LINEAR-QUADRATIC DIFFERENTIAL GAMES: CLOSED LOOP SADDLE POINTS *

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Abstract. The object of this paper is to revisit the results of P. Bernhard (*J. Optim. Theory Appl. 27 (1979), 51–69*) on two-person zero-sum linear quadratic differential games and generalize them to utility functions without positivity assumptions on the matrices acting on the state variable and to linear dynamics with bounded measurable data matrices. The paper specializes to *state feedback* via Lebesgue measurable *affine closed loop strategies* with possible non L^2-integrable singularities. After sharpening our recent results [3] on the characterization of the open loop lower and upper values of the game, it first deals with L^2-integrable closed loop strategies and then with the larger family of strategies that may have non L^2-integrable singularities. A new conceptually meaningful and mathematically precise definition of a closed loop saddle point is introduced to simultaneously handle state feedbacks of the L^2-type and smooth locally bounded ones except at most in the neighborhood of finitely many instant of time. A necessary and sufficient conditions is that the free end problem be *normalizable*. A complete classification of closed loop saddle points is given in terms of the convexity/concavity properties of the utility function and connections are given with the open loop lower value, upper value, and value of the game.

Key words. Linear quadratic differential game, two person, zero sum, saddle point, value of a game, Riccati differential equation, open loop and closed loop strategies, integrable singularities

AMS subject classifications. 91A05, 91A23, 49N70, 91A25

1. Introduction. The object of this paper is to revisit the pioneering work of P. Bernhard [2] on two-person zero-sum linear quadratic differential games and generalize it to utility functions without positivity assumptions on the matrices acting on the state variable and to linear dynamics with bounded measurable data matrices. The paper specializes to *state feedback* via Lebesgue measurable *affine closed loop strategies* with possible non L^2-integrable singularities. After sharpening the recent results of [3] on the characterization of the open loop lower and upper values of the game in § 2, it first deals with L^2-integrable closed loop strategies and then with the larger family of strategies that may have non L^2-integrable singularities.

In § 3 several equivalent necessary and sufficient conditions are given for the existence of a closed loop saddle point with respect to L^2-integrable affine closed loop strategies: the *normality* of the problem; the existence of an $H^1(0,T)$ solution to the associated matrix Riccati differential equation. It was shown in [3] that the existence of a solution to the coupled state-adjoint state system is a necessary condition for the existence of a finite open loop lower value, upper value, or value of the game and that the difference essentially depends on the convexity of the utility function with respect to the control of the minimizing player and on its concavity with respect to the control of the maximizing player. This condition is also necessary for the existence of a closed loop saddle point. It leads to a complete classification in terms on the convexity/concavity properties of the utility function.

§ 4 deals with two delicate issues. The first one is the very definition of a closed

*Submitted on June 28, 2007. This research has been supported by a discovery grant of the National Sciences and Engineering Research Council of Canada.

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loop saddle point in the presence of closed loop strategies with non L^2-integrable singularities. As was pointed out in [2, p. 68 and Remark 5.1] such strategies may lead to conflicting terms that simultaneously blow up in the utility function. Under the positivity assumptions, one may possibly get around this problem by setting the utility function equal to $\pm\infty$, but we don't have them here. So we had to introduce a new conceptually meaningful and mathematically precise definition (cf. Definition 4.5). It says that the original problem can be transformed via feedback in such a way that the new resulting problem has an open loop saddle point at $(0,0)$. The second related issue was to specify the class of affine closed loop strategies (cf. Definition 4.3) in such a way that we could simultaneously handle in the same framework L^2-integrable closed loop strategies and smooth locally bounded ones except at most in the neighborhood of finitely many instant of time as in [2].

It turns out that the classical definition of a closed loop saddle point (cf. Definition 3.2) can be a *degenerate* one when either the open loop lower or upper value of the game is not finite (cf. Theorems 4.5 and 4.6). For instance, Berkovitz [1]'s equivalence Lemma 3.2 may not apply as shown in Example 4.1. The proper point of view of Definition 4.2 is that the two closed loop strategies cannot be chosen independently. They must be linked through the *admissibility condition* of Definition 4.3. This subtle difference fundamentally changes the nature of the problem and makes it different from the classical theory of saddle points with respect to two independent sets. We show that the *normalizability* of the free end problem in the sense of [2, Definition 3.2] is a necessary and sufficient condition for the existence of a closed loop saddle point. This condition is also used to make sense of solutions with singularities to the matrix Riccati differential equation.

In § 4.5, we show that under the convexity-concavity condition, Definitions 3.2 and 4.5 of closed loop saddle points coincide and that closed loop strategies with non L^2-integrable singularities are useless. They naturally occur when either the open loop lower or upper value of the game is not finite. We complete the classification of closed loop saddle points in § 4.6 along with conditions expressed in terms of the convexity/concavity properties of the utility function. We conclude in § 4.7 with an example of a non normalizable problem with finite open loop lower value that can be achieved by state feedback via a solution of the matrix Riccati differential equation.

2. Definitions, notation, and main results.

2.1. System, utility function, values of the game. Given a finite dimensional Euclidean space $\mathbf{R}^d$ of dimension $d \geq 1$, the *norm* and *inner product* will be denoted by $|x|$ and $x \cdot y$, respectively and irrespective of the dimension d of the space. Given $T > 0$, the norm and inner product in $L^2(0,T;\mathbf{R}^n)$ will be denoted $\|f\|$ and (f,g). The norm in the Sobolev space $H^1(0,T;\mathbf{R}^n)$ will be written $\|f\|_{H^1}$.

Consider the following two-player zero-sum game over the finite time interval $[0,T]$ characterized by the quadratic *utility function*

$$C_{x_0}(u,v) \stackrel{\text{def}}{=} Fx(T)\cdot x(T) + \int_0^T Q(t)x(t)\cdot x(t) + |u(t)|^2 - |v(t)|^2\, dt, \tag{2.1}$$

where x is the solution of the linear differential system

$$x'(t) = A(t)x(t) + B_1(t)u(t) + B_2(t)v(t) \quad \text{a.e. in } [0,T], \quad x(0) = x_0, \tag{2.2}$$

$x_0 \in \mathbf{R}^n$ is the *initial state* at time $t = 0$, $u \in L^2(0,T;\mathbf{R}^m)$, $m \geq 1$, is the strategy of the first player and $v \in L^2(0,T;\mathbf{R}^k)$, $k \geq 1$, is the strategy of the second player. We assume that F is an $n \times n$-matrix and that A, B_1, B_2, and Q are matrix-functions of appropriate order that are measurable and bounded almost everywhere in $[0, T]$. Moreover $Q(t)$ and F are symmetrical. It will be convenient to use the following compact notation and drop the a.e. in $[0, T]$ wherever no confusion arises

$$C_{x_0}(u, v) = Fx(T) \cdot x(T) + \int_0^T Qx \cdot x + |u|^2 - |v|^2 \, dt, \tag{2.3}$$

$$x' = Ax + B_1 u + B_2 v \quad \text{in } [0, T], \quad x(0) = x_0. \tag{2.4}$$

The above assumptions on F, A, B_1, B_2, and Q will be used throughout this paper. The transpose of a matrix M will be denoted $M^\top$, the inverse of its transpose $M^{-\top}$, and $R(t)$ will denote the matrix $B_1(t)B_1(t)^\top - B_2(t)B_2(t)^\top$.

Definition 2.1. Let x_0 be an initial state in $\mathbf{R}^n$ at time $t = 0$.

(i) The game is said to achieve its *open loop lower value* (resp. *upper value*) if

$$v^-(x_0) \stackrel{\text{def}}{=} \sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) \tag{2.5}$$

$$\left(\text{resp. } v^+(x_0) \stackrel{\text{def}}{=} \inf_{u \in L^2(0,T;\mathbf{R}^m)} \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u, v)\right) \tag{2.6}$$

is finite. By definition $v^-(x_0) \le v^+(x_0)$.

(ii) The game is said to achieve its *open loop value* if its open loop lower value $v^-(x_0)$ and upper value $v^+(x_0)$ are achieved and $v^-(x_0) = v^+(x_0)$. The *open loop value* of the game will be denoted by $v(x_0)$.

(iii) A pair $(\bar{u}, \bar{v})$ in $L^2(0, T; \mathbf{R}^m) \times L^2(0, T; \mathbf{R}^k)$ is an *open loop saddle point* of $C_{x_0}(u, v)$ in $L^2(0, T; \mathbf{R}^m) \times L^2(0, T; \mathbf{R}^k)$ if for all u in $L^2(0, T; \mathbf{R}^m)$ and all v in $L^2(0, T; \mathbf{R}^k)$

$$C_{x_0}(\bar{u}, v) \le C_{x_0}(\bar{u}, \bar{v}) \le C_{x_0}(u, \bar{v}). \tag{2.7}$$

Definition 2.2. Associate with $x_0 \in \mathbf{R}^n$ the sets

$$V(x_0) \stackrel{\text{def}}{=} \left\{ v \in L^2(0, T; \mathbf{R}^k) : \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) > -\infty \right\}, \tag{2.8}$$

$$U(x_0) \stackrel{\text{def}}{=} \left\{ u \in L^2(0, T; \mathbf{R}^m) : \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u, v) < +\infty \right\}. \tag{2.9}$$

2.2. Properties of the utility function. Recall from [3] that the utility function $C_{x_0}(u, v)$ is infinitely differentiable and that its Hessian of second order derivatives is independent of (u, v). Indeed[1]

$$\frac{1}{2} dC_{x_0}(u, v; \bar{u}, \bar{v}) = Fx(T) \cdot \bar{y}(T) + (Qx, \bar{y}) + (u, \bar{u}) - (v, \bar{v}), \tag{2.10}$$

[1] Given a real function f defined on a Banach space B, the *first directional semiderivative* at x in the direction v (when it exists) is defined as $df(x; v) = \lim_{t \searrow 0} (f(x + tv) - f(x))/t$. When the map $v \mapsto df(x; v) : B \to \mathbf{R}$ is linear and continuous, it defines the *gradient* $\nabla f(x)$ as an element of the dual B^* of B. The *second order bidirectional derivative* at x in the directions (v, w) (when it exists) is defined as $d^2 f(x; v, w) = \lim_{t \searrow 0} (df(x + tw; v) - df(x; v))/t$. When the map $(v, w) \mapsto d^2 f(x; v, w) : B \times B \to \mathbf{R}$ is bilinear and continuous, it defines the *Hessian operator* $Hf(x)$ as a continuous linear operator from B to B^*.

where x is the solution of (2.4) and $\bar{y}$ is the solution of

$$\bar{y}' = A\bar{y} + B_1\bar{u} + B_2\bar{v}, \quad \bar{y}(0) = 0. \tag{2.11}$$

It is customary to introduce the *adjoint system*

$$p' + A^\top p + Qx = 0, \quad p(T) = Fx(T) \tag{2.12}$$

and rewrite expression (2.10) for the gradient in the following form

$$\frac{1}{2} dC_{x_0}(u, v; \bar{u}, \bar{v}) = (B_1^\top p + u, \bar{u}) + (B_2^\top p - v, \bar{v}). \tag{2.13}$$

Hence $dC_{x_0}(\hat{u}, \hat{v}; \bar{u}, \bar{v}) = 0$ for all $\bar{u}$ and $\bar{v}$ if and only if the *coupled system*

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = x_0 \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T). \end{cases} \tag{2.14}$$

has a solution $(\hat{x}, \hat{p})$ in $H^1(0, T; \mathbf{R}^n)^2$ with $(\hat{u}, \hat{v}) = (-B_1^\top \hat{p}, B_2^\top \hat{p})$.

As expected, the Hessian is independent of (u, v)

$$\frac{1}{2} d^2 C_{x_0}(u, v; \bar{u}, \bar{v}; \tilde{u}, \tilde{v}) = F\tilde{y}(T) \cdot \bar{y}(T) + (Q\tilde{y}, \bar{y}) + (\tilde{u}, \bar{u}) - (\tilde{v}, \bar{v}), \tag{2.15}$$

where $\bar{y}$ is the solution of (2.11) and $\tilde{y}$ is the solution of

$$\tilde{y}' = A\tilde{y} + B_1\tilde{u} + B_2\tilde{v}, \quad \tilde{y}(0) = 0. \tag{2.16}$$

In particular, for all x_0, u, v, $\bar{u}$, and $\bar{v}$, $d^2C_{x_0}(u, v; \bar{u}, \bar{v}; \bar{u}, \bar{v}) = 2\, C_0(\bar{u}, \bar{v})$.

2.3. Games with finite open loop lower or upper values. We recall and sharpen the results of [3, Thms 2.2, 2.3, and 2.4] when the open loop lower or upper value of the game is finite for a given initial state x_0. In each case, the global assumption of finiteness for *all* initial state $x_0 \in \mathbf{R}^n$ yields the *uniqueness* of solution (x, p) of the coupled system (2.14) (cf. [3, Thms 2.6, 2.7, and 2.8]).

