DUALITY GAP AND QUADRATIC PROGRAMMING, PART II: DUAL BOUNDS OF A QUADRATIC PROGRAMMING PROBLEM AND APPLICATIONS

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ABSTRACT. We studied problems of quadratic programming : the function to be minimized as well as the constraints are of two categories. We calculated β by an interior-point methods. Applications illustrating this methods are provided.

1. Introduction

Consider the following two quadratic problems of optimization
:

$$(QP)_1 \quad \begin{cases} \alpha_1 := Inf\, q_1(x) = \frac{1}{2}\, x^t A\, x + c^t x \\ \quad\quad \text{subject to } \; Bx \le b \\ \quad\quad\quad x \in \mathbb{R}^n, \end{cases}$$

where :
A is a real symmetric matrix of order n ;
B is an $m \times n$ - real matrix and b a constant vector over $\mathbb{R}^m$.
And

$$(QP)_2 \quad \begin{cases} \alpha_2 := Inf\, q_2(x) = x^t A\, x + b^t\, x + c \\ \quad\quad \text{subject to } \; x^t A_i\, x + b_i^t\, x + c_i = 0\,, \quad 1 \le i \le p \\ \quad\quad\quad x^t A_i\, x + b_i^t\, x + c_i \le 0 \;\;,\;\; p+1 \le i \le p+q, \end{cases}$$

where :
A, A_i $1 \le i \le p+q$ are real symmetric matrices ;
b, b_i $1 \le i \le p+q$ are vectors of $\mathbb{R}^m$;
c, c_i $1 \le i \le p+q$ are real constants.

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In this paper, we will give fundamental properties characterizing the dual function. We define the field on which the problems $(QP)_1$ and $(QP)_2$ admit solutions. Finally explicit algorithms will be developed. They enable us to calculate the dual bound for each problem. The numerical examples illustrating this theory will be given at the end of this paper.

2. Dual Bound of a Quadratic Program

2.1. Case of The Linear Constraints.

Consider the following quadratic problem :

$$(PQ)_1 \quad \begin{cases} \alpha_1 := Inf\, q_1(x) = \frac{1}{2}\, x^t A\, x + c^t\, x \\ \qquad \text{subject to } Bx \leq b \\ \qquad\qquad x \in \mathbb{R}^n, \end{cases}$$

where :

A is a real symmetric matrix of order n ;

B is an $m \times n$ - real matrix and b a constant vector of $\mathbb{R}^m$;

Denote C the set of the constraints defined by

$$C := \{\, x \in \mathbb{R}^n,\ B\, x \leq b \,\}\,.$$

The constraints of $(QP)_1$ are linear thus are qualified.

If A is positive definite (or positive semi-definite), we can give necessary and sufficient conditions of optimality. These conditions are those of Kuhn and Tucker.

Theorem 2.1. *x^* is an optimal solution of $(QP)_1$, if and only, if there exists $\lambda \in \mathbb{R}^m_+$ such that*

$$\begin{cases} A\, x^* + B^t\, \lambda + c = 0 \\ \lambda^t(\, B\, x - b\,) = 0 \\ \lambda \geq 0, \quad x \in \mathbb{R}^n \end{cases}$$

Remark 2.1. *In the absence of the assumption of convexity, we find only the necessary condition of optimality.*

Dual Problem Associated to The Problem (QP)$_1$:

We associate to the problem $(QP)_1$ the Lagrangian defined by :

$$L(\, x,\, \lambda\,) = \frac{1}{2}\, x^t A\, x + c^t x + \lambda^t(\, B\, x - b\,), \quad x \in \mathbb{R}^n\ ,\ \lambda \in \mathbb{R}^m_+.$$

Put

$$h(\,\lambda\,) := \underset{x \in \mathbb{R}^n}{Inf}\, L(\, x,\, \lambda\,).$$

h is called the dual function.

The problem defined by

$$(DQP)_1 \qquad \begin{cases} \beta := \underset{\lambda\in\mathbb{R}^m_+}{Sup}\ \underset{x\in\mathbb{R}^n}{Inf}\ L(x,\lambda) \end{cases}$$

is called the dual problem of $(QP)_1$.

The resolution of this problem is traditional. We can see, for example, ([3], [5]) and their bibliography.

2.2. **Case of The Quadratic Constraints.**

Consider the following quadratic problem :

$$(QP)_2 \qquad \begin{cases} \alpha_2 := Inf\, q_2(x) = x^t A x + b^t x + c \\ \qquad \text{subject to } \ x^t A_i x + b_i^t x + c_i = 0, \qquad 1 \le i \le p \\ \qquad\qquad x^t A_i x + b_i^t x + c_i \le 0, \quad p+1 \le i \le p+q. \end{cases}$$

Denote C the set of the constraints defined by

$$C := \left\{ \begin{array}{c} x \in \mathbb{R}^n : x^t A_i x + b_i^t x + c_i = 0\,,\ 1 \le i \le p \\ \text{and} \\ x^t A_i x + b_i^t x + c_i \le 0\,, \qquad p+1 \le i \le p+q \end{array} \right\}.$$

Suppose that the set of constraints C is non-empty, therefore the value of $(QP)_2$ will be finished.

Definition 2.1. *We call Lagrangian associated to the problem $(QP)_2$, the map*

$$\begin{cases} L : \mathbb{R}^n \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R} \\ \qquad (x,\lambda,\mu) \to L(x,\lambda,\mu) \end{cases}$$

where

$$L(x,\lambda,\mu) = x^t A x + b^t x + c + \sum_{i=1}^{p} \lambda_i\,(x^t A_i x + b_i^t x + c_i) + \sum_{i=p+1}^{p+q} \mu_i\,(x^t A_i x + b_i^t x + c_i).$$

Put

$$\begin{cases} U(\lambda,\,\mu) = \sum_{i=1}^{p}\lambda_i\, A_i + \sum_{i=p+1}^{p+q}\mu_i\, A_i + A \\ \\ b(\lambda,\,\mu) = \sum_{i=1}^{p}\lambda_i\, b_i + \sum_{i=p+1}^{p+q}\mu_i\, b_i + b \\ \\ c(\lambda,\,\mu) = \sum_{i=1}^{p}\lambda_i\, c_i + \sum_{i=p+1}^{p+q}\mu_i\, c_i + c \end{cases}.$$

We obtain :

$$L(x,\,\lambda,\,\mu) = x^t U\,(\lambda,\,\mu)\,x + b^t(\lambda,\,\mu)\,x + c(\lambda,\,\mu).$$

Definition 2.2. *We call dual problem associated to the problem of optimization $(QP)_2$ the problem defined by :*

$$(DQP)_2 \qquad \begin{cases} \beta := \underset{(\lambda,\mu)\in\mathbb{R}^n\times\mathbb{R}^q_+}{Sup} h(\lambda,\,\mu) \end{cases}$$

where

$$\begin{cases} h:\mathbb{R}^p\times\mathbb{R}^q_+ \to \mathbb{R}\cup(-\infty) \\ \\ \quad (x,\,\lambda,\,\mu)\mapsto h(\lambda,\,\mu) = \underset{x\in\mathbb{R}^n}{Inf}\, L(x,\,\lambda,\,\mu) \end{cases}$$

is the dual function.

