g, t)\) yields on the parameter space the map \((g, h) \mapsto (-g, h)\) and leaves invariant this parabola interchanging points from \( g \leq -1 \) (respectively \(-1 < g \leq 0\)) with points on the parabola with \( g \geq 1 \) (respectively \( 0 \leq g < 1 \)). In view of the symmetry with respect to the line \( h = 0 \) we can identify the segment corresponding to \(-1 < g \leq 0\) on the parabola \( 2h = (g^2 + g) \) with the segment corresponding to \(-1 < g \leq 0\) on the parabola \( 2h = -(g^2 + g) \). In view of the above mention arguments we have the following identifications on the upper-half disk: (i) the two segments of the parabola \( 2h = (g^2 + g) \) corresponding to \( g \leq -1 \) and \( g \geq 1 \); (ii) the segment of the parabola \( 2h = (g^2 + g) \) corresponding to \( 0 \leq g < 1 \) with the segment of the parabola \( 2h = -(g^2 + g) \) corresponding to \(-1 < g \leq 0\).

On the line \( g = 0 \) the transformation \((x, y, t) \mapsto (x - 2h + 2h_1, y, t)\) maps the system of parameter \( h \) on the system with \( h_1 \). So the whole half line \( h \geq 0 \) collapses to one point.

On the line \( b = 0 \) in the affine chart corresponding to \( h = 1 \) using the coordinate \( g \), the line \( b = 0 \) becomes the \( g \)-axis. In the resulting equations, via the transformation \((x, y, t) \mapsto (-x, y, t)\) we can change \( g \) to \(-g\) obtaining a system in the same orbit. Moreover we can actually identify via the group action any two systems corresponding to two points \( 0 < g_1 \leq g_2 \) due to the transformation:

\[(x, y, t) \mapsto (g_1x, g_2y, g_2^2t/g_1^2).\]

So we first identify the two quarters of the semicircle \( b = 0 \) keeping the point \([0:0:1]\) fixed. Projecting the cone thus obtained on the disk with circumference the line \( h = 0 \) and placing on this picture the portraits previously obtained for the half disk of Diagram 4.1.4* we obtain Diagram 4.2.4(\textit{M}). with the the above mentioned identifications.

This yields the moduli set for the family (4.12) on which we put via the canonical map into the quotient, the quotient topology.

4.1.5 Configuration 4.13

The systems having the Config. 4.13 are included in the family:

\[ \dot{x} = g(x^2 + g^2), \quad \dot{y} = y[(g-b)x + by - 2bh], \]  

(4.17)

where \([b : g : h] \in \mathbb{P}_2(\mathbb{R})\). We construct now the bifurcation diagram for this canonical form. This diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line \( b = 0 \).
4.1.5.1 The case \( b \neq 0 \) We may assume \( b = 1 \). Then for systems (4.17) calculations yield:

\[
\theta = B_2 = H_7 = 0, \quad \mu_0 = g^2, \quad B_3 = -3g[g^2(g + 1)^2 + 4h^2]x^2y^2,
\]
\[
N = (g^2 - 1)x^2, \quad H_4 = 48g(g - 1)[g^2(g + 1)^2 + 4h^2], \quad H_{10} = -32g^4(g^2 - 1),
\]
\[
G_3 = 288g^2(g - 1)[g^2(g + 1)^2 + 4h^2]. \tag{4.18}
\]

4.1.5.1.1 The subcase \( H_4 \neq 0 \) Then \( g(g - 1)[g^2(g + 1)^2 + 4h^2] \neq 0 \) and this implies \( B_3G_3 \neq 0 \). We shall consider two cases: \( N \neq 0 \) and \( N = 0 \).

1) Assume \( N \neq 0 \). Then \( H_{10}N < 0 \) and according to [26] the phase portrait of a system (4.17) with \( b = 1 \) corresponds to Picture 4.1.3(a) if \( G_3 > 0 \) (i.e. \( g > 1 \)) and to Picture 4.1.3(b) if \( G_3 < 0 \) (i.e. \( g < 1 \)) as it is indicated on Diagram 4.1.5.

![Diagram 4.1.5](image)

2) Suppose now \( N = 0 \). As \( H_4 \neq 0 \) (i.e. \( g \neq 1 \)) we get \( g = -1 \) and then \( H_8 = -2^93^3h^2 < 0 \). So, by [26] we obtain Picture 4.1.3(b).

4.1.5.1.2 The subcase \( H_4 = 0 \) Then \( g(g - 1)[g^2(g + 1)^2 + 4h^2] \) and we shall consider two cases: \( B_3 \neq 0 \) and \( B_3 = 0 \).

1) Assume first \( B_3 \neq 0 \). Then \( g[g^2(g + 1)^2 + 4h^2] \neq 0 \) and this implies \( g = 1 \). In this case we get the family of systems

\[
\dot{x} = x^2 + 1, \quad \dot{y} = y(y - 2h), \tag{4.19}
\]

for which calculations yield:

\[
B_2 = N = H_4 = 0, \quad B_3 = -12(h^2 + 1)x^2y^2, \quad H_1 = 1152(h^2 - 1), \quad H_5 = -6144h^2.
\]
Therefore according to [24] the phase portrait of a system (4.19) corresponds to *Picture 5.4* if \( H_5 \neq 0 \) (then \( H_5 < 0 \)) and to *Picture 5.16* if \( H_5 = 0 \) (i.e. \( h = 0 \) and then \( H_1 < 0 \)).