Theorem 2.1. *The following conditions are equivalent.*

(i) *There exist $\hat{u}$ in $L^2(0, T; \mathbf{R}^m)$ and $\hat{v}$ in $L^2(0, T; \mathbf{R}^k)$ such that*

$$C_{x_0}(\hat{u}, \hat{v}) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \hat{v}) = \sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v). \tag{2.17}$$

(ii) *The open loop lower value $v^-(x_0)$ of the game is finite.*

(iii) *There exists a solution in $H^1(0, T; \mathbf{R}^n)^2$ of the* coupled system *and controls*

$$\begin{cases} x' = Ax - Rp, & x(0) = x_0 \\ p' + A^\top p + Qx = 0, & p(T) = Fx(T), \end{cases} \tag{2.18}$$

$$\hat{u} = -B_1^\top p, \quad \hat{v} = B_2^\top p, \tag{2.19}$$

and the following identities are verified

$$\sup_{v \in V(0)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u, v) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u, 0) = C_0(0.0). \tag{2.20}$$

Remark 2.1. The additional condition $B_2^\top p \in V(x_0)$ that appeared in [3, Thms 2.2 and 2.6] is redundant. To see that, recall that the last identity (2.20) is equivalent to the convexity of the mapping $u \mapsto C_{x_0}(u,v)$. By [3, Thm 3.1] the convexity plus a solution of the coupled system (2.18) yields that

$$\inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, B_2^\top p) = C_{x_0}(-B_1^\top p, B_2^\top p) > -\infty \quad \Rightarrow B_2^\top p \in V(x_0).$$

Similarly, the additional condition $-B_1^\top p \in U(x_0)$ that appeared in [3, Thms 2.3 and 2.7] is also redundant.

Theorem 2.2. *The following conditions are equivalent.*

(i) *There exist* $\hat{u}$ *in* $L^2(0,T;\mathbf{R}^m)$ *and* $\hat{v}$ *in* $L^2(0,T;\mathbf{R}^k)$ *such that*

$$C_{x_0}(\hat{u},\hat{v}) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat{u},v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v). \quad (2.21)$$

(ii) *The open loop upper value* $v^+(x_0)$ *of the game is finite.*

(iii) *There exists a solution* $(x,p) \in H^1(0,T;\mathbf{R}^n)^2$ *of the* coupled system (2.18) *and controls* $(\hat{u},\hat{v})$ *given by* (2.19)*, and the following identities are verified*

$$\inf_{u\in U(0)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(u,v) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(0,v) = C_0(0,0). \quad (2.22)$$

3. L^2-integrable closed loop strategies. We generalize classical results to L^2-integrable affine closed loop feedback strategies for general F and $Q(t)$ under the assumptions of § 2.1 on the matrix functions A, B_1, B_2, Q, and F. We also give a classification of the possible cases in terms of the open loop properties of lower value, upper value and value of the game and the convexity/concavity of the utility function.

3.1. Definitions and main results.

Definition 3.1. The *class of L^2 affine closed loop strategies* is defined as follows:

$$\Phi \stackrel{\text{def}}{=} \left\{ \phi : [0,T]\times\mathbf{R}^n \to \mathbf{R}^m \;\middle|\; \begin{array}{l}\text{such that } x\mapsto \phi(t,x) \text{ is affine and}\\ t\mapsto\phi(t,x) \text{ belongs to } L^2(0,T;\mathbf{R}^m)\end{array}\right\}$$

$$\Psi \stackrel{\text{def}}{=} \left\{ \psi : [0,T]\times\mathbf{R}^n \to \mathbf{R}^k \;\middle|\; \begin{array}{l}\text{such that } x\mapsto \psi(t,x) \text{ is affine and}\\ t\mapsto\psi(t,x) \text{ belongs to } L^2(0,T;\mathbf{R}^k)\end{array}\right\}.$$

We say that ϕ or ψ is an L^2 *linear closed loop strategy* if ϕ or ψ is linear in x.

Remark 3.1. To each $\phi\in\Phi$ (resp. $\psi\in\Psi$) we can associate an $L^2(0,T;\mathbf{R}^m)$-vector function u and an $m\times n$ matrix L^2-function U such that $\phi(t,x) = u(t)+U(t)x$ (resp. an $L^2(0,T;\mathbf{R}^k)$-vector function v and a $k\times n$ matrix L^2-function V such that $\psi(t,x) = v(t)+V(t)x$). The matrix functions U and V may have singularities, but they are globally L^2-integrable. As a result the fundamental matrix associated with the L^2-matrix function $A+B_1U+B_2V$ will be invertible everywhere in $[0,T]$. Therefore for all $\phi\in\Phi$ and $\psi\in\Psi$ the closed loop system

$$\begin{aligned} x' &= Ax + B_1\phi(x) + B_2\psi(x), \quad x(0)=x_0\\ x' &= (A+B_1U+B_2V)x + B_1u + B_2v, \quad x(0)=x_0 \end{aligned} \quad (3.1)$$

has a unique solution in $H^1(0,T;\mathbf{R}^n)$. This means that all pairs $(\phi,\psi)\in\Phi\times\Psi$ are *admissible*, and, a fortiori, all pairs of the form (ϕ,v) or (u,ψ) are admissible for all $u\in L^2(0,T;\mathbf{R}^m)$ and $v\in L^2(0,T;\mathbf{R}^k)$.

Definition 3.2. (i) Given $x_0 \in \mathbf{R}^n$, we say that $(\phi^*, \psi^*) \in \Phi \times \Psi$ is *a closed loop saddle point* of $C_{x_0}(\phi, \psi)$ in $\Phi \times \Psi$ if there exists a pair $(\phi^*, \psi^*) \in \Phi \times \Psi$ such that for all $\phi \in \Phi$ and $\psi \in \Psi$

$$C_{x_0}(\phi^*, \psi) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(\phi, \psi^*). \tag{3.2}$$

(ii) We say that $(\phi^*, \psi^*) \in \Phi \times \Psi$ is *a global closed loop saddle point* of $C_{x_0}(\phi, \psi)$ in $\Phi \times \Psi$ if there exists a pair $(\phi^*, \psi^*) \in \Phi \times \Psi$ such that for all $x_0 \in \mathbf{R}^n$ and for all $\phi \in \Phi$ and $\psi \in \Psi$ the inequalities (3.2) are verified.

By definition $C_{x_0}(\phi^*, \psi^*)$ is finite. Thus the saddle point is not "degenerate" in the sense of [2]. The "global version" is better adapted to closed loop strategies. The interest in a closed loop strategy associated with a single initial state is rather limited.

Given any two pairs (ϕ_1, ψ_1) and (ϕ_2, ψ_2) achieving a closed saddle point, the mixed pairs (ϕ_1, ψ_2) and (ϕ_2, ψ_1) are admissible and also achieve a saddle point. Hence the value of the closed loop saddle point is unique (cf. [1]).

Lemma 3.1. *Given $x_0 \in \mathbf{R}^n$, for all pairs $(\phi_1^*, \psi_1^*) \in \Phi \times \Psi$ and $(\phi_2^*, \psi_2^*) \in \Phi \times \Psi$ verifying* (3.2), *$C_{x_0}(\phi_1^*, \psi_1^*) = C_{x_0}(\phi_2^*, \psi_2^*)$.*

We quote Berkovitz [1]'s equivalence Lemma.

Lemma 3.2. *Given $x_0 \in \mathbf{R}^n$, the following statements are equivalent:*

(i) $(\phi^*, \psi^*) \in \Phi \times \Psi$ *is* a closed loop saddle point *of $C_{x_0}(\phi, \psi)$ in $\Phi \times \Psi$;*

(ii) *there exists a pair $(\phi^*, \psi^*) \in \Phi \times \Psi$ such that for all $u \in L^2(0, T; \mathbf{R}^m)$ and all $v \in L^2(0, T; \mathbf{R}^k)$*

$$C_{x_0}(\phi^*, v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(u, \psi^*). \tag{3.3}$$

Theorem 3.1. *The following statements are equivalent.*

(i) $(\phi^*, \psi^*) \in \Phi \times \Psi$ *is* a global closed loop saddle point *of $C_{x_0}(\phi, \psi)$ in $\Phi \times \Psi$.*

(ii) *For all $x_0 \in \mathbf{R}^n$ there exist a unique solution in $H^1(0, T; \mathbf{R}^n)^2$ of*

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = x_0 \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T) \end{cases} \tag{3.4}$$

and L^2-matrices U_* and V_* of appropriate orders such that for all $x_0 \in \mathbf{R}^n$

$$\hat{u} = -B_1^\top \hat{p} = U_* \hat{x}, \quad \hat{v} = B_2^\top \hat{p} = V_* \hat{x}. \tag{3.5}$$

(iii) *For all $s \in [0, T[$, there exists a unique $H^1(s, T)$ solution of the coupled matrix system*

$$\begin{cases} \widehat{X}_s' = A\widehat{X}_s - R\widehat{\Lambda}_s, & \widehat{X}_s(s) = I \\ \widehat{\Lambda}_s' + A^\top \widehat{\Lambda}_s + Q\widehat{X}_s = 0, & \widehat{\Lambda}_s(T) = F\widehat{X}_s(T). \end{cases} \tag{3.6}$$

By convention, set $\widehat{X}_T(T) = I$ and $\widehat{\Lambda}_T(T) = F$.

(iv) $\det X(t) \neq 0$ *everywhere in $[0, T]$, where (X, Λ) is the $H^1(0, T)$ solution of the matrix differential system*

$$\begin{cases} X' = AX - R\Lambda, & X(T) = I \\ \Lambda' + A^\top \Lambda + QX = 0, & \Lambda(T) = F. \end{cases} \tag{3.7}$$

(v) *There exists a symmetrical solution P with elements in $H^1(0, T)$ to the matrix Riccati differential equation*

$$P' + PA + A^* P - PRP + Q = 0, \quad P(T) = F. \tag{3.8}$$

In particular $C_{x_0}(\phi^*, \psi^*) = P(0)x_0 \cdot x_0$ *and the closed loop strategies are given by*

$$\phi^*(t,x) = -B_1^\top(t)P(t)x = U_*(t)x \text{ and } \psi^*(t,x) = B_2^\top(t)P(t)x = V_*(t)x. \tag{3.9}$$

Proof. (i) $\Rightarrow$ (ii). Let $\hat{x}$ be the trajectory corresponding to the pair (ϕ^*, ψ^*) and denote by $(\hat{u}, \hat{v}) = (\phi^*(x), \psi^*(x))$ the corresponding control pair. Let $U_*(t)$ and $V_*(t)$ be the respective matrices and $u_*(t)$ and $v_*(t)$ be the respective vectors such that $\varphi^*(t,x) = U_*(t)x + u_*(t)$ and $\psi^*(t,x) = V_*(t)x + v_*(t)$. Then

$$\hat{x}' = (A + B_1 U_* + B_2 V_*)\hat{x} + B_1 u_* + B_2 v_*, \quad \hat{x}(0) = x_0. \tag{3.10}$$

For all $u \in L^2(0,T;\mathbf{R}^m)$ and $v \in L^2(0,T;\mathbf{R}^k)$, the pair $(\phi^* + u, \psi^* + v) \in \Phi \times \Psi$ and

$$C_{x_0}(\phi^*, \psi^* + v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(\phi^* + u, \psi^*). \tag{3.11}$$

Introduce the notation $c_{x_0}(u,v)$ for the utility function $C_{x_0}(\phi^* + u, \psi^* + v)$:

$$c_{x_0}(u,v) \stackrel{\text{def}}{=} Fx(T) \cdot x(T) + \int_0^T Qx \cdot x + |U_* x + u_* + u|^2 - |V_* x + v_* + v|^2 \, dt,$$

and denote by x the solution of the corresponding state equation

$$x' = (A + B_1 U_* + B_2 V_*)x + B_1(u_* + u) + B_2(v_* + v), \quad x(0) = x_0. \tag{3.12}$$

Then the closed loop saddle point inequalities (3.11) become open loop saddle point inequalities for system (3.12) and the new quadratic utility function $c_{x_0}(u,v)$:

$$\forall u \in L^2(0,T;\mathbf{R}^m) \text{ and } v \in L^2(0,T;\mathbf{R}^k), \quad c_{x_0}(0,v) \le c_{x_0}(0,0) \le c_{x_0}(u,0) \tag{3.13}$$

and the pair $(0,0)$ achieves that saddle point. By [3, Lemma 3.1] $c_{x_0}(u,v)$ is convex-concave and $dc_{x_0}(0,0;u,v) = 0$ for all u and v. In particular, the coupled system

$$\begin{gathered}
\hat{x}' = (A + B_1 U_* + B_2 V_*)\hat{x} + B_1 u_* + B_2 v_*, \quad \hat{x}(0) = x_0, \\
\begin{cases} \hat{p}' + (A + B_1 U_* + B_2 V_*)^\top \hat{p} + Q\hat{x} + U_*^\top (U_* \hat{x} + u_*) - V_*^\top (V_* \hat{x} + v_*) = 0, \\ \hat{p}(T) = F\hat{x}(T) \end{cases} \\
0 = -B_1^\top \hat{p} - (U_* \hat{x} + u_*) \text{ and } 0 = B_2^\top \hat{p} - (V_* \hat{x} + v_*)
\end{gathered} \tag{3.14}$$

has a solution $(\hat{x}, \hat{p})$ in $H^1(0,T;\mathbf{R}^n)^2$. After substitution, it can be rewritten as

$$\begin{aligned}
&\hat{x}' = A\hat{x} - R\hat{p}, \quad \hat{x}(0) = x_0, \\
&\hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, \quad \hat{p}(T) = F\hat{x}(T).
\end{aligned} \tag{3.15}$$

By assumption this is true for all $x_0 \in \mathbf{R}^n$. But, when system (3.15) has a solution for all x_0, its solution is unique (cf, [3, § 2.6, pp. 760–761]). As a result for $x_0 = 0$, $(\hat{x}, \hat{p}) = (0,0)$ and from identities (3.14)

$$0 = -B_1^\top \hat{p} - (U_* \hat{x} + u_*) \text{ and } 0 = B_2^\top \hat{p} - (V_* \hat{x} + v_*) \quad \Rightarrow u_* = 0 \text{ and } v_* = 0.$$

Hence the feedback controls are of the form $\hat{u} = U_* \hat{x} = -B_1^\top \hat{p}$ and $\hat{v} = V_* \hat{x} = B_2^\top \hat{p}$.

