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An Extension of Jack's Lemma to Polynomials of Fixed Degree

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Abstract. We study an extension of Jack's lemma to polynomials of fixed degree. Our result is based on a class of bound-preserving operators over polynomials introduced by Ruscheweyh.

Keywords. Jack's lemma, bound-preserving operators, extremal polynomials.

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1. Introduction and statement of the results

Let $r > 0$ and $\mathbb{D}_r$ denote the disc $\{z \in \mathbb{C} \mid |z| < r\}$. We write $\mathbb{D}_1 = \mathbb{D}$, the open unit disc and $\overline{\mathbb{D}}_r$ the closure of $\mathbb{D}_r$. Given a subset X of the complex plane $\mathbb{C}$ we also write $\mathcal{H}(X)$ the set of functions analytic in an open neighbourhood of X.

Jack's lemma may be formulated as follows: let $f \in \mathcal{H}(\mathbb{D})$, $0 < r < 1$ and $z_r \in \partial\mathbb{D}_r$ with

$$|f(z_r)| = \max_{|z| \le r} |f(z)| > 0.$$

Then

$$z_r \frac{f'(z_r)}{f(z_r)} \ge 0. \tag{1}$$

The result appeared in a paper by Jack [11] who attributed the proof to Clunie. The result is clearly equivalent to an older statement due to Loewner [13, p. 373]. Bak and Newman [2, p. 73] claim that (1) belongs to Erdös. A somewhat weaker statement is due to Hayman [10]. Ruscheweyh has shown [15] that the lemma follows from Julia's theorem. Yet, Miller and Mocanu [12] provided a detailed and analytical proof of (1) together with the statement

$$\operatorname{Re}\left(1 + z_r \frac{f''(z_r)}{f'(z_r)}\right) \ge z_r \frac{f'(z_r)}{f(z_r)}. \tag{2}$$

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The fact that the inequality is strict in (1) for any non-constant f has been proved by Bak and Newman [2] and also in [7]. Various refinements of the lemma, for example concerning univalent functions or taking into account certain Taylor coefficients of the function f, have been obtained by Dubinin and Kim [6] and also in [7]. Moreover, Jack's lemma keeps on being reinvented even up to present days as seen from [4, Lemma 2.5].

This is possibly explained by the basically geometrical nature of the results expressed by (1) and (2). Indeed a purely (almost!) geometrical proof of Jack's lemma can be obtained. Assuming first that $f(rz)$ is not a multiple of a finite Blaschke product (this means in particular that f is non-constant), it is easily seen that the set

$$\left\{ z_r \mid |z_r| = r \text{ and } |f(z_r)| = \max_{|z|\le r} |f(z)| \right\}$$

is finite and each such $z_r := re^{i\theta_r}$ is isolated. In particular,

$$\frac{d}{d\theta} \ell n |f(re^{i\theta})| = 0 \quad \text{if } \theta = \theta_r, \tag{3}$$

$$\frac{d^2}{d\theta^2} \ell n |f(re^{i\theta})| \le 0 \quad \text{if } \theta = \theta_r, \tag{4}$$

and

$$\frac{d}{d\theta} \arg f(re^{i\theta}) > 0 \quad \text{if } \theta \text{ is close enough to } \theta_r. \tag{5}$$

The conditions (3) and (4) are respectively first and second order conditions related to the maximality of $|f(z_r)|$ while (5) expresses the more or less evident fact that the curve

$$\{ f(re^{i\theta}) \mid \theta \text{ is close enough to } \theta_r \}$$

is "starlike" with respect to the origin. Mild computations now show that (3) and (5) imply (1) while (2) is a simple consequence of (4)! In the remaining case where $f(rz)$ is a finite Blaschke product, we obtain the remarkable fact that

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) = z\frac{f'(z)}{f(z)} \ge 0, \quad \text{for all } z \in \partial\mathbb{D}_r.$$