2) Suppose now \( B_3 = 0 \). Then either \( g = 0 \) or \( g = -1 \) and \( h = 0 \).

a) If \( \mu_0 \neq 0 \) then \( g \neq 0 \) and we get \( g = -1, h = 0 \) and calculations yields: \( B_3 = N = 0 \), \( H_1 = -2304 < 0 \). So, by [24] the phase portrait of the system (4.17) (with \( b = 1 = -g \) and \( h = 0 \)) corresponds to *Picture 6.2*.

b) In the case \( \mu_0 = 0 \) (i.e. \( g = 0 \)) we get the family (4.15) of degenerate systems with the phase portrait given by *Picture D.3*.

![Diagram 4.1.5](https://via.placeholder.com/150)

### 4.1.5.2 The case \( b = 0 \).

Then from (4.17) we get the following family of systems:

\[
\dot{x} = g(x^2 + g^2), \quad \dot{y} = gxy, \quad (4.20)
\]

possessing the infinite line filled with singularities. For these systems calculations yield:

\[
C_2 = 0, \quad H_{10} = 0, \quad H_{12} = -8g^{10}x^2, \quad H_{11} = -192g^8x^4
\]

and hence, if \( H_{12} \neq 0 \) we have \( H_{11} < 0 \). According to [22] in this case the phase portrait of systems (4.3) corresponds to *Picture C.6*.

For \( H_{12} = 0 \) we get the trivial system \( \dot{x} = 0, \dot{y} = 0 \) the phase portrait of which is given (symbolically) by *Picture D.0*.

We now observe that the bifurcation Diagram 4.1.5 is symmetrical with respect to the line \( h = 0 \). In fact the transformation \((x, y, t) \mapsto (-x, -y, -t)\) induces on the parameters the transformation \((b, g, h) \mapsto (b, g, -h)\) and hence the points on the lower side of the diagram are identified with their symmetrical ones on the the upper-half plane. Therefore we can limit ourselves to the case \( h \geq 0 \) and thus we can discard the lower side of this diagram.

It can easily be checked that inside the upper-half disk and outside the line \( g = 0 \) we cannot identify two distinct points under the group action. On the line \( g = 0 \) the transformation \((x, y, t) \mapsto (x - 2h + 2h_1, y, t)\) maps the system of parameter \( h \) on the system with \( h_1 \). So the whole half line \( h \geq 0 \) collapses to one point.

On the line \( b = 0 \) in the affine chart corresponding to \( h = 1 \) using the coordinate \( g \), the line \( b = 0 \) becomes the \( g \)-axis. In the resulting equations, via the transformation \((x, y, t) \mapsto (-x, y, t)\) we can change \( g \) to \(-g\) obtaining a system in the same orbit. Moreover we can actually identify via the group action any two systems corresponding to two points \( 0 < g_1 \leq g_2 \) on the line \( b = 0, h = 1 \).
4.1.6 The case previously obtained for the half disk of Diagram 4.2.5 we obtain Diagram 4.2.5 obtained on the disk with circumference the line \( h \) for the Picture 4.22(b) corresponds to Picture 4.22(a) corresponding to Picture 4.22. So, if \( N \neq 0 \) we have \( g(1-h) \neq 0 \) and then \( \mu_0 B_3 H_1 \neq 0 \).

1) Assume first \( N \neq 0 \). Then according to [26] the phase portrait of a system (4.21) with \( b = 1 \) corresponds to Picture 4.22(a) if \( H_1 > 0 \) (i.e. \( g > 0 \)) and to Picture 4.22(b) if \( H_1 < 0 \) (i.e. \( g < 0 \)) as it is indicated on Diagram 4.1.6.

2) Suppose now \( N = 0 \). As \( H_4 \neq 0 \) (i.e. \( g \neq 1 \)) we get \( g = -1 \) and then by [26] the conditions for the Picture 4.22(b) are fulfilled.

4.1.6.1 The case \( b \neq 0 \) We may assume \( b = 1 \). Then for systems (4.21) calculations yield:

\[
\begin{align*}
\theta &= B_2 = H_7 = H_{10} = 0, \quad \mu_0 = g^2, \quad B_3 = -12gh^2 x^2 y^2, \\
N &= (g^2 - 1)x^2, \quad H_1 = 1152gh^2, \quad H_8 = 0, \quad H_4 = 192(g-1)h^2.
\end{align*}
\]

4.1.6.1.1 The subcase \( H_4 \neq 0 \) This implies \( g(g-1)h \neq 0 \) and then \( \mu_0 B_3 H_1 \neq 0 \).

1) Assume first \( N \neq 0 \). Then according to [26] the phase portrait of a system (4.21) with \( b = 1 \) corresponds to Picture 4.22(a) if \( H_1 > 0 \) (i.e. \( g > 0 \)) and to Picture 4.22(b) if \( H_1 < 0 \) (i.e. \( g < 0 \)) as it is indicated on Diagram 4.1.6.

2) Suppose now \( N = 0 \). As \( H_4 \neq 0 \) (i.e. \( g \neq 1 \)) we get \( g = -1 \) and then by [26] the conditions for the Picture 4.22(b) are fulfilled.