(ii) $\Rightarrow$ (iii). By assumption for all $x_0 \in \mathbf{R}^n$, the coupled system (3.15) has a unique solution $(\hat{x}, \hat{p})$. By linearity of the solution of system (3.4) with respect to x_0, there exist $H^1(0,T)$ matrices $(\widehat{X}, \widehat{\Lambda})$ solution of the matrix system

$$\begin{aligned}
&\widehat{X}' = A\widehat{X} - R\widehat{\Lambda}, \quad \widehat{X}(0) = I, \\
&\widehat{\Lambda}' + A^\top \widehat{\Lambda} + Q\widehat{X} = 0, \quad \widehat{\Lambda}(T) = F\widehat{X}(T).
\end{aligned} \tag{3.16}$$

But the conditions $U_* \hat{x} = -B_1^\top \hat{p}$ and $V_* \hat{x} = B_2^\top \hat{p}$ for all x_0 implies that $U_* \widehat{X} = -B_1^\top \widehat{\Lambda}$ and $V_* \widehat{X} = B_2^\top \widehat{\Lambda}$ and $\widehat{X}$ is also the unique solution of the equation

$$\widehat{X}' = (A + B_1 U_* + B_2 V_*)\widehat{X}, \quad \widehat{X}(0) = I. \tag{3.17}$$

Since the elements of the matrix function $A + B_1 U_* + B_2 V_*$ are L^2 functions, the associated fundamental matrix solution $\Phi(t,s)$ is invertible, $\widehat{X}(t)x_0 = \hat{x}(t) = \Phi(t,0)x_0$, and, a fortiori, $\widehat{X}(t) = \Phi(t,0)$ is invertible in $[0,T]$. In particular, for all $s \in [0,T[$, $(\widehat{X}_s(t), \widehat{\Lambda}_s(t)) = (\widehat{X}(t)\widehat{X}(s)^{-1}, \widehat{\Lambda}(t)\widehat{X}(s)^{-1})$ is a solution of system (3.6).

(iii) $\Rightarrow$ (iv) First observe that $\widehat{X}_s(T)$ is invertible for all $s \in [0,T]$. For $s < T$, let $h \neq 0$ be such that $\widehat{X}_s(T)h = 0$. The pair $(x_s(t), p_s(t)) = (\widehat{X}_s(t)h, \widehat{\Lambda}_s(t)h)$ is solution of the system $x_s' = Ax_s - Rp_s$, $x_s(s) = h$ and $p_s' + A^\top p_s + Qx_s = 0$, $p_s(T) = Fx_s(T)$ with $(x_s(T), p_s(T)) = (0,0)$. Hence $(x_s, p_s) = (0,0)$ and $h = x_s(s) = 0$, a contradiction. For all $0 \leq s \leq t \leq T$, $\widehat{X}_s(T) = \widehat{X}_t(T)\widehat{X}_s(t)$ and $\det \widehat{X}_s(t) \neq 0$. Defining the matrix functions $(X(t), \Lambda(t)) = (\widehat{X}_0(t)\widehat{X}_0(T)^{-1}, \widehat{\Lambda}_0(t)\widehat{X}_0(T)^{-1})$, they are solution of the matrix differential system (3.6) and necessarily $\det X(t) \neq 0$ everywhere in $[0,T]$.

(iv) $\Rightarrow$ (v) Since $X(t)$ is invertible for all $t \in [0,T]$, then $P(t) = \Lambda(t)X(t)^{-1}$ is a matrix of $H^1(0,T)$ functions. Moreover, P is symmetrical since $\Lambda^\top X$ is by computing the derivative of $\Lambda^\top X - X^\top \Lambda$. Finally P is solution of the Riccati matrix differential equation (3.8). By uniqueness of the solution (x,p) of (3.4), $(x(t), p(t)) = (\widehat{X}_0(t)x_0, \widehat{\Lambda}_0(t)x_0) = (X(t)\widehat{X}_0(T)x_0, \Lambda(t)\widehat{X}_0(T)x_0)$. Hence $p(t) = P(t)x(t)$, $\hat{u}(t) = -B_1^\top p(t) = -B_1^\top P(t)x(t)$, and $\hat{v}(t) = B_2^\top p(t) = B_2^\top P(t)x(t)$.

(v) $\Rightarrow$ (i). Let $x \in H^1(0,T;\mathbf{R}^n)$ be the solution of

$$x' = Ax + B_1 u + B_2 v, \quad x(0) = x_0 \tag{3.18}$$

and let P be an $H^1(0,T)$ solution of the matrix Riccati differential equation (3.8). By the classical argument in of Bernhard [2], we get

$$C_{x_0}(u,v) = P(0)x_0 \cdot x_0 + \int_0^T \left|u + B_1^\top Px\right|^2 - \left|v - B_2^\top Px\right|^2 \, dt.$$

Choose the closed loop linear strategies $\phi^*(t,x) = -B_1^\top(t)P(t)x$ and $\psi^*(t,x) = B_2^\top(t)P(t)x$. Then for all $v \in L^2(0,T;\mathbf{R}^k)$ and all $u \in L^2(0,T;\mathbf{R}^m)$

$$\begin{aligned} C_{x_0}(\phi^*, \psi^*) &= P(0)x_0 \cdot x_0 \\ C_{x_0}(u, \psi^*) &= P(0)x_0 \cdot x_0 + \int_0^T \left|u + B_1^\top Px\right|^2 dt \geq P(0)x_0 \cdot x_0 = C_{x_0}(\phi^*, \psi^*) \\ C_{x_0}(\phi^*, v) &= P(0)x_0 \cdot x_0 - \int_0^T \left|v - B_2^\top Px\right|^2 dt \leq P(0)x_0 \cdot x_0 = C_{x_0}(\phi^*, \psi^*). \end{aligned}$$

By Lemma 3.2 (ii), the linear pair (ϕ^*, ψ^*) is a global closed loop saddle point. Finally, the pair of closed loop strategies $\phi^*(t,x) = -B_1^\top(t)P(t)x = U_*(t)x$ and $\psi^*(t,x) = B_2^\top(t)P(t)x = V_*(t)x$ yields the global closed loop saddle point $P(0)x_0 \cdot x_0$. □

3.2. Classification of closed loop saddle points. One of the necessary conditions for the existence of a closed loop saddle point is the existence of a solution to the coupled system in $(\hat{x}, \hat{p})$ that turns out to also be a necessary condition for the finiteness open loop lower value, upper value, or value of the game and the difference essentially depends on the convexity of the utility function with respect to u and on its concavity with respect to v. This leads to the following natural classification in terms of the convexity and concavity properties of the utility function.

Theorem 3.2. *Assume that* $(\phi^*, \psi^*) \in \Phi \times \Psi$ *is a closed loop saddle point of* $C_{x_0}(\phi, \psi)$.

(a) $v(x_0)$ *is finite if and only if* $C_{x_0}(u, v)$ *is convex in* u *and concave in* v.

(b) $v^-(x_0)$ *is finite and* $v^+(x_0) = +\infty$ *if and only if* $C_{x_0}(u, v)$ *is convex in* u *and not concave in* v.

(c) $v^+(x_0)$ *is finite and* $v^-(x_0) = -\infty$ *if and only if* $C_{x_0}(u, v)$ *is concave in* v *and not convex in* u.

(d) $v^-(x_0) = -\infty$ *and* $v^+(x_0) = +\infty$ *if and only if* $C_{x_0}(u, v)$ *is not convex in* u *and not concave in* v.

(e) $v^-(x_0) = v^+(x_0) = +\infty$ *cannot occur.*

(f) $v^-(x_0) = v^+(x_0) = -\infty$ *cannot occur.*

In the first three cases $C_{x_0}(\phi^*, \psi^*)$ *is equal to* $v(x_0)$, $v^-(x_0)$, *and* $v^+(x_0)$, *respectively.*

We need the following lemma.

Lemma 3.3. (i) *For all* $v \in L^2(0, T; \mathbf{R}^k)$

$$\inf_{\phi\in\Phi} C_{x_0}(\phi, v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) \tag{3.19}$$

$$\sup_{\psi\in\Psi} \inf_{\phi\in\Phi} C_{x_0}(\phi, \psi) \geq \sup_{v\in L^2(0,T;\mathbf{R}^k)} \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v). \tag{3.20}$$

(ii) *For all* $u \in L^2(0, T; \mathbf{R}^m)$

$$\sup_{\psi\in\Psi} C_{x_0}(u, \psi) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u, v) \tag{3.21}$$

$$\inf_{\phi\in\Phi} \sup_{\psi\in\Psi} C_{x_0}(\phi, \psi) \leq \inf_{u\in L^2(0,T;\mathbf{R}^m)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u, v). \tag{3.22}$$

Proof of Lemma 3.3. We only need to prove (i). Since $L^2(0, T; \mathbf{R}^m) \subset \Phi$,

$$\inf_{\phi\in\Phi} C_{x_0}(\phi, v) \leq \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v).$$

Conversely, given the pair (ϕ, v), let $x \in H^1(0, T; \mathbf{R}^n)$ be the solution of the system

$$x' = Ax + B_1\phi(x) + B_2 v, \quad x(0) = x_0$$

and let $u = \phi(x) \in L^2(0, T; \mathbf{R}^m)$. This implies that

$$C_{x_0}(\phi, v) = C_{x_0}(u, v) \geq \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v)$$

$$\Rightarrow \inf_{\phi\in\Phi} C_{x_0}(\phi, v) \geq \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) \Rightarrow \inf_{\phi\in\Phi} C_{x_0}(\phi, v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v).$$

The second inequality follows from the fact that $L^2(0, T; \mathbf{R}^k) \subset \Psi$. □

Proof of Theorem 3.2. From inequality (3.20), $v^-(x_0) \leq C_{x_0}(\phi^*, \psi^*) < +\infty$ and case (e) cannot occur. Similarly, from inequality (3.22), $v^+(x_0) \geq C_{x_0}(\phi^*, \psi^*) > -\infty$ and case (f) cannot occur. Therefore we are left with the first four cases.

