Weakly Nonlinear Analysis of Rapidly Decelerated Channel Flow[*]

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Abstract

An amplitude evolution equation for the most unstable mode of a flow between two parallel planes is derived under the quasi-steady assumption. The base flow is rapidly decelerated so that the total fluid flux through the cross-section of the channel is equal to zero. The derivation is based on the method of multiple scales which is widely used in weakly nonlinear theory. It is shown that the equation has complex coefficients and is of Ginzburg–Landau type. Explicit formulas for the calculation of the coefficients of this equation are presented. Results of numerical calculations show that the instability is subcritical

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Résumé

On dérive l'équation d'évolution de l'amplitude pour le mode le plus instable d'un écoulement entre deux plans parallèles sous l'hypothèse de quasi-constance. L'écoulement de base subit une décélération rapide produisant un flux à écoulement total nul dans une coupe transversale du canal. La dérivation se fait par la méthode d'échelles multiples souvent employée dans la théorie de la non-linéarité faible. On montre que l'équation d'évolution du type Ginzburg–Landau est à coefficients complexes. On présente des formules explicites pour le calcul des coefficients de cette équation. Les calculs numériques montrent que l'instabilité est sous-critique.

1 Introduction

A rapidly decelerated laminar flow in a plane channel bounded by two infinite parallel plates is considered in the present paper. Unsteady flows subject to rapid deceleration are important in many applications such as flows in natural gas pipelines, the design and analysis of water supply systems and blood flow in arteries. Transient flows are often generated in hydraulic devices or water supply systems by rapid changes in valve settings or pump shutdowns. Cavitation, pitting and corrosion can occur as a result of rapid pressure changes in unsteady flows [1].

An atherosclerosis plaque development in regions where shear stress changes direction can be triggered by unsteady blood flow [2]. These examples are characterized by fast changes of flow characteristics (such as velocity and pressure) during short time intervals as a result of rapid acceleration or deceleration of the flow and the appearance of instabilities in the flow. Therefore understanding the flow structure and its stability in rapidly changing unsteady flows is an essential part of proper design and analysis of water supply systems.

Linear stability analysis can be used to predict when a particular flow becomes unstable, thus providing a marginal stability curve and the critical values of the parameters at the threshold. However, the linear stability theory cannot describe the evolution of unstable modes above the threshold. Weakly nonlinear theories are used [3], [4] in order to study further development of instability.

Spatio-temporal dynamics of complex flows is often described in fluid mechanics by relatively simple amplitude evolution equations. One of the popular dynamical models is the complex Ginzburg–Landau equation (GLE) [5], [6]. The Ginzburg–Landau model is of great interest due to applications to onset of wave-pattern forming instabilities. The attractiveness of the complex GLE is based on the following important properties of the model. First, the model is relatively simple and allows one to include physical effects such as diffusion and nonlinearity. Second, integrating the equation numerically is a relatively straightforward process which makes it an effective tool to study spatio-temporal characteristics of complex flows. The complex GLE can exhibit a rich variety of solutions depending on the values of its coefficients [7].

Examples include the transition in a wake behind a cylinder given in [8] where it is shown that the evolution of the most unstable mode in an interval of the Reynolds numbers around the critical value is described by the Landau equation. The complex GLE is used in [9] to analyze the dynamics of the flow behind bluff rings. The coefficients of the GLE are computed from experimental data. Good quantitative agreement is found between experimental data and the results of numerical modeling. Complex spatio-temporal flow patterns observed in the wakes of a row of 16 circular cylinders placed close to each other in an incoming flow are analyzed in [10]. Spatio-temporal characteristics of the flow are recorded and the data are used to evaluate the coefficients of the GLE. The validity of the model is assessed by reproducing experimentally observed flow patterns from the model. The effectiveness of a closed-loop control strategy for the stabilization of an unstable bluff-body flow is analyzed in [11] by means of the complex GLE.

In all the examples in [8]–[11], the GLE (or the Landau equation) is used as a phenomenological model equation. In the present paper we derive the GLE from the Navier–Stokes equations for the important practical case of a rapidly decelerated laminar flow in a plane channel.