The main result of this paper is an extension of Jack's lemma suitable for complex polynomials of a given degree; let $\mathcal{P}_n$ denote the class of complex polynomials of degree at most $n \ge 2$ and let also

$$\mathcal{R} = \{ F \in \mathcal{H}(\mathbb{D}) \mid F(0) = 1 \text{ and } \operatorname{Re} F(z) > \tfrac{1}{2}, z \in \mathbb{D} \}.$$

For $q \in \mathcal{P}_{n-1} \cap \mathcal{R}$, let us define the monic polynomial $\tilde{q} \in \mathcal{P}_n$ with $\tilde{q}(0) = 0$ by

$$\tilde{q}(z) := z^n \overline{q(1/\bar{z})}.$$

Let finally $f \star g(z) := \sum_{k=0}^{\infty} a_k(f)a_k(g)z^k$ denote the Hadamard product (or convolution) of two functions $f(z) = \sum_{k=0}^{\infty} a_k(f)z^k$ and $g(z) := \sum_{k=0}^{\infty} a_k(g)z^k$ in $\mathcal{H}(\mathbb{D})$; clearly $f \star g$ also belongs to $\mathcal{H}(\mathbb{D})$. Our main result is

Theorem 1. *Let $p \in \mathcal{P}_n$, p non-constant, $q \in \mathcal{P}_{n-1} \cap \mathcal{R}$, $\theta \in [0, 2\pi)$ and $z \in \partial\mathbb{D}$ such that*

$$|p \star (q + e^{i\theta}\tilde{q})(z)| = |p|_{\mathbb{D}} = \max_{u\in\mathbb{D}} |p(u)|. \tag{6}$$

Then $p \star \tilde{q}(z) \neq 0$ and

$$0 \leq e^{-i\theta}\frac{p \star q(z)}{p \star \tilde{q}(z)}, \tag{7}$$

with equality if and only if p is a monomial of degree n.

The "Fejér" polynomials $q_n(z) := \sum_{k=0}^{n}(1-k/n)z^k$ belong [14, p. 508] to $\mathcal{P}_{n-1}\cap\mathcal{R}$ and a straightforward computation yields

$$q_n(z) + \tilde{q}_n(z) = \sum_{k=0}^{n} z^k.$$

Theorem 1 with $\theta = 0$ and $q \equiv q_n$ amounts to

$$0 \leq z\frac{p'(z)}{p(z)} \leq n \tag{8}$$

if $|z| = 1$ and $|p(z)| = |p|_{\mathbb{D}}$ for $p \in \mathcal{P}_n$. Further equality holds in (8) only if p is constant or else p is a monomial of degree n. This clearly shows that Theorem 1 is an extension of Jack's lemma for polynomials. In turn, it is not hard to obtain Jack's lemma for general analytic functions once it is known to be true for polynomials of arbitrary degrees.

1.1. Proof of Theorem 1. According to a fairly general result [16, p. 128] due to Ruscheweyh (this result extends and/or unifies several classical inequalities as Visser's or Bernstein's for algebraic polynomials), we have for p, q as in the hypothesis

$$|p \star q(z)| + |p \star \tilde{q}(z)| \leq |p|_{\mathbb{D}}, \quad z \in \overline{\mathbb{D}}$$

and therefore by (6)

$$|p|_{\mathbb{D}} = |p \star (q + e^{i\theta}\tilde{q})(z)| \leq |p \star q(z)| + |p \star \tilde{q}(z)| = |p|_{\mathbb{D}},$$

i.e.,

$$|p \star q(z) + e^{i\theta}p \star \tilde{q}(z)| = |p \star q(z)| + |p \star \tilde{q}(z)| = |p|_{\mathbb{D}}. \tag{9}$$

We further remark that $p \star \tilde{q}(z) \neq 0$; otherwise

$$|p \star q|_{\mathbb{D}} = |p \star q(z)| = |p|_{\mathbb{D}}$$

but this is ruled out for any non-constant p by a recent result [9, Lemma 5] due to Fournier and Lesage. It now follows clearly from (9) that

$$\left|1+e^{-i\theta}\frac{p\star q(z)}{p\star\tilde{q}(z)}\right| = 1+\left|\frac{p\star q(z)}{p\star\tilde{q}(z)}\right|,$$

which is equivalent to (7).