(b) From the first part of the proof of Theorem 3.1, system (3.15) has a solution and identities (3.14) are verified:

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, \quad \hat{x}(0) = x_0, \\ \hat{p}' + A^\top\hat{p} + Q\hat{x} = 0, \quad \hat{p}(T) = F\hat{x}(T), \end{cases} \tag{3.23}$$

$$0 = -B_1^\top\hat{p} - (U_*\hat{x} + u_*) \text{ and } 0 = B_2^\top\hat{p} - (V_*\hat{x} + v_*). \tag{3.24}$$

Using the controls $(\hat{u}, \hat{v}) = (U_* \hat{x} + u_*, V_* \hat{x} + v_*) = (-B_1^\top \hat{p}, B_2^\top \hat{p})$, the above system can be rewritten as follows

$$\begin{cases} \hat{x}' = A\hat{x} - B_1 B_1^\top \hat{p} + B_2 \hat{v}, & \hat{x}(0) = x_0, \qquad \hat{u} = -B_1^\top \hat{p}, \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T). \end{cases}$$

If $C_{x_0}(u, v)$ is convex in u, this implies that $\hat{u}$ is a minimizer of $C_{x_0}(u, \hat{v})$ over u (cf. for instance [3, Thm 3.1]). Therefore

$$\sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) \geq \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v}).$$

But, by construction of $(\hat{u}, \hat{v})$, $C_{x_0}(\phi^*, \psi^*) = C_{x_0}(\hat{u}, \hat{v})$. Combining the above inequality with inequality (3.20) in Lemma 3.3, we get

$$v^-(x_0) = \sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v})$$

and hence the finiteness of $v^-(x_0)$. If, in addition, $C_{x_0}(u, v)$ was concave in v, then by [3, Thms 2.5 and 2.4] the value of the game and hence $v^+(x_0)$ would be finite and this would contradicts our assumption. Conversely, if $v^-(x_0)$ is finite, then the mapping $u \mapsto C_{x_0}(u, v)$ is convex ([3, Thm 2.2 (iii), last part of identity (2.35) and Remark 2.2]). If $v^+(x_0)$ was also finite, then by [3, Thms 2.5 and 2.4 (iii)] $C_{x_0}(u, v)$ would be concave in v in contradiction with our assumption.

The proof of (c) is dual to the proof of (b). The proof of (a) is similar to the one of parts (b) and (c). Case (d) is the complement of all the other cases. So it can only occur when $C_{x_0}(u, v)$ is neither convex in u, nor concave in v. □

Remark 3.2. If the problem is normal and $F \geq 0$ and $Q(t) \geq 0$, then $v^-(x_0)$ is finite and $v^-(x_0) \geq 0$ for all $x_0 \in \mathbf{R}^n$, and necessarily $P(0)x_0 \cdot x_0 \geq 0$ for all $x_0 \in \mathbf{R}^n$.

4. The curse of singularities. We now extend the definitions and results of the previous section to Lebesgue measurable feedback matrices with singularities that are not necessarily L^2-integrable in any neighborhood of the singularity. How should the families Φ and Ψ of L^2 affine closed loop strategies be extended while preserving the assumption of *normalizability* of [2] that makes sense of a non $H^1(0, T)$-solution P to the matrix Riccati differential equation? It is clear that the choice of the space of solutions of the matrix Riccati differential equation and the specification of the families Φ and Ψ are closely related.

It has been known that the solution of the scalar Riccati differential equation can exhibit singularities that are not *movable branch points* at least when the coefficients are smooth functions of t (cf. Ince [4, §12.51, p. 293]). Another interesting property is that "the general solution of the Riccati equation is expressible rationally in terms of any three distinct particular solutions, and also that the anharmonic ratio of any four solutions is constant. It also shows that the general solution is a rational function of the constant of integration (cf. Ince [4, §2.15, p. 23 and §12.51, p. 294])."

This result was extended to the $n \times n$ ($n \geq 2$) solution of the matrix Riccati differential equation by Sorine and Winternitz [6] but with five particular solutions in the general case and four in the *symplectic case* that corresponds to our assumptions on the data matrices. They also give some thoughts to the space of solutions: "For smooth coefficients A, B, C, and D in the MRE (6)[2] the solution space consists of

[2]Sorine and Winternitz [6] consider solutions W of the general matrix Riccati differential equation $W' = A + WB + CW + WDW, \quad W(T) = W_0$.

meromorphic matrices: the matrix elements may have first-order poles, the positions of which depends on the initial conditions. In other words, the MRE (6) has the Painlevé property [4]: the solutions have no moving critical points, i.e., no branch points or essential singularities, the positions of which depend on the initial conditions (cf. [6, pp. 271-272])."

4.1. Bernhard [2]'s conditions in the free end case. In the free end case with $F \geq 0$ and $Q(t) \geq 0$, the necessary and sufficient condition of P. Bernhard [2, Thm 3.1] for the existence of a non-degenerate closed loop saddle point in the sense of [2, Definition 2.3 and Remark 5.1] reduces to the following three properties:

(ii) $X(t)$ is invertible except possibly at isolated points in $[0,T]$, where (X,Λ) is the unique $H^1(0,T)$ matrix solution of

$$\begin{cases} X' = AX - R\Lambda, \quad X(T) = I \\ \Lambda' + A^\top \Lambda + QX = 0, \quad \Lambda(T) = F \end{cases} \tag{4.1}$$

(iii) $x_0 \in \operatorname{Im} X(0)$

(iv) for all $t \in [0,T]$, $P(t) \geq 0$,

where P is defined in terms of Λ and the pseudo inverse of X as follows

$$P(t) = \Lambda(t)\, X(t)^\dagger, \quad X(t)^\dagger \stackrel{\text{def}}{=} \begin{cases} \left[X(t)^\top X(t)\right]^{-1} X(t)^\top, & \text{if } X(t) \neq 0, \\ \text{arbitrary}, & \text{if } X(t) = 0, \end{cases} \tag{4.2}$$

and $\left[X(t)^\top X(t)\right]^{-1}$ is the inverse of $X(t)^\top X(t)$ as a matrix from $\operatorname{Im} X(t)^\top$ onto itself.

Condition (ii) defines the matrix function $P(t)$ a.e. in $[0,T]$ and gives a meaning to a solution of the Riccati differential equation via the solution (Λ, X) of system (4.1). The positivity of F and $Q(t)$ makes the utility function $C_{x_0}(u,v)$ convex in u and this leads to the positivity of $P(t)$ (cf. Remark 3.2). Our objective is to relax those positivity assumptions as in Theorem 3.1. Without them some of the competitive terms in the utility function may simultaneously blow up making it difficult to set the utility function equal to $\pm\infty$ (cf. [2, p. 68 and Remark 5.1]). Moreover, non L^2-integrable singularities in the closed loop strategies invalidate the equivalence (ii) of Lemma 3.2 when either the open loop lower or upper value of the game is not finite. So the very definition of a closed loop saddle point has to be properly revisited and the family of pairs of admissible strategies is no longer $\Phi \times \Psi$ but a subspace S of an enlarged product space $\widetilde{\Phi} \times \widetilde{\Psi}$ containing $\Phi \times \Psi$.

4.2. Normalizability and its consequences. Given the matrices A, B_1, B_2, Q, and F verifying the conditions of § 2.1, system (4.1) always has a unique $H^1(0,T)$ solution (X,Λ) and the definition of [2] still makes sense.

Definition 4.1. The problem (2.1)–(2.2) is *normalizable* if $\det X(t) \neq 0$ a.e. in $[0,T]$, where (X,Λ) is the $H^1(0,T)$ solution of the matrix differential system (4.1).

However, the normalizability property relies on the fact that the state equation can be solved backward in finite dimension. This would not be the case for infinite dimensional evolution systems. Yet, we have the following equivalent property in finite dimension that would be more natural in infinite dimension.

Lemma 4.1. *The following properties are equivalent.*

(i) *The problem* (2.1)–(2.2) *is normalizable.*

(ii) *For almost all $s \in [0,T[$, the matrix differential system*

$$\begin{cases} \widehat{X}_s' = A\widehat{X}_s - R\widehat{\Lambda}_s, & \widehat{X}_s(s) = I \\ \widehat{\Lambda}_s' + A^\top \widehat{\Lambda}_s + Q\widehat{X}_s = 0, & \widehat{\Lambda}_s(T) = F\widehat{X}_s(T), \end{cases} \tag{4.3}$$

has a unique solution $(\widehat{X}_s, \widehat{\Lambda}_s)$ with elements in $H^1(s,T)$. By convention we set $\widehat{X}_T(T) = I$ and $\widehat{\Lambda}_T(T) = F$.

For all $s \in [0,T]\backslash Z$ and $t \in [s,T]\backslash Z$, $\det \widehat{X}_s(t) \neq 0$ and $P(s) = \Lambda(s)X(s)^{-1} = \widehat{\Lambda}_s(s)$, where Z denotes the zero measure set of instants t at which $\det X(t) = 0$.

Remark 4.1. This equivalence should be compared with (iii) in Theorem 3.1. It says that the decoupling operator $P(s)$ can be defined a.e. as $\widehat{\Lambda}_s(s)$ and that the invariant embedding with respect to almost all initial times can still be done as in [3].

Proof. (i) $\Rightarrow$ (ii). Denote by Z the set of times where $X(t)$ is not invertible. For $s \in [0,T]\backslash Z$, it is easy to check that the new pair of matrices $(\widehat{X}_s(t), \widehat{\Lambda}_s(t)) = (X(t)X(s)^{-1}, \Lambda(t)X(s)^{-1})$ are $H^1(s,T)$ solutions of system (4.3). Hence $P(s) = \Lambda(s)X(s)^{-1} = \widehat{\Lambda}_s(s)$. Moreover, by definition, $\widehat{X}_s(t)$ is inversible for all $t \in [s,T]\backslash Z$.

(ii) $\Rightarrow$ (i). Let Z' be the zero measure set of initial times for which system (4.3) of Definition 4.1 has no solution. For all $s \in [0,T]\backslash Z'$, $\widehat{X}_s(T)$ is invertible. Indeed, if there exists $h \neq 0$ such that $\widehat{X}_s(T)h = 0$, then the pair $(x,p) = (\widehat{X}_s h, \widehat{\Lambda}_s h)$ would be solution of the system $x' = Ax - Rp$, $x(T) = 0$, $p' + A^\top p + Qx = 0$, $p(T) = Fx(T) = 0$. Hence $(x,p) = (0,0)$ and $0 = x(s) = h$, a contradiction. But for all $s \in [0,T]\backslash Z'$ and $t \in [s,T]\backslash Z'$, $\widehat{X}_s(T) = \widehat{X}_t(T)\widehat{X}_s(t)$ and $\det \widehat{X}_s(T) = \det \widehat{X}_t(T) \det \widehat{X}_s(t)$. Therefore, for all $t \in [s,T]\backslash Z'$, $\det \widehat{X}_s(t) \neq 0$. Define for $s \in [0,T]\backslash Z'$ the new matrices $(X_s(t), \Lambda_s(t)) = (\widehat{X}_s(t)\widehat{X}_s(T)^{-1}, \widehat{\Lambda}_s(t)\widehat{X}_s(T)^{-1})$. They are solution of system (4.1) in $[s,T]$. Hence X_s is the restriction of X to $[s,T]$, $\widehat{X}_s(t) = X(t)\widehat{X}_s(T)^{-1}$, $\det X(t) \neq 0$ in $[0,T]\backslash Z'$, and $Z' = Z$. □

Starting with the normalizability property, we now proceed in a constructive way to identify the appropriate definition of a closed loop saddle point in the presence of non L^2-integrable singularities in the closed loop strategies. The first observation is that Bernhard [2]'s condition (ii) is both equivalent to $\det X(t) \neq 0$ a.e. in $[0,T]$ and to the stronger property that the number of zeros of $\det X(t)$ is finite.

Lemma 4.2. *Let (X,Λ) be the $H^1(0,T)$ matrix solution of* (4.1). *Then $\det X(t) \neq 0$ almost everywhere in $[0,T]$ if and only if $\det X(t) \neq 0$ except on a finite set Z of instants in $[0,T)$.*

Proof. The determinant of $X(t)$ is a continuous function of t. Hence its set of zeros Z is a closed and even compact subset of $[0,T]$. In dimension one a closed set is the at most countable union of disjoint closed intervals. By assumption, $\det X(t) \neq 0$ a.e. in $[0,T]$ implies that Z is the union of at most a countable number of points. If Z has an infinite number of points, then there exists a sequence $\{t_n\}$ in Z and an accumulation point $t_0 \in Z$, $t_n \neq t_0$ such that $t_n \to t_0$. But this is impossible since around a point of Z there is always an open interval (a,b) such that $t_0 \in (a,b)$ and $(a,b) \cap [0,T]\backslash\{t_0\} \subset [0,T]\backslash Z$. $T \notin Z$ since $X(T) = I$. The converse is obvious. □

The normalizability property gives a precise meaning to the *closed loop system* and to a solution of the matrix Riccati differential equation.

Lemma 4.3. *Assume that the problem* (2.1)–(2.2) *is normalizable and denote by Z the finite set of times at which $\det X(t) = 0$. Then*

(i) *$P(s) = \Lambda(s)X(s)^{-1}$ is uniquely defined and symmetrical for all $s \in [0,T]\backslash Z$.*

(ii) *X is the unique $H^1(0,T)$ solution of the closed loop matrix equation*

$$X' = (A - RP)X, \quad X(T) = I, \tag{4.4}$$

and $\det X(t) \neq 0$ *in* $[0,T]\backslash Z$.