Consider a fully developed Poiseuille flow in a plane channel between two infinite parallel plates $\tilde{y} = -h$ and $\tilde{y} = h$. At time $\tilde{t} = 0$ the flow is suddenly blocked so that the total fluid flux through the cross-section of the channel is equal to zero for $\tilde{t} > 0$. The velocity vector is assumed to have only one non-zero component, $\widetilde{U}(\tilde{y}, \tilde{t})$, which depends only on the transverse coordinate $\tilde{y}$ and time $\tilde{t}$. In this case, the system of Navier–Stokes equations reduces to the following equation

$$\frac{\partial \widetilde{U}}{\partial \tilde{t}} = -\frac{1}{\rho}\frac{\partial \widetilde{P}}{\partial \tilde{x}} + \nu\,\frac{\partial^2 \widetilde{U}}{\partial \tilde{y}^2}, \tag{1}$$

where ρ and ν are the density and the kinematic viscosity of the fluid, respectively, $\widetilde{P}$ is the pressure and $\tilde{x}$ is the longitudinal coordinate. The function $\widetilde{U}$ satisfies the following initial and boundary conditions:

$$\widetilde{U}|_{\tilde{y}=\pm h} = 0, \qquad \widetilde{U}|_{\tilde{t}=0} = \frac{1}{2\mu}\frac{d\widetilde{P}_0}{d\tilde{x}}\left(\tilde{y}^2 - h^2\right), \tag{2}$$

where $\mu = \rho\nu$ is the dynamic viscosity of the fluid and $d\widetilde{P}_0/d\tilde{x}$ is a constant pressure gradient in the $\tilde{x}$ direction before deceleration. The total fluid flux through the cross-section of the channel is zero for all $\tilde{t} > 0$:

$$\int_{-h}^{h} \widetilde{U}(\tilde{y}, \tilde{t})\, d\tilde{y} = 0. \tag{3}$$

Problem (1)–(3) is solved by a Pohlhausen type technique in [12] and by the method of matched asymptotic expansions in [13]. Calculations performed in [12] show that the velocity profiles contain inflection points and are therefore potentially unstable for small perturbations. Methods of the linear stability theory are used in [14] to analyze the stability of the base flow. Strictly speaking, the method of normal modes which is used in the linear stability analysis of steady flows cannot be applied to the problem of rapid deceleration of channel flow since the base flow is unsteady. However, the asymptotic analysis in [14] assesses the use of a quasi-steady assumption for large Reynolds numbers. In particular, it is shown that

$$\lambda_{\text{unsteady}} = \lambda_{\text{quadi-unsteady}} + O(1/\operatorname{Re}),$$

where $\lambda_{\text{unsteady}}$ is the unsteady (i.e., actual) growth rate of the unstable perturbation and $\lambda_{\text{quasi-steady}}$ is the growth rate obtained by means of the quasi-steady assumption. Under the quasi-steady assumption the base flow velocity profiles are "frozen" so that the time variable for the base flow is considered as a parameter. The validity of the quasi-steady assumption is justified in [15] where an initial value problem is used to solve full linearized disturbance equations numerically. Calculations performed in [15] indicate that the growth rates of perturbations for oscillatory pipe flows deviate considerably from the results of the quasi-steady approach. On the other hand, it is shown in [15] that the results from the quasi-steady theory are in good agreement with the initial value problem approach for the case of rapidly decelerated channel and pipe flows. Therefore the quasi-steady assumption is adopted in the present study.

Linear stability calculations performed in [14] show that the flow, $\widetilde{U}$, is unstable in a wide range of Reynolds numbers. A marginal stability curve and critical values of the parameters of the problem (the critical Reynolds number, the critical wavenumber and the wave speed) are obtained in [14] for different values of time. However, the linear theory cannot predict the evolution of the most unstable mode. Methods of weakly nonlinear theory are often used in order to take into account the evolution of the most unstable perturbation analytically. A direct application of the method of multiple scales [16] usually leads to model amplitude evolution equations (for example, the complex Ginzburg–Landau equation, see [5]–[6]). Consider a Reynolds number which is slightly above the critical value. If attention is restricted to a neighborhood of the critical point $(\operatorname{Re}_c, \alpha_c, c_c)$, where Re is the Reynolds number, α is the wavenumber, c is the wave speed and the subscript c indicates the critical values of the parameters, then the growth rates of a perturbation will be small in the vicinity of the critical point. Assuming that the amplitude is a slowly varying function of the longitudinal coordinate and time, one can derive an amplitude evolution equation for the most unstable mode. This approach is used, for example, in [3] for the case of a plane Poiseuille flow. The weakly nonlinear analysis of a problem related to the generation of waves by wind in performed in [17]. Recently an amplitude evolution equation for the most unstable mode was derived in [18] for the case of a rapidly decelerated flow in a pipe. Two examples of weakly nonlinear analysis for shallow flows can be found in [19] and [20]. In all the above mentioned cases the amplitude evolution equation is found to be the complex Ginzburg–Landau equation.

