An $L^2$ Inequality for Real Polynomials

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Abstract

Let $f(x) := \sum_{\nu=0}^{n} a_{\nu} x^{\nu}$ be a polynomial of degree at most $n$ such that $\int_{-1}^{1} (1-x^2)^{\lambda-1/2} \{ f(x) \}^2 \, dx = 1$, where $\lambda > -\frac{1}{2}$. In this paper we obtain the sharp upper bound for $|a_n| + \varepsilon |a_{n-1}|$ for any given $\varepsilon > 0$.

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1. Introduction and notation

Let $E$ be a Lebesgue measurable subset of the real line of positive measure. Furthermore, let $W : E \to [0, \infty)$ be a non-negative Lebesgue integrable function. For any given $p \in [1, \infty)$, we associate with any $f$ belonging to the linear space $P_n$ of all real polynomials of degree at most $n$ the quantity

$$
\|f\|_p := \left( \int_E W(x) |f(x)|^p \, dx \right)^{1/p}.
$$

(1)

It may be noted that the quantity $\|f\|_p$, just introduced, defines a norm on $P_n$. This because $1 \leq p < \infty$ and because a polynomial of degree at most $n$ cannot have more than $n$ zeros without being identically zero.

Let $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ be an arbitrary polynomial of degree at most $n$ such that $\int_{-1}^1 W(x) |f(x)|^2 \, dx = 1$, where $W(x)$ is non-negative on $[-1, 1]$ and the integrals $\int_{-1}^1 x^\nu W(x) \, dx$, $\nu = 0, 1, \ldots, n$ exist.

Now, let us recall some facts about the polynomials $\{P_n^{(\lambda)}\}$ orthogonal with respect to the weight function $W(x) := (1 - x^2)^{\lambda - \frac{1}{2}}$, $\lambda > -\frac{1}{2}$. These polynomials are called ultraspherical, and $P_n^{(\lambda)}$ is the same as the Jacobi polynomial $P_n^{(\alpha, \beta)}$ with $\alpha = \beta = \lambda - \frac{1}{2}$. If we take the normalization (see [5, p. 58]):

$$
P_n^{(\lambda)}(1) = \left( \frac{n + \alpha}{n} \right),
$$

the coefficient of the highest term $x^n$ becomes

$$
p_{n,n} = p_{n,n}^{(\lambda)} = \frac{\Gamma(2n + \alpha + \beta + 1)}{2^n n! \Gamma(n + \alpha + \beta + 1)} = \frac{\Gamma(2n + 2\lambda)}{2^n n! \Gamma(n + \lambda + 1)}.
$$

(2)

Since the coefficients of $P_n^{(\lambda)}$ are quite cumbersome we shall find it convenient to refer to them by certain well defined symbols. So, we write

$$
P_n^{(\lambda)}(x) = p_{n,n}^{(\lambda)} x^n - p_{n-2,n}^{(\lambda)} x^{n-2} + p_{n-4,n}^{(\lambda)} x^{n-4} + \ldots.
$$

It is well known that (see [5, p. 68])

$$
\|P_n^{(\lambda)}\|_2^2 := \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} (P_n^{(\lambda)}(x))^2 \, dx = \frac{2^{2\lambda - 1} (\Gamma(\lambda + n + 1/2))^2}{(n + \lambda) n! \Gamma(n + 2\lambda)}.
$$

(3)

It is sometimes more convenient to normalize the ultraspherical polynomials so that their weighted $L^2$ norm on $[-1, 1]$ be equal to 1. We shall do this by introducing the polynomial

$$
P_n^{(\lambda)*}(x) := \sqrt{\frac{n! (n + \lambda) \Gamma(n + 2\lambda)}{2^{2\lambda - 1} (\Gamma(\lambda + n + 1/2))^2}} P_n^{(\lambda)}(x) \quad (n = 0, 1, 2, \ldots)
$$


$$
= p_{n,n}^{\ast, (\lambda)} x^n - p_{n-2,n}^{\ast, (\lambda)} x^{n-2} + p_{n-4,n}^{\ast, (\lambda)} x^{n-4} + \ldots.
$$

(4)

For sake of simplicity, we shall hereafter write $p_{n,n}^{\ast, (\lambda)}$, $p_{n-2,n}^{\ast, (\lambda)}$, $p_{n-4,n}^{\ast, (\lambda)}$ instead of $p_{n,n}^{(\lambda)}$, $p_{n-2,n}^{(\lambda)}$, $p_{n-4,n}^{(\lambda)}$.