4.1.6.1.2 The subcase \( H_4 = 0 \) Then \( g(g-1)h = 0 \) and we shall consider two cases: \( B_3 \neq 0 \) and \( B_3 = 0 \).

1) Assume first \( B_3 \neq 0 \). Then \( gh \neq 0 \) and this implies \( g = 1 \), i.e. \( N = 0 \). In this case we get the family of systems

\[
\begin{align*}
\dot{x} &= x^2, \quad \dot{y} = y(y - 2h),
\end{align*}
\]

for which calculations yield:

\[
\begin{align*}
B_2 &= N = H_4 = 0, \quad B_3 = -12h^2 x^2 y^2, \quad H_1 = 1152h^2, \quad H_5 = 0.
\end{align*}
\]

Hence since \( B_3 \neq 0 \) we have \( H_1 > 0 \) and according to [22] the phase portrait of systems (4.23) corresponds to Picture 5.12.

2) Suppose now \( B_3 = 0 \). Then \( gh = 0 \) and we shall consider two cases: \( \mu_0 \neq 0 \) and \( \mu_0 = 0 \).

a) If \( \mu_0 \neq 0 \) then \( g \neq 0, \ h = 0 \) and we get the family of systems

\[
\begin{align*}
\dot{x} &= gx^2, \quad \dot{y} = y(y + (g-1)x),
\end{align*}
\]

for which calculations yields:

\[
\begin{align*}
B_3 &= \theta = H_1 = 0, \quad N = (g^2 - 1)x^2.
\end{align*}
\]

So, if \( N \neq 0 \) by [24] the phase portrait of the system (4.24) corresponds to Picture 5.8.

Assuming \( N = 0 \) (i.e. \( g = \pm 1 \)), in both cases according to [24] we get Picture 6.5.

b) Supposing \( \mu_0 = 0 \), i.e. \( g = 0 \), we get the family (4.15) of degenerate systems with the phase portrait given by Picture D.3.
4.1.6.2 The case \( b = 0 \) Then from (4.21) we get the following family of degenerate systems:

\[
\dot{x} = gx^2, \quad \dot{y} = gxy, \tag{4.25}
\]

possessing the affine line \( x = 0 \) fulfilled with singularities. If \( g \neq 0 \) then the singular point \((0, 0)\) of the respective linear system is a star node. So, considering the existence of singular line \( x = 0 \) we get Picture D.7.

For \( g = 0 \) we get the trivial system \( \dot{x} = 0, \dot{y} = 0 \) the phase portrait of which is given (symboli-
We now observe that the bifurcation Diagram 4.1.6 is symmetrical with respect to the line $h = 0$. In fact the transformation $(x, y, t) \mapsto (-x, -y, -t)$ induces on the parameters the transformation $(b, g, h) \mapsto (b, g, -h)$ and hence the points on the lower side of the diagram are identified with their symmetrical ones on the upper-half plane. Therefore we can limit ourselves to the case $h \geq 0$ and thus we can remove the lower side of this diagram (see Diagram 4.1.6$^*$).

We now look if we can identify via the group action points which lie inside the half disk ($h \geq 0$). The transformation $(x, y, t) \mapsto (hx, hy, t/h)$ with $h \neq 0$ induces on the parameter plane the transformation $(g, h) \mapsto (g, 1)$. So all the points $(g, h)$ for which $h > 0$ can be identified via the group action with the points on the line $h = 1$.

On the line $h = 0$ the transformation $(x, y, t) \mapsto (x, x - y, -t)$ induces on this line the transformation $(g, 0) \mapsto (-g, 0)$. So we can identify via the group action these two points.

On the line $b = 0$ our systems become (4.25) for $g \neq 0$ and the system $\dot{x} = 0$, $\dot{y} = 0$ for $h \neq 0$. In the first case via the time rescaling we may consider $g = 1$. Hence, on the affine chart $g \neq 0$ we can identify via the group action all the points on the line $b = 0$ with the point $[0 : 1 : 0]$.

Due to the identifications mentioned above we see that the quotient set of the family via the group action will be pictured on three lines: $h = 0$, $h = b$ and $b = 0$, all three passing through the point $[0 : 1 : 0]$. We view these three lines as three circles with one point in common which we draw on the Diagram 4.2.6($\mathfrak{M}$). We draw with dots the circle $b = 0$ and half of the circle $h = 0$, as on these circles we have the identifications mentioned above.

In this way we obtain Diagram 4.2.6($\mathfrak{M}$) which is the moduli set for the family of systems (4.21). On this set we put the quotient topology of the systems via the group action.

### 4.1.7 Configurations 4.16, 4.17 and 4.34

The family of systems:

$$
\dot{x} = h + gx, \quad \dot{y} = -bxy + by^2,
$$

where $[b : g : h] \in \mathbb{P}_2(\mathbb{R})$. We construct now the bifurcation diagram for this canonical form. This diagram will be drawn on the projective plane viewed on the disk with opposite points on the
circumference identified. We place on the circumference the line \( b = 0 \).

4.1.7.1 **The case** \( b = 1 \). Then for systems (4.26) calculations yield:

\[
\theta = B_2 = H_7 = \mu_0 = 0, \quad B_3 = -3(g^2 - h)x^2y^2, \quad N = -x^2,
\]
\[
H_6 = 64(h - g^2)x^5(2x - y), \quad H_9 = -576g^4h^2, \quad H_{10} = -8g^2, \quad G_2 = -3456g^2h.
\] (4.27)

**4.1.7.1.1 The subcase** \( H_9 \neq 0 \) Then \( gh \neq 0 \) and this implies \( G_2 \neq 0 \). We shall consider two cases: \( B_3 \neq 0 \) and \( B_3 = 0 \).

