Center Conditions for a Class of Polynomial Differential Systems

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Abstract

We classify non-degenerate centers of systems of the form

$$\dot{x} = P_3(x)y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2,$$

where the $P_i(x)$ are polynomials in x, y over $\mathbf{R}$. We show that such systems fall naturally into two classes: those with Darboux first integrals, and those which arise from simpler systems via singular algebraic transformations.

Résumé

Nous classifions les systèmes ayant un centre non dégénéré et qui sont de la forme

$$\dot{x} = P_3(x)y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2,$$

où $P_i(x)$ sont des polynômes en x, y sur $\mathbf{R}$. Nous montrons que tous ces systèmes sont de deux types: ceux avec intégrales premières de Darboux et ceux qui sont engendrés par des systèmes plus simples à l'aide de transformations algébriques singulières.

1. Introduction

One of the most intriguing aspects of the dynamics of real planar polynomial vector fields is the close relationship between the algebraic nature of the defining equations and their qualitative dynamics. For example, the existence of algebraic curves which are union of trajectories can have important implications for the existence of centers, *i.e.* isolated singularities surrounded by closed orbits, or for the existence of limit cycles.

Let p be a singularity which is a center for the linear part of the system at p. Then p is either a center or it is a focus in which case p is called a *weak focus*. A theorem of Poincaré in [13] says that p is a center if and only if the system has a nonconstant analytic first integral in the neighborhood of p.

We would like to express the existence of a first integral around a singularity in terms of mechanisms which relate to the essential algebraic nature of the systems considered. It seems that the first article where several mechanisms for producing centers were discussed is [22] where Żołądek mentioned three such mechanisms: searching for 1) a Darboux first integral or 2) a Darboux-Schwarz-Christoffel first integral or 3) generating centers by *rational reversibility.*

In this article we shall consider two mechanisms for producing centers. The first mechanism is by producing a system which is symmetric with respect to a line, by pulling back a non-singular differential equation along a map of an algebraic nature. Firstly, we recall that given a singular point p which is a center or a focus, of a system

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P, $Q \in \mathbf{R}[x,y]$, we can show that p is a center by showing that we have a symmetry with respect to a line through p. By this we actually mean that the system is invariant under a reflection with respect to that line and to the reversal of time. We may assume the point p to be at the origin and the line of symmetry to be the x-axis. Then this symmetry with respect to the x-axis is expressed by the condition:

$$(P(x,-y), Q(x,-y)) = (-P(x,y), Q(x,y)).$$

The following example illustrates construction of a system with a center on a line of symmetry. We start with the system without singularities:

$$\frac{d\bar{x}}{dt} = 2\bar{y}, \qquad \frac{d\bar{y}}{dt} = -1,$$

whose first integral is $F(\bar{x},\bar{y}) = \bar{y}^2 + \bar{x}$. The $\bar{y}$-axis has a contact point with the trajectory through the origin, i.e. the curve $\bar{y}^2 + \bar{x} = 0$. Now consider the map

$$\pi : \mathbf{R}^2 \to \mathbf{R}_{\geq 0} \times \mathbf{R} \hookrightarrow \mathbf{R}^2,$$

taking (x,y) to $(x^2,y) = (\bar{x},\bar{y})$. We construct a symmetric system by pulling back along this map the form $\bar{\omega} = dF = 2\bar{y}\,d\bar{y} + d\bar{x}$ associated to the above system to obtain the form $w = \pi^*\bar{\omega} = 2y\,dy + 2x\,dx$ associated to the system

$$\frac{dx}{dt} = -y, \qquad \frac{dy}{dt} = x.$$

This system has a center at the origin, it has the first integral $x^2 + y^2$, and it has a symmetry in the y-axis. The map above discards the left side of the phase portrait in the plane $(\bar{x},\bar{y})$ and creates a symmetric system by "folding" the right side of its phase portrait onto the left side while smoothing the curves on the y-axis.

The above example illustrates a mechanism for producing systems with center by pulling back nonsingular differential equations via the polynomial map $(x,y) \mapsto (x^2,y)$ with the result of obtaining symmetric differential systems.

This mechanism, which is of an algebraic nature, can be generalized by using more general maps than the one above but which are still of an algebraic nature.

To use the full power of this mechanism it is convenient to work over the complex field. More precisely, each differential system (1) over $\mathbf{R}$ generates a complex differential system and we shall frequently make use of this complex system in order to study the real one.

We shall say that a system is *algebraically reducible* at a singular point p if we can find a map $(x,y) \mapsto (\bar{x},\bar{y}) = (f(x,y), g(x,y))$ with f and g analytic functions (real or complex) in the neighborhood of p which are also algebraic over $\mathbf{C}(x,y)$, such that the differential equation $P\,dy - Q\,dx = 0$ associated to our system (1) is the pull-back of a differential equation $\bar{P}\,d\bar{y} - \bar{Q}\,d\bar{x} = 0$ without singularities. Since this last equation is nonsingular it has a local first integral and this first integral can be pulled back to a first integral of our system, producing a center. We will also use the term *rationally reducible* if the map above can be chosen to be rational.

We shall say that a system is *algebraically reversible* at a singular point p if we can find a map $(x, y) \mapsto (\bar{x}, \bar{y}) = (f(x, y), g(x, y))$ with f and g analytic functions (real or complex) in the neighborhood of p such that they are algebraic over $\mathbf{C}(x, y)$ and such that the Jacobian of f and g at p is negative. It is not hard to show that an algebraically reversible system is also algebraically reducible.

We also have another mechanism for proving that a system possesses a center. This mechanism is based on Darboux' paper [8] in which he made an extensive study of the integration of algebraic differential equations in terms of algebraic particular solutions of such equations.

Dulac proved integrability for quadratic systems with center in [9] by using a case-by-case discussion and various ad-hoc methods of integration in each case. In only one of the cases he proved integrability by using the method of Darboux. Dulac worked with a notion of center for polynomial systems over $\mathbf{C}$ and he defined a center as a singularity with non-zero eigenvalues λ_1, λ_2 such that $\lambda = \lambda_1/\lambda_2$ is a negative rational number. In [9] he only considered the case when $\lambda = -1$ and worked with the canonical form $dx/dt = x + ..., dy/dt = -y + ...$

Darboux' method of integration was applied in [19, 16] to prove in a uniform way for all cases, that quadratic systems (respectively cubic symmetric systems) with centers are integrable. Poincaré's theorem together with the method of integration of Darboux thus yields another way of producing centers by methods of an algebraic nature.

The first integrals obtained via the method of Darboux are functions of the form

$$\exp{(D/E)} \prod C_i^{\alpha_i}, \tag{$*$}$$

where D, E and the C_i are polynomials in $\mathbf{C}[x, y]$ and $\alpha_i \in \mathbf{C}$. Each of the C_i defines an algebraic curve which is a particular solution of the system. Darboux called *invariant algebraic curve* (respectively *algebraic solution*) of a differential equation $Qdx - Pdy = 0$, an algebraic curve $f(x, y) = 0$ over $\mathbf{C}$ such that $Df = fk$ for some $k \in \mathbf{C}[x, y]$ (respectively an algebraic invariant curve $f(x, y) = 0$ with f an irreducible polynomial in x, y over $\mathbf{C}$).