Definition 2.3. *1) We call $edom(h)$ the set of existence of a minimizer for the lagrangian L, the set defined by*

$$edom(h) := \left\{ \begin{array}{c} (\lambda,\,\mu)\in\mathbb{R}^p\times\mathbb{R}^q \text{ such that there exists } x_{\lambda,\,\mu} \text{ such that} \\ h(\lambda,\,\mu) = L(x_{\lambda,\,\mu},\,\lambda,\,\mu) \end{array} \right\}$$

2) We call $Uedom(h)$ the set of existence of one minimizer for the Lagrangian L, the set defined by

$$Uedom(h) := \left\{ \begin{array}{c} (\lambda,\,\mu)\in\mathbb{R}^p\times\mathbb{R}^q \text{ such that there exists one } x_{\lambda,\mu} \text{ such that} \\ h(\lambda,\,\mu) = L(x_{\lambda,\,\mu},\,\lambda,\,\mu) \end{array} \right\}.$$

Definition 2.4. *We define*

$$E := \{\,(\lambda,\,\mu)\in\mathbb{R}^p\times\mathbb{R}^q : U(\lambda,\,\mu) \text{ is positive definite}\}.$$

$$E^* := \{(\lambda,\,\mu)\in\mathbb{R}^p\times\mathbb{R}^q : U(\lambda,\,\mu) \text{ is positive semi-definite}\}.$$

Proposition 2.2. *We have*
a) The set E is open in $\mathbb{R}^p \times \mathbb{R}^q$;
b) the set E^* is the set of limit points of E.

Proposition 2.3. *We have*
a) $Uedom(h) = E$.
b) $edom(h) \subset E^*$.

Proof. *a*)We have

$$2U(\lambda,\, \mu) = \nabla^2 L(x_{\lambda,\mu},\, \lambda,\, \mu)$$

independently of x. Thus if $U(\lambda,\, \mu)$ is positive definite then $L(\,.,\, \lambda,\, \mu)$ is strictly convex coercive and achieves its minimizer in one point. Reciprocally : If $x(\lambda,\, \mu)$ is such that the minimizer of $L(.,\, \lambda,\, \mu)$ is achieved in one point $x_{\lambda,\mu}$, then $U(\lambda,\, \mu)$ is positive definite. Indeed ; let us suppose $U(\lambda,\, \mu)$ positive non-definite. If $U(\lambda,\, \mu)$ is not positive semi-definite, the minimizer of $L(.,\, \lambda,\, \mu)$ is $-\infty$. That contradicted the assumption of existence of the minimizer. If $U(\lambda,\, \mu)$ is only positive semi-definite, $L(.,\, \lambda,\, \mu)$ is only convex and we supposed that it achieved its minimizer in $x_{\lambda\,,\mu}$, thus

$$\nabla_x L(x_{\lambda,\mu},\, \lambda,\, \mu)^t = 0.$$

However

$$\nabla_x L(x_{\lambda,\mu},\, \lambda,\, \mu)^t = 2U(\lambda,\, \mu)\, x_{\lambda,\mu} + b(\lambda,\, \mu)$$

and thus for all $x^* \in Ker(U(\lambda,\, \mu))$,we will have

$$\nabla_x L(x_{\lambda,\mu} + x^*,\, \lambda,\, \mu)^t - 2U(\lambda,\, \mu)(x_{\lambda,\mu} + x^*) - b(\lambda,\, \mu) = 0.$$

Thus by convexity, $L(.,\, \lambda,\, \mu)$ achieved its minimizer in all the points $x_{\lambda,\mu} + x^*$ which contradicts the uniqueness of the minimizer.

b) Suppose, now, that $(\lambda,\, \mu) \in edom(h)$; there exists thus at least one point x_0 in which the minimizer of $L(.,\, \lambda,\, \mu)$ is achieved, therefore we will have

$$\nabla_x L(x_0,\, \lambda,\, \mu)^t = 0$$

and $\nabla^2_{xx} L(x_0,\, \lambda,\, \mu)$ is positive semi-definite. Or

$$2U(\lambda,\, \mu) = \nabla^2_{xx} L(x_0,\, \lambda,\, \mu).$$

We conclude, therefore, that the positive definite Hessian matrix and the uniqueness of the minimizer are two equivalent assumptions in the quadratic case. ∎

Proposition 2.4. *The two sets E and E^* are convex.*

Proof. Let $(\lambda,\, \mu)$ and $(\lambda^*,\, \mu^*)$ be two couples of elements in E and α be a real parameter such that : $0 < \alpha < 1$. We have by linearity of U in $(\lambda,\, \mu)$ and for all $x \in \mathbb{R}^n$ and $x \neq 0$,

$$x^t U(\alpha(\lambda,\, \mu) + (1-\alpha)(\,\lambda^*,\, \mu^*\,))\, x = \alpha\, x^t U(\lambda,\, \mu)\, x + (1-\alpha)\, x^t U(\lambda^*,\, \mu^*)\, x > 0$$

because $x^t U(\lambda,\, \mu)\, x$ and $x^t U(\lambda^*,\, \mu^*)\, x$ are strictly positive, by assumption, for all $x \in \mathbb{R}^n$ and $x \neq 0$. ∎

Proposition 2.5. *We have*

- the set
$$E_+ := \{\, (\lambda,\, \mu) \in E,\ \ \mu > 0\,\}$$
is an open convex ;
- the set
$$E_+^* := \{\, (\lambda,\, \mu) \in E^*,\ \ \mu \geq 0\,\}$$
is the closing of E_+.

3. Study of The Dual Function h

We will study the behaviour of the dual function h over $\mathbb{R}^p \times \mathbb{R}^q$. We start with a proposition characterizing the function h.

Proposition 3.1. *([4]) Let $(\lambda^*,\, \mu^*)$ be an element of $int(Uedom(\,h\,))$, if there exists a neighbourhood $V(\lambda^*,\, \mu^*)\ \subset Uedom(h)$ such that the function which with to all $(\lambda,\, \mu) \in V(\lambda^*,\, \mu^*)$ makes correspond the one minimizer $x(\lambda,\, \mu)$ of $L(\,.,\, \lambda,\, \mu)$ is limited, then the function h is of class C^1 in $(\lambda^*,\, \mu^*)$ and its differential is :*

$$\nabla h(\lambda^*,\, \mu^*)\ = (f_1(x), ...,\ \ f_p(x),\ \ g_1(x), ...,\ g_q(x))^t_{x(\lambda^*,\mu^*)}.$$

To study the comportment of h, we distinguish two cases :

3.1. Study on The Set E.

Proposition 3.2. *For all $(\lambda,\, \mu) \in E$, the only minimizer of $L(\,.,\, \lambda,\, \mu)$ is equal to $x(\lambda,\, \mu)$ where*

$$x(\lambda,\, \mu) = -\frac{1}{2}\, U^{-1}(\lambda,\, \mu)\, b(\lambda,\, \mu)$$

and we have

$$h(\lambda,\, \mu) = -\frac{1}{4}\, b^t(\lambda,\, \mu)\, U^{-1}(\lambda,\, \mu)\, b(\lambda,\, \mu) + c(\lambda,\, \mu).$$

Proof. For all $(\lambda,\ \mu) \in\ E$, $L(x,\ \lambda,\ \mu)$ is strictly convex and coercive from where the existence and the uniqueness of $x(\lambda,\ \mu)$. Moreover $L(x,\ \lambda,\ \mu)$ is minimal in $x(\lambda,\ \mu)$ if and only, if

$$\nabla_x\ h(x(\lambda,\ \mu),\ \lambda,\ \mu)\ = 0$$

Or

$$\nabla_x L(x(\lambda,\ \mu),\ \lambda,\ \mu) = 2U(\lambda,\ \mu)\, x(\lambda,\ \mu) + b(\lambda,\ \mu)$$

from where

$$x(\lambda,\ \mu) = -\frac{1}{2}\, U^{-1}(\lambda,\ \mu)\, b(\lambda,\ \mu).$$

Thus

$$h(\lambda,\ \mu) = L(x(\lambda,\ \mu),\ \lambda,\ \mu).$$

Substitute $x(\lambda,\ \mu)$ by its value, it results that :

$$h(\lambda,\ \mu) = -\frac{1}{4}\, b^t(\lambda,\ \mu) U^{-1}(\lambda,\ \mu)\, b(\lambda,\ \mu) + c(\lambda,\ \mu)$$