1) Assume first \( B_3 \neq 0 \). Then \( g^2 - h \neq 0 \) and according to [26] the phase portrait of a system (4.26) with \( b = 1 \) corresponds to Picture 4.16(a) if \( G_2 > 0 \) (i.e. \( h > 0 \)) and to Picture 4.16(b) if \( G_2 < 0 \) (i.e. \( h < 0 \)) as it is indicated on Diagram 4.1.7.

2) Suppose now \( B_3 \neq 0 \). In this case we obtain \( h = g^2 \) and from (4.27) we get \( H_6 = 0 \). So, according to [22] the phase portrait of systems (4.26) in this case corresponds to Picture 5.7.

**4.1.7.1.2 The subcase** \( H_9 = 0 \). Then \( gh = 0 \) and we shall consider two cases: \( H_{10} \neq 0 \) and \( H_{10} = 0 \).

1) If \( H_{10} \neq 0 \) then from (4.27) we obtain \( g \neq 0 \) and then \( h = 0 \). Considering (4.27) according to [26] we arrive to Picture 4.17.

2) Assume \( H_{10} = 0 \). In this case we have \( g = 0 \) and systems (4.26) become

\[
\dot{x} = h, \quad \dot{y} = -xy + y^2,
\] (4.28)
for which we calculate: \( B_3 = -3hx^2y^2 \) and \( H_4 = -48h \).

If \( B_3 \neq 0 \) then \( H_4 \neq 0 \) and according to [26] the phase portrait of a system (4.28) corresponds to Picture 4.34(a) if \( H_4 < 0 \) (i.e. \( h > 0 \)) and to Picture 4.34(b) if \( H_4 > 0 \) (i.e. \( h < 0 \)).

In the case \( B_3 = 0 \) (i.e. \( h = 0 \)) we get the degenerate system (4.28) the phase portrait of which is given by Picture D.3.

4.1.7.2 The case \( b = 0 \) Then from (4.26) we get the following family of degenerate systems:

\[
\begin{align*}
\dot{x} &= h + gx, \\
\dot{y} &= 0,
\end{align*}
\]  

(4.29)

possessing the affine singular line \( gx + h = 0 \) if \( g \neq 0 \). Since for systems (4.26) we have \( b^2 + g^2 + h^2 \neq 0 \) then we get the phase portrait Picture DL.1 if \( g \neq 0 \) and Picture L.0 if \( g = 0 \).

We now observe that the bifurcation Diagram 4.1.7 is symmetrical with respect to the line \( g = 0 \). In fact the transformation \((x, y, t) \mapsto (-x, -y, -t)\) induces on the parameters the transformation \((b, g, h) \mapsto (b, -g, h)\) and hence the points on the right-hand side of the diagram are identified with their symmetrical ones on the left-hand side of the plane. Therefore we can limit ourselves to the case \( g \geq 0 \) and thus we can discard the left-hand side of this diagram (see Diagram 4.1.7*).

![Diagram 4.1.7*](image)

We now look if we can identify via the group action points which lie inside the half disk \((g \geq 0)\). The transformation \((x, y, t) \mapsto (gx, gy, t/g)\) with \( g \neq 0 \) induces on the parameter plane the transformation \((g, h) \mapsto (1, h/g^2)\). So all the points \((g, h)\) for which \( g > 0 \) can be identified via the group action with the points on the line \( g = 1 \).

On the line \( g = 0 \) and \( h \neq 0 \) the transformation \((x, y, t) \mapsto (|h|^{1/2}x, |h|^{1/2}y, |h|^{-1/2}t)\) induces on this line the transformation \((0, h) \mapsto (0, \text{sign}(h))\). So we can identify via the group action any system \((0, h)\) with one of the two points: \((0, 1)\) or \((0, -1)\).

On the line \( b = 0 \) our systems become (4.29). For \( g \neq 0 \) the transformation \((x, y, t) \mapsto (x - h/g, y, t/g)\) the systems (4.29) are identified with \( \dot{x} = x, \quad \dot{y} = 0 \). In the case \( g = 0 \) (then \( h \neq 0 \)) via the time rescaling the systems can be identified with \( \dot{x} = 1, \quad \dot{y} = 0 \).
Due to the identifications mentioned above we see that the quotient set of the family via the group action will be pictured on three lines: $g = 0$, $g = b$ and $b = 0$, all three passing through the point $[0 : 0 : 1]$. We view these three lines as three circles with one point in common which we draw on the Diagram 4.1.7$(\mathfrak{M})$. We draw with dots the circles $b = 0$ and $g = 0$, as on these circles we have the identifications mentioned above.

Due to the identifications mentioned above we see that the quotient set of the family via the group action will be pictured on three lines: $h = 0$, $h = b$ and $b = 0$, all three passing through the point $[0 : 1 : 0]$. We view these three lines as three circles with one point in common which we draw on the Diagram 4.2.7$(\mathfrak{M})$. We draw with dots the circle $b = 0$ and half of the circle $h = 0$, as on these circles we have the identifications mentioned above.