In view of Poincaré's result, in order to show that a singularity which is a center for its linearization is in fact a center we must look for a local nonconstant analytic first integral and in particular we may look for a first integral of the above form. Such functions are called *Darboux* functions. However, these functions are not sufficient for describing the local behavior around a center as simple examples of systems with center exist which have Liouvillian but not Darboux first integrals (see [17,22]). Żołądek mentioned some special kinds of Liouvillian first integrals, *i.e.* called *Schwarz-Christoffel* and *Daboux-Hyperelliptic* first integrals in [22].

A natural question which arises is the following:

> *Is it true that all the weak foci which are centers of real planar polynomial vector fields have either a Darboux integrating factor or are algebraically reversible?*

Remark. For complex systems there seems to be yet another way of producing systems with centers in the sense of Dulac for singularities whose quotient of eigenvalues is -1 and which have a local nonconstant analytic first integral. In this case, there are examples of systems which can be transformed to Ricatti-like equations, and whose integrals can be given explicitly in terms of hyperelliptic functions. A cubic example can be found in [14]. There are no known examples of this phenomena however for real centers.

Given a class of real polynomial systems depending analytically on a finite number of parameters, each with a critical point which is a center for its linearization (which we may consider fixed for the whole class), the conditions to be satisfied by a system in order to have a center at this point can be determined algorithmically from Poincaré's theorem. The set of these conditions is denumerable but by the Hilbert's basis theorem, it is sufficient that a finite number of them be satisfied. In the case where the class depends polynomially on the parameters, these conditions are also polynomial.

Although computing the conditions is quite straight forward, reducing them to a finite basis is computationally much harder. Indeed, the computational difficulties are insurmountable even in the case of relatively simple cubic systems (see [11] for a typical example of massive expression swell in the symbolic computations). Nevertheless, it has been possible to solve these conditions in several interesting cases. Examples of these calculations can be found in many papers (for example, several sub cases of the Kukles' system have be analyzed this way in [6,10,11]). For each of these, the conjecture holds true.

The purpose of this paper is to obtain complete conditions for a center in an infinite dimensional class of systems. The class we consider are the generalized Liénard systems:

$$\dot{x} = P_3(x)y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2, \tag{1.1}$$

where the $P_i(x)$ are polynomials in x, with $P_3(0) \neq 0$, $P_0(0) = 0$ and $P_3(0)P_0'(0) < 0$. Such systems include, the Kukles system with $a_7 = 0$, and, after a change of variable, the quadratic systems.

Such systems were first considered by Cherkas [1], who found necessary and sufficient conditions for a center. However, in practice from the conditions obtained it is not easy to get the explicit form of the systems with center even for systems of small degree [7]. Furthermore, it is not clear, in the general case, how the conditions obtained relate to the types of first integrals mentioned above. We show here how the centers of (1.1) arise from either a Darboux first integral of the form (*) or from a simple form of algebraic reducibility or reversibility.

In a previous paper [4], it was shown that all polynomial Liénard systems with centers can be considered as arising through a simple type of rational reducibility. In the present paper, the systems seem to exhibit more of the complexity that we would expect from the complete class of polynomial systems. How far these results can be generalized towards the class of all polynomial systems is unknown at the moment, although there are results of Cherkas which allow some systems with higher degree nonlinearities in y to be dealt with in a similar fashion [3].

In Section 2 we outline the two main results that we need for the classification. In Section 3 we consider the case where $P_3(x)$ is constant. We do this is because the results we obtain are stronger in this case, and we are able to outline the method of proof without becoming entangled in the details of the more general case. The full case is treated in Section 4. A summary of the results can be found in Theorems 3 and 4 of these last two sections, with brief comments on how such centers could be distinguished computationally.

This paper is based on an earlier preprint with the same title by the first author which was reworked into its current more expanded version.

2. Preliminaries

We shall give here the two main results on which this work is based. The first is a result due to Cherkas (Theorem 1 below is essentially Lemma 3 of [1]), which enables us to give an algebraic description of the necessary and sufficient conditions for a center.

Theorem 1. *The Liénard system*

$$\dot{x} = y, \qquad \dot{y} = g(x) + f(x)y, \tag{2.1}$$

with $f(x)$ and $g(x)$ real analytic in a neighborhood of zero with $g(0) = 0$ and $g'(0) < 0$, has a center at the origin if and only if there exists a real analytic function $z(x)$, defined in a neghborhood of the origin, with $z(0) = 0$ and $z'(0) = -1$ such that

$$F(x) = F(z(x)), \quad \text{and} \quad G(x) = G(z(x)), \tag{2.2}$$

where

$$F(x) = \int_0^x f(\xi)\, d\xi, \qquad G(x) = \int_0^x g(\xi)\, d\xi. \tag{2.3}$$

To stress a certain symmetry in (2.2) we shall write (2.2) as $F(x) = F(z)$ and $G(x) = G(z)$ meaning of course here that $z = z(x)$.

The other result concerns algebraic solutions to transcendental equations and is due to Rosenlicht [15].

Theorem 2. *Let $(k,')$ be a differential field of characteristic zero with differential extension field $(K,')$ with the same field of constants and such that k is algebraically closed in K, i.e. all elements in K which are algebraic over k also lie in k. We also assume that K is a finite algebraic extension of $k(t)$, where t is transcendental over k and such that $t' \in k$. Suppose that*

$$\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' \in k,$$

where $c_1, \dots c_n$ are constants of k which are linearly independent over $\mathbf{Q}$ and $u_1, \dots u_n$ and v are in K. Then $u_1, \dots, u_n \in k$ and $v = ct + d$, with c a constant of k and $d \in k$.

Remarks: In our applications, we shall take $k = \mathbf{C}$ to ensure that k is algebraically closed. In particular, all elements of k are constants.

3. The System with P_3 constant

In this section we consider the system (1.1) with P_3 a constant. Since $P_3(0) \neq 0$ by hypothesis, we can arrange that $P_3 \equiv 1$ by scaling the time and therefore consider the simplified system

$$\dot{x} = y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2, \tag{3.1}$$

We prove the following theorem for this system.

Theorem 3. *A system of the form* (3.1) *with* $P_0(0) = 0$ *and* $P_0'(0) < 0$, *which has a center at the origin satisfies one of the following (possibly overlapping) conditions.*

(i) The system is algebraically reducible via the map $(x, y) \mapsto (x, y^2)$ *and thus it has a symmetry in the* x*-axis;*
(ii) the system is algebraically reducible via map $(x, y) \mapsto (r(x), y)$ *for some polynomial* $r(x) = x^2 + O(x^3)$ *over* $\mathbf{R}$*;*
(iii) there is a local first integral of Darboux type.

Proof: Several steps will be needed for the proof.

