The full study of planar quadratic differential systems possessing exactly one line of singularities, finite or infinite

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Abstract

In this article we make a full study of the class of real quadratic differential systems \( \frac{dx}{dt} = p(x, y) \), \( \frac{dy}{dt} = q(x, y) \) where \( p, q \) are polynomials in \( x, y \) over \( \mathbb{R} \), \( \text{max}(\deg p, \deg q) = 2 \), \( \gcd(p, q) = 1 \) and such that all points at infinity in the Poincaré compactification are singularities.

We prove that all such systems have invariant affine lines of total multiplicity 3 and give all their Configurations. We show that all these systems are integrable via the method of Darboux. We construct all their topologically distinct phase portraits, give invariant necessary and sufficient conditions in terms of their 12 coefficients for the realization of each one of them and give representatives of the orbits under the action of the affine group and time rescaling. We cover the quotient space with two subspaces and give bifurcation diagrams for each one of them.

Résumé

Dans cet article nous faisons l’étude complète de la classe de tous les systèmes différentiels réels quadratiques \( \frac{dx}{dt} = p(x, y) \), \( \frac{dy}{dt} = q(x, y) \), \( \gcd(p, q) = 1 \), où \( p, q \) sont des polynômes en \( x, y \), \( \text{max}(\deg p, \deg q) = 2 \), \( \gcd(p, q) = 1 \), tels que tous les points à l’infini dans la compactification de Poincaré sont singuliers. Nous prouvons que tous ces systèmes ont des lignes affines invariantes de multiplicité totale 3 et nous donnons toutes leurs Configurations. Nous montrons que ces systèmes sont intégrables par la méthode de Darboux. Nous construisons leurs portraits de phases, donnons des conditions nécessaires et suffisantes, en termes de leurs 12 coefficients, pour la réalisation de chacun de ces portraits et nous donnons des représentatifs de leurs orbites sous l’action du groupe affine réel et des homothéties du temps. Nous donnons les diagrammes de bifurcations pour deux sous-espaces de l’espace quotient dont la réunion couvre tout l’espace.
1 Introduction

We consider here real planar differential systems of the form

$$\frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y),$$

(1.1)

where $p, q \in \mathbb{R}[x, y]$, i.e. $p, q$ are polynomials in $x, y$ over $\mathbb{R}$, their associated vector fields

$$\tilde{D} = p(x, y) \frac{\partial}{\partial x} + q(x, y) \frac{\partial}{\partial y}$$

(1.2)

and differential equations

$$q(x, y)dx - p(x, y)dy = 0.$$  

(1.3)

We call degree of a system (1.1) (or of a vector field (1.2) or of a differential equation (1.3)) the integer $m = \max(\deg p, \deg q)$. In particular we call quadratic a differential system (1.1) with $m = 2$.

Each such differential system generates a complex differential system when the variables range over $\mathbb{C}$. In (cf. [7]) Darboux gave a method of integration of complex differential equations (1.3) using invariant algebraic curves (see Definition 1.5).

Poincaré was enthusiastic about the work of Darboux [7], which he called "admirable" in [18]. This method of integration was applied to prove that several families of systems (1.1) are integrable. For example in [23] it was applied to show in a unified way, unlike previous proofs which used ad hoc methods (see for example [24]) the integrability of planar quadratic systems possessing a center.

A brief and easily accessible exposition of the method of Darboux can be found in the survey article [22].

In this article we consider another class of quadratic differential systems (1.1), namely the class of all quadratic systems for which all points at infinity are singularities of (1.1). We use the Poincaré compactification of (1.1) on the sphere and also the compactification of (1.3) in the real projective plane $\mathbb{P}_2(\mathbb{R})$ (for details see [27, 28]).

