Prime Factorization of Entire Functions*

Xinhou Hua†‡ Rémi Vaillancourt†§

CRM-3211

January 2006

*This research was partially supported by the Natural Sciences and Engineering Research Council of Canada and the Centre de recherches mathématiques of the Université de Montréal.

†Department of Mathematics and Statistics, University of Ottawa, Ottawa, ON K1N 6N5, Canada

‡hua@mathstat.uottawa.ca

§remi@uottawa.ca

Abstract

Let n be a prime number and let $f(z)$ be a transcendental entire function. Then it is proved that both $[f(z)+cz]^n$ and $[f(z)+cz]^{-n}$ are uniquely factorizable for any complex number c, except for a countable set in $\mathbb{C}$.

Key words and phrases: entire function, unique factorization

Mathematical Subject Classification: 30D05

Resumen

Sea n un número primo y $f(z)$ una función entera transcendental. Entonces ambos $[f(z) + cz]^n$ y $[f(z) + cz]^{-n}$ se factorizan de manera única para cualquier número complejo c, excepto para un conjunto numerable in $\mathbb{C}$.

Résumé

Soit n un entier premier et $f(z)$ une fonction entière transcendente. Alors $[f(z) + cz]^n$ et $[f(z) + cz]^{-n}$ admettent une factorisatiton unique pour presque tout $c \in \mathbb{C}$ sauf pour un ensemble dénombrable en $\mathbb{C}$.

1 Introduction

The fundamental theorem of elementary number theory states that every integer $n \geq 2$ can be expressed uniquely as the product of primes in the form

$$n = p_1^{m_1} \cdots p_k^{m_k}, \qquad \text{for} \quad k \geq 1,$$

with distinct prime factors $p_1, \ldots, p_k$ and corresponding exponents $m_1 \geq 1, \ldots, m_k \geq 1$ uniquely determined by n. For example, $2700 = 2^2 3^3 5^2$.

In 1922, Ritt ([14]) generalized this theorem to polynomials. To state his result, we introduce the following concepts.

Let $F(z)$ be a nonconstant meromorphic function. A decomposition

$$F(z) = f(g(z)) = f \circ g(z) \tag{1}$$

will be called a factorization of $F(z)$ with $f(z)$ and $g(z)$ being the left and right factors of $F(z)$, respectively, where $f(z)$ is meromorphic and $g(z)$ is entire ($g(z)$ may be meromorphic when $f(z)$ is rational) (see [2], [4], [19]).

A function $F(z)$ is said to be prime (pseudo-prime) if $F(z)$ is nonlinear and every factorization of the form (1) implies that either $f(z)$ is fractional linear or $g(z)$ is linear (either $f(z)$ is rational or $g(z)$ is a polynomial).

Example 1 *$e^z + z$ is prime.*

This is stated by Rosenbloom [15] and proved by Gross [3].

Example 2 *$(\cos z)e^{az+b} + p(z)$ is prime, where a $(\neq 0)$ and b are constants, and $p(z)$ is a nonconstant polynomial.*

This was conjectured by Gross–Yang [5] and proved by Hua [7].

Suppose that a function $F(z)$ has two prime factorizations

$$F(z) = f_1 \circ \cdots \circ f_m(z) = g_1 \circ \cdots \circ g_n(z),$$

i.e., f_i $(i = 1, \ldots, m)$ and g_j $(j = 1, \ldots, n)$ are prime functions. If $m = n$ and if there exist linear functions L_j $(j = 1, \ldots, n-1)$ such that

$$f_1(z) = g_1 \circ L_1^{-1}, \quad f_2(z) = L_1 \circ g_2 \circ L_2^{-1}, \quad \ldots, \quad f_n(z) = L_n^{-1} \circ g_n(z),$$

then the two factorizations are called equivalent. If any two prime factorizations of $F(z)$ are equivalent, then $F(z)$ is called uniquely factorizable. In particular, for an entire function $F(z)$, if any two prime entire factorizations of $F(z)$ are equivalent, then $F(z)$ is called uniquely factorizable in the entire sense.

Ritt [14] proved the following result.

Proposition 1 *Let $p(z)$ be a nonlinear polynomial. If $p(z)$ has two prime factorizations*

$$p(z) = p_1 \circ \cdots \circ p_m(z) = q_1 \circ \cdots \circ q_n(z),$$

where p_i $(i = 1, \ldots, m)$ and q_j $(j = 1, \ldots, n)$ are polynomials, then $m = n$. Moreover, one factorization can be changed to another one by a sequence of applications of any of the following three ways:

1. *replace p_i and p_{i+1} by $p_i \circ L$ and $L^{-1} \circ p_{i+1}$, respectively;*
2. *alternate p_i and p_{i+1} when both are Chebychev polynomials;*
3. *replace z^k and $z^s h(z^k)$ by $z^s h(z)^k$ and z^k, respectively, where $h(z)$ is a polynomial, and s and k are natural numbers.*

Example 3 $z^{10} + 1 = (z^5 + 1) \circ z^2 = (z^2 + 1) \circ z^5$.