Let us now tackle the equality case in (7), which amounts to $p\star q(z)=0$. It then follows from (9) that

$$|p\star\tilde{q}(z)| = |p\star\tilde{q}|_{\mathbb{D}} = |p|_{\mathbb{D}}$$

and, because $|p\star\tilde{q}|_{\mathbb{D}} = |\tilde{p}\star q|_{\mathbb{D}}$ and $|\tilde{p}|_{\mathbb{D}} = |p|_{\mathbb{D}}$, we obtain

$$|\tilde{p}\star q|_{\mathbb{D}} = |\tilde{p}|_{\mathbb{D}}.$$

Again by the above quoted result of Fournier and Lesage, we obtain that the polynomial $\tilde{p}$ is constant, i.e. p is a monomial of exact degree n.

We shall end this section by giving an example of application of Theorem 1. The polynomials

$$Q_n(z) := 1+\sum_{k=1}^{n-1}\frac{1+\alpha}{2(k+\alpha)}z^k$$

are known to belong to $\mathcal{P}_{n-1}\cap\mathcal{R}$ for $-1<\alpha\leq\alpha_{\mathrm{MAX}}=4.567\ldots$. This is a result due to several mathematicians and the historical details may be found in [1, p. 10]. We may therefore state

Corollary 2. *Let $P(z):=\sum_{k=0}^{n}A_k(P)z^k$ be non-constant in $\mathcal{P}_n$ such that*

$$\left|A_0(P)+\sum_{k=1}^{n-1}\left(\frac{1}{k+\alpha}+\frac{e^{i\theta}}{n-k+\alpha}\right)\frac{1+\alpha}{2}A_k(P)Z^k+e^{i\theta}A_n(P)Z^n\right| = |P|_{\mathbb{D}}$$

for certain Z in $\partial\mathbb{D}$, θ real and admissible α. Then

$$0\leq e^{-i\theta}\frac{A_0(P)+\sum_{k=1}^{n-1}(1+\alpha)/2(k+\alpha)A_k(P)Z^k}{\sum_{k=1}^{n-1}(1+\alpha)/2(n-k+\alpha)A_k(P)Z^k+A_n(P)Z^n}$$

with equality if and only if $A_0(P)=A_1(P)=\cdots=A_{n-1}(P)=0$.

2. Some existential questions

Clearly Theorem 1 will be of some interest only if there exist "enough" polynomials $q\in\mathcal{P}_{n-1}\cap\mathcal{R}$ and $p\in\mathcal{P}_n$ such that for some real θ,

$$|p\star(q+e^{i\theta}\tilde{q})|_{\mathbb{D}} = |p|_{\mathbb{D}}. \tag{10}$$

For example, it is not clear at all which polynomials P satisfy the hypothesis of Corollary 2. It is easy to see that given $q\in\mathcal{P}_{n-1}\cap\mathcal{R}$, any polynomial of the type $p(z)=A+Bz^n$ will satisfy (10). We also know that for any admissible q,

we have a representation (see [14, Theorem 14.2.4] and [8, Theorem 1] for two different proofs)

$$q(z) + e^{i\theta}\tilde{q}(z) = \sum_{j=0}^{n-1} \frac{\ell_j}{1 - e^{(i\theta)/n} w_j z} + o(z^n) \tag{11}$$

where $w_j = e^{(2ij\pi)/n}$, $\ell_j = 2(\operatorname{Re} q(\bar{w}_j e^{-i\theta/n}) - \frac{1}{2})/n$ and $o(z^n)$ is a function of the type $z^{n+1}g(z)$ where $g \in \mathcal{H}(\mathbb{D})$. Using (11), it has been shown [8] that if all coefficients ℓ_j are strictly positive, except possibly for one coefficient when $n \geq 2$, then any polynomial p satisfying (10) must be of the type $p(z) \equiv A + Bz^n$ (clearly, this shall be the case when $\inf_{z\in\mathbb{D}} \operatorname{Re} q(z) > \frac{1}{2}$). On the other hand if

$$\ell_j = 0 \quad \text{for all } j \text{ except one,} \tag{12}$$

any polynomial $p \in \mathcal{P}_n$ satisfies (10).