In this way we obtain Diagram 4.2.6$(\mathfrak{M})$ which is the moduli set for the family of systems (4.21). On this set we put the quotient topology of the systems via the group action.

### 4.1.8 Configuration 4.18

According to Theorem 3.1 all the systems having the Configuration 4.18 are included in the family:

\[
\begin{align*}
\dot{x} &= c(c + d) + cx + dy, \\
\dot{y} &= -xy + y^2.
\end{align*}
\]

with $[c : d] \in \mathbb{P}_1(\mathbb{R})$, for which we calculate: $B_3 = \theta = 0$, $H_7 = -4d$, $\mu_2 = c(c + d)(y - x)y$, $L = 8y(y - x)$.

1) If $H_7 \neq 0$ then according to [26] the phase portrait corresponds to Picture 4.18(a) if $\mu_2L > 0$ (i.e. $c(c + d) > 0$) and to Picture 4.18(b) if $\mu_2L < 0$ (i.e. $c(c + d) < 0$). So, for $cd(c + d) \neq 0$ we end up with phase portraits Picture 4.18(a) and Picture 4.18(a) as we indicated on Diagram 4.1.8.

If $c(c + d) = 0$ then systems (4.30) become degenerate and we get Picture D.13 if $c = 0$ and the Picture D.14 if $c = -d$.

2) Assume now $H_7 = 0$, i.e. $d = 0$. In this case for systems (4.30) we have $\mu_0 = 0$, $H_6 = 0$, $N = -x^2 \neq 0$ and according to [24] the phase portrait of these systems with $d = 0$ and $c \neq 0$ corresponds to Picture 5.7.
We consider the projective line as a circle with opposite points on diameters identified. Then the bifurcation diagram is indicated in Diagram 4.2.8. We observe that under the action of the affine group and time rescaling we cannot identify points \([c_1 : d_1]\) and \([c_2 : d_2]\), unless these are opposite points on the circle.

Removing the left hand-side of this diagram and identifying the north and south poles we obtain the Diagram 4.2.8(\(\mathfrak{M}\)).

### 4.2 Systems with the divisor \(D_S(C, Z) = \omega_1^c + \omega_2^c + \omega_3\)

There are the systems for which we have \(\eta < 0\) (see Table 1, page 8) and each such system has the configuration of invariant lines corresponding to one of the configurations \(\text{Config. 4.}j, j=2,6,7,8,27\) (see [23]). We shall examine these configurations one by one.

#### 4.2.1 Configuration 4.2

According to Theorem 3.2 all the systems having the Configuration 4.2 are included in the family:

\[
\begin{align*}
\dot{x} &= gx^2 + (h + b)xy, \\
\dot{y} &= h[g^2 + (h + b)^2] + (g^2 + b^2 - h^2)x + 2ghy - bx^2 + gxy + hy^2,
\end{align*}
\]

(4.31)

where \([b : g : h] \in \mathbb{P}_2(\mathbb{R})\). We construct now the bifurcation diagram for this canonical form. This diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line \(b = 0\).

#### 4.2.1.1 The case \(b \neq 0\).

Then we may consider \(b = 1\) and following the Table 2 of [26] we compute invariants necessary for each one of the Pictures 4.2(u), \(u \in \{a, b, c, d\}\):

\[
\begin{align*}
\theta &= 8(h + 1)[g^2 + (h - 1)^2], \quad \mu_0 = -h[g^2 + (h + 1)^2], \quad B_3 = 0, \\
G_1 &= 2g(h + 1)[g^2 + (3h + 1)^2], \quad H_7 = 4(h + 1)^2[g^2 + (h - 1)^2], \\
N &= (g^2 - 2h + 2)x^2 + 2g(h + 1)xy + (h^2 - 1)y^2.
\end{align*}
\]

(4.32)
4.2.1.1 The subcase $\mu_0 \neq 0$. We consider two possibilities: $\theta \neq 0$ and $\theta = 0$.

1) $\theta \neq 0$. This implies $H_1 \neq 0$ and according to [26] the phase portrait of a system (4.31) with $b = 1$ corresponds to Picture 4.2(a) (respectively Picture 4.2(b); Picture 4.2(c); Picture 4.2(d)) if $\mu_0 > 0$, $G_1 \neq 0$ (respectively $\mu_0 > 0$, $G_1 = 0$; $\mu_0 < 0$, $G_1 \neq 0$; $\mu_0 < 0$, $G_1 = 0$). So, if $\theta \mu_0 \neq 0$, i.e. $h(h+1)[g^2 + (h-1)^2][g^2 + (h+1)^2] \neq 0$ then we end up with the phase portraits above indicated placed inside the circle of Diagram 4.2.1.

Diagram 4.2.1

2) $\theta = 0$. Then $(h+1)[g^2 + (h-1)^2] = 0$ and we shall consider two cases: $N \neq 0$ and $N = 0$.

a) Assume first $N \neq 0$. Then considering (4.32) we get $h = -1$ and we have:

$$B_3 = \theta = H_1 = 0, \quad \mu_0 = g^2 \neq 0, \quad N = (g^2 + 4)x^2 \neq 0.$$ 

So, according to [24] the phase portrait of systems (4.31) in this case corresponds to Picture 5.10.

b) Suppose now $N = 0$. From (4.32) we obtain $h = 1$, $g = 0$ and then we have $B_3 = N = 0$, $H_1 = 36864 > 0$. Hence, by [22] we get Picture 6.4.