In the present paper a complex Ginzburg–Landau equation is derived under the quasi-steady assumption for the flow between two parallel infinite plates. The base flow in this case is unsteady and is given as the solution of (1)–(3). The linear stability calculations presented in [14] show that the flow (1)–(3) is linearly unstable for a certain range of the Reynolds numbers. Assuming that Re is slightly larger than the critical value, Re_c, and using the methods of the weakly nonlinear theory we derive an amplitude evolution equation for the most unstable mode. It is shown that the evolution equation is the complex Ginzburg–Landau equation whose coefficients depend on the solution of the linearized stability problem.

2 The Ginzburg–Landau model

Spatio-temporal dynamics of complex open flows above the threshold often can be described by a complex GLE of the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} + \mu |A|^2 A, \tag{4}$$

where $\sigma = \sigma_r + i\sigma_i$, $\delta = \delta_r + i\delta_i$ and $\mu = \mu_r + i\mu_i$ are complex coefficients. It is shown in [7] that, depending on the values of the coefficients, equation (4) possesses a rich variety of solutions, including some solutions which may even look chaotic. An excellent review of the properties of the Ginzburg–Landau equation (4) is given in [5]. The three terms on the right-hand side of equation (4) correspond to linear amplification, diffusion and nonlinear saturation, respectively The physical meaning of the coefficients of (4) is the following. The real part of σ, namely, σ_r, represents the rate of amplification of an unstable perturbation. The angular frequency of oscillation is given by σ_i. The dependence of the instability growth rate and oscillation frequency on the wavelength is reflected by the coefficients δ_r and δ_i, respectively. The nonlinearities tend to saturate the instability if $\mu_r < 0$. From a physical point of view, this means that there exists another equilibrium state after the flow loses stability. Examples of such equilibrium states are, for example, the Rayleigh–Bénard convection between two parallel plates which are maintained at different constant temperatures and the Taylor–Couette flow between two rotating circular cylinders (see [21]). Such a situation is referred to as "supercritical instability" in the hydrodynamic stability literature.

On the other hand, if $\mu_r > 0$, then higher order terms on the right-hand side of (4) are also important and (4) is much less informative. Such a case is known as "subcritical instability". One example of subcritical instability is given in [3] for the case of a plane Poiseuille flow. Note that μ_r in equation (4) is usually referred to as the Landau constant in the literature.

3 Derivation of the Ginzburg–Landau equation

A two-dimensional viscous incompressible flow in a plane channel can be described by the following dimensionless equation

$$(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y = \frac{1}{\mathrm{Re}}\,\Delta^2\psi, \tag{5}$$

where $\mathrm{Re} = U_{\max}h/\nu$, $U_{\max}$ is the maximum velocity of the undisturbed flow and $\psi(x, y)$ is the stream function defined by the relations

$$u = \psi_y, \qquad v = -\psi_x.$$

Consider a perturbed solution to (5) in the form

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots, \tag{6}$$

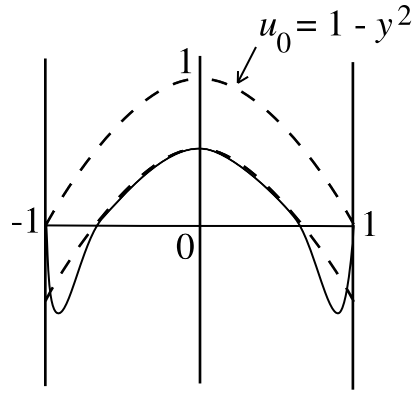

Figure 1: Schematic picture of velocity profiles for rapidly decelerated channel flow.

where ε is a small parameter and $\partial_y \psi_0 = U$ is the dimensionless solution to (1)–(3). The solution $U(y, t)$ can be found by the method of the Laplace transform and has the form

$$U(y,t) = -\frac{4}{3} \sum_{n=1}^{\infty} \frac{\beta_n \cos(\beta_n y) - \sin \beta_n}{\beta_n^2 \sin \beta_n} \, e^{-\beta_n^2 t}, \tag{7}$$

where β_n are the roots of the equation $\tan \beta_n = \beta_n$.