We shall prove first that (12) can occur essentially only if $q \equiv q_n$. Indeed, (12) is equivalent with

$$q(z) + e^{i\theta}\tilde{q}(z) = \frac{1}{1 - e^{i\theta/n} w_j z} + o(z^n), \quad |z| < 1$$

for some j and θ; we may assume without any loss of generality that $j = \theta = 0$, i.e.,

$$q(z) + \tilde{q}(z) \equiv \sum_{k=0}^{n} z^k.$$

This implies that

$$\operatorname{Re}\left(e^{-in\varphi} q(e^{2i\varphi})\right) = \frac{\sin(n+1)\varphi}{2\sin\varphi} = \operatorname{Re}\left(e^{-in\varphi} q_n(e^{2i\varphi})\right)$$

for all $\varphi \in [0, 2\pi)$. This in turn means that the trigonometric polynomial of degree at most $n-2$

$$t_n(\varphi) := ie^{-in\varphi}\big(q(e^{2i\varphi}) - q_n(e^{2i\varphi})\big)$$

is real and admits a representation

$$t_n(\varphi) = \sum_{k=-n}^{n} a_k e^{ik\varphi}, \quad \text{where} \quad a_k = \bar{a}_{-k} \ (0 \leq k \leq n).$$

It follows that

$$q(z) - q_n(z) \equiv -iR(z)$$

where $R \in \mathcal{P}_n$ and $R(z) \equiv z^n \overline{R(1/\bar{z})}$. Since for such a self inversive polynomial R,

$$\cos\left(n\frac{\varphi}{2}\right) \operatorname{Im}\left(R(e^{i\varphi})\right) \equiv \sin\left(n\frac{\varphi}{2}\right) \operatorname{Re}\left(R(e^{i\varphi})\right), \quad \varphi \in [0, 2\pi),$$

we obtain, upon using the well-known identity

$$\operatorname{Re} q_n(e^{i\varphi}) = \frac{1}{2} + \frac{1}{2}\left(\frac{\sin(n\varphi)/2}{n\sin\varphi/2}\right)^2, \quad \varphi \in [0, 2\pi),$$

that

$$\operatorname{Re} q(e^{i\varphi_j}) - \tfrac{1}{2} = \operatorname{Re} q(e^{i\varphi_j}) - \operatorname{Re} q_n(e^{i\varphi_j}) = \operatorname{Im} R(e^{i\varphi_j}) = 0$$

if $\varphi_j = (2j\pi)/n$, $j \in \{1, 2, \ldots, n-1\}$. The positive trigonometric polynomial $\operatorname{Re} q(e^{i\varphi}) - \frac{1}{2}$, of degree at most $n-1$, therefore has double zeros precisely at the $n-1$ double zeros of $\operatorname{Re} q_n(e^{i\varphi}) - \frac{1}{2}$. It is now easy to show that $q(z) \equiv q_n(z)$.

It follows from (11) that

$$|p \star (q + e^{i\theta}\tilde{q})|_{\mathbb{D}} = \left|p(z) \star \left(q(z) + e^{i\theta}\tilde{q}(z)\right)\right| = |p|_{\mathbb{D}}$$

for some $z \in \partial\mathbb{D}$ if and only if

$$p(e^{(i\theta)/n} w_j z) = |p|_{\mathbb{D}} e^{i\varphi}$$

for a fixed real φ and all j, $0 \le j \le n-1$, such that $\ell_j \neq 0$ in the representation (11). The mere existence of non-trivial extremal polynomials p for (10) depends in some sense on the behaviour of the maximum modulus of p on the closed unit disc. We are not able to elucidate this behaviour in its full generality but we have partial results indicating that extremal polynomials for (10) are numerous.