The goal of this article is to present a full study of this class by:

- proving that all systems in this class have invariant affine lines of total multiplicity three;

- constructing all configurations of invariant lines (see Definition 2.7);

- proving that all systems in this class are integrable via the method of Darboux yielding polynomial inverse integrating factors which split into linear factors over $\mathbb{C}$ and (elementary) Darboux first integrals (see Definitions 1.3 and 1.8 below);

- constructing all the topologically distinct phase portraits of the systems in this class (we have 28 such phase portraits);

- giving invariant (under the action of the affine group and time rescaling) necessary and sufficient conditions, in terms of the twelve coefficients of the systems, for each specific phase portrait to be realized;

- listing all the orbits of the action of the affine group and time rescaling, by determining representatives for each one of them;

- determining the representatives of the orbits of their associated projective differential equations under the action of the real projective group $PGL(2, \mathbb{R})$;

- giving bifurcations diagrams for the systems in this class and the corresponding moduli spaces.
The long term goal of this work and of the work done in [30] is to provide us with specific data to be used along with similar material for higher degree curves, for the purpose of solving some classical, very old problems among them the problem of Poincaré, stated by Poincaré in 1891 [18, 19]. This problem asks us to determine necessary and sufficient conditions for a planar polynomial differential system (1.1) or vector field (1.2) or differential equation (1.3) to have a rational first integral. Although much work has been done to solve it, this problem is still open. Partial results on this problem appeared in [11], [4] and [5].

We first need to recall some basic notions. In what follows we shall denote by $\mathbb{K}$ the field of real or of complex numbers depending on the situation.

**Definition 1.1.** A first integral on an open set $U \subseteq \mathbb{R}^2$ (respectively $U \subseteq \mathbb{C}^2$) of a real (respectively complex) system (1.1) is a $C^1$ function $F : U \rightarrow \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ (respectively $\mathbb{K} = \mathbb{C}$) such that for all solutions $(x(t), y(t))$ of (1.1) with $t$ in an open disk $\mathcal{D}_0$ in $\mathbb{K}$, such that $(x(t), y(t)) \in U$ for all $t \in \mathcal{D}_0$, we have $F(x(t), y(t)) = \text{constant}$.

**Remark 1.1.** We note that such a $C^1$ function $F : U \rightarrow \mathbb{K}$ is a first integral on $U$ of (1.1) if and only if for all solutions $(x(t), y(t))$ of (1.1) with $t$ in an open disk $\mathcal{D}_0$ in $\mathbb{K}$, such that $(x(t), y(t)) \in U$ for all $t \in \mathcal{D}_0$, we have $\frac{dF(x(t), y(t))}{dt} = 0$ for all $t \in \mathcal{D}_0$, or equivalently

$$\tilde{D}F \equiv p(x, y)\frac{\partial F}{\partial x} + q(x, y)\frac{\partial F}{\partial y} = 0. \quad (1.4)$$

**Definition 1.2.** An integrating factor of an equation (1.3) on an open subset $U$ of $\mathbb{R}^2$ is a $C^1$ function $R(x, y) \neq 0$ such that the 1-form

$$\omega = Rg(x, y)dx - Rp(x, y)dy$$

is exact, i.e. there exist a $C^1$ function $F : U \rightarrow \mathbb{K}$ on $U$ such that

$$\omega = dF. \quad (1.5)$$

**Remark 1.2.** We observe that if $R$ is an integrating factor on $U$ of (1.3) then the function $F$ such that $\omega = Rgdx - Rpdy = dF$ is a first integral of the equation $w = 0$ (or a system (1.1)). In this case we necessarily have on $U$:

$$\frac{\partial (Rg)}{\partial y} = -\frac{\partial (Rp)}{\partial x} \quad (1.6)$$

and developing the above equality we obtain

$$\frac{\partial R}{\partial x}p + \frac{\partial R}{\partial y}q = -R\left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)$$

or equivalently

$$\tilde{D}R = -R \text{ div } \tilde{D}. \quad (1.7)$$

Hence $R(x, y)$ is an integrating factor on $U$ of (1.3) if and only if (1.6) (respectively (1.7)) holds.