From the theory of orthogonal polynomials we know how large the highest coefficient $a_n$ of $f$ can be. In fact,

$$
|a_n| \leq p_{n,n}^{\ast} \left( \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} \{f(x)\}^2 \, dx \right)^{\frac{1}{2}}.
$$

(5)

We may just as well wish to know the sharp upper bound for any other coefficient of $f$. It was proved by Pierre and Rahman (see [2] or [3, Chapter 16]) that if $f$ takes only real values for real values of $x$, then

$$
|a_{n-2k}| + |a_{n-2k-1}| \leq \max_{-1 \leq x \leq 1} |f(x)| \left( k = 0, 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \right),
$$

(6)
where \( t_{n-2k,n} \) denotes the coefficient of \( x^{n-2k} \) in the Maclaurin development of \( T_n \), the Chebyshev polynomials of the first kind of degree \( n \). It is also reasonable to ask how large \( |a_j| + |a_k| \) can be for any given \( j \) and \( k \) such that \( 0 \leq j \neq k \leq n \). Such a question rarely has a simple answer. Little is known about analogous problems for polynomials \( f \) of degree at most \( n \) such that \( \int_{-1}^{1} |f(x)|^p \, dx = 1 \), where \( p \) is a given positive number. Rogosinski [4] presents a theory, which applies to the case where \( p \geq 1 \). Therein he characterizes the polynomials (called extremals) for which the desired upper bound in any of these problems is attained. This could be helpful in putting one’s finger on such a polynomial.

Let \((\lambda_0, \ldots, \lambda_n)\) be a vector in \( \mathbb{R}^{n+1} \). Then

\[
    f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu \mapsto \lambda_0 a_0 + \cdots + \lambda_n a_n \tag{7}
\]

defines a continuous linear functional \( \mathcal{I} \) on the normed linear space \( \mathcal{P}_n \) of all real polynomials of degree at most \( n \) with \( \|f\|_p^\dagger \) as defined in (1). Rogosinski [4] proved the following theorem.

**Theorem A.** As above, let \( \mathcal{P}_n \) be the linear space of all polynomials of degree at most \( n \). Furthermore, let \( \mathcal{I} \) be the functional on \( \mathcal{P}_n \) defined by the vector \((\lambda_0, \ldots, \lambda_n)\) \( \in \mathbb{R}^{n+1} \) in the sense that \( \mathcal{I}(f) = \sum_{\nu=0}^{n} \lambda_\nu a_\nu \) if \( f(x) = \sum_{\nu=0}^{n} a_\nu x^\nu \). In addition, let

\[
    \|\mathcal{I}\| = \|\mathcal{I}\|_{\mathcal{P}_n} := \sup \left\{ \mathcal{I}(f) : \|f\|_p^\dagger \leq 1 \right\} . \tag{8}
\]

Then \( \mathcal{I} \) has a given polynomial \( F \in \mathcal{P}_n \) with \( \|F\|_p^\dagger = 1 \) as extremal, i.e. \( \mathcal{I}(F) = \|\mathcal{I}\| \), if and only if

\[
    \lambda_k = \|\mathcal{I}\| \int_E x^k W(x) \cdot \text{sign} F(x) \cdot |F(x)|^{p-1} \, dx \quad (k = 0, 1, \ldots, n) . \tag{9}
\]

In the case where \( p = 2 \), condition (9) reduces to

\[
    \lambda_k = \|\mathcal{I}\| \int_E x^k W(x) F(x) \, dx \quad (k = 0, 1, \ldots, n) . \tag{9'}
\]

In view of (6), one may wonder if the quantity appearing in the left-hand side of (5) can be replaced by \( |a_n| + |a_{n-1}| \). The answer is no at least for \( \lambda = \frac{1}{2} \) as the inequality (see[1, p. 26])

\[
    |a_n| + \varepsilon |a_{n-1}| \leq \sqrt{\left(1 + \varepsilon^2 \frac{n^2}{4n^2 - 1}\right) \frac{2n + 1}{2} \frac{(2n)!}{2^n(n!)^2} \left( \int_{-1}^{1} |f(x)|^2 \, dx \right)^{1/2}} \tag{10}
\]

shows, where the estimate is sharp for each \( \varepsilon \).

It is natural to wonder what happens for other value of \( \lambda \). With the help of Theorem A, we find the sharp analogue of (10) for any \( \lambda > -\frac{1}{2} \). This is presented in Theorem 1 below.

**2. The main result and its proof.**

The following result is the sharp version of (10) for any \( \lambda > -\frac{1}{2} \).

**Theorem 1.** For any non-zero \( \lambda > -\frac{1}{2} \), let \( P_{n,n}^{\lambda \varepsilon}(x) \) be as in (4), where

\[
    p_{n,n}^{\lambda \varepsilon}(\lambda) = \frac{\Gamma(2\lambda + 2n)}{2^{n+\lambda-1/2}\Gamma(\lambda + n + 1/2)} \sqrt{\frac{n + \lambda}{n! \Gamma(n + 2\lambda)}} . \tag{11}
\]

Then for any real polynomial \( f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu \) of degree \( n \) and any \( \varepsilon \geq 0 \), we have

\[
    |a_n| + \varepsilon |a_{n-1}| \leq \sqrt{1 + \varepsilon^2 \frac{n(n + 2\lambda - 1)}{4(n + \lambda - 1)(n + \lambda)} p_{n,n}^{\lambda \varepsilon} \|f\|_2^\dagger} , \tag{12}
\]
where \( \|f\|_2^\dagger := (\int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} \{f(x)\}^2 \, dx)^{1/2} \). For any given \( \varepsilon \geq 0 \), the inequality becomes an equality for constant multiples of \( f(x) := P_n^{(\lambda)}(x) + \varepsilon P_{n-1}^{(\lambda)}(x) \).