(iii) *For almost all $s \in [0,T]\backslash Z$, $\widehat{X}_s$ is the unique $H^1(s,T)$ solution of the closed loop matrix differential equation*

$$\widehat{X}_s' = (A - RP)\widehat{X}_s, \quad \widehat{X}_s(s) = I, \tag{4.5}$$

and $\det \widehat{X}_s(t) \neq 0$ *in* $[s,T]\backslash Z$.

(iv) *P verifies the matrix Riccati differential equation*

$$P' + PA + A^\top P - PRP + Q = 0, \quad P(T) = F$$

in $[0,T]\backslash Z$, PX is an $H^1(0,T)$ solution of the matrix system

$$(PX)' + A^\top (PX) + QX = 0, \quad (PX)(T) = F$$

and, for all $s \in [0,T]\backslash Z$, $P\widehat{X}_s$ is an $H^1(s,T)$ solution of the matrix system

$$(P\widehat{X}_s)' + A^\top(P\widehat{X}_s) + Q\widehat{X}_s = 0, \quad (P\widehat{X}_s)(T) = F\widehat{X}_s(T).$$

Proof. (i) It is easy to verify that the derivative of the matrix function $X^\top\Lambda - \Lambda^\top X$ is null and that $(X^\top\Lambda - \Lambda^\top X)(T) = 0$. Hence $X^\top\Lambda = \Lambda^\top X$ and $P = \Lambda X^{-1} = (\Lambda X^{-1})^\top = X^{-\top}\Lambda^\top = P^\top$ a.e. in $[0,T]$.

(ii) The second statement follows from the fact that, by definition of P, $X' = AX - R\Lambda = AX - R\Lambda X^{-1}X = (A - RP)X$.

(iii) Equation (4.5) for $\widehat{X}_s$ follows from identity $\widehat{\Lambda}_s(t) = \widehat{\Lambda}_t(t)\widehat{X}_s(t) = P(t)\widehat{X}_s(t)$.

(iv) Again by definition of P and the identity $\Lambda = PX$. ☐

The introduction of the closed loop strategies amounts to a change in the state variable via the transformation $X(t)$.

Lemma 4.4. *Assume that problem (2.1)–(2.2) is normalizable. Let (X,Λ) be the solution of system (4.1) and P be defined by (4.2).*

(i) *Associate with $y_0 \in \mathbf{R}^n$ the function $\hat{x}(t) = X(t)y_0$ and $\hat{p}(t) = \Lambda(t)y_0$. They are $H^1(0,T)$ solutions of the coupled system*

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = X(0)y_0 \\ \hat{p}' + A^\top\hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T), \end{cases} \tag{4.6}$$

and

$$\hat{u} = -B_1^\top P\hat{x} = -B_1^\top\hat{p} \in L^2(0,T;\mathbf{R}^m) \tag{4.7}$$

$$\hat{v} = B_2^\top P\hat{x} = B_2^\top\hat{p} \in L^2(0,T;\mathbf{R}^k). \tag{4.8}$$

(ii) *For all $y_0 \in \mathbf{R}^n$, $u \in L^2(0,T;\mathbf{R}^m)$ such that $|X|^{-1}u \in L^2(0,T;\mathbf{R}^k)$, and $v \in L^2(0,T;\mathbf{R}^m)$ such that $|X|^{-1}v \in L^2(0,T;\mathbf{R}^k)$, the system*

$$y' = X^{-1}(B_1u + B_2v), \quad y(0) = y_0, \tag{4.9}$$

has a unique solution in $H^1(0, T\mathbf{R}^n)$, $x \stackrel{\text{def}}{=} Xy$ *is the unique solution in* $H^1(0,T;\mathbf{R}^n)$ *of the system*

$$x' = Ax + B_1(-B_1^\top Px + u) + B_2(B_2^\top Px + v), \quad x(0) = X(0)y_0, \tag{4.10}$$

and $Px = \Lambda y$. *In particular*

$$x(t) = X(t)\left[y_0 + \int_0^t X^{-1}(B_1 u + B_2 v)\, ds\right]. \tag{4.11}$$

Proof. (i) By definition of $\hat{x}$ and $\hat{p}$. As for the identities for $\hat{u}$ and $\hat{v}$, they follow from the definition of $\hat{p}$ and P: $\hat{p} = \Lambda y_0 = \Lambda X^{-1} X y_0 = P\hat{x}$.

(ii) By assumption on u and v, the right-hand side of (4.9) belongs to $L^2(0,T;\mathbf{R}^n)$ and its solution y belongs to $H^1(0,T;\mathbf{R}^n)$. By direct computation of the derivative of $x = Xy$, it is easy to check that x is solution of system (4.10). □

We are now ready to prove the following result that sheds some light on the choice of a definition of the closed loop saddle point in the presence of closed loop strategies with non L^2-integrable singularities.

Theorem 4.1. *Assume that problem* (2.1)–(2.2) *is normalizable. Let* (X,Λ) *be the solution of system* (4.1) *and* P *be defined by* (4.2). *Consider the linear closed loop strategies*

$$\phi^*(t,x) = -B_1^\top(t)P(t)x \text{ and } \psi^*(t,x) = B_2^\top(t)P(t)x, \tag{4.12}$$

where P *is defined by* (4.2). *Then, for all* $x_0 \in \operatorname{Im} X(0)$, $u \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X|^{-1}u \in L^2(0,T;\mathbf{R}^k)$, *and* $v \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X|^{-1}v \in L^2(0,T;\mathbf{R}^k)$, *there exists a unique solution in* $H^1(0,T;\mathbf{R}^n)$ *to the state equation* (4.10) *and*

$$C_{x_0}(\phi^*, \psi^* + v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(\phi^* + u, \psi^*). \tag{4.13}$$

In particular, for all $x_0 \in \operatorname{Im} X(0)$, $C_{x_0}(\phi^*,\psi^*) = \Lambda(0)y_0 \cdot x_0$ *for some* $y_0 \in \mathbf{R}^n$ *such that* $x_0 = X(0)y_0$, *and this value is independent of the choice of* y_0 *such that* $x_0 = X(0)y_0$.

Proof. By Lemma 4.4, for all $y_0 \in \mathbf{R}^n$, $u \in L^2(0,T;\mathbf{R}^m)$ such that $|X|^{-1}u \in L^2(0,T;\mathbf{R}^k)$, and $v \in L^2(0,T;\mathbf{R}^m)$ such that $|X|^{-1}v \in L^2(0,T;\mathbf{R}^k)$, there exists a unique solution in $H^1(0,T;\mathbf{R}^n)$ to the state equation (4.10) rewritten in the form

$$x' = (A - RP)x + B_1 u + B_2 v, \quad x(0) = X(0)y_0,$$

or, after the change of variable $y = X^{-1}x$ of (cf. (4.9) of Lemma 4.4)

$$y' = X^{-1}(B_1 u + B_2 v), \quad y(0) = y_0.$$

and the utility function $C_{x_0}(\phi^* + u, \psi^* + v)$ is a function of u and v. Recall that $Px = \Lambda X^{-1}Xy = \Lambda y$ and hence Px is an $H^1(0,T;\mathbf{R}^n)$ function. Differentiate the inner product $\Lambda y \cdot x$ as follows

$$\begin{aligned}
&\frac{d}{dt}\Lambda y \cdot x = \Lambda' y \cdot x + \Lambda y' \cdot x + \Lambda y \cdot x' \\
&= -[A^\top \Lambda + QX]y \cdot x + \Lambda X^{-1}(B_1 u + B_2 v) \cdot x + \Lambda y \cdot [(A - RP)x + B_1 u + B_2 v] \\
&= -\left[Qx \cdot x + |-B_1^\top Px + u|^2 - |B_2^\top Px + v|^2\right] + |u|^2 - |v|^2
\end{aligned}$$

and

$$C_{x_0}(\phi^*+u,\psi^*+v) = \Lambda(0)y_0\cdot x_0 + \int_0^T |u|^2 - |v|^2\,dt \quad \Rightarrow C_{x_0}(\phi^*,\psi^*) = \Lambda(0)y_0\cdot x_0$$
$$\Rightarrow \forall u,v, \quad C_{x_0}(\phi^*,\psi^*+v) \le C_{x_0}(\phi^*,\psi^*) \le C_{x_0}(\phi^*+u,\psi^*)$$

and $C_{X(0)y_0}(\phi^*,\psi^*) = \Lambda(0)y_0\cdot X(0)y_0 = \Lambda(0)y_0\cdot x_0$.

As for the uniqueness of the value of the saddle point, assume that there exists y_0^1 and y_0^2 such that $X(0)y_0^1 = x_0 = X(0)y_0^2$. Then $y_0^2 - y_0^1 \in \ker X(0)$. By the symmetry in Lemma 4.3, $X^\top(0)\Lambda(0)(y_0^2-y_0^1) = \Lambda(0)X(0)(y_0^2-y_0^1) = 0$. Hence

$$\Lambda(0)y_0^2\cdot x_0 - \Lambda(0)y_0^1\cdot x_0 = \Lambda(0)(y_0^2-y_0^1)\cdot x_0 = X(0)^\top\Lambda(0)(y_0^2-y_0^1)\cdot y_0^1 = 0$$

and the value of the saddle point only depends on $x_0 = X(0)y_0$. □

4.3. Closed loop strategies and saddle points in presence of non L^2-integrable singularities. In order to accommodate strategies with non L^2-integrable singularities, we first enlarge the sets of strategies Φ and Ψ.

Definition 4.2. The *class of affine closed loop strategies* is defined as follows:

$$\widetilde{\Phi} \stackrel{\text{def}}{=} \left\{ \phi:[0,T]\times\mathbf{R}^n \to \mathbf{R}^m \;\middle|\; \begin{array}{l} \text{such that } x\mapsto\phi(t,x) \text{ is affine,} \\ t\mapsto\phi(t,x) \text{ is Lebesgue measurable, and} \\ t\mapsto\phi(t,0) \text{ belongs to } L^2(0,T;\mathbf{R}^m) \end{array} \right\}$$
$$\widetilde{\Psi} \stackrel{\text{def}}{=} \left\{ \psi:[0,T]\times\mathbf{R}^n \to \mathbf{R}^k \;\middle|\; \begin{array}{l} \text{such that } x\mapsto\psi(t,x) \text{ is affine,} \\ t\mapsto\psi(t,x) \text{ is Lebesgue measurable, and} \\ t\mapsto\psi(t,0) \text{ belongs to } L^2(0,T;\mathbf{R}^k) \end{array} \right\}.$$

We say that ϕ (resp. ψ) is a *linear closed loop strategy* if ϕ (resp. ψ) is linear in x.

To any $(\phi,\psi)\in\widetilde{\Phi}\times\widetilde{\Psi}$ are associated measurable matrices $U(t)$ and $V(t)$ and L^2 vector functions u and v such that $\phi(t,x) = U(t)x+u(t)$ and $\psi(t,x) = V(t)x+v(t)$. At that level of generality, an *admissibility condition* on the pair $(\phi,\psi)\in\widetilde{\Phi}\times\widetilde{\Psi}$ is required to make sense of a solution of the underlying differential equation.

Definition 4.3. We say that the pair $(\phi,\psi)\in\widetilde{\Phi}\times\widetilde{\Psi}$ belongs to S or simply that (ϕ,ψ) is an *admissible pair* if the associated matrix differential equation

$$X' = (A+B_1U+B_2V)X, \quad X(T) = I \tag{4.14}$$

has a unique $H^1(0,T)$ solution such that $\det X(t)\neq 0$ a.e. in $[0,T]$, UX and VX are $L^2(0,T)$ matrices, $|X|^{-1}u\in L^2(0,T;\mathbf{R}^m)$, and $|X|^{-1}v\in L^2(0,T;\mathbf{R}^k)$.

Remark 4.2. As in Lemma 4.2, it can be proved that the condition $\det X(t)\neq 0$ a.e. in $[0,T]$ is equivalent to $\det X(t)\neq 0$ except at a finite number of points in $[0,T]$. Therefore, the matrix $A+B_1U+B_2V = X'X^{-1}$ is the product of an L^2-matrix function and a continuous matrix function with possible non L^2-integrable singularities at a finite number of times. Since A, B_1, and B_2 are L^∞-matrix functions, it implies that the feedback matrix functions U and V will have properties similar to $X'X^{-1}$ and hence possible non L^2-integrable singularities at a finite number of times.

As a consequence, given an admissible pair $(\phi,\psi)\in S$ and $y_0\in\mathbf{R}^n$, the equation

$$x' = Ax + B_1\phi(x) + B_2\psi(x), \quad x(0) = X(0)y_0,$$

has a solution in $H^1(0,T;\mathbf{R}^n)$ such that

$$u = \phi(x) \in L^2(0,T;\mathbf{R}^m) \text{ and } v = \psi(x) \in L^2(0,T;\mathbf{R}^k).$$

As for normalizability, Definition 4.3 is equivalent to the following definition.