However, Ritt's result cannot be extended to rational functions.

Example 4 $z^3 \circ \frac{z^2-4}{z-1} \circ \frac{z^2+2}{z+1} = \frac{z(z-8)^3}{(z+1)^3} \circ z^3$.

This example was given by Michael Zieve (see [1]).

For transcendental functions, the diverse cases are very complex. For example, e^z can have infinitely many nonlinear factors.

Example 5 *For any integer n,*

$$e^z = z^2 \circ z^3 \circ \cdots \circ z^n \circ e^{z/n!}.$$

The following example shows that transcendental entire functions can have non-equivalent prime factorizations (see [10]).

Example 6

$$z^2 \circ \left(ze^{z^2}\right) = \left(ze^{2z}\right) \circ z^2.$$

Of course, there are functions which are uniquely factorizable. The following example is given by Urabe [17].

Example 7 *For any two nonconstant polynomials $p(z)$ and $q(z)$,*

$$(z + e^{p(e^z)}) \circ (z + q(e^z))$$

is uniquely factorizable.

The following result, proved by Hua [6], shows that, for a given function, we can construct uncountably many uniquely factorizable functions.

Proposition 2 *Let $f(z)$ be a transcendental entire function and $n \geq 3$ be a prime number. Then both $f(z^n) - cz^n$ and $(z^n - c)f(z^n)$ are uniquely factorizable for any complex number c except for a countable set.*

In this paper, we prove the following two results.

Theorem 1 *Let $f(z)$ be a transcendental entire function and $n \geq 3$ be a prime number. Then $[f(z)-cz]^n$ is uniquely factorizable for any complex number c except for a countable set.*

Theorem 2 *Let $f(z)$ be a transcendental entire function and $n \geq 3$ be a prime number. Then $[f(z) - cz]^{-n}$ is uniquely factorizable for any complex number c except for a countable set.*

2 Some Lemmas

The following lemmas will be used in the proof of the theorems.

Lemma 1 ([4]) *Suppose that $p(z)$ is a nonconstant polynomial and $g(z)$ is entire. Then $p(g(z))$ is periodic if and only if $g(z)$ is periodic.*

Lemma 2 ([11]) *Let $f(z)$ be a transcendental entire function. Then for any complex number c except for a countable set, $f(z) - cz$ is prime.*

Remark. So far, there is no example with countably infinite exceptions. In [13], it is proved that there is at most one exception for $f(z) = g(e^z)$, where $g(z)$ is an entire function satisfying $\max_{|z|=r} |g(z)| \leq e^{Kr}$ for a positive constant K. In [8] and [18], some other functions $f(z)$ are studied.

Lemma 3 ([12]) *Let $f(z)$ be a transcendental entire function. We denote by $\nu(a, f)$ the least order of almost all zeros of $f(z) - a$, where "almost all" means all with possibly finite exceptions. Then*

$$\sum_{a\neq\infty} \left(1 - \frac{1}{\nu(a, f)}\right) \leq 1.$$

Lemma 4 ([16]) *Let $f(z)$ and $g(z)$ be prime entire functions. Assume that both $f(z)$ and $F(z) = f(g(z))$ are non-periodic. Then $F(z)$ is uniquely factorizable if and only if $F(z)$ is uniquely factorizable in the entire sense.*

Lemma 5 *Let $f(z)$ be a nonconstant meromorphic function. Then $f(z)-cz$ is non-periodic for any complex number c with at most one exception.*

Proof of Lemma 5. Suppose there exist two different numbers c and d such that $f(z) - cz$ and $f(z) - dz$ are periodic with period u and v, respectively. Then $f'(z)$ is periodic and $f'(z + u) = f'(z) = f'(z + v)$. Let w be the period of $f'(z)$. Then there exist two nonzero integers m and k such that $u = mw$ and $v = kw$. This implies that $u = \frac{m}{k}v$. Hence

$$\begin{aligned} f(z) - cz &= f(z + ku) - c(z + ku) \\ &= f(z + mv) - c(z + ku) \\ &= f(z + mv) - d(z + mv) + d(z + mv) - c(z + ku) \\ &= f(z) - dz + d(z + mv) - c(z + ku) \\ &= f(z) - cz + dmv - cku. \end{aligned}$$

Therefore $dmv = cku$, and so, $d = c$, which is a contradiction. □

The following lemma is a simple version of the so-called Borel Unicity Theorem which can be found in [2] and [4].