Step 1 We perform the change of variables used by Cherkas

$$y = Y \exp\left(\int_0^x P_2(\xi)\, d\xi\right).$$

to arrive, after renaming the variable Y as y at the system

$$\dot{x} = y, \qquad \dot{y} = g(x) + f(x)y.$$

Here

$$g(x) = P_0(x) \exp(-2\int_0^x P_2(\xi)\, d\xi), \qquad f(x) = P_1(x) \exp(-\int_0^x P_2(\xi)\, d\xi). \tag{3.2}$$

We note that the transformation of Cherkas changed the system into one which is polynomial in y but with coefficients in a Liouvillian differential field extension of $(\mathbf{C}(x), d/dx)$ generated by adjoining the exponentials of integrals in (3.2). On the other hand the above transformation reduced to the first degree, the polynomial in y in the right side of the second differential equation, making it possible to apply the Theorem of Cherkas.

Step 2 We now apply Cherkas' Theorem.

It is easy to verify that the conditions on $P_0(x)$ given above imply the hypotheses of Theorem 1 on $g(x)$. Therefore the conclusion of Theorem 1 tells us that (2.1) has a center at the origin if and only if there is a real analytic function $z(x)$ in the neighborhood of the origin, with $z(0) = 0$ and $z'(0) = -1$ which simultaneously satisfies

$$\int_0^x f(\xi)\, d\xi = \int_0^z f(\xi)\, d\xi, \qquad \int_0^x g(\xi)\, d\xi = \int_0^z g(\xi)\, d\xi,$$

or equivalently,

$$f(x)\, dx = f(z)\, dz, \qquad g(x)\, dx = g(z)\, dz. \tag{3.3}$$

We first dismiss the trivial case where $f(x)$ vanishes identically, as this implies that P_1 is identically zero. The origin is, in this case, a center by symmetry in the x axis. Alternatively, the system can be algebraically reduced to the system

$$\dot{\bar{x}} = 1, \qquad \dot{\bar{y}} = 2P_0(\bar{x}) + 2P_2(\bar{x})\bar{y}$$

by the map $(x, y) \mapsto (\bar{x}, \bar{y}) = (x, y^2)$. This means that the first integral defined at the origin of this system in $(\bar{x}, \bar{y})$ can be pulled back to a first integral in a neighborhood of the origin of (3.1), giving a center.

Thus we shall assume from now on that f and g do not vanish identically. We shall also exclude the case where P_2 vanishes identically in (3.1) as this has been covered in [4]. In fact this case is just a special case of Step 3 below.

From (3.3) we obtain

$$g(x)/f(x) = g(z)/f(z), \tag{3.4}$$

as local meromorphic functions in x around the origin (in the right side of (3.4) z is actually $z(x)$) and hence $z(x)$ satisfies the equation

$$\frac{P_0(x)}{P_1(x)}\exp\left(-\int_0^x P_2(\xi)\,d\xi\right)=\frac{P_0(z)}{P_1(z)}\exp\left(-\int_0^z P_2(\xi)\,d\xi\right). \tag{3.5}$$

As a real local analytic function, $z(x)$ may be considered as an element of $\mathbf{R}\{\{x\}\}$ which determines an element of $\mathbf{C}\{\{x\}\}$ and hence a local complex analytic function which we also denote by $z(x)$. We now divide our investigation into two cases: the first case is when $z(x)$ is algebraic over $\mathbf{C}(x)$, and the second when $z(x)$ is transcendental over $\mathbf{C}(x)$.

Step 3 z(x) is algebraic over $\mathbf{C}(x)$: In this case we can apply the results of Theorem 2. First note that $\int P_2(x)\,dx$ is a non-constant polynomial, and that the equation (3.5) gives

$$\frac{P_0(x)/P_1(x)}{P_0(z)/P_1(z)}\exp\left(-\int_0^x P_2(\xi)\,d\xi+\int_0^z P_2(\xi)\,d\xi\right)=1. \tag{3.6}$$

Thus we have

$$\Psi(x)=R_1(x,z(x))e^{R_2(x,z(x))}=1, \tag{3.7}$$

where

$$R_1(x,y)=\frac{P_0(x)/P_1(x)}{P_0(y)/P_1(y)}\in\mathbf{C}(x,y),$$
$$R_2(x,y)=\textstyle\int_0^y P_2(\xi)\,d\xi-\int_0^x P_2(\xi)\,d\xi\in\mathbf{C}[x,y].$$

$\overline{R_1}(x)=R_1(x,z(x))$ and $\overline{R_2}(x)=R_2(x,z(x))$ therefore lie in the algebraic differential field extension $(\mathbf{C}(x)[z(x)],d/dx)$ of $(\mathbf{C}(x),d/dx)$ generated by $z(x)$. Below we denote the derivations in the two fields by $'$. Considering now the expression for Ψ'/Ψ from (3.7) we obtain:

$$(\overline{R_1})'/\overline{R_1}+(\overline{R_2})'=0. \tag{3.8}$$

Thus, by Theorem 2 where we take $k=\mathbf{C}$ and $K=\mathbf{C}(x)[z]$, we get that $R_1(x,z(x))$ is a constant and $R_2(x,z(x))$ is of the form $cx+d$, for constants c and d. Substituting back into (3.8), we see that c must vanish. Furthermore, it is clear that at $x=0$, $R_2(x,z(x))$ vanishes, and hence $d=0$. Lastly, from (3.7), $R_1(x,z(x))=1$.

Thus, we have arrived at the following equations

$$P_0(x)/P_1(x)=P_0(z)/P_1(z),\qquad \int_0^x P_2(\xi)\,d\xi=\int_0^z P_2(\xi)\,d\xi. \tag{3.9}$$

Consider the subfield F of $\mathbf{R}(x)$ formed by all rational functions $S(x)$ such that $S(x)=S(z(x))$. By Lüroth's theorem, the field is isomorphic to $\mathbf{R}(r/s)$, for some function $r(x)/s(x)$ with $r(x),s(x)\in\mathbf{R}[x,y]$. Without loss of generality, we can choose the degree of r to be greater than the degree of s, with r and s coprime. Hence, we can write

$$\int_0^x P_2(\xi)\,d\xi=\phi(r(x)/s(x)), \tag{3.10}$$

for some rational function ϕ over $\mathbf{R}$, in one variable.

Now, working over $\mathbf{C}$, the right hand side of (3.10) can be written as

$$\prod_{i=1}^q(\alpha_i r+\beta_i s)\Big/\prod_{i=1}^q(\gamma_i r+\delta_i s),$$

If $(\alpha_i r+\beta_i s)$ shares a common factor with $(\gamma_j r+\delta_j s)$, these two polynomials must differ by a constant, whence we can assume that the fraction above allows no further cancellations. Since the left hand side of (3.10) is a polynomial, $\prod(\gamma_i r+\delta_i s)$ must be a constant, and hence the denominator has no dependence on r, and s must be a constant. Without loss of generality, we can take $s(x)=1$ and $r(0)=0$.