In the previous proposition, we have proved that the function $x(\lambda,\ \mu)$ which with to all $(\lambda,\ \mu) \in E$ makes correspond the only minimizer of $L(\,.,\ \lambda,\ \mu)$ is of class C^1. In particular this function is limited over one neighborhood $V(\lambda,\ \mu) \subset E$ containing $(\lambda,\ \mu)$. Since E is open, then acoording to Proposition 3.1 the function h is differentiable on E. ∎

Proposition 3.3. *The function h is continuous and twice differentiable on E, and*

$$\frac{\partial h}{\partial \xi_i}(\lambda,\ \mu) = \tfrac{1}{4}\, b^t(\lambda,\ \mu) U^{-1}(\lambda,\ \mu)\, A_i U^{-1}(\lambda,\ \mu)\, b(\lambda,\ \mu)$$
$$-\tfrac{1}{2}\, b^t(\lambda,\ \mu) U^{-1}(\lambda,\ \mu)\, b_i + c_i, \quad 1 \le i \le p+q$$

$$\frac{\partial^2 h}{\partial \xi_i \partial \xi_j}(\lambda,\ \mu) = -\tfrac{1}{2}\, b^t(\lambda,\ \mu)\, U^{-1}(\lambda,\ \mu)\, A_j\, U^{-1}(\lambda,\ \mu)\, A_i\, U^{-1}(\lambda,\ \mu)\, b(\lambda,\ \mu)$$
$$+\ \tfrac{1}{2}\, b^t(\lambda,\ \mu) U^{-1}(\lambda,\ \mu)\, A_j\, U^{-1}(\lambda,\ \mu)\, b_i - \tfrac{1}{2}\, b_i^t U^{-1}(\lambda,\ \mu)\, b_j$$
$$+\ \tfrac{1}{2}\, b^t(\lambda,\ \mu)\, U^{-1}(\lambda,\ \mu)\, A_i\, U^{-1}(\lambda,\ \mu)\, b_j, \quad 1 \le i, j \le p+q.$$

3.2. Study Outside The Set $E : (\mathbb{R}^p \times \mathbb{R}^q \setminus E)$.

Before making this study, we need to the following notations :

- denote $Im(U(\lambda,\, \mu)) := \{\, y \in \mathbb{R}^n, \exists\, x \in \mathbb{R}^n \,:\, U(\lambda,\, \mu)\, x = y \,\}$ the image of $\mathbb{R}^n$ by $U(\lambda,\, \mu)$;
- denote $Ker(U(\lambda,\, \mu)) := \{\, x \in \mathbb{R}^n \,:\, U(\lambda,\, \mu)\, x = 0\}$ the core ;
- denote Pr_{Im} the operator of orthogonal projection on $Im(U(\lambda,\, \mu))$;
- denote $U_{Im}U(\lambda,\, \mu)$ the operator

$$Pr_{Im} \circ U_{|\, ImU(\lambda,\mu)}\,(\lambda,\, \mu)$$

where $U_{|\, ImU(\lambda,\mu)}\,(\lambda,\, \mu)$ is the restriction of $U(\lambda,\, \mu)$ to $Im(U(\lambda,\, \mu))$ in the canonical basis of $\mathbb{R}^n$.

Moreover, we need to the following results resulting from the linear algebra and the geometry.

Proposition 3.4. *Let $U : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a real symmetric linear operator. We have the following decomposition :*

$$\mathbb{R}^n = Ker\,(U) \oplus Im(U).$$

Proposition 3.5. *Let $U : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a non-zero and real positive semi-definite symmetric linear operator. The operator*

$$U_{Im} = Pr_{Im(U\,)} \circ U_{|\, Im(U\,)}$$

is positive definite of $Im(U\,)$ in $Im(U\,)$.

Proposition 3.6. *Let A be an $(n \times n)$-real positive definite symmetric matrix and let B be an $(n \times p)$-matrix of row p, then B^tAB is positive definite.*

Proposition 3.7. *Let $Fr(E)$ be the boundary of E $(Fr(E) = E^* \setminus E)$.*

a) For all $(\lambda,\, \mu)$ not belongs to E^* (i.e. $U(\lambda,\, \mu)$ is not positive semi-definite) we have : $h(\lambda,\, \mu) = -\infty$.

b) For all $(\lambda,\, \mu)$ belongs to $Fr(E)$, the two following statements are equivalent :

b_1) $h(\lambda,\, \mu) > -\infty$. b_2) $b\,(\lambda,\, \mu)$ is orthogonal to $Ker(U(\lambda,\, \mu))$.

c) In this last case, we have

$$h(\lambda,\, \mu) = -\frac{1}{4}\, Pr_{Im} b^t(\lambda,\, \mu)\, U_{Im}^{-1}(\lambda,\, \mu)\, Pr_{Im}(b(\lambda,\, \mu)) + c(\lambda,\, \mu).$$

Proof. a) If $U(\lambda,\, \mu)$ is not positive semi-definite, then there exists a strictly negative eigenvalue r of $U(\lambda,\, \mu)$ and an unitary eigenvector v_r associated and we have

$$L(\alpha\, v_r,\, \lambda,\, \mu) = (\alpha\, v_r)^t\, U(\lambda,\, \mu)\, (\alpha\, v_r) + b^t(\lambda,\, \mu)(\alpha\, v_r) + c(\lambda,\, \mu) \rightarrow -\infty, \text{when } \alpha \rightarrow \infty.$$

Thus

$$\underset{x\in\mathbb{R}^n}{Inf}\, L(x,\,\lambda,\,\mu) = -\infty.$$

b) We have

$$h(\lambda,\,\mu) = \underset{x\in\mathbb{R}^n}{Inf}\, L(x,\,\lambda,\,\mu),$$

according to Proposition 3.4, for all $x \in \mathbb{R}^n$, there exists uniqueness $x_{Im} \in Im(U(\lambda,\,\mu))$, and uniqueness $x_K \in Ker(U(\,\lambda,\,\mu))$ such that

$$x = x_{Im} + x_K,$$

thus

$$h(\lambda,\,\mu) = \underset{x\in\mathbb{R}^n}{Inf}\left\{(x_{Im} + x_K\,)^t\, U(\lambda,\,\mu)(x_{Im} + x_K) + b^t(\lambda,\,\mu)(x_{Im} + x_K) + c(\lambda,\,\mu)\,\right\}$$

Then

$$h(\lambda,\,\mu) = \underset{x_{Im}\in Im(U(\lambda,\mu))}{Inf}\left\{x_{Im}^t\, U(\lambda,\,\mu)\, x_{Im} + b^t(\lambda,\,\mu)\, x_{Im}\right\} + c(\lambda,\,\mu)$$

$$+ \underset{x_K\in Ker(U(\lambda,\mu))}{Inf}\left\{\, b^t(\lambda,\,\mu)\, x_K\,\right\}.$$

However

$$\underset{x_K\in Ker(U(\lambda,\mu))}{Inf}\left\{\, b^t(\lambda,\,\mu)\, x_K\,\right\} = \begin{cases} 0 & \text{if} \quad b(\lambda,\,\mu) \in Ker(U^{\perp}(\lambda,\,\mu)) \\ -\infty & \text{if} \quad b(\lambda,\,\mu) \notin Ker(U^{\perp}(\lambda,\,\mu)) \end{cases}$$

Thus the proposition holds since $U_{Im}(\lambda,\,\mu)$ is positive definite (see Proposition 3.5).

c) From *b*), we have

$$h(\lambda,\,\mu) = \underset{x\in\mathbb{R}^n}{Inf}\left\{\, x^t\, U_{Im}(\lambda,\,\mu)\, x + Pr_{Im}(b^t(\lambda,\,\mu))\, x + c(\lambda,\,\mu)\right\}$$

and since $U_{Im}(\lambda,\,\mu)$ is positive definite according to Proposition 3.2, we obtain

$$x(\lambda,\,\mu) = -\frac{1}{2}\, U_{Im}^{-1}(\lambda,\,\mu) + Pr_{Im}(b(\lambda,\,\mu)).$$