Typical velocity profiles for the case of a suddenly blocked channel flow are shown in Fig. 1. The upper dashed line corresponds to a fully developed plane Poiseuille flow before deceleration. A sudden closure of the channel results in an instantaneous deceleration of the flow (lower dashed line). This process is essentially inviscid, i.e., the flow velocity at any point in the channel becomes smaller than the velocity of the undisturbed flow by the amount which is equal to the mean velocity of the undisturbed flow. However, viscous boundary layer starts to develop near the walls since the condition of zero velocity should be satisfied at channel walls. As can be seen from Fig. 1, the core region of the flow is not affected by the deceleration (at least for sufficiently small times) but the profiles change rather quickly near the walls.

Substituting (6) into (5) and keeping only the linear terms with respect to ε, we obtain the following equation which governs the linear stability of the flow:

$$L\psi_1 = 0, \tag{8}$$

where

$$L\eta = \eta_{xxt} + \eta_{yyt} + U\eta_{xxx} + U\eta_{yyx} - U_{yy}\eta_x - \frac{1}{\mathrm{Re}}\left(\eta_{xxxx} + 2\eta_{xxyy} + \eta_{yyyy}\right). \tag{9}$$

Using the method of normal modes and assuming the solution to (8) to be of the form

$$\psi_1(x, y, t) = \varphi_1(y) \exp[i\alpha(x - ct)] + \mathrm{c.c} \tag{10}$$

where α is the wavenumber, c is the wave speed of a perturbation, and the commonly used fluid dynamics short-hand notation $c.c$ means the 'complex conjugate' of the first term on the right-hand side of the equation where it appears, we obtain the Orr–Sommerfeld equation in the form

$$\varphi_1'''' - (2\alpha^2 + i\alpha U \,\mathrm{Re})\varphi_1'' + (i\alpha^3 U \,\mathrm{Re} + i\alpha \,\mathrm{Re}\, U_{yy} + \alpha^4)\varphi_1 = -i\alpha \,\mathrm{Re}\; c(\varphi_1'' - \alpha^2 \varphi_1). \tag{11}$$

The boundary conditions are

$$\varphi_1(\pm 1) = 0, \qquad \varphi_1'(\pm 1) = 0. \tag{12}$$

The boundary value problem (11)–(12) is an eigenvalue problem which determines the critical values of the parameters Re, α and c. Recall that the base flow velocity, U, is a function of y and t, but

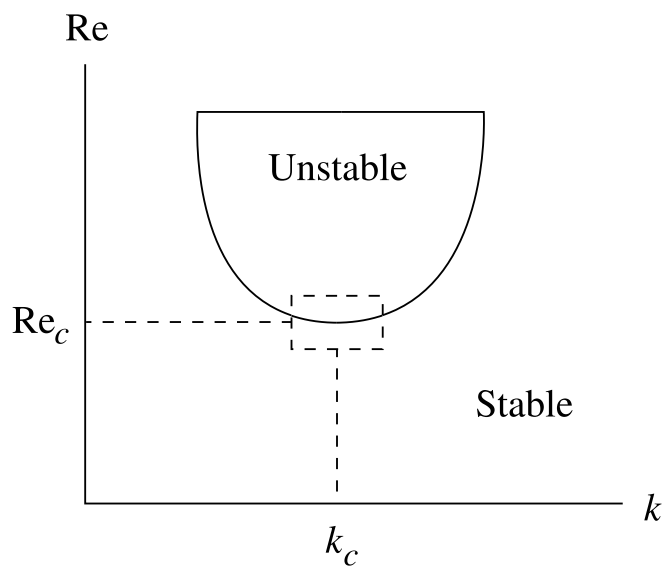

Figure 2: Schematic diagram of the critical Reynolds numbers versus k. The dashed rectangle shows the region where the weakly nonlinear theory is applicable.

we adopted here the quasi-steady assumption. Thus, t is considered as a parameter in $U = U(y,t)$. It is shown in [18] that the quasi-steady assumption is justified if the rate of change of the base flow velocity with respect to time is smaller than the growth rate of the perturbation. Numerical results presented in [18] indicate that the quasi-steady assumption is appropriate for the case of suddenly blocked pipe flows. In addition, the numerical solution of full linearized disturbance equations (solved as an initial value problem) in [15] showed that the calculated growth rates are in good agreement with the quasi-steady theory. Therefore, the quasi-steady assumption is adopted in the present study.