The method of Darboux, which is briefly described below, yields first integrals or integrating factors for complex differential equations (1.3) of the form:

$$F = f_1(x, y)^{\lambda_1} \cdots f_s(x, y)^{\lambda_s}, \quad f_i \in \mathbb{C}[x, y], \quad \lambda_i \in \mathbb{C}, \quad (1.8)$$

with $f_i$ irreducible over $\mathbb{C}$. It is clear that in general the expression (1.8) makes sense only for points $(x, y) \in \mathbb{C}^2 \setminus \{\{f_1(x, y) = 0\} \cap \cdots \cap \{f_s(x, y) = 0\}\}$.

The above expression (1.8) yields a many-valued (or multivalued) function in

$$\mathcal{U} = \mathbb{C}^2 \setminus \{\{f_1(x, y) = 0\} \cap \cdots \cap \{f_s(x, y) = 0\}\}.$$
For every point \((x_0, y_0) \in U\), there is a neighborhood of \((x_0, y_0)\) such that on this neighborhood the expression (1.8) yields an analytic function when choosing arbitrarily one of the many values of the logarithms of \(f_i(x, y), i = 1, \ldots, s\) in a small neighborhood of \(f_i(x_0, y_0)\). So for specific open subsets \(U \in \mathbb{C}^2\) the expression (1.8) is actually an analytic function. The function \(F\) in (1.8) is a first integral of (1.1) on \(U\) in case \(\bar{D}F = 0\).

The proper way of continuing this discussion is by introducing a bit of differential algebra. Differential algebra is a field of mathematics which was created by Joseph Fels Ritt and Ellis Kolchin. Ritt was interested in the transcendental functions defined by differential equations with algebraic coefficients and he published a book *Differential equations from the algebraic standpoint* in the Colloquium Series of the AMS [20]. A second book by him *Differential algebra* [21] appeared in the same series. The term *Differential Algebra* was introduced by Ellis Kolchin (see [3]). The interested reader may also consult [13] and [12].

We are concerned here with differential field extensions of the differential field \(\left(\mathbb{C}(x, y), \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\). For example \(f(x, y)^{1/2}\) is an expression of the form (1.8), when \(f \in \mathbb{C}[x, y]\). This function belongs to the differential field extension \(\left(\mathbb{C}(x, y), \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\) obtained by adjoining to \(\mathbb{C}(x, y)\) a root of the equation
\[
u^2 - f(x, y) = 0.
\]
In this way we obtain an algebraic differential field extension of \(\left(\mathbb{C}(x, y), \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\). The expression (1.8) belongs to a differential field extension which in general is not necessarily algebraic.

**Definition 1.3.** A system (1.1) or a vector field (1.2) or a differential equation (1.3) has an inverse integrating factor \(V \neq 0\) in a differential field extension \(K\) of \(\left(\mathbb{C}(x, y), \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\) if \(V^{-1}\) is an integrating factor of system (1.1) (or (1.2) or (1.3)), i.e. \(\bar{D}V = V \div \bar{D}\) (see (1.7)).

**Definition 1.4.** A first integral of (1.1) (or (1.2) or (1.3)) is a function \(F\) in a differential field extension \(K\) of \(\left(\mathbb{C}(x, y), \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\) such that \(\bar{D}F = 0\).

In 1878 Darboux introduced the notion of invariant algebraic curve in [7].

**Definition 1.5 (Darboux [7]).** An affine algebraic curve \(f(x, y) = 0, f \in \mathbb{C}[x, y], \deg f \geq 1\) is invariant for an equation (1.3) if and only if \(f | \bar{D}f\) in \(\mathbb{C}[x, y]\), i.e. \(k = \frac{\bar{D}f}{f} \in \mathbb{C}[x, y]\). In this case \(k\) is called the cofactor of \(f\).

**Definition 1.6 (Darboux [7]).** An algebraic solution of an equation (1.3) is an algebraic curve \(f(x, y) = 0, f \in \mathbb{C}[x, y]\) (deg \(f \geq 1\)) with \(f\) an irreducible polynomial over \(\mathbb{C}\).