It holds the formula for $h(\lambda,\,\mu)$. ∎

3.3. **Sufficient Conditions for Zero Duality Gap.**

3.3.1. ***Case 1: Equalities Constraints.***

Theorem 3.8. *If the maximizer of h is achieved in λ^* of $E = (E_+)$, we have*
*) $\nabla_x h(\lambda^*) = (x^t(\lambda^*) A_i x(\lambda^*) + b_i^t x(\lambda^*) + c_i)^t_{i=1,\dots,p} = 0$.
*) $x(\lambda^*) \in C$ *and the maximizer of q_2 on C is achieved in $x(\lambda^*)$:*

$$\alpha^* := q_2(x(\lambda^*)) = \underset{x\in C}{Inf}\, q_2(x).$$

Proof. According to Proposition 3.3, we know that h is differentiable in λ^* and

$$\nabla h(\lambda^*) = (x^t(\lambda^*) A_i x(\lambda^*) + b_i^t x(\lambda^*) + c_i)^t_{i=1,\dots,p} \tag{3.1}$$

According to Theorem 6.13, we have ([1]), we have

$$\nabla h(\lambda^*) = 0 \tag{3.2}$$

So, by (3.1) and (3.2)

$$x^t(\lambda^*) A_i x(\lambda^*) + b_i^t x(\lambda^*) + c_i = 0, \qquad 1 \le i \le p$$

which implies $x(\lambda^*) \in C$ and $\alpha^* = q_2(x(\lambda^*))$.
According to the Proposition 7.2, we have

$$\alpha^* \le \underset{x\in C}{Inf}\, q_2(x),$$

thus

$$\alpha^* = q_2(x(\lambda^*)) = \underset{x\in C}{Inf}\, q_2(x).$$

∎

3.4. **Case 2 : Inequalities Constraints.**

Theorem 3.9. *If the maximizer of h on $\mathbb{R}^q_+$ is achieved in μ^* of E, then we have*

a) $\dfrac{\partial h}{\partial \mu_j}(\mu^*) = x^t(\mu^*) A_j x(\mu^*) + b_j^t x(\mu^*) + c_j \le 0$ *if* $\mu_j^* = 0, \quad 1 \le j \le q.$

b) $\dfrac{\partial h}{\partial \mu_j}(\mu^*) = x^t(\mu^*) A_j x(\mu^*) + b_j^t x(\mu^*) + c_j = 0$ *if* $\mu_j^* > 0, \quad 1 \le j \le q.$

c) $x(\mu^*) \in C$ *and the minimizer of q_2 is achieved in $x(\mu^*)$:*

$$\alpha^* := q_2(x(\mu^*)) = \underset{x\in C}{Inf}\, q_2(x).$$

Proof. According to the Proposition 3.2, we know that h is differentiable in μ^*, and

$$\frac{\partial h}{\partial \mu_j}(\mu^*) = x^t(\mu^*)\,A_j\,x(\mu^*) + b_j^t\,x(\mu^*) + c_j, \qquad 1 \leq j \leq q \tag{3.3}$$

The set of admissibles μ is $D = \mathbb{R}_+^q$, Theorem 6.13 ([1]) gives for all admissible direction d :

$$\nabla h^t(\mu^*)\,d \leq 0. \tag{3.4}$$

By (3.3) and (3.4), we have

$$(x^t(\mu^*)\,A_j\,x(\mu^*) + b_j^t\,x(\mu^*) + c_j)\,d_j \leq 0 \qquad 1 \leq j \leq q.$$

If $\mu_j^* > 0$,then the directions $(0, ..., 0, 1, 0, ..., 0)$ and $(0, ..., 0, -1, 0, ..., 0)$ are both admissibles. Then

$$x^t(\mu^*)\,A_j\,x(\mu^*) + b_j^t\,x(\mu^*) + c_j = 0.$$

If $\mu_j^* = 0$, only $(0, ..., 0, 1, 0, ..., 0)$ is admissible, therefore

$$x^t(\mu^*)\,A_j\,x(\mu^*) + b_j^t\,x(\mu^*) + c_j \leq 0$$

which implies $x(\mu^*) \in C$ and $\alpha^* = q_2(x(\mu^*))$.

By the Proposition 7.2 ([1]), we have

$$\alpha^* \leq \underset{x\in C}{Inf}\, q_2(x),$$

Thus

$$\alpha^* = q_2(x(\mu^*)) = \underset{x\in C}{Inf}\, q_2(x).$$

∎

4. Method of Resolution: Dual Bound Calculus By Interior-Point Method

4.1. Interior-Point Method Rule.

Let β be the following maximization problem :

$$\beta := \underset{x\in adh(C)}{Sup}\ f(x)$$

where $C \subset \mathbb{R}^n$ is an open set and f concave function on $adh(C)$ and twice continuously differentiable on C.

Suppose that one has a concave function P on $adh(C)$ and this last is twice continuously differentiable on C, such that for all $\xi \in adh(C) \setminus C$, $P(\xi) = -\infty$.

For all $\gamma > 0$, the function

$$g(x) = f(x) + \gamma\, P(x)$$

will be concave and will tend to $-\infty$ on the boundary of C.

Then, for all $\gamma > 0$, an algorithm of the Newton type will converge to

$$x(\gamma) = \arg\max\{\, f(x) + \gamma\, P(x), \qquad x \in \mathbb{R}^n \,\}\,.$$

We will prove that, under some conditions, the function $x(\gamma)$ is continuous into zero and converges to the solution of the problem (P).

The idea of the method is thus to solve a sequence of penalized problems:

$$(P_\gamma) \qquad \beta_\gamma := \max_{x \in adh(C)} \{\, f(\,x\,) + \gamma\, P(x)\,\}$$

on starting with a sufficiently large value of γ so that the maximum is enough far from the boundary and thus to avoid the problems due to the bad conditioning.

Then, we decrease γ and construct a sequence of maximizer $x(\gamma)$ which will tend to $x(0)$ while remaining further possible from the boundary since P equal $-\infty$ on the boundary.

4.2. **Choice of Penalty Function.**

To choose the penalty function, we base our argument on the works of G.Sonnevend ([7], 1990), Nesterov-A.Nemirovsky ([6], 1994) and Ch.Ferrier([4], 1997) which studied the function :

$$g(X) = \ln(\det(X))$$

where X is a positive definite and real symmetric matrix.

Before passing to some essential results thereafter, we have the following properties of the function $\ln(\det(A))$, where A is a real positive definite and symmetric matrix :

Proposition 4.1. *Let $\mathcal{M}^n$ be the set of the real symmetrical square matrices and $\mathcal{M}^n_+$ the subset of $\mathcal{M}^n$ of the positive definite matrices. Then the function $\ln(\det)$ is C^2 on $\mathcal{M}^n_+$ and thus we have :*

$$\nabla \ln\det(A).H = Trace(A^{-1}H),$$

$$\nabla^2 \ln\det(A).H.K = -\,Trace(A^{-1}K\,A^{-1}H).$$

Proof. We will denote $(A)_i$ the i^{th} column of matrix A and $(A)_{i,i}$ the i^{th} element of the diagonal of A.