We shall need two known results

Lemma 3. [3, p. 101] *Given integers $1 \le K \le N$ and distinct nodes $\{\zeta_k\}_{k=1}^K$, there exists $q \in \mathcal{P}_N \cap \mathcal{R}$ such that for any $z \in \partial\mathbb{D}$,*

$$\operatorname{Re} q(z) = \tfrac{1}{2} \Longleftrightarrow z \in \{\zeta_k\}_{k=1}^K$$

Lemma 4. [5, p. 559] *Given an integer $1 \le K$ and two sets of distinct nodes $\{\zeta_k\}_{k=1}^K$ and $\{\zeta'_k\}_{k=1}^K$ in $\partial\mathbb{D}$, there exists a univalent polynomial p such that*

$$|p(z)| < 1 \quad \textit{if} \quad z \in \partial\mathbb{D} \backslash \{\zeta_k\}_{k=1}^K$$

and

$$p(\zeta_k) = \zeta'_k, \quad 1 \le k \le K.$$

Let $K \ge 1$, θ real and $\{m_k\}_{k=0}^{K-1}$ a finite sequence of positive integers such that $1 \le m_0 < m_2 < \cdots < m_{K-1}$ with $m := \sum_{k=0}^{K-1} m_k$. By Lemma 4, there exists a polynomial P such that

$$|P|_{\mathbb{D}} = 1 \quad \text{and} \quad P\left(e^{i(\theta/K + (2k\pi)/K)}\right) = e^{2i\pi(m_k)/m}, \qquad 0 \le k < K,$$

(we shall not use the fact that P is univalent). Assuming $P \in \mathcal{P}_N$ for some $N \ge 1$, we set $p(z) := P(z)^m \in \mathcal{P}_{Nm} \subset \mathcal{P}_{KNm}$ and $n := KNm$; clearly

$$p(\zeta_k) = 1 = |p|_{\mathbb{D}}, \quad \zeta_k := \frac{\theta}{K} + \frac{2k\pi}{K}, \qquad 0 \le k \le K-1. \tag{13}$$

Further, by Lemma 4, there exists $q \in \mathcal{P}_{n-1} \cap \mathcal{R}$ such that

$$\operatorname{Re} q(e^{i\varphi}) = \frac{1}{2} \Longleftrightarrow \varphi = \frac{-\theta mN}{n} - \frac{2j\pi}{n}; \quad 0 \le j \le n-1, j \text{ not a multiple of } mN.$$

Therefore by (11),

$$q(z) + e^{imN\theta}\tilde{q}(z) = \sum_{j=0}^{n-1} \frac{\ell_j}{1 - e^{i\varphi_j} z} + o(z^n) \tag{14}$$

where

$$\ell_j > 0 \Longleftrightarrow j = kNm, \quad 0 \le k < K$$

and

$$e^{i\varphi_j} = \zeta_k, \quad 0 \le k < K.$$

It now follows from (13) and (14) that

$$|p \star q(1) + e^{imN\theta} p \star \tilde{q}(1)| = 1 = |p|_{\mathbb{D}}.$$

There are of course other (and perhaps less cumbersome) ways of constructing extremal polynomials. Let for example N, K be large enough integers and $n := NK$. By Lemma 3, there exists $q \in \mathcal{P}_{n-1} \cap \mathcal{R}$ such that

$$\operatorname{Re} q(z) = \tfrac{1}{2} \text{ for some } z \in \partial\mathbb{D} \Longleftrightarrow z = \bar{w}_j, \quad 0 \le j < n, \ N \text{ does not divide } j.$$

Then we have

$$q(z) + \tilde{q}(z) = \sum_{k=1}^{K-1} \frac{\lambda_k}{1 - W_k z} + o(z^n), \quad \lambda_k > 0 \ (0 \le k \le K-1)$$

where $\{W_k\}_{k=0}^{K-1}$ is the set of distinct Kth roots of unity. Let now $P \in \mathcal{P}_N$ with positive coefficients and $p(z) := P(z^K) \in \mathcal{P}_n$. It follows that

$$|q \star p(z) + \tilde{q} \star p(z)|_{z=1} = \sum_{k=0}^{\infty} \lambda_k P(W_k^K) = P(1) = p(1) = |p|_{\mathbb{D}}$$

i.e., for that choice q and $\theta = 0$, any such p is extremal.

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