From (3.9), we must also have

$$P_0(x)/P_1(x) = \psi_0(r/s)/\psi_1(r/s) = \psi_0(r)/\psi_1(r)$$

for some polynomials in one variable ψ_0 and ψ_1 over $\mathbf{R}$ with $(\psi_0, \psi_1) = 1$. From the equality above we get

$$P_0(x)/\psi_0(r(x)) = P_1(x)/\psi_1(r(x)) = K(x)$$

with $K(x)$ a rational function in x over $\mathbf{R}$. We then have:

$$P_0(x) = K(x)\psi_0(r(x)), \qquad P_1(x) = K(x)\psi_1(r(x)), \tag{3.11}$$

which implies that $K(x)$ is a polynomial over $\mathbf{R}$.

Using the expression for P_1 in (3.11) and replacing it into the second part of (3.2) we have:

$$f(x) = K(x)\psi_1(r(x))\exp(-\int_0^x P_2(\xi)\,d\xi).$$

Substituting the above in the first part of (3.3) we obtain:

$$K(x)\psi_1(r(x))\exp(-\int_0^x P_2(\xi)\,d\xi)\,dx =$$

$$K(z)\psi_1(r(z))\exp(-\int_0^z P_2(\xi)\,d\xi)\,dz.$$

But $r(x) = r(z(x))$ and using the second part of (3.9) we obtain:

$$K(x)\,dx = K(z)\,dz.$$

However, from $r(x) = r(z)$, we also have $r'(x)\,dx = r'(z)\,dz$, and hence,

$$K(x)/r'(x) = K(z)/r'(z).$$

So $K(z)/r'(z)$ is in the field F. This implies that $K(x) = r'(x)\chi(r(x))$ for some rational function χ over $\mathbf{R}$ in one variable, which in view of the preceding equality must be a polynomial by a comparison of the degrees of r and r'. Since $r(x) = r(z)$ with $z'(0) = -1$, we must have $r'(0) = 0$. However, from the expression of $P_0(x)$ in (3.11) and using the expression $K(x) = r'(x)\chi(r(x))$ we get that $P_0'(0) = r''(0)\chi(0)\psi(0)$ and hence $r''(0) \neq 0$, and without loss of generality, we can take $r''(0) = 1$. We also have $\chi(0)\psi_0(0) \neq 0$.

Putting together this information, there exist polynomials A_0, A_1 and A_2 such that

$$P_0(x) = A_0(r(x))r'(x), \qquad P_1(x) = A_1(r(x))r'(x), \qquad P_2(r(x)) = A_2(r(x))r'(x),$$

with $A_0(r(x)) = \chi(r(x))\psi(r(x))$ and hence $A_0(0) < 0$. The system is then algebraically reducible. Indeed the map $(x, y) \mapsto (\bar{x}, \bar{y}) = (r(x), y)$ reduces (3.1) to the system

$$\dot{\bar{x}} = \bar{y}, \qquad \dot{\bar{y}} = A_0(\bar{x}) + A_1(\bar{x})\bar{y} + A_2(\bar{x})\bar{y}^2.$$

This system is nonsingular at the origin since the conditions on P_0 imply that $A_0(0) < 0$.

Alternatively, the center can be seen to be given by a reversing transformation

$$(x, y, t) \mapsto (\bar{x}, y, -t),$$

where $r(x) = r(\bar{x})$, $\bar{x}(0) = 0$ and $\bar{x}'(0) < 0$. The case when P_2 vanishes identically follows the same pattern [4].

Step 4 z(x) is transcendental over C(x): In this case, we first consider some consequences of (3.4). Differentiating (3.4) we obtain

$$(g/f)'(x)\,dx = (g/f)'(z)\,dz$$

where here and also in the equality below, $'$ denotes the derivative as function in one variable. This gives:

$$\left[\frac{1}{f}\left(\frac{g}{f}\right)'\right](x) = \left[\frac{1}{f}\left(\frac{g}{f}\right)'\right](z),$$

which gives

$$\left[\frac{P_2}{P_1}\left(\frac{P_0}{P_1}\right) - \frac{1}{P_1}\left(\frac{P_0}{P_1}\right)'\right](x) = \left[\frac{P_2}{P_1}\left(\frac{P_0}{P_1}\right) - \frac{1}{P_1}\left(\frac{P_0}{P_1}\right)'\right](z).$$

Since this is an algebraic equation between z and x and since both x and z are transcendental over $\mathbf{C}$ then both sides of the above equality must be a constant c. Hence we consider the equality in $\mathbf{R}(x)$:

$$P_2 P_0 P_1 + P_0 P_1' - P_1 P_0' = c P_1^3. \tag{3.12}$$

To simplify this relation further, we now split up P_0 and P_1 over $\mathbf{C}$ into their linear factors. We write

$$P_1(x) = q_1 \prod_{i=1}^{n} l_i(x)^{\alpha_i}, \qquad P_0(x) = q_0 \prod_{i=1}^{n} l_i(x)^{\beta_i}, \qquad (\alpha_i + \beta_i \geq 1, \quad \forall i),$$

where the l_i are of the form $x - k_i$ for each i, with the k_i distinct, and the q_i are constants. Equation (3.12) then gives

$$P_2 q_0 q_1 \prod_{i=1}^{n} l_i^{\alpha_i+\beta_i} + q_0 q_1 \prod_{i=1}^{n} l_i^{\alpha_i+\beta_i-1} \sum_{i=1}^{n} (\alpha_i - \beta_i) K_i = c q_1^3 \prod_{i=1}^{n} l_i^{3\alpha_i}, \tag{3.13}$$

where K_i is the product of all the l_k except l_i. If $c = 0$, then we have that $\alpha_i = \beta_i$ for all i, and hence $P_2 \equiv 0$. However, we have already left aside this case at it was considered in [4]. Thus, $c \neq 0$, and from (3.13) we see that for all i, $3\alpha_i \geq \alpha_i + \beta_i - 1$. That is to say $\beta_i \leq 2\alpha_i + 1$. Let K be the product of all the l_i's. We thus have

$$P_2 q_0 q_1 K + q_0 q_1 \sum_{i=1}^{n} (\alpha_i - \beta_i) K_i = c q_1^3 \prod_{i=1}^{n} l_i^{2\alpha_i - \beta_i + 1}. \tag{3.14}$$

If $2\alpha_i - \beta_i + 1 > 0$, then $\alpha_i = \beta_i$. Thus we can split the l_i into two categories: those with $\alpha_i = \beta_i$ and those with $\beta_i = 2\alpha_i + 1$. For ease of working, we shall drop the reference to the respective ranges of the suffixes of these two classes, and denote the elements of the first class by $\bar{l}_i$, and those of the second by $\bar{\bar{l}}_i$. We also denote the product of the $\bar{\bar{l}}_i$'s by $\bar{\bar{L}}$, and $\bar{\bar{L}}/\bar{\bar{l}}_j$ by $\bar{\bar{L}}_j$. Finally, we take $M = \prod \bar{l}_i^{\alpha_i}$. Thus,

$$\begin{aligned} P_0 &= q_0 \prod \bar{l}_i^{\alpha_i} \prod \bar{\bar{l}}_i^{2\alpha_i+1} = q_0 M \prod \bar{\bar{l}}_i^{2\alpha_i+1}, \\ P_1 &= q_1 \prod \bar{l}_i^{\alpha_i} \prod \bar{\bar{l}}_i^{\alpha_i} = q_1 M \prod \bar{\bar{l}}_i^{\alpha_i}. \end{aligned} \tag{3.15}$$