Darboux showed that if an equation (1.3) possesses a sufficient number of such invariant algebraic solutions \(f_i(x, y) = 0, f_i \in \mathbb{C}[x, y], i = 1, 2, \ldots, s\) then the equation has a first integral of the form (1.8).

**Definition 1.7.** An expression of the form \(F = e^{G(x, y)}, G(x, y) \in \mathbb{C}(x, y)\) (i.e. \(G\) is a rational function over \(\mathbb{C}\)), is an exponential factor for a system (1.1) or for an equation (1.3) if and only if \(k = \frac{\bar{D}F}{F} \in \mathbb{C}[x, y]\). In this case \(k\) is called the cofactor of the exponential factor \(F\).

**Definition 1.8.** We say that a system (1.1) or an equation (1.3) has a Darboux first integral (respectively Darboux integrating factor) if it admits a first integral (respectively integrating factor) of the form
\[e^{G(x, y)} \prod_{i=1}^{s} f_i(x, y)^{\lambda_i},\]
where \(G(x, y) \in \mathbb{C}(x, y)\) and \(f_i \in \mathbb{C}[x, y]\), \(\deg f_i \geq 1, i = 1, 2, \ldots, s\), \(f_i\) irreducible over \(\mathbb{C}\) and \(\lambda_i \in \mathbb{C}\).
Proposition 1.1 (Darboux [7]). If an equation (1.3) has an integrating factor (or first integral) of the form \( F = \prod_{i=1}^{s} f_i^{\lambda_i} \) then \( \forall i \in \{1, \ldots, s\}, f_i = 0 \) is an algebraic invariant curve of (1.3).

In [7] Darboux proved the following remarkable theorem of integrability using invariant algebraic solutions of differential equation (1.3):

Theorem 1.1 (Darboux [7]). Consider a differential equation (1.3) with \( p, q \in \mathbb{C}[x, y] \). Let us assume that \( m = \max(\deg p, \deg q) \) and that the equation admits \( s \) algebraic solutions \( f_i(x, y) = 0, \ i = 1, 2, \ldots, s \) \( (\deg f_i \geq 1) \). We have:

I. If \( s \geq m(m + 1)/2 + 1 \) then there exists \( (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s \setminus \{0\} \) such that
\[
F = \prod_{i=1}^{s} f_i(x, y)^{\lambda_i}
\]
is a first integral of (1.3).

II. If \( s = m(m + 1)/2 \) then there exists \( (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s \setminus \{0\} \) such that
\[
R = \prod_{i=1}^{s} f_i(x, y)^{\lambda_i}
\]
\( R \) is a first integral or an integrating factor of (1.3).

Remark 1.3. We stated the theorem for an equation (1.3) but clearly we could have stated it for a vector field \( \tilde{D} \) (1.2) or for a polynomial differential system (1.1). We point out that Darboux’s work was done for differential equations in the complex projective space. The above formulation is an adaptation of his theorem for the complex affine space.

In [11] Jouanolou proved the following theorem which improves part I of Darboux’s Theorem.

Theorem 1.2 (Jouanolou [11]). Consider a polynomial differential equation (1.3) and assume that it has \( s \) algebraic solutions \( f_i(x, y) = 0, \ i = 1, 2, \ldots, s \) \( (\deg f_i \geq 1) \). Suppose that \( s \geq m(m + 1)/2 + 2 \). Then there exists \( (n_1, \ldots, n_s) \in \mathbb{Z}^s \setminus \{0\} \) such that \( F = \prod_{i=1}^{s} f_i(x, y)^{n_i} \) is a first integral of (1.3), hence \( F \) is a rational function over \( \mathbb{C} \) \( (F \in \mathbb{C}(x, y)) \).