Definition 4.4. We say that the pair $(\phi,\psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ belongs to S or simply that (ϕ,ψ) is an *admissible pair* if, for almost all $s \in [0,T[$, the matrix differential equation

$$X_s' = (A + B_1U + B_2V)X_s, \quad X_s(s) = I \tag{4.15}$$

has a unique $H^1(s,T)$ solution, UX_s and VX_s are $L^2(s,T)$ matrices, $|X_s|^{-1}u \in L^2(s,T;\mathbf{R}^m)$, and $|X_s|^{-1}v \in L^2(s,T;\mathbf{R}^k)$.

In view of inequalities (4.13), we now introduce the following definition that says that the original problem can be changed via feedback in such a way that the new resulting problem has an open loop saddle point at $(0,0)$. It will still be referred to as a *closed loop saddle point*. Its connection with Definition 3.2 is not completely obvious, but it will be clarified later on.

Definition 4.5. (i) Given $x_0 \in \mathbf{R}^n$, $(\phi^*,\psi^*) \in S$ is *a closed loop saddle point* of C_{x_0} if there exists a unique solution in $H^1(0,T;\mathbf{R}^n)$ to the state equation

$$\hat{x}' = (A + B_1U_* + B_2V_*)\hat{x} + B_1u_* + B_2v_*, \quad \hat{x}(0) = x_0, \tag{4.16}$$

and for all $u \in L^2(0,T;\mathbf{R}^m)$ such that $|X|^{-1}u \in L^2(0,T;\mathbf{R}^m)$, and all $v \in L^2(0,T;\mathbf{R}^k)$ such that $|X|^{-1}v \in L^2(0,T;\mathbf{R}^k)$

$$C_{x_0}(\phi^*,\psi^*+v) \le C_{x_0}(\phi^*,\psi^*) \le C_{x_0}(\phi^*+u,\psi^*). \tag{4.17}$$

(ii) We say that $(\phi^*,\psi^*) \in S$ is an $X(0)$*-global closed loop saddle point* of C_{x_0} if for all $x_0 \in \operatorname{Im} X(0)$ inequalities (4.17) are verified.

Remark 4.3. We shall see in the proof of Theorem 4.2 that the existence of a solution to the state equation (4.16) implies that $x_0 \in \operatorname{Im} X(0)$.

Remark 4.4. The subspaces $\mathcal{U} = \{u \in L^2(0,T;\mathbf{R}^m) : |X|^{-1}u \in L^2(0,T;\mathbf{R}^m)\}$ and $\mathcal{V} = \{v \in L^2(0,T;\mathbf{R}^k) : |X|^{-1}v \in L^2(0,T;\mathbf{R}^k)\}$ are dense in $L^2(0,T;\mathbf{R}^m)$ and $L^2(0,T;\mathbf{R}^k)$, respectively. Define $w_n(t) = 1$ if $|X(t)| \ge 1/n$ and 0 if $|X(t)| < 1/n$. For any $u \in L^2(0,T;\mathbf{R}^n)$, the sequence $\{u_n = u\, w_n\} \subset \mathcal{U}$ converges to u in $L^2(0,T;\mathbf{R}^n)$ by Lebesgue Dominated Convergence Theorem. So inequalities (4.17) must be verified on dense subsets of $L^2(0,T;\mathbf{R}^m)$ and $L^2(0,T;\mathbf{R}^k)$. They define an open loop saddle point in $\mathcal{U} \times \mathcal{V}$ after a change of the state variable via the transformation $X(t)$.

The fact that the closed loop strategies ϕ and ψ of Definition 4.2 with possible non L^2-integrable singularities need to be linked through the admissibility condition S of Definition 4.3 fundamentally changes the nature of the problem. This subtle difference prevents the use of the nice classical results of the theory of saddle points with respect to two fixed independent sets Φ and Ψ. For instance, two pairs $(\phi_1,\psi_1) \in S$ and $(\phi_2,\psi_2) \in S$ cannot be mixed: (ϕ_1,ψ_2) and (ϕ_2,ψ_1) need not belong to S as shown in Example 4.1 for the pairs $(\phi^*,\psi^*) \in S$ and $(0,0) \in S$. In particular, property (ii) of Berkovitz's equivalence Lemma 3.2 is not verified for $u = 0$ or $v = 0$.

Example 4.1. Consider the example from [2, Example 5.1, p. 67]:

$$x'(t) = (2-t)\,u(t) + t\,v(t), \text{ a.e. in } [0,2], \quad x(0) = x_0 \tag{4.18}$$

$$C_{x_0}(u,v) = \frac{1}{2}|x(2)|^2 + \int_0^2 |u(t)|^2 - |v(t)|^2\,dt. \tag{4.19}$$

Here $A = 0$, $B_1(t) = 2 - t$, $B_2(t) = t$, $F = 1/2$, $Q = 0$, and $R = B_1 B_1^* - B_2 B_2^* = 4(1 - t)$. The Riccati equation reduces to

$$P' - 4(1-t)P^2 = 0, \quad P(2) = 1/2 \quad \Rightarrow P(t) = \frac{1}{2(t-1)^2}.$$

Its solution is positive and blows up at $t = 1$. The open loop lower value of the game is $v^-(x_0) = (x_0)^2/2$ and the open loop upper value of the game is $v^+(x_0) = +\infty$ for all $x_0 \in \mathbf{R}$. The closed loop strategies have a singularity in $t = 1$:

$$\phi^*(t,x) = -\frac{2-t}{2(t-1)^2}x \text{ and } \psi^*(t,x) = \frac{t}{2(t-1)^2}x. \tag{4.20}$$

Yet the state x is an $H^1(0,2)$ function and the controls $\hat{u}$ and $\hat{v}$ are L^2-functions:

$$x(t) = x_0(t-1)^2, \quad \hat{u}(t) = -(2-t)\frac{1}{2}x_0 \text{ and } \hat{v}(t) = t\,\frac{1}{2}x_0. \tag{4.21}$$

Moreover, $X(t) = (t-1)^2$ and by Theorem 4.1 we have a closed loop saddle point in the sense of Definition 4.5.

In general for the pair (ϕ^*, v), both the resulting L^2-norms of the state x and the control $u = \phi^*(x)$ will blow up even when $v = 0$:

$$x'(t) = -\frac{(2-t)^2}{2(t-1)^2}x(t) + t\,v(t) \text{ in } [0,2], \quad x(0) = x_0, \tag{4.22}$$

$$\text{for } v = 0, \quad x(t) = e^{\frac{1}{2}\left[\frac{1}{t-1} - (t-1)\right]}|t-1|\,x_0, \tag{4.23}$$

where $x(1^-) = 0$ and $x(1^+) = \infty$. So the equivalent condition (ii) of Berkovitz's Lemma 3.2 is not verified.

4.4. Necessary and sufficient condition. We now complete the analysis of § 4.2 by proving that the normalizability condition is a necessary and sufficient condition for the existence of a closed loop saddle point in S.

Theorem 4.2. *The following statements are equivalent.*

(i) *Problem* (2.1)–(2.2) *is* normalizable.

(ii) *There exists a pair of closed loop strategies* $(\phi^*, \psi^*) \in S$ *that achieves an* $X(0)$-global closed loop saddle point *of* C_{x_0}.

For all $x_0 \in \operatorname{Im} X(0)$ *the feedback strategies associated with* C_{x_0} *are given by*

$$\phi^*(t,x) = -B_1^\top(t)P(t)x \text{ and } \psi^*(t,x) = B_2^\top(t)P(t)x, \tag{4.24}$$

where P *is defined by* (4.2), *and the value of the closed loop saddle point by*

$$C_{x_0}(\phi^*, \psi^*) = \Lambda(0)y_0 \cdot x_0 \tag{4.25}$$

for some $y_0 \in \mathbf{R}^n$ *such that* $x_0 = X(0)y_0$ *and this value is independent of the choice of* y_0 *such that* $x_0 = X(0)y_0$.

Proof. (i) $\Rightarrow$ (ii) By Theorem 4.1. In addition, the value of $C_{x_0}(\phi^*, \psi^*)$ only depends on $x_0 \in \operatorname{Im} X(0)$ and not on the choice of y_0 such that $x_0 = X(0)y_0$.

(ii) $\Rightarrow$ (i) Denote by U_* and V_* and u_* and v_* the matrices and vectors associated with the pair $(\phi^*, \psi^*) \in S$. By Definition 4.3, there exists a solution $\overline{X}$ to the matrix differential equation (4.14)

$$\overline{X}' = (A + B_1 U_* + B_2 V_*)\overline{X}, \quad \overline{X}(T) = I, \tag{4.26}$$

such that $\det \overline{X}(t) \neq 0$ a.e. in $[0,T]$, $u_* \in L^2(0,T;\mathbf{R}^m)$ such that $|\overline{X}|^{-1} u_* \in L^2(0,T;\mathbf{R}^k)$, and $v_* \in L^2(0,T;\mathbf{R}^m)$ such that $|\overline{X}|^{-1} v_* \in L^2(0,T;\mathbf{R}^k)$.

Consider the solution $\hat{x} \in H^1(0,T;\mathbf{R}^n)$ of the state equation (4.16) corresponding to the admissible closed loop pair (ϕ^*, ψ^*) and choose $y_0 = \hat{x}(T) - \int_0^T \overline{X}^{-1}(B_1 u_* + B_2 v_*)\, dt$. Let $0 \leq t_1 < t_2 < \cdots < t_N < T$ be the finite set Z of points where $\det \overline{X}(t) = 0$. On $(t_N, T]$, the function $\hat{y} = \overline{X}^{-1}\hat{x}$ is the solution in $H^1(t_N, T)$ of

$$\hat{y}' = \overline{X}^{-1}(B_1 u_* + B_2 v_*), \quad \hat{y}(T) = \hat{x}(T).$$

Therefore $\lim_{t \searrow t_N} (\overline{X}^{-1}\hat{x})(t) = \hat{y}(t_N)$. By proceeding successively in a finite number of steps[3] on each interval (t_{i-1}, t_i) we finally get that $\overline{X}^{-1}\hat{x} = \hat{y}$ and that $\lim_{t \searrow 0} (\overline{X}^{-1}\hat{x})(t) = y_0$. This implies that $x_0 = \overline{X}(0) y_0$ and that $x_0 \in \operatorname{Im} \overline{X}(0)$.

Since, by assumption, we have a closed loop saddle point for all x_0 in $\operatorname{Im} X(0)$, for all $y_0 \in \mathbf{R}^n$, $u \in L^2(0,T;\mathbf{R}^m)$ such that $|X|^{-1}u \in L^2(0,T;\mathbf{R}^m)$, and $v \in L^2(0,T;\mathbf{R}^k)$ such that $|X|^{-1}v \in L^2(0,T;\mathbf{R}^k)$, $(\phi^* + u, \psi^* + v) \in S$ and the solution x of the state equation corresponding to $(\phi^* + u, \psi^* + v)$

$$x' = (A + B_1 U_* + B_2 V_*)x + B_1(u_* + u) + B_2(v_* + v), \quad x(0) = X(0) y_0$$

belongs to $H^1(0,T;\mathbf{R}^n)$. Moreover, by analogy with Lemma 4.4 (ii), $y = \overline{X}^{-1} x$ is the solution in $H^1(0,T;\mathbf{R}^n)$ of the differential equation

$$y' = \overline{X}^{-1}\left[B_1(u_* + u) + B_2(v_* + v)\right], \quad y(0) = y_0. \tag{4.27}$$

We now proceed as in the proof of Theorem 3.1 by introducing the new quadratic utility function $c_{x_0}(u,v) = C_{x_0}(\phi^* + u, \psi^* + v)$, but expressed in terms of the state variable y rather than x. From the closed loop saddle point inequalities

$$c_{x_0}(0,v) \leq c_{x_0}(0,0) \leq c_{x_0}(u,0) \tag{4.28}$$

for all $u \in L^2(0,T;\mathbf{R}^m)$ such that $|X|^{-1}u \in L^2(0,T;\mathbf{R}^m)$ and $v \in L^2(0,T;\mathbf{R}^k)$ such that $|X|^{-1}v \in L^2(0,T;\mathbf{R}^k)$, the pair $(0,0)$ achieves an open loop saddle point. By [3, Lemma 3.1], $c_{x_0}(u,v)$ is convex-concave and $dc_{x_0}(0,0;u,v) = 0$ for all u and v. A direct computation yields

$$\begin{aligned} \frac{1}{2} dc_{x_0}(0,0;u,v) = & F\hat{y}(T) \cdot z(T) + \int_0^T Q\overline{X}\,\hat{y} \cdot \overline{X}\,z + (U_*\overline{X}\,\hat{y} + u_*) \cdot (U_*\overline{X}\,z + u) \\ & \qquad - (V_*\overline{X}\,\hat{y} + v_*) \cdot (V_*\overline{X}\,z + v)\, dt, \\ \hat{y}' = \overline{X}^{-1}&(B_1 u_* + B_2 v_*), \quad \hat{y}(0) = y_0, \quad z' = \overline{X}^{-1}(B_1 u + B_2 v), \quad z(0) = 0. \end{aligned}$$