**Proof.** Our proof of this result uses the case \( p = 2 \) of Theorem A. For any \( \delta \in \mathbb{R} \) let \( f_\delta(x) := P_n^{(\lambda)}(x) + \delta P_{n-1}^{(\lambda)}(x) \). Then, because

\[
\int_{-1}^1 x^k (1 - x^2)^{\lambda - \frac{1}{2}} P_n^{(\lambda)}(x) \, dx = 0 \quad (k = 0, \ldots, n - 1),
\]

we have

\[
\left( \|f_\delta\|_2^\dagger \right)^2 := \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} \{f_\delta(x)\}^2 \, dx
\]

\[
= \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} \{P_n^{(\lambda)}(x)\}^2 \, dx + \delta^2 \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} \{P_{n-1}^{(\lambda)}(x)\}^2 \, dx
\]

\[
= 1 + \delta^2,
\]

so that

\[
f_\delta(x) := \frac{1}{\sqrt{1 + \delta^2}} f_\delta(x) = \frac{1}{\sqrt{1 + \delta^2}} \left\{ P_n^{(\lambda)}(x) + \delta P_{n-1}^{(\lambda)}(x) \right\}
\]

is a polynomial of degree \( n \) with \( \|f_\delta\|_2^\dagger = 1 \). Because of the orthogonality property of \( P_n^{(\lambda)} \), we have

\[
\int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} x^n f_\delta^*(x) \, dx = \frac{1}{\sqrt{1 + \delta^2}} \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} x^n P_n^{(\lambda)}(x) \, dx
\]

\[
= \frac{1}{\sqrt{1 + \delta^2}} \frac{1}{P_{n,n}^*},
\]

\[
\int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} x^{n-1} f_\delta^*(x) \, dx = \frac{\delta}{\sqrt{1 + \delta^2}} \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} x^{n-1} P_{n-1}^{(\lambda)}(x) \, dx
\]

\[
= \frac{\delta}{\sqrt{1 + \delta^2}} \frac{1}{P_{n-1,n-1}^*},
\]

and

\[
\int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} x^k f_\delta^*(x) \, dx = 0 \quad (k = 0, \ldots, n - 2).
\]

Then with \( A := \sqrt{1 + \delta^2} P_{n,n}^* \), we have

\[
\begin{cases}
A \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} x^k f_\delta^*(x) \, dx = 0 \quad (k = 0, \ldots, n - 2) \\
A \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} x^{n-1} f_\delta^*(x) \, dx = \delta \frac{P_{n,n}^*}{P_{n-1,n-1}^*} = \delta \sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}} \\
A \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} x^n f_\delta^*(x) \, dx = 1
\end{cases}
\]

Comparing (13) with (9’) we readily see that \( f_\delta^* \) is extremal for the functional

\[
\mathcal{I} : \sum_{\nu=0}^n a_\nu x^\nu \mapsto a_n + \delta \sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}} a_{n-1}
\]

on the normed linear space \( \mathcal{P}_n \) of all polynomials \( f(x) := \sum_{\nu=0}^n a_\nu x^\nu \) of degree at most \( n \) with \( \|f\|_2^\dagger := \left( \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} \{f(x)\}^2 \, dx \right)^{1/2} \), and that the norm of \( \mathcal{I} \) is \( A := \sqrt{1 + \delta^2} P_{n,n}^* \). Thus

\[
a_n + \delta \sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}} a_{n-1} \leq \sqrt{1 + \delta^2} P_{n,n}^*.
\]
Here \( \delta \) may be positive or negative, and so

\[
|a_n| + |\delta|\sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}}|a_{n-1}| \leq \sqrt{1 + \delta^2} p^*_{n,n}.
\]

Now replace \(|\delta|\sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}}\) by \(\varepsilon\); we obtain (12).

\[\square\]

**Remark.** Note that, (12) can be seen as a generalization of (10), we just have to put \(\lambda = \frac{1}{2}\). Dividing the two sides of inequality (12) by \(\varepsilon\) and letting \(\varepsilon\) tend to \(\infty\), we obtain

\[
|a_{n-1}| \leq \sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}} p^*_{n,n} \|f\|_2^{1/2} = p^*_{n-1,n-1} \left( \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} \{f(x)\}^2 \, dx \right)^{1/2},
\]

where the inequality becomes an equality for constant multiples of \(P^*_{n-1}\).

Taking \(\varepsilon = 0\) we find the classical inequality \(|a_n| \leq p^*_{n,n}(\|f\|_2^{1/2})^{1/2}\). The sharpness of (12) shows that the left-hand side of this last inequality cannot be replaced by \(|a_n| + |a_{n-1}|\). As a matter of fact, it cannot be replaced by \(|a_n| + \varepsilon|a_{n-1}|\) for any positive \(\varepsilon\).

**References**