Theorem 3.1. Consider the class \( QS_{C_2=0} \) of all real quadratic differential systems (2.1) with \( C_2(a, x, y) = yp_2(x, y) - xq_2(x, y) \equiv 0 \) and \( \gcd(p, q) = 1 \). Then:

- Every system in this class has invariant affine lines of total multiplicity three (see Definition 2.3).
- The quotient space of \( QS_{C_2=0} \) under the action of the real affine group and time rescaling is formed by (i) a set of five orbits and (ii) a set of four one-parameter families of orbits.
- We give in Table 1 nine Configurations of invariant lines corresponding to the equivalence classes (i) and (ii). The systems \( (C_2, i), \ i = 1, \ldots, 9 \) form a system of representatives of the orbits of \( QS_{C_2=0} \) under the group action. A differential system \( (S) \) in \( QS_{C_2=0} \) is in the orbit of a system belonging to \( (C_2, i) \) if and only if the corresponding invariant conditions in Table 1 are satisfied.

Theorem 3.2. Every system (2.1) in \( QS_{C_2=0} \) has a polynomial inverse integrating factor which is always the product of all polynomials defining the invariant lines, each one appearing in the product with its respective multiplicity (see Table 2). The system has a Darboux first integral in all the cases. Furthermore the Darboux first integral is of the form:

- \( \alpha \) \( \prod_{i=1}^{3} [f_i(x, y)]^{\lambda_i} \) if and only if the lines \( f_i \) are distinct (or \( f_i \)'s are simple, \( i = 1, 2, 3 \));
- \( \beta \) \( e^{C(x, y)} f_1^{\lambda_1} f_2^{\lambda_2} \) if and only if we have two distinct lines, one of them with multiplicity 2;
- \( \gamma \) it is a rational function if and only if we have a single invariant affine line with multiplicity 3.
Theorem 3.3. I. The systems in $\text{QS}_{c_2=0}$ are classified by 5 integer-valued invariants which are collected in the sequence $(N_c, N_a, d_{g,e}, O, J_f)$. We give the full classification under the group action in Diagram 1. According to the possible values of this invariant we obtain the types of the configurations of invariant lines and all the corresponding phase portraits of such systems. The necessary and sufficient conditions in $\mathbb{R}^{12}$ for the realization of each one of the eleven phase portraits are given in Diagram 2.

II. We have exactly 11 topologically distinct phase portraits (appearing in Diagram 3) in the class $\text{QS}_{c_2=0}$:

- Among these we have exactly four phase portraits each one with exactly one real singularity and such that this singularity is a strong focus, respectively a center, a node or an elliptic point;
- There are exactly two phase portraits without real singularities and exactly two with exactly one real singularity which is a saddle-node;
- The remaining three phase portraits have either two or three real singularities: two phase portraits with a saddle and either one or two nodes, and one with two nodes only.

Theorem 6.1. Let $(S)$ be an arbitrary system in $\text{QS}_{c_2=0}$ and let $a \in \mathbb{R}^{12}$ be the 12-tuple of its coefficients. Then there exists a small vicinity $U \ni a$, $U \subset \mathbb{R}^{12}$ such that:

i) for every point $\hat{a} \in U$ the corresponding system $(\hat{S})$ does not have invariant lines of total multiplicity $\geq 5$;

ii) $\exists \hat{a} \in U$ such that the corresponding system $(\hat{S})$ has invariant lines of total multiplicity four.

Remark 1.4. The phase portraits for the class $\text{QS}_{c_2=0}$ were given before in [9]. However, the issues raised by us here such as: presence of affine invariants lines of total multiplicity three, Darboux integrability, invariant necessary and sufficient conditions for the realization of each phase portrait, the structure of the quotient space under the group action, etc. were not considered in [9].

Some (but not all) of the phase portraits we obtain here were also obtained in [15] where the authors listed in particular, phase portraits of quadratic systems possessing exactly three affine invariant lines in generic position. We note here that in [15] it is claim that in Figure 1 and 2 are listed all topologically distinct such portraits. We observe however that the portraits 41 and 44 in Figure 2 are in fact topologically equivalent. A similar situation occurs for the portraits 45 and 46.