The function ln(det) is C^2 on $\mathcal{M}_+^n$ as composed of function C^2, then
:

$$\nabla \ln\det(A).H = \tfrac{d}{dt}(\ln\det(A+tH))_{t=0} = \tfrac{d}{dt}(\ln\det(A) + \ln\det(I+tA^{-1}H))_{t=0}$$

$$= \tfrac{d}{dt}(\ln\det(I+t\,A^{-1}H))_{t=0}$$

$$= (\tfrac{1}{\det(I+t\,A^{-1}H)})_{t=0}\ \tfrac{d}{dt}(\det(I+t\,A^{-1}H))_{t=0},$$

by derivation of the determinant and by taking $t=0$ we obtain :

$$\nabla \ln\det(A).H = \sum_{i=1}^{n}(\det((I)_1, ..., (I)_{i-1}, (A^{-1}H)_i, (I)_{i+1}, ..., (I)_n))$$

$$= \sum_{i=1}^{n}(A^{-1}H)_{i,i}.$$

By the same way for the second derivative, we have

$$\nabla^2 \ln\det(A).H.K = \tfrac{d}{dt}(trace((A+t\,K)^{-1}H))_{t=0}$$

$$= -(\,trace((A+t\,K)^{-1}K(A+t\,K)^{-1}H))_{t=0}$$

$$= -\,trace(A^{-1}K\,A^{-1}H).$$

∎

Proposition 4.2. *Let A and B be two symmetric matrices and C a symmetrical and positive definite matrix , we have then*

$$AB = ACC^{-1}B \quad \text{and} \quad \rho(AB) \le \rho(AC)\,\rho(C^{-1}B).$$

Moreover, we have

$$\rho(AB) = \rho(C^{1/2}ABC^{-1/2}).$$

Proposition 4.3. *The function* ln(det) *is strictly concave on* $\mathcal{M}_+^n$.

Proof. According to Proposition 4.1 for all matrices $A \in \mathcal{M}_+^n$ and $H \in \mathcal{M}^n$,we have :

$$\nabla^2 \ln\det(A).H.H = -trace(A^{-1}H\,A^{-1}H).$$

According to Proposition 4.2, that is to say

$$S = A^{-\frac{1}{2}}H\,A^{-\frac{1}{2}},$$

we have $S^t = S$ and

$$trace(A^{-1}H\, A^{-1}H) = trace(S^t S) = \| \, S \, \|^2_{Fr},$$

where$\| \, . \, \|_{Fr}$ is the norm of Frobenius. Thus if H is non-zero, then

$$\nabla^2 \ln \det(A).H.H < 0$$

and consequently $\ln(\det)$ is strictly concave. ∎

Proposition 4.4. *Let* $P \, : \, E_+ \, = \, \{ \, (\lambda, \, \mu) \in E, \, \mu > 0 \, \} \, \longrightarrow \, \mathbb{R}$ *be a definite function by :*

$$P : E_+ \to \mathbb{R}$$

$$(\lambda, \, \mu) \to P(\lambda, \, \mu) := \ln(\det(U(\lambda, \, \mu))) + \sum_{i=p+1}^{p+q} \ln(\mu_i).$$

We have

a_1) $\lim\limits_{(\lambda,\mu)\to(\lambda^*,\mu^*)} P(\lambda, \, \mu) = -\infty,$ *where* $(\lambda^*, \, \mu^*) \in adh(E_+) \setminus E_+.$

a_2) *P is twice continuously differentiable on E_+, its derivative and its Hessian are written as follows :*

$$\frac{\partial P}{\partial \lambda_i}(\lambda, \, \mu) = trace(U^{-1}(\lambda, \, \mu) \, A_i),$$

$$\frac{\partial P}{\partial \mu_i}(\lambda, \, \mu) = trace(U^{-1}(\lambda, \, \mu) \, A_i) + \frac{1}{\mu_i},$$

$$\frac{\partial^2 P}{\partial \lambda_i \, \partial \lambda_j}(\lambda, \, \mu) = - \, trace(U^{-1}(\lambda, \mu) \, A_j U^{-1}(\lambda, \, \mu) \, A_i)),$$

$$\frac{\partial^2 P}{\partial \lambda_i \, \partial \mu_j}(\lambda, \, \mu) = - \, trace(U^{-1}(\lambda, \, \mu) \, A_j U^{-1}(\lambda, \, \mu) \, A_i)),$$

$$\frac{\partial^2 P}{\partial \mu_i \, \partial \mu_j}(\lambda, \, \mu) = - \, trace(U^{-1}(\lambda, \, \mu) \, A_j U^{-1}(\lambda, \, \mu) \, A_i)) + \mathcal{X}_{i,j} \, \frac{1}{\mu_i},$$

where

$$\mathcal{X}_{i,j} = \begin{cases} 1 & if \;\; i = j, \\ 0 & otherwise. \end{cases} \; .$$

Proof. a_1) This first statement is obvious, since by assumption, we have :

$$\lim_{(\lambda,\mu)\to(\lambda^*,\mu^*)} P(\lambda, \, \mu) = - \, \infty, \quad \text{where} \quad (\lambda^*, \, \mu^*) \in adh(E_+) \setminus E_+.$$

a_2) Prove the second statement. According to Proposition 4.1, the map, which with a real positive definite and square symmetric matrix

B, makes correspond $\ln\det(B)$ is C^2 and admits for differential :

$$\nabla \ln\det(B).C = trace(B^{-1}C).$$

Since

$$\nabla(\sum_{i=p+1}^{p+q} \ln(\mu_i)).h = \sum_{i=p+1}^{p+q} \frac{h_i}{\mu_i}.$$

By the theorem of derivation of the composed functions, we have :

$$\nabla P(\lambda,\, \mu).h = trace(U^{-1}(\lambda,\, \mu)\sum_{i=1}^{p+q} h_i\, A_i) \; + \sum_{i=p+1}^{p+q} \frac{h_i}{\mu_i}.$$

By the same way, it results from this the formula of the Hessian :

$$\nabla^2 P(\lambda,\, \mu).h.k = -\sum_{i,\,j}^{p+q}(k_j\, trace(U^{-1}(\lambda,\, \mu)\, A_j U^{-1}(\lambda,\, \mu)\, A_i)\, h_i + \mathcal{X}_{i,j}\, \frac{h_i k_j}{\mu_i}\,).$$

∎

Proposition 4.5. *We have :*
(p_1) : the function P is concave ;
(p_2) : if the matrices A_i, for $1 \le i \le p+q$ are linearly independent, then the function P is strictly concave.

Proof. (p_1) : This property is obvious.
(p_2) : The function

$$\sum_{i=p+1}^{p+q} \ln(\mu_i)$$

is strictly concave as limited sum of strictly concave functions.

Thus let us study $\ln\det(U(\lambda,\, \mu))$. According to the Proposition 4.3, the function $A \to \ln\det(A)$ is strictly concave on the set of the real positive definite and symmetric matrices. As the function

$$(\lambda,\, \mu) \to U(\lambda,\, \mu)$$

is affine, $\ln\det(U(\lambda,\, \mu))$ is concave.

Moreover, since $\ln\det$ is strictly concave, so that P is not strictly concave, it is necessary and it is enough that there exist $(\lambda,\, \mu)$ and $(\lambda^0,\, \mu^0)$ distinct in E_+, such that $U(\,\lambda,\, \mu\,) = U(\,\lambda^0,\, \mu^0\,)$. What implies that the matrices $A_i, 1 \le i \le p+q$, are linearly dependent. ∎

4.3. **Penalty Function.**

Definition 4.1. *We call penalized function the function G defined by :*

$$G : E_+ \times \mathbb{R}_+ \to \mathbb{R}^n$$

$$(\lambda,\, \mu,\, \gamma) \to G(\lambda,\, \mu,\, \gamma) := h(\lambda,\, \mu) + \gamma\; P(\lambda,\, \mu).$$

It results the following proposition :

Proposition 4.6. *If $adh(E)$ is limited and non-empty, then for all $\gamma > 0$ the function $G(.,\, .,\, \gamma)$ achieved its uniqueness maximizer point on E_+ and the map*

$$\gamma \to (\lambda(\gamma),\, \mu(\gamma)) \; = \arg\max\; {}_{(\lambda,\mu)\in adh(E_+)} G(\lambda,\, \mu,\, \gamma)$$

is C^1.