Lemma 6 *Let $h_0(z), \ldots, h_n(z)$ be rational functions and let $g_1(z), \ldots, g_n(z)$ be nonconstant entire functions such that*

$$\sum_{j=1}^{n} h_j(z)e^{g_j(z)} = h_0(z).$$

Then $h_0 = 0$.

Lemma 7 *Let $f(z)$ be a transcendental entire function. Then*

$$f(z) - cz \neq P(z)e^{f_1(z)}$$

for all $c \in \mathbb{C}$ with at most one exception, where $P(z)$ is a polynomial and $f_1(z)$ is a nonconstant entire function.

Proof of Lemma 7. Suppose to the contrary that there exist two different constants c and d, two polynomials $P_1(z)$ and $P_2(z)$, and two nonconstant entire functions $f_1(z)$ and $f_2(z)$ such that

$$f(z) - cz = P_1(z)e^{f_1(z)}$$

and

$$f(z) - dz = P_2(z)e^{f_2(z)}.$$

Then

$$cz - dz = P_2(z)e^{f_2(z)} - P_1(z)e^{f_1(z)}.$$

By Lemma 6, $cz - dz = 0$; thus $d = c$ which is a contradiction. □

3 Proof of Theorem 1

Let

$$F(z) = [f(z) - cz]^n = z^n \circ (f(z) - cz).$$

Obviously, z^n is non-periodic.

Let

$$Z(f) = \{f(z) : \ f''(z) = 0\}.$$

Then $Z(f)$ is a countable set, and for any $c \notin Z(f)$, $f'(z) - c$ has only simple zeros ([9, Theorem F]). We combine $Z(f)$ and all the exceptions (if any) in Lemmas 1, 2, 5 and 7 to form an exceptional set E. Then E is a countable set which may be empty. For any $c \in \mathbb{C} - E$, we have the following properties:

(P1) The function $F(z)$ is non-periodic;

(P2) The function $f(z) - cz$ is prime;

(P3) $f'(z) - c$ has only simple zeros.

(P4) $f(z) - cz \neq P(z)e^{f_1(z)}$ for any polynomial $P(z)$ and nonconstant entire function $f_1(z)$.

Next we assume $c \in \mathbb{C} - E$.

By Lemma 4, we need only prove that $F(z)$ is uniquely factorizable in the entire sense, which means, we just need to consider entire factors. Assume that

$$F(z) = g(z) \circ h(z), \tag{2}$$

where $g(z)$ and $h(z)$ are nonconstant entire functions. We consider three cases.

Case 1. $g(z)$ has at least two zeros, z_1 and z_2, of order m_1 and m_2, respectively, such that $(n, m_1) = (n, m_2) = 1$, that is, n and m_i $(i = 1, 2)$ have no common factors other than 1. Then by (2) and the fact that n is prime, the order of any zero of $h(z) - z_i$ $(i = 1, 2)$ should be a multiple of n. Hence

$$\nu(z_i, h) \geq n \geq 3 \quad (i = 1, 2),$$

which implies that

$$\sum_{a \neq \infty} \left(1 - \frac{1}{\nu(a, f)}\right) \geq 1 - \frac{1}{3} + 1 - \frac{1}{3} > 1.$$

This is a contradiction to Lemma 3.

Case 2. $g(z)$ has one zero, z_0, of order m such that $(n,m)=1$. Then by (2) and the fact that n is prime, $g(z)$ and $h(z)$ can be written as

$$g(z)=(z-z_0)^r g_1(z)^n, \quad h(z)=z_0+h_1(z)^n, \quad r=m \ (\mathrm{mod}\ n), \tag{3}$$

where $g_1(z)$ and $h_1(z)$ are entire functions. Obviously, $1\le r<n$. Substituting (3) into (2) we have

$$F(z)=h_1(z)^{rn}[g_1(z_0+h_1(z)^n)]^n,$$

which implies that

$$\begin{aligned} f(z)-cz &= uh_1(z)^r g_1(z_0+h_1(z)^n) \\ &= [uz^r g_1(z_0+z^n)]\circ h_1(z), \end{aligned} \tag{4}$$

where u is an n-th root of unity. Since $f(z)-cz$ is prime, we have two subcases as follows.

Case 2.1. Since the left factor $uz^r g_1(z_0+z^n)$ is linear, then $r=1$ and g_1 is a constant. It follows from (3) that $g(z)$ is linear. This is a trivial case.