4.2.1.2 The subcase $\mu_0 = 0$. Then $h[g^2 + (h+1)^2] = 0$. 

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4.2.1.2.1 If \( h = 0 \) then we get the family of degenerate systems

\[
\dot{x} = gx^2 + xy, \quad \dot{y} = (g^2 + 1)x - x^2 + gxy. \tag{4.33}
\]

We observe, that the associated linear systems possess a focus if \( g \neq 0 \) and for \( g = 0 \) the system (4.33) possesses a center. Considering the affine line \( x = 0 \) filled up with singularities, we get Picture D.8 if \( g \neq 0 \) and Picture D.9 if \( g = 0 \).

4.2.1.2.2 Assuming \( g = h + 1 = 0 \) we get the degenerate system \( \dot{x} = 0, \dot{y} = -(x^2 + y^2) \) the phase portrait of which is given by Picture D.10.

4.2.2 The case \( b = 0 \)

Then from (4.31) we get the following family of systems:

\[
\dot{x} = gx^2 + hxy, \quad \dot{y} = h[g^2 + h^2] + (g^2 - h^2)x + 2ghy + gxy + hy^2, \tag{4.34}
\]

possessing the infinite line filled up with singularities. For these systems according to Table 1 of [25] we calculate:

\[
C_2 = 0, \quad H_{10} = 36h^4(g^2 + h^2)^2, \quad H_0 = h^{12}(g^2 + h^2)^4, \quad N_7 = 16gh^3(g^2 + 9h^2).
\]

Hence, if \( H_{10} \neq 0 \) then \( h \neq 0 \) and this implies \( H_0 > 0 \). According to [25] in this case the phase portrait of systems (4.34) corresponds to Picture C.2.2(a) if \( N_7 \neq 0 \) (i.e. \( g \neq 0 \)) and to Picture C.2.2(b) if \( N_7 = 0 \) (i.e. \( g = 0 \)).

For \( H_{10} = 0 \) we get \( h = 0 \) and this leads to degenerate systems \( \dot{x} = gx^2, \dot{y} = gx(g + y) \), where \( g \neq 0 \) due to the condition \( g^2 + h^2 + b^2 \neq 0 \) for the family of systems (4.31). Clearly in this case we get Picture D.4.

We now see that the portraits on the left-hand side (\( g < 0 \)) of Diagram 4.2.1 coincide with those on the right-hand side (\( g > 0 \)) and we wonder if they could be identified via the group action. Indeed, this is the case as we see by using the transformation \((x, y, t) \mapsto (x, -y, -t)\) inducing the map \((b, g, h) \mapsto (b, -g, h)\). Therefore we can limit ourselves to the case \( g \geq 0 \) and thus we can discard the left-hand side of this Diagram. We check if under the group action we can still identify points. It can be easily verified that two systems corresponding to two distinct points inside the half disk cannot lie on the same orbit. Secondly we limit ourselves to the line \( b = 0 \). In the affine chart corresponding to \( g = 1 \) using the coordinates \((b, h)\), the line \( b = 0 \) becomes the h-axis. In the resulting equations, via the transformation \((x, y, t) \mapsto (-x, y, -t)\) we can change \( h \) to \(-h\) obtaining a system in the same orbit. So we can identify the points with \( h \geq 0 \) with those with \( h \leq 0 \). On the other hand one can easily prove that two systems with \( b = 0 \) and \(|h_1| \neq |h_2|\) cannot lie on the same orbit. Projecting the cone obtained by identifying the points with \( h > 0 \) with those with \( h < 0 \) on the disk with circumference the line \( g = 0 \), and placing on this picture the portraits previously obtained for the half disk of Diagram 4.2.1 we obtain Diagram 4.2.1(9R).

4.2.3 Configurations 4.6 and 4.7

According to Theorem 3.2 all the systems having the Configurations 4.6 and 4.7 are included in the family:

\[
\dot{x} = gx^2 + (h + b)xy, \quad \dot{y} = -b + gx + (h - b)y - bx^2 + gxy + hy^2, \tag{4.35}
\]

where \([b : g : h] \in \mathbb{P}_2(\mathbb{R})\). We construct now the bifurcation diagram for this canonical form. This diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line \( b = 0 \).
4.2.3.1 The case $b = 1$. Then for systems (4.35) calculations yield:

$$\theta = 8(h + 1)[g^2 + (h - 1)^2], \quad \mu_0 = -h[g^2 + (h + 1)^2], \quad B_3 = H_7 = 0, \quad N = (g^2 - 2h + 2)x^2 + 2g(h + 1)xy + (h^2 - 1)y^2, \quad H_9 = 2304(h + 1)^8. \quad (4.36)$$

4.2.3.1.1 The subcase $\mu_0 \neq 0$, i.e. $h[g^2 + (h + 1)^2] \neq 0$. We shall consider two cases: $\theta \neq 0$ and $\theta = 0$.

1) Assume $\theta \neq 0$. Then we have $h + 1 \neq 0$ and this implies $H_9 \neq 0$. According to [26] the phase portrait of a system (4.35) with $b = 1$ corresponds to Picture 4.6(a) if $\mu_0 > 0$ (i.e. $h < 0$) and to Picture 4.6(b) if $\mu_0 < 0$ (i.e. $h > 0$). So, if $g(h + 1)[g^2 + (h - 1)^2][g^2 + (h + 1)^2] \neq 0$ then we arrive to the situation given by Diagram 4.2.2.