The critical Reynolds numbers are calculated in [14] by means of a numerical solution of (11)–(12). The structure of the most unstable mode can also be obtained from the solution of (11)–(12) but the linear theory cannot be used to estimate the amplitude of the most unstable mode and to descibe the evolution of the most unstable mode. The next natural step is to use the methods of weakly nonlinear theory (see, for example, [3]) in order to derive the amplitude evolution equation for the case where Re is slightly larger than the critical value, Re_c. Following [3] we assume that

$$\mathrm{Re} = \mathrm{Re}_c(1+\varepsilon^2) \tag{13}$$

and introduce the slow time, τ, and the stretched longitudinal variable, ξ, such that

$$\tau = \varepsilon^2 t, \qquad \xi = \varepsilon(x - c_g t), \tag{14}$$

where c_g is the group velocity.

The weakly nonlinear theory is therefore applied in the vicinity of the critical point (see Fig. 2).

The differential operators $\partial/\partial t$ and $\partial/\partial x$ are then replaced by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi}. \tag{15}$$

The function ψ_1 in (6) is represented in the form

$$\psi_1 = A(\xi,\tau)\varphi_1(y)\exp[i\alpha_c(x - c_c t)] + c.c, \tag{16}$$

where $A(\xi,\tau)$ is a slowly varying amplitude and $\varphi_1(y)$ is an eigenfunction of the linear stability problem (11)–(12) at $\mathrm{Re} = \mathrm{Re}_c$, $\alpha = \alpha_c$ and $c = c_c$. The evolution equation for A is obtained by taking higher terms of the perturbation expansion (6) into account. Substituting (13)–(16) into (6) and collecting the terms of order ε^2 we obtain

$$\begin{aligned} L\psi_2 = {} & c_g(\psi_{1xx\xi} + \psi_{1yy\xi}) - 2\psi_{1x\xi t} - U(3\psi_{1xx\xi} + \psi_{1yy\xi}) \\ & - \psi_{1y}(\psi_{1xxx} + \psi_{1yyx}) + \psi_{1x}(\psi_{1xxy} + \psi_{1yyy}) + U_{yy}\psi_{1\xi} \\ & - \frac{1}{\mathrm{Re}_c} U_{yyy} + \frac{1}{\mathrm{Re}_c}(4\psi_{1xxx\xi} + 4\psi_{1xyy\xi}). \end{aligned} \tag{17}$$

Similarly, substituting (13)–(16) into (6) and collecting the terms of order ε^3 we get

$$\begin{aligned} L\psi_3 = {} & c_g(\psi_{2xx\xi} + \psi_{2yy\xi} + 2\psi_{1x\xi\xi}) - \psi_{1xx\tau} - \psi_{1yy\tau} \\ & - \psi_{1\xi\xi t} - U(3\psi_{2xx\xi} + \psi_{2yy\xi} - 3\psi_{1x\xi\xi}) - \psi_{1y}(\psi_{2xxx} + \psi_{2yyx}) \\ & - \psi_{1y}(3\psi_{1xx\xi} + \psi_{1yy\xi}) + \psi_{1\xi}(\psi_{1xxy} + \psi_{1yyy}) + U_{yy}\psi_{2\xi} \\ & - \psi_{2y}(\psi_{1xxx} + \psi_{1yyx}) + \psi_{2x}(\psi_{1xxy} + \psi_{1yyy}) + \psi_{1x}(\psi_{2xxy} + \psi_{2yyy}) \\ & + 2\psi_{1x}\psi_{1xy\xi} - \frac{1}{\mathrm{Re}_c}(\psi_{1xxxx} + 2\psi_{1xxyy} + \psi_{1yyyy}) \\ & + \frac{1}{\mathrm{Re}_c}(4\psi_{2xxx\xi} + 4\psi_{2xyy\xi} + 6\psi_{1xx\xi\xi} + 2\psi_{1yy\xi\xi}). \end{aligned} \tag{18}$$

The solution to (17) is sought in the form

$$\psi_2 = A^2\varphi_2^{(0)}(y)\exp[2i\alpha_c(x - c_c t)] + AA^*\varphi_2^{(1)}(y) + A_\xi\varphi_2^{(2)}(y)\exp[i\alpha_c(x - c_c t)] + c.c, \tag{19}$$

where A^* is the complex conjugate of A. Substituting (19) and (16) into (17) and collecting the terms proportional to $A^2\exp[2i\alpha_c(x - c_c t)]$ we obtain the following equation for the function $\varphi_2^{(0)}(y)$:

$$\begin{aligned} & -\frac{1}{\mathrm{Re}_c}\left[\varphi_{2yyyy}^{(0)} - 8\alpha_c^2\varphi_{2yy}^{(0)} + 16\alpha_c^4\varphi_2^{(0)}\right] + 2i\alpha_c\varphi_{2yy}^{(0)}(U - c_c) - 8i\alpha_c^3(U - c_c)\varphi_2^{(0)} \\ & \qquad - 2i\alpha_c U_{yy}\varphi_2^{(0)} = -i\alpha_c\varphi_{1y}\varphi_{1yy} + i\alpha_c\varphi_1\varphi_{1yyy} \end{aligned} \tag{20}$$

with the boundary conditions

$$\varphi_2^{(0)}(\pm 1) = 0, \qquad \varphi_{2y}^{(0)}(\pm 1) = 0. \tag{21}$$

Similarly, collecting the terms proportional to AA^* we get

$$-\frac{1}{\mathrm{Re}_c}\varphi_{2yyyy}^{(1)} = i\alpha_c\left[\varphi_{1y}\varphi_{1yy}^* - \varphi_{1y}^*\varphi_{1yy} + \varphi_1\varphi_{1yyy}^* - \varphi_1^*\varphi_{1yyy}\right] - \frac{1}{\mathrm{Re}_c}U_{yyy} \tag{22}$$

with the boundary conditions

$$\varphi_2^{(1)}(\pm 1) = 0, \qquad \varphi_{2y}^{(1)}(\pm 1) = 0. \tag{23}$$

In a similar manner, we obtain the equation for the function $\varphi_2^{(2)}$:

$$\begin{aligned} & -\frac{1}{\mathrm{Re}_c}\left[\varphi_{2yyyy}^{(2)} - 2\alpha_c^2\varphi_{2yy}^{(2)} + \alpha_c^4\varphi_2^{(2)}\right] + i\alpha_c(U - c_c)(\varphi_{2yy}^{(2)} - \alpha_c^2\varphi_2^{(2)}) - i\alpha_c U_{yy}\varphi_2^{(2)} \\ & \qquad = c_g(\varphi_{1yy} - \alpha_c\varphi_1) - 2\alpha_c^2 c_c\varphi_1 + 3\alpha_c^2 U\varphi_1 - U\varphi_{1yy} + \varphi_1 U_{yy} + \frac{4i\alpha_c}{\mathrm{Re}}(\varphi_{1yy} - \alpha_c^2\varphi_1) \end{aligned} \tag{24}$$

with the boundary conditions

$$\varphi_2^{(2)}(\pm 1) = 0, \qquad \varphi_{2y}^{(2)}(\pm 1) = 0. \tag{25}$$

Comparing (11)–(12) and (24)–(25) we see that the solution to (24)–(25), namely, the function $\varphi_2^{(2)}$, is resonantly forced since the homogeneous equation which corresponds to (24) is satisfied at $\mathrm{Re} = \mathrm{Re}_c$, $\alpha = \alpha_c$ and $c = c_c$. Thus, (24)–(25) has a solution if and only if the right-hand side of (24) is orthogonal to all the eigenfunctions of the corresponding adjoint problem. The adjoint operator, L^a, and the adjoint eigenfunction, φ_1^a, are defined as follows:

$$\int_{-1}^{1} \varphi_1^a L(\varphi_1)\, dy = \int_{-1}^{1} \varphi_1 L^a(\varphi_1^a)\, dy = 0. \tag{26}$$

The adjoint eigenfuction is the solution of the equation

$$\varphi_{1yyyy}^a - (2\alpha^2 + i\alpha U\,\mathrm{Re})\varphi_{1yy}^a - 2i\alpha U_y\,\mathrm{Re}\,\varphi_{1y}^a + (i\alpha^3 U\,\mathrm{Re} + \alpha^4)\varphi_1^a = -i\alpha\,\mathrm{Re}\,c(\varphi_{1yy}^a - \alpha^2\varphi_1^a) \tag{27}$$

with the boundary conditions

$$\varphi_1^a(\pm 1) = 0, \qquad \varphi_{1y}^a(\pm 1) = 0. \tag{28}$$

Note that the critical values, Re_c, α_c and c_c, are the same for problems (11)–(12) and (26)–(27).