Furthermore, from (3.14),

$$P_2 q_0 q_1 \bar{\bar{L}} - q_0 q_1 \sum (1 + \alpha_i) \bar{\bar{L}}_i = c q_1^3 \prod \bar{l}_i^{\alpha_i} = c q_1^3 M, \tag{3.16}$$

and from (3.16) we get

$$M = \frac{q_0}{q_1^2 c}\left(P_2 \bar{\bar{L}} - \sum (1 + \alpha_i) \bar{\bar{L}}_i\right). \tag{3.18}$$

Finally, we can construct a first integral of the system using the geometric method of integration of Darboux. First we observe that, in view of (3.15), P_0/P_1 is a polynomial. Indeed $P_0/P_1 = (q_0/q_1)(\prod \bar{\bar{l}}_i^{\alpha_i+1})$.
For $k \in \mathbf{C}$ we define

$$C_k(x, y) = y + k(P_0/P_1)(x) = y + k\frac{q_0}{q_1} \prod \bar{\bar{l}}_i^{\alpha_i+1}.$$

We seek invariant algebraic curves in the family of curves $C_k = 0$. A curve $C_k = 0$ is an invariant algebraic curve of the differential system (3.1) if and only if $DC_k/C_k \in C[x, y]$, where D is the operator

$$y\frac{\partial}{\partial x} + (P_0 + P_1 y + P_2 y^2)\frac{\partial}{\partial y}.$$

We determine the condition on the constant k such that this be satisfied.

We first compute

$$DC_k = k(P_0/P_1)'y + P_0 + P_1y + P_2y^2.$$

Using (3.12) we have that

$$(P_0/P_1)' = (P_0'P_1 - P_0P_1')/P_1^2 = P_2P_0/P_1 - cP_1,$$

and hence

$$DC_k = P_2y^2 + k(P_0P_2/P_1 - cP_1)y + P_0 + P_1y. \qquad (a)$$

We search for polynomials $A_k(x)$ and $B_k(x)$ such that

$$DC_k = C_k(A_ky + B_k) = (y + kP_0/P_1)(A_ky + B_k) \qquad (b).$$

Clearly, we have from (a) and (b) that

$$A_k = P_2, \qquad B_k = P_1/k,$$

and

$$-ckP_1 + P_1 = P_1/k,$$

yielding

$$ck^2 - k + 1 = 0.$$

Depending on the value of c this equation has one or two distinct solutions. If $c = 1/4$ it has only one solution $k = 2$, and if $c \neq 1/4$ it has two distinct solutions k_1 and k_2. Thus

$$DC_k = C_k(P_2y + P_1/k) = C_k\left(P_2y + \frac{q_1M}{k}\prod \bar{\bar{l}}_i^{\alpha_i}\right).$$

where $k = 2$ in case $c = 1/4$ and $k = k_j$, $j = 1, 2$ in case $c \neq 1/4$.

Before considering each one of these two cases we observe that the system (3.1) admits the following expression

$$C = \exp\left(\int_0^x P_2(x)\,dx\right),$$

as an exponential factor, i.e. $DC/C \in \mathbf{C}[x, y]$. Indeed, we have:

$$DC = C\,(P_2y).$$

We now consider the two possible cases. First let us suppose $c \neq 1/4$. In this case we construct a Darboux first integral from these three functions C, C_{k_1} and C_{k_2} of the form

$$\left(y + k_1\frac{q_0}{q_1}\prod \bar{\bar{l}}_i^{\alpha_i+1}\right)^{r_1}\left(y + k_2\frac{q_0}{q_1}\prod \bar{\bar{l}}_i^{\alpha_i+1}\right)^{r_2}\exp\left(r_3\int_0^x P_2(x)\,dx\right),$$

It is immediately verified that if we take $r_1 = 1$, $r_2 = -k_2/k_1$ and $r_3 = -1 + k_2/k_1$ then we have the linear combination of their corresponding cofactors

$$r_1(P_2y + P_1/k_1) + r_2(P_2y + P_1/k_2) + r_3P_2y = 0$$

In the case $c = 1/4$ we only have one invariant algebraic curve, i.e. $C_2 = 0$. We recall that we also have the exponential factor C. We now consider the expression

$$\tilde{C} = \exp(P_0/(P_1y + 2P_0)) = \exp\left(\frac{(q_0/q_1)\prod \bar{\bar{l}}_i^{\alpha_i+1}}{y + 2(q_0/q_1)\prod \bar{\bar{l}}_i^{\alpha_i+1}}\right),$$

then $\tilde{C}$ is another Darboux exponential factor. Indeed calculations yield:

$$D\tilde{C} = \tilde{C}(-P_1/4) = \tilde{C}\left(-\frac{1}{4}q_1M\prod \bar{\bar{l}}^{\alpha_i}\right).$$

In this case we construct a Darboux first integral by using the curve $C_2 = y + 2(P_0/P1)(x)$ and the two exponential factors C and $\tilde{C}$ defined above. This first integral is of the form

$$(C_2)^{r_1} C^{r_2} (\tilde{C})^{r_3}.$$

It is easy to see that if we take $r_1 = 2$, $r_2 = -1$ and $r_3 = 1$ we obtain the following linear combination of their corresponding cofactors:

$$r_1(-P_1/4) + r_2(P_2 y) + r_3(P_2 y + P_1/2) = 0$$

These integrals are well defined and holomorphic in a neighborhood of the origin, and hence, by Poincaré's result the origin is a center.

Remarks: It is now easy to see how we can determine computationally whether a particular system of the form (3.1) has a non-degenerate center or not. First we check whether $P_1 \equiv 0$ (case (i)). Secondly, we check whether (3.12) holds for some constant c (case (iii)). Finally we seek a non-trivial common generating polynomial of the form $x^2 + \cdots$ for the polynomials

$$\int_0^x P_0(x)\,dx, \qquad \int_0^x P_1(x)\,dx, \qquad \int_0^x P_2(x)\,dx.$$

4. The General System

We now consider the general case,

$$\dot{x} = P_3(x) y, \qquad \dot{y} = P_0(x) + P_1(x) y + P_2(x) y^2, \tag{4.1}$$

where P_3 is a polynomial with $P_3(0) \neq 0$. Without loss of generality, we can take $P_3(0) > 0$. The conditions on P_0 for the origin to be of focal type are then the same as Section 3, namely $P_0(0) = 0$ and $P_0'(0) < 0$.