Case 2.2. The right factor $h_1(z)$ is linear. Let $h_1(z)=az+b$ $(a,b\in\mathbb{C}, a\neq 0)$. By (4),

$$f(z)-cz=u(az+b)^r g_1[z_0+(az+b)^n)]. \tag{5}$$

If $g_1(z)$ has a zero, then by differentiating (5) we see that $f'(z)-c$ has a zero of order $n-1\ge 2$, which is a multiple zero of $f'(z)-c$. This contradicts (P3). Therefore $g_1(z)$ has no zero. This implies that there exists a nonconstant entire function $g_2(z)$ such that $g_1(z)=e^{g_2(z)}$. By (5),

$$f(z)-cz=u(az+b)^r e^{g_2[z_0+(az+b)^n)]},$$

which contradicts (P4).

Case 3. The order of any zero of $g(z)$ is a multiple of n. Then there exists an entire function $g_2(z)$ such that

$$g(z)=g_2(z)^n. \tag{6}$$

It follows from (2) that

$$[f(z)-cz]^n=[g_2\circ h(z)]^n,$$

and so,

$$f(z)-cz=ug_2(z)\circ h(z)$$

for an n-th root of unity, u. Since $f(z)-cz$ is prime, we have two subcases.

Case 3.1. The left factor $ug_2(z)$ is linear. It follows from (6) that $g(z)=z^n\circ L(z)$ for a linear function $L(z)$. Therefore we get an equivalent factorization.

Case 3.2. The right factor $h(z)$ is linear. This is a trivial case.

The proof is complete. □

4 Proof of Theorem 2

Assume that

$$[f(z)-cz]^{-n}=g(z)\circ h(z),$$

where $g(z)$ is a nonconstant meromorphic function and $h(z)$ is a nonconstant entire function. Then we have

$$[f(z)-cz]^n=\frac{1}{g(z)}\circ h(z).$$

Now, since the left-hand side is entire, the conclusion follows from Lemma 4 and Theorem 1. □

5 Open Questions

Question 1 *Can n be 2 in Theorems 1 and 2?*

Question 2 *What kind of rational functions are uniquely factorizable?*

Question 3 *Is $(z+e^{e^z})\circ(z+e^{e^z})$ uniquely factorizable?*

References

[1] W. Bergweiler, *An example concerning factorization of rational functions* (with correction), Exposition. Math., **11** (1993) 281–283.

[2] C. T. Chuang and C. C. Yang, *Fix-points and Factorization of Meromorphic Functions*, World Scientific, Singapore, 1990.

[3] F. Gross, *On factorization of meromorphic functions*, Trans. Amer. Math. Soc., **131** (1968) 215–222.

[4] F. Gross, *Factorization of Meromorphic Functions*, U.S. Government Printing Office, Washington, D.C., 1972.

[5] F. Gross and C. C. Yang, *Further results on prime entire functions*, Trans. Amer. Math. Soc., **192** (1974), 347–355.

[6] X. H. Hua, *Uniquely factorizable entire functions*, J. Math. Anal. Appl. **153** (1990), 11–17.

[7] X. H. Hua, *Proof of Gross–Yang's conjecture concerning factorization of entire functions*, Complex Variables Theory Appl. **14** (1990), 65–70.

[8] L. W. Liao and C. C. Yang, *On factorization of entire functions of bounded type*, Ann. Acad. Sci. Fenn. **29** (2004), 345–356.

[9] T. W. Ng, *Permutable entire functions and their Julia sets*, Math. Proc. Cambridge Philos. Soc. **131** (2001), 129?-138.

[10] T. W. Ng and C. C. Yang, *On the composition of a prime transcendental function and a prime polynomial*, Pacific J. Math. **193** (2000), 131–141.

[11] Y. Noda, *On the factorization of entire functions*, Kodai Math. J. **4** (1981), 480–494.

[12] M. Ozawa, *On uniquely factorizable entire functions*, Kodai Math. Sem. Rep. **28** (1977), 342–360.

[13] M. Ozawa and K. Sawada, *A theorem on factorization of certain entire functions*, Complex Variables Theory Appl. **34** (1997), 181–185.

[14] J. F. Ritt, *Prime and composite polynomials*, Trans. Amer. Math. Soc. **23** (1922), 51–66.

[15] P. C. Rosenbloom, *The fix-points of entire functions*, Medd. Lunds Univ. Mat. Sem. Tome supplémentaire, (1952), 186–192.

[16] G. D. Song, *On unique factorizability of composite entire functions*, Kodai Math. J. **10** (1987), 285–291.

[17] H. Urabe, *Some further results on factorization of entire functions*, Research Reports of the Nevanlinna Theory and its Applications II, Nippon Institute of Technology (NIT), 1998, p. 94.

[18] X. L. Wang and C. C. Yang, *On the factorization of a certain class of entire functions*, Indian J. Pure Appl. Math. **33** (2002), 1727–1741.

[19] C. C. Yang, *Factorization theory of meromorphic functions*, Lecture Notes in Pure and Applied Math. (edited by C. C. Yang), Marcel Dekker, 1982.