The group velocity, c_g, is determined from the solvability condition for equation (24) and is given by

$$c_g = \frac{I_1}{I_2}, \tag{29}$$

where

$$I_1 = \int_{-1}^{1} \varphi_1^a \left[2\alpha_c^2 c_c\varphi_1 - 3\alpha_c^2 U\varphi_1 + U\varphi_{1yy} - \varphi_1 U_{yy} - \frac{4i\alpha_c}{\mathrm{Re}_c}(\varphi_{1yy} - \alpha_c^2\varphi_1)\right] dy$$

and

$$I_2 = \int_{-1}^{1} \varphi_1^a(\varphi_{1yy} - \alpha_c^2\varphi_1)\, dy.$$

Finally, the amplitude evolution equation for A is obtained from the solvability condition for equation (18) and has the form of the complex Ginzburg–Landau equation (4) where

$$\sigma = \frac{\sigma_1}{\gamma_1}, \qquad \delta = \frac{\delta_1}{\gamma_1}, \qquad \mu = \frac{\mu_1}{\gamma_1}. \tag{30}$$

The coefficients γ_1, σ_1, δ_1 and μ_1 are given by

$$\gamma_1 = \int_{-1}^{1} \varphi_1^a(\varphi_{1yy} - \alpha_c^2\varphi_1)\, dy, \tag{31}$$

$$\sigma_1 = -\frac{1}{\mathrm{Re}_c}\int_{-1}^{1} \varphi_1^a(\varphi_{1yyyy} - 2\alpha_c^2\varphi_{1yy} + \alpha_c^4\varphi_1)\, dy, \tag{32}$$

$$\begin{aligned}\delta_1 = \int_{-1}^{1} \varphi_1^a\Bigg[&\left(c_g - U + \frac{4i\alpha_c}{\mathrm{Re}_c}\right)\varphi_{2yy}^{(2)} + \left(-\alpha_c^2 c_g - 2\alpha_c^2 c_c + 3\alpha_c^2 U + U_{yy} - \frac{4i\alpha_c}{\mathrm{Re}_c}\right)\varphi_2^{(2)} \\ &+ \frac{2}{\mathrm{Re}_c}\varphi_{1yy} + \left(2i\alpha_c c_g + i\alpha_c c_c - 3i\alpha_c U - \frac{6\alpha_c^2}{\mathrm{Re}_c}\right)\varphi_1\Bigg] dy,\end{aligned} \tag{33}$$

$$\begin{aligned}\mu_1 = -i\alpha_c\int_{-1}^{1} \varphi_1^a\Big[&-\varphi_1^*\varphi_{2yyy}^{(0)} - 2\varphi_{1y}^*\varphi_{2yy}^{(0)} + (3\alpha_c^2\varphi_1^* + \varphi_{1yy}^*)\varphi_{2y}^{(0)} \\ &+ (-10\alpha_c^2\varphi_{1y}^* + 2\varphi_{1yyy}^*)\varphi_2^{(0)} + \varphi_1\varphi_{2yyy}^{(1)} - (\varphi_{1yy} - \alpha_c^2\varphi_1)\varphi_{2y}^{(1)}\Big] dy.\end{aligned} \tag{34}$$

4 Numerical example

The coefficients of the Ginzburg–Landau equation are evaluated numerically in this section. First, the linear stability problem (11)–(12) is solved by means of a pseudospectral collocation method based on Chebyshev polynomials. The solution to (11)–(12) is sought in the form

$$\varphi(y) = \sum_{k=0}^{N} a_k (1-y^2)^2 T_k(y), \tag{35}$$

where $T_k(y)$ is the Chebyshev polynomial of degree k. The form of (35) guarantees that the boundary conditions (12) are satisfied automatically. The collocation points y_j are

$$y_j = \cos\frac{\pi j}{N}, \qquad j = 0, 1, \dots, N. \tag{36}$$

Substituting (35) into (11)–(12) and evaluating the derivatives of the function $\varphi(y)$at the collocation points (36) we obtain a generalized eigenvalue problem of the form

$$(A - \lambda B)a = 0, \tag{37}$$

where A and B are complex-valued matrices,

$$a = (a_1 a_2 \dots a_N)^T,$$

and the superscript T denotes the matrix transpose.

Note that the solution of the form (37) is more convenient than the one obtained by traditional collocation methods [22] for two reasons: first, the matrix B in (37) is not singular, and second, the fact that function (35) satisfies the boundary conditions automatically reduces the condition number of the matrices in this method [23].

Problem (37) is solved numerically by means of the IMSL routine DGVCCG. High precision is necessary to calculate the coefficients of the Ginzburg–Landau equation. Therefore, the number of collocation points, N, is fixed at $N = 100$ in the present study. Numerical results indicate that such value of N is sufficient to calculate the coefficients to four decimal places. In order to illustrate the procedure, we have chosen the base velocity profile which corresponds to the dimensionless time $t = 0.0001$. The corresponding critical values of the parameters of the problem are $\mathrm{Re}_c = 2853.024$, $\alpha_c = 1.1527$ and $c_c = 0.35903$. Note that c_c is the real part of c and the imaginary part of c is of order 10^{-8} in this case.