Theorem 4. *A system of the form (4.1) with $P_0(0) = 0$, $P_3(0) > 0$ and $P_0'(0) < 0$, which has a center at the origin satisfies one of the following (possibly overlapping) conditions.*

(i) The system is algebraically reducible via the map $(x, y) \mapsto (x, y^2)$ and thus it has a symmetry in the x-axis;
(ii) the system is algebraically reducible via the map $(x, y) \mapsto (M(x)^{1/r}, yR(x)^{1/q})$, where $M(x)$ is a rational function in x over $\mathbf{R}$ with $M(x)^{1/r}$ of order 2 at $x = 0$, and $R(0) \neq 0$;
(iii) there is a local first integral of Darboux type.

Proof: The steps are similar to those of Theorem 3.

Step 1. In the differential equation $P\,dy - Q\,dx = 0$ we perform the change of variables of Cherkas

$$y = Y \exp\left(\int_0^x P_2(x)/P_3(x)\,dx\right).$$

and we obtain a differential equation in x, Y whose associated differential system, after renaming Y as y is:

$$\dot{x} = y, \qquad \dot{y} = g(x) + f(x) y,$$

where

$$g(x) = \frac{P_0(x)}{P_3(x)} \exp\left(-2\int_0^x \frac{P_2(\xi)}{P_3(\xi)}\,d\xi\right), \quad f(x) = \frac{P_1(x)}{P_3(x)} \exp\left(-\int_0^x \frac{P_2(\xi)}{P_3(\xi)}\,d\xi\right).$$

We note that, since $P_3(0) \neq 0$, the functions f and g are analytic at the origin, and that the conditions on $P_0(x)$ imply the hypothesis of Theorem 1.

Step 2. Hence we have a center if and only if there exists an analytic function $z(x)$, with $z(0) = 0$ and $z'(0) < 0$ which simultaneously satisfies

$$f(x)\,dx = f(z)\,dz, \qquad g(x)\,dx = g(z)\,dz, \tag{4.2}$$

If $f(x)$ vanishes identically, then there is again a center at the origin by symmetry in the x-axis. We therefore exclude this case from further consideration.

From (4.2) we have $g(x)/f(x) = g(z)/f(z)$, and so

$$\frac{P_0(x)}{P_1(x)} \exp(-\int_0^x P_2(x)/P_3(x)\,dx) = \frac{P_0(z)}{P_1(z)} \exp(-\int_0^z P_2(z)/P_3(z)\,dz). \tag{4.3}$$

As an element of $\mathbf{R}\{\{x\}\}$, $z(x)$ generates also an element of $\mathbf{C}\{\{x\}\}$ which as before we also denote by $z(x)$. We again divide the search into two cases.

Step 3. z(x) is algebraic over $\mathbf{C}(x)$. First note that we can write $\int P_2(x)/P_3(x)\,dx$ in the form

$$-\int_0^x P_2(x)/P_3(x)\,dx = \sum_{i=0}^{n} \alpha_i \log(R_i(x)) + R(x), \tag{4.4}$$

via the partial fraction expansion of $P_2(x)/P_3(x)$ over $\mathbf{C}$, where the R_i and R are rational functions of x over $\mathbf{C}$, well defined at $x = 0$ and such that $R_i(0) = 1$ for all i and $R(0) = 0$. Furthermore, splitting the α_i into their real and imaginary parts, and taking the average of the right hand side of (4.4) with its complex conjugate, we can assume without loss of generality that $R(x)$ is real and for each i, if α_i is real then $R_i(x)$ is real, and if α_i is complex then $\alpha_i = i\beta_i$ for some real β_i and $\bar{R_i}(x) = 1/R_i(x)$.

Any linear dependence $\sum k_i\alpha_i = 0$ over $\mathbf{Q}$ resolves itself into real and imaginary dependencies, and it is clear that we can use such a dependency to reduce the number of coefficients in (4.4) by a suitable choice of the $R_i(x)$, still retaining the properties in the paragraph above.

We can therefore assume that the α_i are linearly independent over $\mathbf{Q}$. If one of the α_i is rational, then denote the corresponding R_i by R_0 and decrease n by 1; otherwise we take $R_0 = 1$. By taking a suitable power of R_0 if necessary, we can assume that $\alpha_0 = 1/q$ for some integer q. From the properties of the $R_i(x)$ above, $R_0(x)$ must have real coefficients.

Rearranging, we take α_i real for $i = 0, \ldots, r$ and $\alpha_i = i\beta_i$ with β_i real for $i = r + 1, \ldots, n$.

We now can rewrite (4.3) in the form

$$\frac{P_0(x)}{P_1(x)} R_0(x)^{\alpha_0} \prod_{i=1}^{n} R_i(x)^{\alpha_i} \exp(R(x)) = \frac{P_0(z)}{P_1(z)} R_0(z)^{\alpha_0} \prod_{i=1}^{n} R_i(z)^{\alpha_i} \exp(R(z))$$

Rearranging, we get

$$S_0(x, z(x))^{1/q} \prod_{i=1}^{n} S_i(x, z(x))^{\alpha_i} e^{S(x,z(x))} = 1, \tag{4.5}$$

where

$$S_0(x, z) = \frac{P_0(x)^q R_0(x)/P_1(x)^q}{P_0(z)^q R_0(z)/P_1(z)^q}, \quad S_i(x, z) = \frac{R_i(x)}{R_i(z)}, \quad S(x, z) = R(x) - R(z),$$

with $i > 0$ above and where the α_i and $\alpha_0 = 1/q$ are linearly independent over $\mathbf{Q}$. We denote

$$\tilde{S_0}(x) = S_0(x, z(x)), \tilde{S_i}(x) = S_i(x, z(x)), \tilde{S}(x) = S(x, z(x)).$$

Differentiating (4.5) with respect to x, we obtain

$$(1/q)\tilde{S_0}'/\tilde{S_0} + \sum \alpha_i \tilde{S_i}'/\tilde{S_i} + \tilde{S}' = 0, \tag{4.6}$$

By Theorem 2, taking K to be the algebraic function field $K = \mathbf{C}(x)[z(x)]$ generated by $z(x)$ over $\mathbf{C}(x)$, we have $\tilde{S_i} \in \mathbf{C}$, $(i = 0, 1, \ldots)$, and $\tilde{S}(x) = cx + d$, with $c, d \in \mathbf{C}$. Substituting back into (4.6), we see that $c = 0$. Lastly, as $\tilde{S}(0) = S(0, z(0)) = R(0) - R(z(0)) = 0$, we see that $d = 0$, so $\tilde{S} = 0$ and $\tilde{S_i} = 1$, $(i = 1, 2, \ldots)$, thus from (4.5) we see that $\tilde{S_0} = 1$; whence we arrive at the following equations:

$$\frac{P_0(x)^q R_0(x)}{P_1(x)^q} = \frac{P_0(z)^q R_0(z)}{P_1(z)^q}, \quad R(x) = R(z), \quad R_i(x) = R_i(z)\ (i > 0). \tag{4.7}$$

Once again, consider the subfield of $\mathbf{R}(x)$ of all rational functions P over $\mathbf{R}$, such that $P(x) = P(z(x))$. Application of Lüroth's theorem gives a generator for this field, say $M(x) \in \mathbf{R}(x)$. Now, $P(x) = P(z(x))$ implies that $\bar{P}(x) = \bar{P}(z(x))$ since $z(x)$ can be considered as an element of $\mathbf{R}\{\{x\}\}$. Hence, if $P(x) = P(z(x))$, then both the real and imaginary parts of $P(x)$ must be expressible as rational functions of $M(x)$.