By introducing the solution $\pi \in H^1(0,T;\mathbf{R}^n)$ of the adjoint equation

$$\pi' + \overline{X}^\top Q\overline{X}\,\hat{y} + \overline{X}^\top U_*^\top (U_*\overline{X}\,\hat{y} + u_*) - \overline{X}^\top V_*^\top (V_*\overline{X}\,\hat{y} + v_*) = 0, \quad \pi(T) = F\hat{y}(T),$$

[3] This technique is an alternative to the use of linearly constrained end-state games in [2].

we get

$$0 = \frac{1}{2} dc_{x_0}(0,0;u,v) = \int_0^T (B_1^\top \overline{X}^{-\top}\pi + U_* \overline{X}\,\hat{y} + u_*)\cdot u + (B_2^\top \overline{X}^{-\top}\pi - V_* \overline{X}\,\hat{y} - v_*)\cdot v\, dt$$

$$\Rightarrow |X|(B_1^\top \overline{X}^{-\top}\pi + U_* \overline{X}\,\hat{y} + u_*) = 0 \text{ and } |X|(B_2^\top \overline{X}^{-\top}\pi - V_* \overline{X}\,\hat{y} - v_*) = 0.$$

Since $\overline{X}(t)$ is invertible a.e. in $[0,T]$, the controls are given by

$$\begin{aligned} 0 &= -B_1^\top \overline{X}^{-\top}\pi - (U_* \overline{X}\,\hat{y} + u_*) \quad &\Rightarrow -B_1^\top \overline{X}^{-\top}\pi \in L^2(0,T;\mathbf{R}^m) \\ 0 &= B_2^\top \overline{X}^{-\top}\pi - (V_* \overline{X}\,\hat{y} + v_*) \quad &\Rightarrow B_2^\top \overline{X}^{-\top}\pi \in L^2(0,T;\mathbf{R}^k) \end{aligned} \tag{4.29}$$

and this yields the following system of equations

$$\begin{cases} \pi' + \overline{X}^\top Q \overline{X}\,\hat{y} - \overline{X}^\top (U_*^\top B_1^\top + V_*^\top B_2^\top)\overline{X}^{-\top}\pi = 0, \quad \pi(T) = F\hat{y}(T), \\ \hat{y}' = -\overline{X}^{-1}\left[R\Lambda^{-\top}\pi + (B_1 U_* + B_2 V_*)\overline{X}\,\hat{y}\right], \quad \hat{y}(0) = y_0. \end{cases} \tag{4.30}$$

Denote by $\hat{p}$ the function $\overline{X}^{-\top}\pi$. On the interval $(t_N, T]$, it is readily checked that $\hat{p}$ is a solution in $H^1(t_N,T;\mathbf{R}^n)$ of the equation

$$\hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, \quad \hat{p}(T) = F\hat{x}(T) \tag{4.31}$$

and hence $\lim_{t\searrow t_N}(\overline{X}^{-\top}\pi)(t) = \hat{p}(t_N)$. By proceeding successively in a finite number of steps on each interval (t_{i-1}, t_i) we get that $\overline{X}^{-1}\pi = \hat{p} \in H^1(0,T;\mathbf{R}^n)$ is the solution of equation (4.31). Finally, $(\hat{x},\hat{p}) \in H^1(0,T;\mathbf{R}^n)^2$ is solution of the coupled system

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, \quad \hat{x}(0) = X(0)y_0, \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, \quad \hat{p}(T) = F\hat{x}(T). \end{cases} \tag{4.32}$$

Furthermore, the linear coupled system in the state variables $(\hat{y},\pi)$ is similar to the one at the end of the first part of the proof of Theorem 3.1. Since there is a solution for all initial conditions $y_0 \in \mathbf{R}^n$, its solution is unique. In particular for $y_0 = 0$, $(\hat{y},\pi) = (0,0)$ and from identities (4.29)

$$0 = -B_1^\top \overline{X}^{-\top}\pi - (U_* \overline{X}\ 0 + u_*) \text{ and } 0 = B_2^\top \overline{X}^{-\top}\pi - (V_* \overline{X}\ 0 + v_*).$$

Thence the functions u_* and v_* are null. Therefore $\hat{x}(t) = \overline{X}(t)y_0$, $\hat{x}(T) = y_0$, and

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, \quad \hat{x}(T) = y_0, \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, \quad \hat{p}(T) = F\hat{x}(T) = Fy_0. \end{cases}$$

Since $\hat{x}(t) = \overline{X}(t)y_0$, by linearity of $(\hat{x},\hat{p})$ with respect to y_0, we can associate with $\hat{p}$ an $H^1(0,T)$ matrix function $\overline{\Lambda}$ solution of the system

$$\begin{cases} \overline{X}' = A\overline{X} - R\overline{\Lambda}, \quad \overline{X}(T) = I, \\ \overline{\Lambda}' + A^\top \overline{\Lambda} + Q\overline{X} = 0, \quad \overline{\Lambda}(T) = F, \end{cases} \tag{4.33}$$

where we already know that $\det \overline{X}(t) \neq 0$ a.e. in $[0,T]$. By uniqueness $(\overline{X},\overline{\Lambda}) = (X,\Lambda)$ and the problem is normalizable. $\square$

4.5. Closed loop saddle points when $v(x_0)$ is finite. We first study closed loop saddle points when $v(x_0)$ is finite.

Theorem 4.3. *Given $x_0 \in \mathbf{R}^n$, the following statements are equivalent.*

(i) *C_{x_0} has a closed loop saddle point $(\phi^*, \psi^*) \in S$ and $C_{x_0}(u,v)$ is convex in u and concave in v.*

(ii) *C_{x_0} has a closed loop saddle point $(\phi^*, \psi^*) \in \Phi \times \Psi$ and $C_{x_0}(u,v)$ is convex in u and concave in v* (case (a) of Theorem 3.2).

(iii) *C_{x_0} has an open loop saddle point.*

Proof. (i) $\Rightarrow$ (iii) From the proof of Theorem 4.2, there exists a solution $(\hat{x}, \hat{p})$ in $H^1(0,T;\mathbf{R}^n)^2$ to the coupled system (4.32). Since $C_{x_0}(u,v)$ is convex in u and concave in v, C_{x_0} has an open loop saddle point by [3, Thm 2.4].

(iii) $\Rightarrow$ (ii) Denote by $(\hat{u}, \hat{v})$ the pair achieving the open loop saddle point. By [3, Thm 2.4], $C_{x_0}(u,v)$ is convex in u and concave in v. By definition of $\Phi \times \Psi$, $(\hat{u}, \hat{v}) \in \Phi \times \Psi$. By inequalities (3.19) and (3.20) in Lemma 3.3,

$$\inf_{\phi \in \Phi} C_{x_0}(\phi, \hat{v}) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v})$$

$$\sup_{\psi \in \Psi} \inf_{\phi \in \Phi} C_{x_0}(\phi, \psi) \geq v^-(x_0) \geq \inf_{\phi \in \Phi} C_{x_0}(\phi, \hat{v}) = C_{x_0}(\hat{u}, \hat{v}).$$

By inequalities (3.21) and (3.22) in Lemma 3.3,

$$\sup_{\psi \in \Psi} C_{x_0}(\hat{u}, \psi) = \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat{u}, v) = C_{x_0}(\hat{u}, \hat{v})$$

$$\inf_{\phi \in \Phi} \sup_{\psi \in \Psi} C_{x_0}(\phi, \psi) \leq v^+(x_0) \leq \sup_{\psi \in \Psi} C_{x_0}(\hat{u}, \psi) = C_{x_0}(\hat{u}, \hat{v}).$$

Hence, there exists $(\hat{u}, \hat{v}) \in \Phi \times \Psi$ such that for all $\phi \in \Phi$ and all $\psi \in \Psi$

$$C_{x_0}(\hat{u}, \psi) \leq C_{x_0}(\hat{u}, \hat{v}) \leq C_{x_0}(\phi, \hat{v}).$$

By Lemma 3.2, $(\hat{u}, \hat{v})$ is a closed loop saddle point in $\Phi \times \Psi$.

(ii) $\Rightarrow$ (i) By assumption, C_{x_0} is convex-concave and $(\phi^*, \psi^*) \in \Phi \times \Psi \subset S$. By definition of a closed loop saddle point in $\Phi \times \Psi$, for all $\phi \in \Phi$ and all $\psi \in \Psi$

$$C_{x_0}(\phi^*, \psi) \leq C_{x_0}(\phi^*, \psi^*) \leq C_{x_0}(\phi, \psi^*),$$

the associated matrices U_* and V_* are L^2-matrices, and u_* and v_* are L^2-vectors. Therefore the state equation (4.16) has a unique solution in $H^1(0,T;\mathbf{R}^n)$. Moreover, for all $u \in L^2(0,T;\mathbf{R}^n)$, $\phi = \phi^* + u \in \Phi$ and for all $v \in L^2(0,T;\mathbf{R}^k)$, $\psi = \psi^* + v \in \Psi$, and hence $(\phi^*, \psi^*) \in S$ is a closed loop saddle point in S. □

We also have a global version of the previous theorem.

Theorem 4.4. *The following statements are equivalent.*

(i) *C_{x_0} has an $X(0)$-global closed loop saddle point $(\phi^*, \psi^*) \in S$, $X(0) = \mathbf{R}^n$, and $C_{x_0}(u,v)$ is convex in u and concave in v.*

(ii) *C_{x_0} has a global closed loop saddle point $(\phi^*, \psi^*) \in \Phi \times \Psi$ and $C_{x_0}(u,v)$ is convex in u and concave in v.*

(iii) *For all $x_0 \in \mathbf{R}^n$, C_{x_0} has an open loop saddle point.*

Proof. (i) $\Rightarrow$ (iii) From the equivalence of (i) and (iii) in Theorem 4.3.

(iii) $\Rightarrow$ (ii) By [3, Thm 2.4], $C_{x_0}(u,v)$ is convex in u and concave in v. By [3, Thm 2.9], there exists a unique symmetrical solution to the Riccati equation with elements in $H^1(0,T)$. By Theorem 3.1, C_{x_0} has a global closed loop saddle point $(\phi^*, \psi^*) \in \Phi \times \Psi$.

(ii) $\Rightarrow$ (i) By Theorem 3.1 (iv), $X(t)$ is invertible for all $[0,T]$, $\operatorname{Im} X(0) = \mathbf{R}^n$, and the problem is normalizable. By Theorem 4.2, C_{x_0} has an $X(0)$-global closed loop saddle point in S and $X(0) = \mathbf{R}^n$. □

4.6. Closed loop saddle points when either $v^-(x_0)$ or $v^+(x_0)$ is not finite. From Theorem 4.3, when the value of the game $v(x_0)$ is finite, the closed loop strategy in S can be trivially chosen as $(\phi^*(t,x), \psi^*(t,x)) = (\hat{u}(t), \hat{v}(t))$; from Theorem 4.4, when the value of the game $v(x_0)$ is finite for all $x_0 \in \mathbf{R}^n$, the global closed loop strategy is equal to the L^2-integrable closed loop strategy $(\phi^*, \psi^*) = (-B_1^\top P\hat{x}, B_2^\top P\hat{x}) \in \Phi \times \Psi$, where P is the $H^1(0,T)$ solution of the Riccati differential equation.

The conclusion is that *closed loop strategies with non L^2-integrable singularities will only occur when either $v^-(x_0)$ or $v^+(x_0)$ is not finite.* We now complete the classification of closed loop saddle points in S along the lines of Theorem 3.2 in terms of the u-convexity and v-concavity of the utility function. The *non degenerate* cases are: (a) $v(x_0)$ finite; (b) $v^-(x_0)$ finite and $v^+(x_0) = +\infty$; (c) $v^-(x_0) = -\infty$ and $v^+(x_0)$ finite. The *degenerate* cases are: (d) $v^-(x_0) = -\infty$ and $v^+(x_0) = +\infty$; (e) $v^-(x_0) = v^+(x_0) = +\infty$; (f) $v^-(x_0) = v^+(x_0) = -\infty$.

The new Definition 4.5 of a closed loop saddle point was introduced to accommodate closed loop strategies with non L^2-integrable singularities, but its relation to the open loop values is not as immediate as in the L^2-integrable case of Theorem 3.2. Yet, a complete classification can be obtained when either $v^-(x_0)$ or $v^+(x_0)$ is not finite.