To prove this proposition, we need the following lemma :

Lemma 4.7. *If $adh(E_+)$ is limited and non-empty, then the matrices A_i, $1 \leq i \leq p+q$ are linearly independent.*

Proof. We prove this by the absurdity. If there exist one couple $(\lambda,\, \mu)$ such that :

$$\sum_{i=1}^{p} \lambda_i\, A_i + \sum_{i=p+1}^{p+q} \mu_i\, A_i = 0$$

then, for all couple $(\lambda^0,\, \mu^0) \in E_+$ and for all $\sigma \in \mathbb{R}^n$ we will have $(\lambda^0 + \sigma\,\lambda,\, \mu^0 + \sigma\,\mu) \in E_+$, because

$$A + \sum_{i=1}^{p} (\lambda_i^0 + \sigma\,\lambda_i)\, A_i + \sum_{i=p+1}^{p+q} (\mu_i^0 + \sigma\,\mu_i)\, A_i = \; A + \sum_{i=1}^{p} \lambda_i^0\, A_i + \sum_{i=p+1}^{p+q} \mu_i^0\, A_i$$

is, by assumption, positive semi-definite. ∎

Proof. (**of Proposition** 4.6) Let $\gamma > 0$ and let $(\lambda_k,\, \mu_k) \in adh(E_+)$ such that

$$G(\lambda_k,\, \mu_k,\, \gamma) \to \underset{(\lambda,\mu)\in adh(E_+)}{Sup} G(\lambda,\, \mu,\, \gamma).$$

For $\gamma > 0$ enough large one has $(\lambda_k,\, \mu_k) \in \mathbb{K} \subset adh(E_+)$ with $\mathbb{K}$ compact, because the boundary of E_+ is compact and $G(.,\, .,\, \gamma)$ tends to -∞ when one approaches the boundary. Thus, the sequence $(\lambda_k,\, \mu_k)$ has limit points $(\lambda_\infty,\, \mu_\infty)$.

Since $G(.,\, .,\, \gamma\,)$ is upper semi-continuous, for all limit point, one has :

$$G(\lambda_\infty,\, \mu_\infty,\, \gamma) = \underset{(\lambda,\mu)\in adh(E_+)}{Sup} G(\lambda,\, \mu, \gamma).$$

However, according to the Lemma 4.1 the matrices A_i, $1 \le i \le p+q$ are linearly independent, therefore according to Proposition 4.5 P is strictly concave and since h is concave, $G(.,\, .,\, \gamma)$ is strictly concave on E_+.

Thus the maximizer of $G(.,\, .,\, \gamma)$ is achieved in uniqueness point and all limit points are equal.

Moreover strict concavity of $G(.,\, .,\, \gamma)$ involves the inversion of its Hessian matrix. One can, therefore, apply the theorem of implicit functions to the function :

$$(\lambda,\, \mu,\, \gamma) \to \nabla G(\lambda,\, \mu,\, \gamma)$$

at the point $(\lambda_\infty,\, \mu_\infty)$. Which give us the existence of one neighborhood $W(\gamma)$, one neighborhood $W(\lambda_\infty,\, \mu_\infty)$ and a uniqueness map of class $C^1, (\lambda(\gamma),\, \mu(\gamma))$ of $W(\gamma)$ in $W(\lambda_\infty,\, \mu_\infty)$ such that :

$$G(\lambda(\gamma), \mu(\gamma),\, \gamma) = G(\lambda_\infty,\, \mu_\infty\,, \gamma)$$

and for all $\gamma^{'} \in W(\gamma)$:

$$\nabla G(\lambda(\gamma^{'}),\, \mu(\gamma^{'}),\, \gamma^{'}) = 0 \tag{4.1}$$

However, since $G(.,.,\gamma^{'})$ is strictly concave, expression (4.1) implies that $G(.,.,\gamma^{'})$ is maximal in $(\lambda(\gamma^{'}),\, \mu(\gamma^{'}))$ and that this maximum is single, therefore the function

$$\gamma \to (\lambda(\gamma),\, \mu(\gamma)) = \arg\max{}_{(\lambda,\mu)\in adh(E_+)} G(\lambda,\, \mu,\, \gamma)$$

is of class C^1. ∎

4.4. **Convergence of The Interior-Points Method** .

Under reasonable assumptions, we will show that this interior-points method converges.

To solve the problem :

$$(PD) \qquad \beta := \underset{(\lambda,\mu)\in adh\ (E_+)}{Sup} h(\lambda,\, \mu)$$

we chose to solve the perturbed problem namely the sequence of the following problems :

$$(DP)_\gamma \qquad \beta_\gamma = \underset{(\lambda,\mu)\in\mathbb{R}^p\times\mathbb{R}^q}{Sup} G(\lambda,\, \mu,\, \gamma),$$

where

$$G(\lambda,\, \mu,\, \gamma) = \begin{cases} h(\lambda,\, \mu) + \gamma\ P(\lambda,\, \mu) & \text{if } (\lambda,\, \mu) \in E_+ \\ -\infty & \text{otherwise.} \end{cases}$$

Denote

- $(\lambda_\gamma,\, \mu_\gamma)$, for $\gamma > 0$, points where the maximum of $G(.,\, .,\, \gamma\,)$ is achieved when such points exist ;
- $(\lambda_0,\, \mu_0)$, for $\gamma = 0$, points where the maximum of h is achieved when such points exist.

Now, we can state the convergence theorem which is as follows :

Theorem 4.8. *If*
(H_1) : $Int(adh(E_+)) \neq \emptyset$;
(H_2) : *h is continuous on E_+* ;
(H_3) : *P is continuous on E_+ and $P(\lambda,\, \mu) \to -\infty$ when $(\lambda,\, \mu)$ tends to the boundary of E_+* ;
and if one of the two following conditions is satisfied :
(c_1) : *$adh(E_+)$ is limited ;*
(c_2) : *there exists one $a > 0$ such that for $\gamma < a$,*

$$\lim_{\|(\lambda,\mu)\|\to\infty} G(\lambda,\, \mu,\, \gamma\,) = -\infty\ ;$$

then, when the penalty coefficient γ tends to 0, one has :
1) *the sequence $(\lambda_\gamma,\, \mu_\gamma)$ admits, at least, one limit point value and all limit point value of the sequence $(\lambda_\gamma,\, \mu_\gamma)$ is a global minimizer of the problem (DP) ;*
2) *the product $\gamma\ P(\lambda,\, \mu)$ tends to 0.*

Proof. According to (c_1), i.e. if $adh(E_+)$ is limited, then by Theorem 5.1 ([1], Theorem 5.1, p.9), P is upper bounded by $M \in \mathbb{R}$ and h achieved its maximum in, at least, one point $(\lambda_0,\, \mu_0)$.

If there is not the case, according to (c_2), there exists $a > 0$ such that for all $\gamma < a$, one has :

$$\lim_{\|(\lambda,\mu)\|\to\infty} G(\lambda,\, \mu,\, \gamma) = -\infty,$$

thus for all $\gamma < a$, the maximum of $G(.,\, .,\gamma)$ on $adh(E_+)$ will be achieved in at least one point and the set of these optima will be bounded.

Thus, there exists one compact $C \subset \mathbb{R}^p \times \mathbb{R}^q$ dependent of a such that for all $\gamma < a$, all $(\lambda_\gamma,\, \mu_\gamma)$ belongs to C.

Moreover, one can take C convex and of non-empty interior since $adh(E_+)$ is convex and of non-empty interior.

By Theorem 5.1 ([1], Theorem 5.1, p. 9), the function P is upper bounded on C, we will denote M this bound as in the case of $adh(E_+)$ bounded.