Thus, we can now write

$$R(x) = \psi(M(x)), \qquad R_i(x) = \chi_i(M(x)), \qquad (i = 1, 2, \dots) \tag{4.8}$$

where ψ and the χ_i, $i = 0, \dots, r$ are rational functions in x over $\mathbf{R}$. For $i = r+1, \dots, n$, χ_i lies in $\mathbf{C}(x)$; however, in this case, we know that $\bar{R}_i(x) = 1/R_i(x)$, and hence $\bar{\chi_i} = 1/\chi_i$, since $M(x)$ is real.

Without loss of generality, we can choose $M(x) = r(x)/s(x)$ with $r(0) = 0$ and $s(0) \neq 0$ and $r, s \in \mathbf{R}(x)$. Thus χ_i and ψ are well-defined and χ_i is non-zero at 0 from the corresponding conditions on R_i.

Since $M(x) = M(z)$, we have $M'(x)\,dx = M'(z)\,dz$ (where $M'(z)$ just means the derivative of M with respect to z), and therefore, $f(x)/M'(x) = f(z)/M'(z)$ and similarly for g, from (4.2). Now using the fact that

$$\exp(-\int_0^x P_2(x)/P_3(x)\,dx) = e^{R(x)} \prod_{i=1}^n R_i(x)^{\alpha_i}$$

and the relations given in (4.7), we obtain

$$\frac{P_0(x)R_0(x)^{2\alpha_0}}{P_3(x)M'(x)} = \frac{P_0(z)R_0(z)^{2\alpha_0}}{P_3(z)M'(z)}, \qquad \frac{P_1(x)R_0(x)^{\alpha_0}}{P_3(x)M'(x)} = \frac{P_1(z)R_0(z)^{\alpha_0}}{P_3(z)M'(z)}. \tag{4.9}$$

Whence, we deduce

$$\frac{P_0(x)/(P_3(x)M'(x))}{(P_1(x)/(P_3(x)M'(x)))^2} = \frac{P_0(z)/(P_3(z)M'(z))}{(P_1(z)/(P_3(z)M'(z)))^2},$$

which implies that,

$$\frac{P_0(x)/P_3(x)M'(x)}{(P_1(x)/P_3(x)M'(x))^2} = \xi(M(x)), \tag{4.10}$$

for some rational function $\xi \in \mathbf{R}(x)$.

Clearly, any power series $m(x) \in \mathbf{R}[[x]]$, for which $m(x) = m(z(x))$ must have the first non-zero terms of the expansion to be even since $z'(0) < 0$. Thus we take the first order terms of the series expansion of $M(x)$ to be $2k$, $k > 0$. In fact, we can choose M so that $r = x^{2k} + \cdots$ and $s = 1 + \cdots$ without loss of generality.

Let p be the order of $P_1(x)$ at $x = 0$, and consider the second equation of (4.9) as an equation with each side expanded in power series at $x = 0$. From the argument above, we must have $p + 1 - 2k$ even (recall that R_0 and P_3 do not vanish at $x = 0$), and hence p is odd. Similarly, evaluating the orders of the terms of equation (4.10) gives

$$2 - 2k - 2(p + 1 - 2k) = 2k\bar{\xi},$$

where $\bar{\xi}$ is the order at 0 of ξ. Hence from the above equality we get $p = k(1 - \bar{\xi}) = kl$. As p is odd so both k and l are odd so we write $l = 1 - \bar{\xi} = 2v + 1$, $v \neq 0$ and hence $\bar{\xi} = -2v$ and we write

$$\xi(x) = x^{-2v}\xi_0(x),$$

for some rational function ξ_0 over $\mathbf{R}$ with $\xi_0(0) \neq 0$. In fact $\xi_0(0) < 0$ from (4.10) and the hypotheses on $P(0)$ and $P(3)$.

Returning to (4.9), by taking q-th powers (to obtain rational functions with the property that $P(x) = P(z(x))$) we see that there are rational functions ζ and η over $\mathbf{R}$ such that

$$R_0(x)^2\left(\frac{P_0(x)}{P_3(x)M'(x)}\right)^q = \zeta(M(x)), \qquad R_0(x)\left(\frac{P_1(x)}{P_3(x)M'(x)}\right)^q = \eta(M(x)).$$

Comparing the orders in the second of these equations, we have

$$q(kl + 1 - 2k) = 2k\bar{\eta},$$

where $\bar{\eta}$ is the order at 0 of η. From the above equality we deduce that q is divisible by k and so we put $q = ku$ for some integer u. Thus $\bar{\eta}$ is given by

$$\bar{\eta} = u(vk + (1-k)/2) = vq + u(1-k)/2.$$

Now, we write $\eta(x) = \beta^q x^{(vq+u(1-k)/2)}\eta_0(x)$, where $\eta_0(0) = 1$. Thus

$$P_1(x)/P_3(x) = \beta M'(x)M(x)^v M(x)^{(1-k)/2k}\eta_0(M(x))^{1/q}R_0(x)^{-1/q}$$

and, from (4.10),

$$P_0(x)/P_3(x) = \beta^2 M'(x)M(x)^{(1-k)/k}\eta_0(M(x))^{2/q}R_0(x)^{-2/q}\xi_0(M(x)),$$

We have therefore shown that the system (4.1) can be written in the form (after scaling time by P_3)

$$\begin{aligned} \dot{x} = y, \qquad \dot{y} =& \beta^2 M'(x)(M(x))^{(1-k)/k}\xi_0(M(x))\left(\frac{\eta_0(M(x))}{R_0(x)}\right)^{2/q} \\ &+ y\beta M'(x)(M(x))^v(M(x))^{(1-k)/2k}\left(\frac{\eta_0(M(x))}{R_0(x)}\right)^{1/q} - y^2 H(x), \end{aligned} \tag{4.11}$$

where

$$H(x) = \left(\left(\psi'(M(x)) + \sum_{i=1}^{n}\alpha_i\frac{\chi_i'(M(x))}{\chi_i(M(x))}\right)M'(x) + \frac{R_0'(x)}{qR_0(x)}\right),$$

and therefore the system can be seen to have a center arising from the system

$$\begin{aligned} \dot{\bar{x}} = \bar{y}, \quad \dot{\bar{y}} =& k\beta^2\xi_0(\bar{x}^k) + k\beta\bar{y}\bar{x}^{vk+(k-1)/2} \\ &- \bar{y}^2\left(\psi'(\bar{x}^k) + \sum\alpha_i\frac{\chi_i'(\bar{x}^k)}{\chi_i(\bar{x}^k)} + \frac{\eta_0'(\bar{x}^k)}{q\eta_0(\bar{x}^k)}\right)k\bar{x}^{k-1}, \end{aligned} \tag{4.12}$$

via the singular transformation

$$(\bar{x}, \bar{y}) \mapsto \left(M(x)^{1/k}, y\left(\frac{R_0(x)}{\eta_0(M(x))}\right)^{1/q}\right).$$

Note that (4.12) is real, since for $i = r+1, \ldots, n$ we have $\bar{\chi}_i = 1/\chi_i$ and hence,

$$\alpha_i\frac{\chi_i'}{\chi_i} = \beta_i\left((\chi_i^R)'\chi_i^I - (\chi_i^I)'\chi_i^R\right),$$

where χ_i^R and χ_i^I are the real and imaginary parts of χ_i.

