An L^2 Inequality for Real Polynomials

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Abstract

Let $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ be a polynomial of degree at most n such that $\int_{-1}^{1}(1-x^2)^{\lambda-1/2}\{f(x)\}^2\,\mathrm{d}x = 1$, where $\lambda > -\frac{1}{2}$. In this paper we obtain the sharp upper bound for $|a_n| + \varepsilon\,|a_{n-1}|$ for any given $\varepsilon > 0$.

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1. Introduction and notation

Let E be a Lebesgue measurable subset of the real line of positive measure. Furthermore, let $W : E \to [0, \infty)$ be a non-negative Lebesgue integrable function. For any given $p \in [1, \infty)$, we associate with any f belonging to the linear space $\mathcal{P}_n$ of all real polynomials of degree at most n the quantity

$$\|f\|_p^\dagger := \left\{ \int_E W(x)\, |f(x)|^p \, \mathrm{d}x \right\}^{1/p} . \tag{1}$$

It may be noted that the quantity $\|f\|_p^\dagger$, just introduced, defines a norm on $\mathcal{P}_n$. This because $1 \le p < \infty$ and because a polynomial of degree at most n cannot have more than n zeros without being identically zero.

Let $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ be an arbitrary polynomial of degree at most n such that $\int_{-1}^1 W(x)\, |f(x)|^2 \, \mathrm{d}x = 1$, where $W(x)$ is non-negative on $[-1, 1]$ and the integrals $\int_{-1}^1 x^\nu W(x)\, \mathrm{d}x$, $\nu = 0, 1, \ldots, n$ exist.

Now, let us recall some facts about the polynomials $\{P_n^{(\lambda)}\}$ orthogonal with respect to the weight function $W(x) := (1 - x^2)^{\lambda - \frac{1}{2}}$, $\lambda > -\frac{1}{2}$. These polynomials are called ultraspherical, and $P_n^{(\lambda)}$ is the same as the Jacobi polynomial $P_n^{(\alpha,\beta)}$ with $\alpha = \beta = \lambda - \frac{1}{2}$. If we take the normalization (see [5, p. 58]):

$$P_n^{(\lambda)}(1) = \binom{n + \alpha}{n} ,$$

the coefficient of the highest term x^n becomes

$$p_{n,n} = p_{n,n}(\lambda) = \frac{\Gamma(2n + \alpha + \beta + 1)}{2^n n!\, \Gamma(n + \alpha + \beta + 1)} = \frac{\Gamma(2n + 2\lambda)}{2^n n!\, \Gamma(n + 2\lambda)} . \tag{2}$$

Since the coefficients of $P_n^{(\lambda)}$ are quite cumbersome we shall find it convenient to refer to them by certain well defined symbols. So, we write

$$P_n^{(\lambda)}(x) = p_{n,n} x^n - p_{n-2,\,n} x^{n-2} + p_{n-4,\,n} x^{n-4} + \ldots .$$

It is well known that (see [5, p. 68])

$$\|P_n^{(\lambda)}\|_2^{\dagger\,2} := \int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} (P_n^{(\lambda)}(x))^2 \, \mathrm{d}x = \frac{2^{2\lambda - 1} (\Gamma(\lambda + n + 1/2))^2}{(n + \lambda)\, n!\, \Gamma(n + 2\lambda)} . \tag{3}$$

It is sometimes more convenient to normalize the ultraspherical polynomials so that their weighted L^2 norm on $[-1, 1]$ be equal to 1. We shall do this by introducing the polynomial

$$\begin{aligned} P_n^{(\lambda)*}(x) &:= \sqrt{\frac{n!\, (n + \lambda)\, \Gamma(n + 2\lambda)}{2^{2\lambda - 1} (\Gamma(\lambda + n + \frac{1}{2}))^2}}\, P_n^{(\lambda)}(x) \qquad (n = 0, 1, 2, \ldots) \\ &= p_{n,n}^*(\lambda) x^n - p_{n-2,\,n}^*(\lambda) x^{n-2} + p_{n-4,\,n}^*(\lambda) x^{n-4} + \ldots . \end{aligned} \tag{4}$$

For sake of simplicity, we shall hereafter write $p_{n,n}^*$, $p_{n-2,\,n}^*$, $p_{n-4,\,n}^* \ldots$ instead of $p_{n,n}^*(\lambda)$, $p_{n-2,\,n}^*(\lambda)$, $p_{n-4,\,n}^*(\lambda) \ldots$.