In the case of $adh(E_+)$ bounded, we will take

$$C = adh(E_+) \quad \text{and} \quad a = +\infty.$$

Consider an infinite decreasing sequence γ_k of positive values of $\gamma < a$ ($\gamma_k \to 0$ when $k \to +\infty$ and $a > \gamma_k \geq 0$).

Consider the function

$$P_M : C \to \mathbb{R}$$

defined by

$$P_M(\lambda,\, \mu) = M - P(\lambda,\, \mu),$$

for all $(\lambda,\, \mu) \in C$. One has

$$P_M(\lambda,\, \mu) \leq 0, \;\; \forall(\lambda,\, \mu) \in C.$$

Let

$$G_M(\lambda,\, \mu,\, \gamma) = h(\lambda,\, \mu) + \gamma\; P_M(\lambda,\, \mu).$$

G_M being the translate of G, for all $\gamma < a$ the function $G_M(\lambda,\, \mu,\, \gamma)$ achieved its maximum at the same points $(\lambda_\gamma,\, \mu_\gamma)$ that $G(\lambda,\, \mu,\, \gamma)$.

We have, for all $\gamma > 0$:

$$h(\lambda_0,\, \mu_0) \geq h(\lambda_\gamma,\, \mu_\gamma) \geq h(\lambda_\gamma,\, \mu_\gamma) + \gamma\; P_M(\lambda_\gamma,\, \mu_\gamma)$$

By continuity of h and since all point of C is limit of a sequence of points of the interior of C (C is convex), we have, for all $\varepsilon > 0$, there exists $(\lambda^*,\, \mu^*) \in C$ such that :

$$h(\lambda^*,\, \mu^*) \geq h(\lambda_0,\, \mu_0) - \varepsilon,$$

which implies, for $\gamma < a$, with $\gamma \leq 0$

(4.2)

$$\begin{aligned} h(\lambda_\gamma,\, \mu_\gamma) + \gamma\; P_M(\lambda_\gamma,\, \mu_\gamma) &\geq h(\lambda^*,\, \mu^*) + \gamma\; P_M(\lambda^*,\, \mu^*) \\ &\geq h(\lambda_0,\, \mu_0) - \varepsilon + \gamma\; P_M(\lambda^*, \mu^*) \end{aligned}$$

However since for all $\gamma < a$, the function $G_M(.,\, .,\, \gamma)$ is upper bounded and than

$$G_M(\lambda_\gamma,\, \mu_\gamma,\, \gamma) \geq G_M(\lambda,\, \mu,\, \gamma) \;\; \forall\;\; (\lambda,\, \mu) \in C,$$

the sequence $G_M(\lambda_{\gamma k},\, \mu_{\gamma k},\, \gamma_k)$ is bounded and admits thus limit points noted G_M^*. By the expression (4.3) we have, for all $\varepsilon > 0$:

$$G_M^* \geq h(\lambda_0,\, \mu_0) + \varepsilon,$$

while combining with (4.2) we will have :

$$h(\lambda_0,\, \mu_0) \geq G_M^* \geq h(\lambda_0,\, \mu_0) + \varepsilon$$

this being true for all $\varepsilon > 0$, we obtain that $G_M^* = h(\lambda_0,\, \mu_0)$ and thus

$$\lim_{(k\to\infty)} G_M(\lambda_{\gamma k},\, \mu_{\gamma k},\, \gamma_k) = h(\lambda_0,\, \mu_0),$$
$$\lim_{(k\to\infty)} h(\lambda_{\gamma k},\, \mu_{\gamma k}) = h(\lambda_0,\, \mu_0),$$
$$\lim_{(k\to\infty)} P_M(\lambda_{\gamma k},\, \mu_{\gamma k}) = 0.$$

From where one deduces directly

$$\lim_{(k\to\infty)} G(\lambda_{\gamma k},\, \mu_{\gamma k},\, \gamma_k) = h(\lambda_0,\, \mu_0),$$
$$\lim_{(k\to\infty)} P(\lambda_{\gamma k},\, \mu_{\gamma k}) = 0.$$

Since all the sequence $(\lambda_{\gamma k},\, \mu_{\gamma k})$ is continuous in C which is compact, thus it has limit points $(\lambda^*,\, \mu^*) \in C$. By continuity of h one has :

$$h(\lambda^*,\, \mu^*) = h(\lambda_0,\, \mu_0).$$

Thus all point limit of the sequence $(\lambda_{\gamma k},\, \mu_{\gamma k})$ is a global optimum of h. Then, under the assumptions of theorem, the algorithm will converge. ∎

4.5. **Algorithm.**

1. Take $\lambda_0 \in E$, choose $\gamma_0 > 0$, $\varepsilon > 0$, γ_m and γ_M.

2. $\lambda_0 \to \lambda^c$, $\lambda_0 \to \lambda^p$, $\gamma_0 \to \gamma^c$, $\gamma_M \to \gamma^p$.

3. As long as $\gamma_m < \gamma^c < \gamma^p$,

i. $\alpha \to 1$,

ii. calculus of $\nabla_\lambda G(\lambda^c)$

iv. if $\|\nabla_\lambda G(\lambda^c)\| < \varepsilon$,

$*$. $\gamma^c \to \gamma^p$,one decreases γ^c return to the step 3 .

v. If $\|\nabla_\lambda G(\lambda^c)\| \geq \varepsilon$, calculus of $\nabla^2 G(\lambda^c)$.

$*$. If $\nabla^2 G(\lambda^c)$ is definite positive :

$\lambda^c \to \lambda^p$.

$\lambda^c = \lambda_l^c - \alpha\, \nabla_\lambda^2 G^{-1}(\lambda^c,\, \gamma)\nabla_\lambda G(\lambda^c,\, \gamma)$,

$*$. If $\nabla^2 G(\lambda^c)$ is not definite positive, one decreases α

– If $\alpha < \varepsilon$ one increases λ^c, go to **3**.

– If $\alpha \geq \varepsilon$, $\lambda_k^p \to \lambda_k^c$ and go to **ii**.

4. If $\gamma^c < \gamma_m$ calculus of the maximum of $h(.,\, .)$ by starting of the last maximum calculated.

5. If $\gamma^c > \gamma^p$ to start a step of optimization on $h(.,.)$ by starting of the last maximum calculated to approach the maximum as much as possible.

6. If $\gamma^c > \gamma_M$ failure of the program.

5. Numerical implementation

Example 5.1. $(n = 3,\ m = 2)$

Consider the following problem :

$$\begin{cases} \alpha := Inf\, q(x,\, y,\, z) = x^2 + y^2 - 2z \\ subject\ to\ \begin{cases} x^2 + y^2 + z^2 - 1 = 0 \\ x^2 + y^2 + z^2 - 4x = 0 \end{cases} \end{cases}$$

$\gamma \in [0.05\ , 10]$, $\varepsilon = 10^{-9}$.