Second, we calculate the eigenfuctions of the linear stability problem (11)–(12) and the eigenfunction of the corresponding adjoint problem (27)–(28). The eigenvalues of (27)–(28) are the same as the eigenvalues of (11)–(12), as it should be.

Third, the boundary value problems (20)–(21), (22)–(23) and (24)–(25) are solved by means of the Chebyshev collocation method and the functions $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are used to calculate the coefficients of the Ginzburg–Landau equation (4). The group velocity c_g calculated by (29) must be real. The calculations confirm this fact: the computed value of c_g is $c_g = -0.229730 - 0.000011i$. The coefficients σ, δ and μ of the Ginzburg–Landau equation (4) are

$$\sigma = 0.0244 + 0.0594i, \quad \delta = 0.1805 + 0.2835i, \quad \mu = 33.0005 - 253.5093i.$$

Since the real part of μ (the Landau constant) is positive, a finite equilibrium state is not possible. This means that the disturbances are linearly unstable and grow unbounded; that is, the stability is subcritical. A similar result is obtained in [3] for a weakly nonlinear analysis of plane Poiseuille flow.

Note that the complex GLE is also derived for the case of unsteady pipe flows driven by the controlled motion of a piston [18]. Results of numerical computations in [18] are compared with experimental data in [24] where the flow of an incompressible fluid (water) was generated as follows: the velocity of a piston linearly increases from zero to a constant value of $\tilde{U}_p$ for $0 < \tilde{t} < \tilde{t}_0$, maintains a value of $\tilde{U}_p$ for $\tilde{t}_0 < \tilde{t} < \tilde{t}_1$, decreases linearly to zero for $\tilde{t}_1 < \tilde{t} < \tilde{t}_2$ and maintains a value of zero for $\tilde{t} > \tilde{t}_2$. Four experimental cases for different values of $\tilde{t}_0, \tilde{t}_1, \tilde{t}_2$ and $\tilde{U}_p$ are considered in [24]. The base flow velocity profiles in the deceleration phase contain inflection points. Linear stability calculations in [18] show that the critical Reynolds numbers are rapidly decreasing for small times and, therefore, such flows become unstable. The results also show that, in some cases, secondary structures are found in the experiments while, in other cases, rapid transition to turbulence is observed. The complex coefficients of the corresponding complex GLE are calculated in [18]. Calculations show that for the two experimental cases considered in [24] the real part of the Landau constant is positive (this indicates the existence of a finite equilibrium state after the flow loses stability). Secondary structures are observed also in experiments [24] for the same two cases. Therefore, the complex Ginzburg–Landau model seems to predict (at least, qualitatively) the behavior of the most unstable mode above the threshold.

However, important differences exist between the two cases (a suddenly blocked channel flow considered in the present paper and a flow generated by the controlled motion of a piston in [18], [24]). The base flow before deceleration in the present paper is assumed to be fully developed while the base flow in [18] is time-dependent and is not fully developed. In addition, the fluid flux through the cross-section of the channel in the present paper is instantaneously reduced to zero while the fluid flux is reduced to zero in a finite time for the flow induced by the controlled motion of the piston. Thus, the time history of the base flow profiles plays an essential role in the development of instability in rapidly decelerated flows.

5 Conclusions

Unsteady rapidly decelerated flows in a plane channel bounded by two infinite parallel plates are considered in the present paper. Deceleration is caused by instantaneous closure of the channel at time $t = 0$ so that the total fluid flux through the cross section of the channel is equal to zero for $t \geq 0$. The base flow profiles contain inflection points and are found to be potentially linearly unstable. Weakly nonlinear theory is used in the present paper to derive an amplitude evolution equation for the most unstable mode. The quasi-steady assumption is used for linear stability analysis, that is, the growth rate of perturbations is assumed to be considerably larger than the rate of change of the base flow with respect to time. It is shown that the evolution equation in this case is the complex Ginzburg–Landau equation. The coefficients of the equation are calculated in closed form. Results of numerical calculations indicate that unstable disturbances (computed at $t = 0.0001$) grow without bounds and a finite amplitude state is not possible. The predictions based on the Ginzburg–Landau model (in terms of the presence or absence of secondary structures in the flow after it looses stability) for suddenly blocked channel flow are compared with unsteady flows in a pipe generated by the controlled motion of a piston.

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