From the theory of orthogonal polynomials we know how large the highest coefficient a_n of f can be. In fact,

$$|a_n| \le p_{n,n}^* \left(\int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} \{f(x)\}^2 \, \mathrm{d}x \right)^{\frac{1}{2}} . \tag{5}$$

We may just as well wish to know the *sharp* upper bound for any other coefficient of f. It was proved by Pierre and Rahman (see [2] or [3, Chapter 16]) that if f takes only real values for real values of x, then

$$|a_{n-2k}| + |a_{n-2k-1}| \le |t_{n-2k,n}| \max_{-1 \le x \le 1} |f(x)| \qquad \left(k = 0, 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \right) , \tag{6}$$

where $t_{n-2k,n}$ denotes the coefficient of x^{n-2k} in the Maclaurin development of T_n, the Chebyshev polynomials of the first kind of degree n. It is also reasonable to ask how large $|a_j|+|a_k|$ can be for any given j and k such that $0 \le j \ne k \le n$. Such a question rarely has a simple answer. Little is known about analogous problems for polynomials f of degree at most n such that $\int_{-1}^{1} |f(x)|^p \, \mathrm{d}x = 1$, where p is a given positive number. Rogosinski [4] presents a theory, which applies to the case where $p \ge 1$. Therein he characterizes the polynomials (called *extremals*) for which the *desired upper bound* in any of these problems is attained. This could be helpful in putting one's finger on such a polynomial.

Let $(\lambda_0, \ldots, \lambda_n)$ be a vector in $\mathbb{R}^{n+1}$. Then

$$f(x) := \sum_{\nu=0}^{n} a_\nu \, x^\nu \mapsto \lambda_0 a_0 + \cdots + \lambda_n a_n \tag{7}$$

defines a continuous linear functional $\mathcal{I}$ on the normed linear space $\mathcal{P}_n$ of all real polynomials of degree at most n with $\|f\|_p^\dagger$ as defined in (1). Rogosinki [4] proved the following theorem.

Theorem A. *As above, let $\mathcal{P}_n$ be the linear space of all polynomials of degree at most n. Furthermore, let $\mathcal{I}$ be the functional on $\mathcal{P}_n$ defined by the vector $(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1}$ in the sense that $\mathcal{I}(f) = \sum_{\nu=0}^{n} \lambda_\nu \, a_\nu$ if $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$. In addition, let*

$$\|\mathcal{I}\| = \|\mathcal{I}\|_{\mathcal{P}_n} := \sup \left\{ \mathcal{I}(f) : \|f\|_p^\dagger \le 1 \right\} . \tag{8}$$

Then $\mathcal{I}$ has a given polynomial $F \in \mathcal{P}_n$ with $\|F\|_p^\dagger = 1$ as extremal, i.e. $\mathcal{I}(F) = \|\mathcal{I}\|$, if and only if

$$\lambda_k = \|\mathcal{I}\| \int_E x^k \, W(x) \cdot \operatorname{sign} F(x) \cdot |F(x)|^{p-1} \, \mathrm{d}x \qquad (k = 0, 1, \ldots, n) . \tag{9}$$

In the case where $p = 2$, condition (9) reduces to

$$\lambda_k = \|\mathcal{I}\| \int_E x^k \, W(x) \, F(x) \, \mathrm{d}x \qquad (k = 0, 1, \ldots, n) . \tag{9'}$$

□

In view of (6) , one may wonder if the quantity appearing in the left-hand side of (5) can be replaced by $|a_n| + |a_{n-1}|$. The answer is no at least for $\lambda = \frac{1}{2}$ as the inequality (see[1 , p. 26])

$$|a_n| + \varepsilon |a_{n-1}| \le \sqrt{\left(1 + \varepsilon^2 \frac{n^2}{4n^2 - 1}\right) \frac{2n+1}{2}} \frac{(2n)!}{2^n (n!)^2} \left(\int_{-1}^{1} |f(x)|^2 \, \mathrm{d}x \right)^{1/2} \tag{10}$$

shows , where the estimate is sharp for each ε .

It is natural to wonder what happens for other value of λ . With the help of Theorem A, we find the sharp analogue of (10) for any $\lambda > -\frac{1}{2}$. This is presented in Theorem 1 below.

2. The main result and its proof

The following result is the sharp version of (10) for any $\lambda > -\frac{1}{2}$.

Theorem 1. *For any non-zero $\lambda > -\frac{1}{2}$, let $P_{n,n}^{\lambda *}(x)$ be as in (4) , where*

$$p_{n,n}^* := p_{n,n}^*(\lambda) = \frac{\Gamma(2\lambda + 2n)}{2^{n+\lambda-1/2} \Gamma(\lambda + n + 1/2)} \sqrt{\frac{n+\lambda}{n! \, \Gamma(n + 2\lambda)}} . \tag{11}$$

Then for any real polynomial $f(x) := \sum_{\nu=0}^{n} a_\nu \, x^\nu$ of degree n and any $\varepsilon \ge 0$, we have

$$|a_n| + \varepsilon |a_{n-1}| \le \sqrt{1 + \varepsilon^2 \frac{n(n + 2\lambda - 1)}{4(n + \lambda - 1)(n + \lambda)}} \, p_{n,n}^* \, \|f\|_2^\dagger , \tag{12}$$

where $\|f\|_2^\dagger := (\int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\,\{f(x)\}^2\,\mathrm{d}x)^{1/2}$. *For any given* $\varepsilon \geq 0$, *the inequality becomes an equality for constant multiples of* $f(x) := P_n^{(\lambda)*}(x) + \varepsilon\, P_{n-1}^{(\lambda)*}(x)$.