Alternatively, the system can seen to be symmetric under the reversing transformation $(x, y) \leftrightarrow (x_1, y_1)$, where

$$y\left(\frac{R_0(x)}{\eta_0(M(x))}\right)^{1/q} = y_1\left(\frac{R_0(x_1)}{\eta_0(M(x_1))}\right)^{1/q}, \qquad M(x) = M(x_1),$$

with $x_1(0) = 0$ and $x_1'(0) < 0$.

Example 1. The following system shows that we need to take the case $q > 1$ into account.

$$\dot{x} = y(1+x), \qquad \dot{y} = -x - x^2 - a_2x^3 - \frac{1}{9}a_2x^4 - a(1+\frac{1}{3}x)xy + \frac{2}{3}y^2. \tag{4.13}$$

For this system we have $R_0 = (1+x)^{-2}$, $q = 3$, $M = (x^2 + x^3/9)/(1+x)$ and $\eta_0 = (1+M/3)^{-1} = (1+x)/(1+x/3)^3$.

Example 2. Another system of interest is the following

$$\dot{x} = y(1+x), \qquad \dot{y} = -(6x+7x^2)(1+x) - \lambda y(6x^3+7x^4)(1+x) + \frac{1}{3}y^2, \tag{4.14}$$

with $R_0 = (1+x)^{-1}$, $q = r = 3$ and $M(x) = x^6 + x^7$ which shows that we can in fact have $r > 1$.

Of course it is not proved here that these systems cannot be derived in a different way from a polynomial system via a singular rational transformation, though it seems unlikely. If it is true, then it would indicate that in the study of polynomial systems with centers we need to include systems arising from singular algebraic transformations.

Step 4. $\mathbf{z(x)}$ is transcendental: As in Section 3, differentiating $(g/f)(x) = (g/f)(z)$ and using the first equality in (4.2), we have as in Section 3

$$\left[\frac{1}{f}\left(\frac{g}{f}\right)'\right](x) = \left[\frac{1}{f}\left(\frac{g}{f}\right)'\right](z).$$

This yields:

$$\frac{P_2(x)}{P_1(x)}\left(\frac{P_0(x)}{P_1(x)}\right) - \frac{P_3(x)}{P_1(x)}\left(\frac{P_0(x)}{P_1(x)}\right)' = \frac{P_2(y)}{P_1(y)}\left(\frac{P_0(y)}{P_1(y)}\right) - \frac{P_3(y)}{P_1(y)}\left(\frac{P_0(y)}{P_1(y)}\right)'. \tag{4.15}$$

Both sides of the equation are thus constants, c say. Let T denote P_1/P_0 and U denote P_1/P_3. Then (4.15) gives $P_2/P_3 = cTU - T'/T$. The system (4.1) can now be written in the form

$$\dot{x} = y, \qquad \dot{y} = \frac{U}{T} + yU + y^2(cTU - \frac{T'}{T}). \tag{4.16}$$

Considering the first terms of the series expansion of

$$\int_0^x f(x)\,dx = \int_0^z f(z)\,dz,$$

we must have $P_1(0) = 0$. Furthermore, from the expression for P_2/P_3, it is clear that if $T(x)$ vanishes at $x = 0$ then the hypothesis on P_3 will be violated. Thus $T(x)$ is non-zero at $x = 0$. This of course is in contrast to the algebraic case where P_1 could have a higher order of vanishing than P_0 at $x = 0$.

Consider the curve $y + \lambda T^{-1} = 0$. It is easy to verify that if λ verifies the equation $c\lambda^2 - \lambda + 1 = 0$, this curve defines a union of trajectories. Indeed we have in this case:

$$\frac{d}{dt}(y + \lambda T^{-1}) = (y + \lambda T^{-1})\left((cTU - \frac{T'}{T})y + \frac{U}{\lambda}\right).$$

The cofactor of the invariant curve $y + \lambda T^{-1} = 0$ is $(cTU - \frac{T'}{T})y + \frac{U}{\lambda}$. The above equation in λ has two solutions λ_1, λ_2 for $c \neq 1/4, 0$. In this case we get two such curves which are invariant. We also have as in Section 3 the exponential factor

$$C = \exp\left(\int_0^x P_2(\psi)\,d\psi\right),$$

which verifies $DC = C(cTU - T'/T)y$ and hence its cofactor is $(cTU - T'/T)y$.

From the two invariant curves corresponding to λ_1, λ_2 and the exponential factor C we can construct a first integral of the form

$$(y + \lambda_1 T^{-1})^{r_1}(y + \lambda_2 T^{-1})^{r_2}\exp(r_3\int(cTU - \frac{T'}{T})\,dx) = c,$$

where

$$r_1\left((cTU - \frac{T'}{T})y + \frac{U}{\lambda_1}\right) + r_2\left((cTU - \frac{T'}{T})y + \frac{U}{\lambda_2}\right) + r_3\left((cTU - \frac{T'}{T})y\right) = 0.$$

When $c = 1/4$, we take $\lambda = 2$ and get the trajectory $y + 2T^{-1} = 0$ as well as an exponential factor, $\exp(T^{-1}/(y + 2T^{-1}))$, such that

$$\frac{d}{dt}\exp(T^{-1}/(y + 2T^{-1})) = \exp(T^{-1}/(y + 2T^{-1}))(-U/4).$$

A first integral can then be constructed using the invariant curve $y + 2T^{-1} = 0$ and two exponential factors,

$$C = \exp\left(\int_0^x P_2(\psi)\,d\psi\right),$$

and $\exp(T^{-1}/(y+2T^{-1}))$. In a similar way, when $c=0$ we note that we have an exponential factor $\exp(Ty-\int TU\,dx)$ which satisfies

$$\frac{d}{dt}\exp(Ty-\int TU\,dx)=\exp(Ty-\int TU\,dx)U,$$

from which we can again construct an integrating factor as above.

Remarks: Again, we can check for these conditions computationally. First we see whether P_1 vanishes (case (i)), then we check to see if

$$\frac{P_2(x)}{P_1(x)}\left(\frac{P_0(x)}{P_1(x)}\right)-\frac{P_3(x)}{P_1(x)}\left(\frac{P_0(x)}{P_1(x)}\right)'$$

is a constant (case (iii)). Finally we calculate R and R_i and seek a common generator for the polynomials (4.7) (case (ii)).

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