Theorem 4.5. *Assume that C_{x_0} has a closed loop saddle point $(\phi^*, \psi^*) \in S$ and that $C_{x_0}(u,v)$ is convex in u and not concave in v. Denote by $(\hat{u}, \hat{v}) = (\phi^*, \psi^*)$ the associated controls. Then one of the following two possibilities can occur.*

(i) *$v^-(x_0)$ is finite and $v^+(x_0) = \infty$* (case (b) of Theorem 3.2) *and*

$$v^-(x_0) = \inf_{u \in L^2(0,T;\mathbf{R}^n)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v}) = C_{x_0}(\phi^*, \psi^*). \tag{4.34}$$

(ii) *$v^-(x_0) = v^+(x_0) = +\infty$* (case (e) of Theorem 3.2) *and*

$$\sup_{\psi \in \Psi} \inf_{\phi \in \Phi} C_{x_0}(\phi, \psi) = \inf_{\phi \in \Phi} \sup_{\psi \in \Psi} C_{x_0}(\phi, \psi) = +\infty.$$

Proof. From the proof of Theorem 4.2, there exists a solution $(\hat{x}, \hat{p})$ in $H^1(0,T;\mathbf{R}^n)^2$ to the coupled system (4.32). Since $C_{x_0}(u,v)$ is convex in u, we have

$$v^-(x_0) \geq \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v}) > -\infty$$

for the pair $(\hat{u}, \hat{v}) = (-B_1^\top \hat{p}, B_2^\top \hat{p})$. Moreover since $C_{x_0}(u,v)$ is not concave in v, then $v^+(x_0) = +\infty$. As a result, only two cases can occur: $v^-(x_0)$ finite and $v^+(x_0) = +\infty$ or $v^-(x_0) = v^+(x_0) = +\infty$. The property in (ii) follows from Lemma 3.3. □

Since the cases (c) and (f) are dual of cases (b) and (e), we have the dual result.

Theorem 4.6. *Assume that C_{x_0} has a closed loop saddle point $(\phi^*, \psi^*) \in S$ and that $C_{x_0}(u,v)$ is concave in v and not convex in u. Denote by $(\hat{u}, \hat{v}) = (\phi^*, \psi^*)$ the associated controls. Then one of the following two possibilities can occur.*

(i) *$v^+(x_0)$ is finite and $v^-(x_0) = -\infty$* (case (c) of Theorem 3.2) *and*

$$v^+(x_0) = \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat{u}, v) = C_{x_0}(\hat{u}, \hat{v}) = C_{x_0}(\phi^*, \psi^*). \tag{4.35}$$

(ii) $v^-(x_0) = v^+(x_0) = -\infty$ (case (f) of Theorem 3.2) *and*

$$\sup_{\psi\in\Psi} \inf_{\phi\in\Phi} C_{x_0}(\phi,\psi) = \inf_{\phi\in\Phi} \sup_{\psi\in\Psi} C_{x_0}(\phi,\psi) = -\infty.$$

Remark 4.5. Part (ii) of Theorems 4.5 and 4.6 justifies the terminology "degenerate" for cases (e) and (f), since the closed loop saddle point in $\Phi\times\Psi$ is respectively equal to $+\infty$ and $-\infty$. This is to be compared with [2, Definition 2.3 and Theorem 3.1]. If the problem is normalizable, then the conclusions of Theorems 4.5 and 4.6 hold for all $x_0 \in \operatorname{Im} X(0)$ (cf. Theorem 4.2).

Finally, by complementarity, we have the last case (d) of Theorem 3.2.

Theorem 4.7. *Assume that C_{x_0} has a closed loop saddle point $(\phi^*,\psi^*) \in S$ and that $C_{x_0}(u,v)$ is not convex in u and not concave in v. Then $v^-(x_0) = -\infty$ and $v^+(x_0) = +\infty$.*

4.7. An example of a problem that is not normalizable. We conclude with an example where $\det X(t) = 0$ on an interval of non-zero length. Yet there are solutions to the matrix Riccati differential equation and the open loop lower value of the game is finite for all initial conditions. The associated strategies can be obtained by feedback. So, the condition $\det X(t) \neq 0$ a.e. in $[0,T]$ might not be the most general one and Definitions 4.3 and 4.5 might be further relaxed or generalized.

Example 4.2. Consider the dynamics and utility function in the time interval $[0,3]$

$$x'(t) = b_1(t)\,u(t) + b_2(t)\,v(t), \text{ a.e. in } [0,3], \quad x(0) = x_0, \tag{4.36}$$

$$C_{x_0}(u,v) = \frac{1}{2}|x(3)|^2 + \int_0^3 |u(t)|^2 - |v(t)|^2\,dt, \tag{4.37}$$

where

$$b_1(t) = \begin{cases} 2-t, & 0\le t<1\\ 0, & 1\le t<2\\ 3-t, & 2\le t\le 3\end{cases} \quad b_2(t) = \begin{cases} t, & 0\le t<1\\ 0, & 1\le t<2\\ t-1, & 2\le t\le 3\end{cases} \tag{4.38}$$

Here $A = 0$, $B_1(t) = b_1(t)$, $B_2(t) = b_2(t)$, $F = 1/2$, $Q = 0$, and

$$R(t) = b_1(t)^2 - b_2(t)^2 = \begin{cases} 4(1-t), & 0\le t<1\\ 0, & 1\le t<2\\ 4(2-t), & 2\le t\le 3\end{cases} \tag{4.39}$$

We show that $v^-(x_0) = (x_0)^2/2$ and $v^+(x_0) = +\infty$. For the open loop lower value of the game, the minimization with respect to u has a unique solution for all (x_0, v) since the utility function $u \mapsto C_{x_0}(u,v)$ is convex and bounded below by $-\|v\|^2_{L^2}$. The minimizer is completely characterized by the coupled system

$$\begin{cases} x' = -b_1^2\,p + b_2\,v \text{ a.e. in } [0,3], & x(0) = x_0 \qquad \hat{u} = -b_1\,p\\ p' = 0 \text{ a.e. in } [0,3], & p(3) = \dfrac{1}{2}x(3).\end{cases}$$

From this

$$J^-_{x_0}(v) \stackrel{\text{def}}{=} \inf_{u\in L^2(0,2;\mathbf{R})} C_{x_0}(u,v) = C_{x_0}(\hat{u},v) = \frac{1}{4}\left[x_0 + \int_0^3 b_2\,v\,ds\right]^2 - \int_0^3 |v|^2\,dt.$$

It is readily seen that $J^-_{x_0}$ is concave in v and that the supremum with respect to v of $J^-_{x_0}(v)$ exists. Indeed, from the first order condition: for all v

$$\frac{1}{2} dJ^-_{x_0}(\hat{v}; v) = \frac{1}{4}\left[x_0 + \int_0^3 b_2\, \hat{v}\, ds\right] \int_0^3 b_2\, v\, ds - \int_0^2 \hat{v}\, v(t)\, dt = 0,$$

there is a unique stationary point $\hat{v}(t) = b_2(t)\, x_0/2$, the Hessian is negative

$$\frac{1}{2} d^2 J^-_{x_0}(\hat{v}; v; v) = \frac{1}{4}\left[\int_0^3 b_2\, v\, ds\right]^2 - \int_0^3 |v|^2\, dt \le -\frac{1}{3}\int_0^3 |v|^2\, dt,$$

and the open loop lower value of the game is $v^-(x_0) = J^-_{x_0}(\hat{v}) = (x_0)^2/2$. Moreover,

$$\hat{u}(t) = \begin{cases} -(2-t)\frac{1}{2}x_0, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ -(3-t)\frac{1}{2}x_0, & 2 \le t \le 3 \end{cases} \quad \hat{v}(t) = \begin{cases} t\frac{1}{2}x_0, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ (t-1)\frac{1}{2}x_0, & 2 \le t \le 3 \end{cases} \tag{4.40}$$

However the open loop upper value of the game is $v^+(x_0) = +\infty$ for all $x_0 \in \mathbf{R}$. Indeed pick the sequence of controls $\{v_n\}$, $n \ge 1$, $v_n(t) = 0$ in $[0, 2]$ and $v_n(t) = n$ in $[2, 3]$. The corresponding sequence of states at time $t = 3$ is

$$x_n(3) = x_0 + \int_0^3 b_1\, u\, dt + n\int_2^3 (t-1)\, dt = \left[x_0 + \int_0^3 b_1\, u\, dt\right] + \frac{3}{2}n.$$

Denote by X the square bracket that does not depend on n. Then

$$\begin{aligned} C_{x_0}(u, v_n) &= \frac{1}{2}\left|X + \frac{3}{2}n\right|^2 + \int_0^3 |u|^2\, dt - \int_2^3 n^2\, dt \\ &= \frac{1}{8}n^2 + \frac{3}{2}n\, X + \frac{X^2}{2} + \int_0^3 |u|^2\, dt \to +\infty \text{ as } n \to +\infty. \end{aligned}$$

Thus for all $x_0 \in \mathbf{R}$ and $u \in L^2(0, T; \mathbf{R})$

$$\sup_{v \in L^2(0,T;\mathbf{R})} C_{x_0}(u, v) = +\infty \quad \Rightarrow v^+(x_0) = +\infty.$$

Therefore, $C_{x_0}(u, v)$ has no open loop saddle point. For all x_0 the coupled system

$$\hat{x}' = -R\hat{p}, \quad \hat{x}(0) = x_0 \quad \text{and} \quad \hat{p} = 0, \quad \hat{p} = \frac{1}{2}\hat{x}(3) \tag{4.41}$$

$$\hat{u} = -b_1\hat{p} \text{ and } \hat{v} = b_2\hat{p} \tag{4.42}$$

has a unique solution in $H^1(0, 3)$. The unique solution of system (4.1)

$$\begin{cases} X' = AX - R\Lambda, \quad X(T) = I \\ \Lambda' + A^\top\Lambda + QX = 0, \quad \Lambda(T) = F \end{cases} \tag{4.43}$$

is given by

$$X(t) = \left\{\begin{array}{ll} (t-1)^2, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ (t-2)^2, & 2 \le t \le 3 \end{array}\right\}, \quad \Lambda(t) = 1/2 \tag{4.44}$$

and the associated matrix function (4.2) by

$$P_c(t) = \begin{cases} \dfrac{1}{2(t-1)^2}, & 0 \le t < 1 \\ c \text{ (arbitrary)}, & 1 \le t < 2 \\ \dfrac{1}{2(t-2)^2}, & 2 \le t \le 3 \end{cases} \tag{4.45}$$

The problem is not normalizable since $X(t) = 0$ in $[1, 2]$. Yet, the associated optimal strategies are feedback strategies of the usual form $\hat{u} = -b_1 P_c x$ and $\hat{v} = b_2 P_c x$ and the linear closed loop strategies are

$$\phi^*(t, x) = U_c(t)x = -b_1(t)P_c(t)x \text{ and } \psi^*(t, x) = V_c(t)x = b_2(t)P_c(t)x.$$

The function X is also solution of the equation (4.14) of Definition 4.3

$$X' = (b_1 U_c + b_2 V_c)X, \quad X(3) = I.$$

If we adopt the convention that a function $u \in L^2(0, 3)$ such that $|X|^{-1}u \in L^2(0, 3)$ implies that $u = 0$ on $Z = \{t \in [0, 3] : X(t) = 0\} = [1, 2]$ and the same thing for the function v, it can be shown that $dc_{x_0}(0, 0, : u, v) = 0$ for all u and v. However, we have not been able to prove the convexity-concavity of $c_{x_0}(u, v)$ to conclude that the problem has a closed loop saddle point in the sense of Definitions 4.3 and 4.5.

Another issue is the meaning of a solution to the Riccati equation $P' - R\,P^2 = 0$, with final value $P(3) = 1/2$ and a discontinuous function R. What is the effect of a discontinuity in $R(t)$? For instance the following solutions are continuous in $t = 1$

$$P(t) = \begin{cases} \dfrac{\bar{c}}{1 + 2\bar{c}\,(t-1)^2}, & 0 \le t < 1 \\ \bar{c}, & 1 \le t < 2 \\ \dfrac{1}{2(t-2)^2}, & 2 \le t \le 3 \end{cases}$$

for some arbitrary constant $\bar{c} \in \mathbf{R}$. The singularity at $t_2 = 2$ is independent of $\bar{c}$. For $\bar{c} > -1/2$ it is the only singularity. For $\bar{c} \le -1/2$, there is a second singularity at $t_1 = 1 - \sqrt{1/(-2\bar{c})}$ in the interval $[0, 1)$. We also have the solutions (4.45) with a singularity in $t_1 = 1$. Therefore the solution of the Riccati equation is not unique.

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