2) Suppose now $\theta = 0$. Then $(h + 1)[g^2 + (h - 1)^2] = 0$ and we shall consider two subcases: $H_9 \neq 0$ and $H_9 = 0$.

a) If $H_9 \neq 0$ then from (4.36) we have $h + 1 \neq 0$ and hence we get $g = 0$ and $h = 1$. So, from (4.35) we get a single system, for which calculation yields: $B_3 = N = 0$ and $H_1 = -9216 < 0$. According to [24] the phase portrait of this system corresponds to Picture 6.3.

b) Assume now $H_9 = 0$, i.e. $h = -1$. Then we get the family of systems

$$\dot{x} = gx^2, \quad \dot{y} = -1 + g x - 2y - x^2 + gxy - y^2, \quad (4.37)$$

for which we have

$$\theta = B_3 = H_1 = 0, \quad \mu_0 = g^2, \quad N = (g^2 + 4)x^2.$$ 

Since $\mu_0 \neq 0$ according to [24] the phase portrait of systems (4.37) corresponds to Picture 5.10.
4.2.3.1.2 The subcase \( \mu_0 = 0 \). In this case we have \( h[g^2 + (h + 1)^2] = 0 \).

1) If \( h = 0 \) then we get the family of systems

\[
\dot{x} = gx^2 + xy, \quad \dot{y} = -1 + g x + y - x^2 + gxy, \quad \text{(4.38)}
\]

for which we have \( \theta = 8(g^2 + 1) \), \( B_3 = H_7 = \mu_0 = 0 \). So, by [26] the phase portrait of these systems corresponds to Picture 4.7.
2) Assuming \( g = h + 1 = 0 \) we get the degenerate system \( \dot{x} = 0, \dot{y} = -x^2 - (y - 1)^2 \) the phase portrait of which is given by Picture D.10.

### 4.2.3.2 The case \( b = 0 \)

Then from (4.35) we get the following family of degenerate systems:

\[
\dot{x} = x(gx + hy), \quad \dot{y} = (y + 1)(gx + hy), \tag{4.39}
\]

possessing the affine singular line \( gx + hy = 0 \) (as for for systems (4.35) we have \( b^2 + g^2 + h^2 \neq 0 \)). Then we get the phase portrait Picture D.4.

We observe that the transformation \( (x, y) \mapsto (-x, y) \) change the sign of the parameter \( g \) in systems (4.35). Then the portraits on the left-hand side \( (g < 0) \) of Diagram 4.2.2 coincide with those on the right-hand side \( (g > 0) \) and they are identified under the group action. Therefore we can limit ourselves to the case \( g \geq 0 \) and thus we can discard the left-hand side of this Diagram.

Firstly we observe that under the group action we cannot identify systems corresponding to points inside the half disk as it can easily be seen. However on the line \( b = 0 \) in the affine chart corresponding to \( g = 1 \) using the coordinates \((b, h)\), the line \( b = 0 \) becomes the \( h \)-axis. In the resulting equations, via the transformation \((x, y, t) \mapsto (-x, y, -t)\) we can change \( h \) to \(-h\) obtaining a system in the same orbit. Moreover we can actually identify via the group action any two systems corresponding to two points \( 0 \neq |h_1| \neq |h_2| \neq 0 \) on the line \( b = 0 \).

So we first identify the points with \( h \geq 0 \) with those with \( h \leq 0 \). Projecting the cone thus obtained on the disk with circumference the line \( g = 0 \) and placing on this picture the portraits previously obtained for the half disk of Diagram 4.2.2 we obtain Diagram 4.2.2(\(\mathfrak{M}\)). In view of the previous arguments we can identify all the points from the open segment corresponding to Picture D.4 with the point on the disk with phase portrait Picture D.4.

**Remark 4.3.** We observe that the moduli space thus obtained is not Hausdorff. Indeed, the two points obtained via the group action from the segment mentioned above cannot be separated in the topology of this moduli space.

### 4.2.4 Configuration 4.8

According to Theorem 3.2 all the systems having the Configuration 4.8 are included in the family:

\[
\dot{x} = gx^2 + (h + b)xy, \quad \dot{y} = -bx^2 + gxy + hy^2, \tag{4.40}
\]

where \([b : g : h] \in \mathbb{P}_2(\mathbb{R})\). We construct now the bifurcation diagram for this canonical form. This diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line \( b = 0 \).

#### 4.2.4.1 The case \( b = 1 \)

Then for systems (4.40) calculations yield:

\[
\theta = 8(h + 1)[g^2 + (h - 1)^2], \quad \mu_0 = -h[g^2 + (h + 1)^2], \quad B_3 = H_7 = 0,
\]

\[
N = (g^2 - 2h + 2)x^2 + 2g(h + 1)xy + (h^2 - 1)y^2, \quad H_9 = 0. \tag{4.41}
\]

**4.2.4.1.1 The subcase \( \mu_0 \neq 0 \), i.e. \( h[g^2 + (h + 1)^2] \neq 0 \)** and we shall consider two cases: \( \theta \neq 0 \) and \( \theta = 0 \).