The primal-dual solution of this example is :

$(x^*, \lambda^*) = (0.25, 0, 0.968245835, 0.778696116, 0.254099445)$

a. *Unfolding of the algorithm according to the penalty term*

γ_0 *initial*	*iter.nber k*	$x^\star$ *(app.sol.)*	*fopti.*	*temps*
1	*343*	*(0.25000000013,0.0000000000,0.96824383548)*	*-1.873991673*	*0.22*
3	*539*	*(0.24999999995,0.0000000000,0.96824383510)*	*-1.873991673*	*0.33*
5	*839*	*(0.24999999995,0.0000000000,0.96824383510)*	*-1.873991673*	*0.49*
7	*971*	*(0.24999999994,0.0000000000,0.96824383512)*	*-1.873991673*	*0.55*
8	*1178*	*(0.24999999995,0.0000000000,0.96824383511)*	*-1.873991673*	*0.71*
9	*1635*	*(0.24999999995,0.0000000000,0.96824383511)*	*-1.873991673*	*0.94*

b. *Global convergence (independent of initial point)*

λ_0 *intial*	*iter.nber k*	$x^\star$ *(app. sol.)*	*fopti.*	*temps(*
(1,0)	*250*	*(0.24999999995,0.0000000000,0.96824383510)*	*-1.873991673*	*0.17*
(2,2)	*343*	*(0.25000000013,0.0000000000,0.96824383548)*	*-1.873991673*	*0.22*
(-3,4)	*720*	*(0.24999999995,0.0000000000,0.96824383510)*	*-1.873991673*	*0.44*
(7,0)	*538*	*(0.24999999995,0.0000000000,0.96824383510)*	*-1.873991673*	*0.33*

Example 5.2. $(n = 3,\ m = 2)$

Consider the following problem :

$$\begin{cases} \alpha := Inf\, q(x,\, y,\, z) = x^2 + y^2 + z^2 - 2z + 1 \\ subject\ to\ \begin{cases} x^2 + y^2 - 2z = 0 \\ x^2 + y^2 + z^2 - 2x - 2y - 2z + 2 = 0 \end{cases} \end{cases}$$

$\gamma \in [0.05\ , 5]$, $\varepsilon = 10^{-9}$.

The primal-dual solution of this example is :

$(x^*, \lambda^*) = (0.517999356,\ 0.517999356,\ 0.268323349, -0.849790101,\ 0.161428479)$.

The optimal value of $f(x^*) = 1.071997409$.

a. *Unfolding of the algorithm according to the penalty term*

λ_0 *initial*	*iter.nber.k*	x^* *(app.sol.)*	*temps*
0.5	*165*	*(0.5179993559,0.5179993559,0.2683233488)*	*0.11*
1	*212*	*(0.5179993557,0.5179993557,0.2683233487)*	*0.11*
3	*241*	*(0.5179993558,0.5179993558,0.2683233488)*	*0.16*
4	*208*	*(0.5179993557,0.5179993557,0.2683233485)*	*0.11*

b. *Global convergence (independent of initial point)*

λ_0 *initial*	*iter.nber k.*	x^* *(app.sol.)*	*temps*
(0,0)	*134*	*(0.5179993558,0.5179993558,0.2683233488)*	*0.11*
(-5,6)	*236*	*(0.5179993557,0.5179993557,0.2683233487)*	*0.16*
(0,3)	*209*	*(0.5179993556,0.5179993556,0.2683233487)*	*0.11*
(-3,4)	*319*	*(0.5179993533,0.5179993533,0.2683233452)*	*0.22*
(6,1)	*276*	*(0.5179993557,0.5179993557,0.2683233488)*	*0.22*

Example 5.3. $(n = 3,\ m = 2)$

Consider the following problem :

$$\begin{cases} \alpha := Inf\, q(x,\, y,\, z) = x^2 + y^2 + z^2 - 4z + 4 \\ subject\ to \begin{cases} y^2 + z^2 - 4x = 0 \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases} \end{cases}$$

$\gamma \in [0.01\,,\ 4]$, $\varepsilon = 10^{-9}$.

The primal-dual solution of this example is :

$(x^*,\ \lambda^*) = (0.236067977,\ 0,\ 0.971736541,\ 0.217286897,\ 0.840884134)$.

The optimal value of $f(x^*)\ =\ 1.11305383$

a. *Unfolding of the algorithm according to the penalty term*

γ_0 *initial*	*iter.nber k.*	x^* *(app.sol.)*	*temps*
0.1	*62*	*(0.23606797734,0.0000000000,0.97173654232)*	*0.00*
0.5	*69*	*(0.23606797734,0.0000000000,0.97173654232)*	*0.06*
1	*73*	*(0.23606797708,0.0000000000,0.97173654075)*	*0.06*
2	*109*	*(0.23606797730,0.0000000000,0.97173654200)*	*0.06*
3	*142*	*(0.23606797729,0.0000000000,0.97173654202)*	*0.07*

b. *Global convergence (independent of initial point)*

λ_0 *initial*	*iter.nber k*	x^* *(app.sol.)*	*temps*
(0,0)	*62*	*(0.23606797734,0.0000000000,0.97173654232)*	*0.00*
(1,3)	*99*	*(0.23606797708,0.0000000000,0.97173654075)*	*0.06*
(3,-0.5)	*211*	*(0.23606797729,0.0000000000,0.97173654202)*	*0.11*
(2,4)	*109*	*(0.23606797730,0.0000000000,0.97173654200)*	*0.06*

Example 5.4. *(*[4]*)* $(n = 3,\ m = 2)$

Consider the following problem :

$$\begin{cases} \alpha := Inf\, q(x,\, y,\, z) = x^2 + y^2 + z^2 + 22x + 121 \\ subject\ to \begin{cases} x^2 - 3y^2 + z^2 - 6x - 7 = 0 \\ x^2 + y^2 + z^2 - 8y - 20 = 0 \end{cases} \end{cases}$$

$\gamma \in [0.05\ , 10]$, $\varepsilon = 10^{-9}$.

The primal-dual solution of this example is :
$(x^*, \lambda^*) = (-5.966500315, 4.633146082, 0, 0.347943423, 0.320734897)$.
The optimal value of $f(x^*) = 46.80215170$.
a. *Global convergence (independent of initial point).*

λ_0 *initial*	*iter.nber* k	x^* *(app.sol.)*	*temps*
(0,0)	*116*	*(-5.966500315,4.633146082,0.000000000)*	*0.06*
(-2,5)	*325*	*(-5.966500316,4.633146085,0.000000000)*	*0.22*
(0,2)	*183*	*(-5.966500317,4.633146084,0.000000000)*	*0.11*
(-1,4)	*294*	*(-5.966500315,4.633146084,0.000000000)*	*0.17*

Example 5.5. *([4])*$(n = 3, m = 3)$
Consider the following problem :

$$\begin{cases} \alpha := Inf\, q(x, y, z) = x^2 + y^2 + z^2 \\ \text{subject to} \begin{cases} x^2 - x + y + z - 10 = 0 \\ y^2 + x + z - y - 10 = 0 \\ z^2 + x + y - z - 10 = 0 \end{cases} \end{cases}$$

$\gamma \in [0.001, 5]$, $\varepsilon = 10^{-6}$.
The primal-dual solution of this example is :
$(x^*, \lambda^*) = (-0.843826043, -0.843826046, -0.843826050, 2.701558149, 2.701558180, 2.7015$
a. *Global convergence (independent of initial point)*

λ_0 *intial*	*iter.nber* k	$x^\star$ *(app.sol.)*	*fopti.*	*temps(*
(1,0,0)	*331*	*(- 0.843826043,- 0.843826046,- 0.843826050)*	*13.4570731*	*0.33*
(2,3,4)	*590*	*(- 0.843826042,- 0.843826046,- 0.843826048)*	*13.4570731*	*0.55*
(0,-0.5,0)	*328*	*(- 0.843826045,- 0.843826044,- 0.843826049)*	*13.4570731*	*0.33*
(2,0,-0.5)	*383*	*(- 0.843826044,- 0.843826040,- 0.843826048)*	*13.4570731*	*0.38*

6. Conclusions

According to our numerical tests, we found that when we increase the initial penalty coefficient, the iteration count increases and we arrive at the same approximate solution. This situation has a theoretical explanation i.e. the coefficient must be sufficiently large not to cross the boundary.

For a considerable number of examples our algorithm converges to the same solution starting from all point belongs to inside the field.

The reduction in step of penalty and step of displacement has an influence on the unfolding of the algorithm (iteration number and time).

If we reduce the precision in this algorithm we obtain a little appreciable solutions with a time and reduced iteration number.

As conclusion on this algorithm its convergence is independent of initial point, it is expensive (iteration number, time) which is due to the Newton procedure.

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