Proof. Our proof of this result uses the case $p = 2$ of Theorem A. For any $\delta \in \mathbb{R}$ let $f_\delta(x) := P_n^{(\lambda)*}(x) + \delta\, P_{n-1}^{(\lambda)*}(x)$. Then, because

$$\int_{-1}^1 x^k\,(1-x^2)^{\lambda-\frac{1}{2}} P_n^{(\lambda)*}(x)\,\mathrm{d}x = 0 \qquad (k = 0, \ldots, n-1)\,,$$

we have

$$\begin{aligned}\Big(|f_\delta\|_2^\dagger\Big)^2 &:= \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\,\{f_\delta(x)\}^2\,\mathrm{d}x \\ &= \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\{P_n^{(\lambda)*}(x)\}^2\,\mathrm{d}x + \delta^2 \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\{P_{n-1}^{(\lambda)*}(x)\}^2\,\mathrm{d}x \\ &= 1+\delta^2\,,\end{aligned}$$

so that

$$f_\delta^*(x) := \frac{1}{\sqrt{1+\delta^2}} f_\delta(x) = \frac{1}{\sqrt{1+\delta^2}}\left\{P_n^{(\lambda)*}(x) + \delta\, P_{n-1}^{(\lambda)*}(x)\right\}$$

is a polynomial of degree n with $\|f_\delta^*\|_2^\dagger = 1$. Because of the orthogonality property of $P_n^{(\lambda)*}$, we have

$$\begin{aligned}\int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\; x^n\, f_\delta^*(x)\,\mathrm{d}x &= \frac{1}{\sqrt{1+\delta^2}} \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\; x^n\, P_n^{(\lambda)*}(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{1+\delta^2}}\,\frac{1}{p_{n,n}^*}\,,\end{aligned}$$

$$\begin{aligned}\int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\; x^{n-1}\, f_\delta^*(x)\,\mathrm{d}x &= \frac{\delta}{\sqrt{1+\delta^2}} \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\; x^{n-1}\, P_{n-1}^{(\lambda)*}(x)\,\mathrm{d}x \\ &= \frac{\delta}{\sqrt{1+\delta^2}}\,\frac{1}{p_{n-1,n-1}^*}\,,\end{aligned}$$

and

$$\int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\; x^k\, f_\delta^*(x)\,\mathrm{d}x = 0 \qquad (k = 0, \ldots, n-2)\,.$$

Then with $A := \sqrt{1+\delta^2}\, p_{n,n}^*$, we have

$$\left.\begin{array}{r} A\int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\; x^k\, f_\delta^*(x)\,\mathrm{d}x = 0 \quad (k = 0, \ldots, n-2) \\ A\int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\; x^{n-1}\, f_\delta^*(x)\,\mathrm{d}x = \delta\,\frac{p_{n,n}^*}{p^*_{n-1,n-1}} = \delta\sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}} \\ A\int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\; x^n\, f_\delta^*(x)\,\mathrm{d}x = 1 \end{array}\right\}. \tag{13}$$

Comparing (13) with (9′) we readily see that f_δ^* is extremal for the functional

$$\mathcal{I}\;:\;\sum_{\nu=0}^n a_\nu x^\nu \mapsto a_n + \delta\sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}}\,a_{n-1}$$

on the normed linear space $\mathcal{P}_n$ of all polynomials $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ of degree at most n with $\|f\|_2^\dagger := \Big(\int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}}\{f(x)\}^2\,\mathrm{d}x\Big)^{1/2}$, and that the norm of $\mathcal{I}$ is $A := \sqrt{1+\delta^2}\, p_{n,n}^*$. Thus

$$a_n + \delta\sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}}\,a_{n-1} \leq \sqrt{1+\delta^2}\, p_{n,n}^*\,.$$

Here δ may be positive or negative, and so

$$|a_n| + |\delta|\sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}}|a_{n-1}| \le \sqrt{1+\delta^2}\, p^*_{n,n}\,.$$

Now replace $|\delta|\sqrt{\dfrac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}}$ by ε; we obtain (12). $\square$

Remark . Note that, (12) can be seen as a generalization of (10), we just have to put $\lambda = \frac{1}{2}$. Dividing the two sides of inequality (12) by ε and letting ε tend to ∞, we obtain

$$\begin{aligned} |a_{n-1}| &\le \sqrt{\frac{4(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}}p^*_{n,n}\,\|f\|_2^{\dagger\,1/2} \\ &= p^*_{n-1,n-1}\left(\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}\{f(x)\}^2\,\mathrm{d}x\right)^{1/2}, \end{aligned}$$

where the inequality becomes an equality for constant multiples of P^*_{n-1}.

Taking $\varepsilon = 0$ we find the classical inequality $|a_n| \le p^*_{n,n}(\|f\|_2^\dagger)^{1/2}$. The sharpness of (12) shows that the left-hand side of this last inequality cannot be replaced by $|a_n| + |a_{n-1}|$. As a matter of fact, it cannot be replaced by $|a_n| + \varepsilon\,|a_{n-1}|$ for *any* positive ε.

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