1) Assume first \( \theta \neq 0 \). Then we have \( h + 1 \neq 0 \) and according to [26] the phase portrait of a system (4.40) with \( b = 1 \) corresponds to Picture 4.8(a) if \( \mu_0 > 0 \) (i.e. \( h < 0 \)) and to Picture 4.8(b)
if $\mu_0 < 0$ (i.e. $h > 0$). So, if $g(h+1)[g^2 + (h-1)^2][g^2 + (h+1)^2] \neq 0$ then we arrive to the situation given by Diagram 4.2.3.

2) Admit now $\theta = 0$. Then $(h+1)[g^2 + (h-1)^2] = 0$.

a) If $h = -1$ then we get the family of systems

$$\dot{x} = gx^2, \quad \dot{y} = -x^2 + gxy - y^2,$$  \hspace{1cm} (4.42)

for which we have

$$\theta = B_3 = H_1 = 0, \quad \mu_0 = g^2, \quad N = (g^2 + 4)x^2.$$  

Since $\mu_0 \neq 0$ according to [24] the phase portrait of systems (4.37) corresponds to Picture 5.10.

b) Assume now $g = 0$ and $h = 1$. So, from (4.40) we get a single system, for which calculation yields: $B_3 = N = H_1 = 0$. So, according to [24] the phase portrait of this system corresponds to Picture 6.6.

4.2.4.1.2 The subcase $\mu_0 = 0$. In this case we have $h[g^2 + (h+1)^2] = 0$.

1) If $h = 0$ then we get the family of degenerate systems

$$\dot{x} = x(gx + y), \quad \dot{y} = x(-x + gy).$$  \hspace{1cm} (4.43)

We observe that for the respective linear systems the point $(0,0)$ is a focus if $g \neq 0$ (Picture D.11) and it is a center if $g = 0$ (Picture D.12).

2) Assuming $g = h + 1 = 0$ we get the degenerate system $\dot{x} = 0$, $\dot{y} = -(x^2 + y^2)$ the phase portrait of which is given by Picture D.10.
4.2.4.2 The case $b = 0$

Then from (4.40) we get the following family of degenerate systems:

$$\begin{align*}
\dot{x} &= x(gx + hy), \\
\dot{y} &= y(gx + hy),
\end{align*}$$

possessing the affine singular line $gx + hy = 0$ (as for systems (4.40) we have $b^2 + g^2 + h^2 \neq 0$). Then we get the phase portrait

![Diagram 4.2.3(M)](image)

We now see that the portraits on the left-hand side ($g < 0$) of Diagram 4.2.3 coincide with those on the right-hand side ($g > 0$) and we wonder if they could be identified via the group action. Indeed, this is the case as we see by using the transformation $(x, y, t) \mapsto (-x, y, t)$. Therefore we can limit ourselves to the case $g \geq 0$ and thus we can discard the left-hand side of this Diagram.

Firstly we observe that under the group action we cannot identify systems corresponding to points inside the half disk as it can easily be shown. However on the line $b = 0$ in the affine chart corresponding to $g = 1$ using the coordinates $(b, h)$, the line $b = 0$ becomes the $h$-axis. In the resulting equations, via the transformation $(x, y, t) \mapsto (x, -y, t)$ we can change $h$ to $-h$ obtaining a system in the same orbit. Moreover we can actually identify via the group action any two systems corresponding to two points $|h_1| \neq |h_2|$ on the line $b = 0$.

So we first identify the points with $h \geq 0$ with those with $h \leq 0$. Projecting the cone thus obtained on the disk with circumference the line $g = 0$ and placing on this picture the portraits previously obtained for the half disk of Diagram 4.2.3 we obtain Diagram 4.2.3(M). In view of the previous arguments we can identify all the points from the segment corresponding to Picture D.7 with the point on the disk with phase portrait Picture D.7.

4.2.5 Configuration 4.27

According to Theorem 3.2 all the systems having the Configuration 4.27 are included in the family:

$$\begin{align*}
\dot{x} &= 2cx + 2dy, \\
\dot{y} &= c^2 + d^2 - x^2 - y^2
\end{align*}$$

(4.45)

with $[c : d] \in \mathbb{P}_1(\mathbb{R})$, for which we calculate: $B_3 = \theta = 0$, $N = x^2$, $H_7 = 32d$, $G_1 = 16c$.

1) If $H_7 \neq 0$ then according to [26] the phase portrait corresponds to Picture 4.27(a) if $G_1 \neq 0$ (i.e. $c \neq 0$) and to Picture 4.27(b) if $G_1 = 0$ (i.e. $c = 0$).
2) Assume now $H_7 = 0$, i.e. $d = 0$. In this case for systems (4.45) we have $\mu_0 = 0$, $H_6 = 0$ and according to [24] the phase portrait of these systems with $d = 0$ and $c \neq 0$ corresponds to Picture 5.9.

We consider the projective line as a circle with opposite points on diameters identified. Then the bifurcation diagram is indicated in Diagram 4.2.4. We observe that under the action of the affine group and time rescaling we cannot identify points $[c_1: d_1]$ and $[c_2: d_2]$, unless these are opposite points on the circle.

Discarding the left hand-side of this diagram and identifying the north and south poles we obtain the Diagram 4.2.4$(\mathcal{M})$. 

![Diagram 4.2.4](image-url)  

![Diagram 4.2.4(M)](image-url)
References