2 Preliminary statements and definitions

Consider real differential systems of the form:

\[ (S) \quad \begin{cases} \frac{dx}{dt} = p_0(a) + p_1(a, x, y) + p_2(a, x, y) \equiv p(a, x, y), \\ \frac{dy}{dt} = q_0(a) + q_1(a, x, y) + q_2(a, x, y) \equiv q(x, y), \end{cases} \quad (2.1) \]

where $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ and

\begin{align*}
    p_0 &= a_{00}, & p_1(x, y) &= a_{10}x + a_{01}y, & p_2(x, y) &= a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\
    q_0 &= b_{00}, & q_1(x, y) &= b_{10}x + b_{01}y, & q_2(x, y) &= b_{20}x^2 + 2b_{11}xy + b_{02}y^2.
\end{align*}

Let us introduce the notation:

Notation 2.1. $\mathbb{R}[a, x, y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$. Let us denote by $a = (a_{00}, a_{10}, \ldots, b_{02})$ a point in $\mathbb{R}^{12}$. Each particular system (2.1) yields an ordered 12-tuple $a$ formed by its coefficients.
2.1 Divisors associated to invariant lines configurations

Notation 2.2. Let
\[ P(X, Y, Z) = p_0(a)Z^2 + p_1(a, X, Y)Z + p_2(a, X, Y) = 0, \]
\[ Q(X, Y, Z) = q_0(a)Z^2 + q_1(a, X, Y)Z + q_2(a, X, Y) = 0. \]

We denote \( \sigma(P, Q) = \{ w \in \mathbb{P}_2(\mathbb{C}) \mid P(w) = Q(w) = 0 \} \).

Definition 2.1. We consider formal expressions \( D = \sum n(w)w \) where \( n(w) \) is an integer and only a finite number of \( n(w) \) are nonzero. Such an expression is called: i) a zero-cycle of \( \mathbb{P}_2(\mathbb{C}) \) if all \( w \) appearing in \( D \) are points of \( \mathbb{P}_2(\mathbb{C}) \); ii) a divisor of \( \mathbb{P}_2(\mathbb{C}) \) if all \( w \) appearing in \( D \) are irreducible algebraic curves of \( \mathbb{P}_2(\mathbb{C}) \); iii) a divisor of an irreducible algebraic curve \( \mathcal{C} \) in \( \mathbb{P}_2(\mathbb{C}) \) if all \( w \) in \( D \) belong to the curve \( \mathcal{C} \). We call degree of the expression \( D \) the integer \( \deg(D) = \sum n(w) \). We call support of \( D \) the set \( \text{Supp}(D) \) of \( w \) appearing in \( D \) such that \( n(w) \neq 0 \).

Notation 2.3. Let us denote by
\[ \text{QS} = \left\{ (S) \mid \begin{array}{c} (S) \text{ is a system (2.1) such that } \gcd(p(x, y), q(x, y)) = 1 \\ \text{and } \max(\deg(p(x, y)), \deg(q(x, y))) = 2 \end{array} \right\}; \]
\[ \text{QSL} = \left\{ (S) \in \text{QS} \mid (S) \text{ possesses at least one invariant affine line or the line at infinity with multiplicity at least two} \right\}. \]

In this section we shall assume that systems (2.1) belong to \( \text{QS} \).

Definition 2.2. Let \( C(X, Y, Z) = YP(X, Y, Z) - XP(X, Y, Z) \).
\[ D_S(P, Q) = \sum_{w \in \sigma(P, Q)} I_w(P, Q)w; \]
\[ D_S(C, Z) = \sum_{w \in \{Z=0\}} I_w(C, Z)w \text{ if } Z \nmid C(X, Y, Z); \]
\[ D_S(P, Q; Z) = \sum_{w \in \{Z=0\}} I_w(P, Q)w; \]
\[ \hat{D}_S(P, Q, Z) = \sum_{w \in \{Z=0\}} \left( I_w(C, Z), I_w(P, Q) \right) w, \]

where \( I_w(F, G) \) is the intersection number (see, [8]) of the curves defined by homogeneous polynomials \( F, G \in \mathbb{C}[X, Y, Z] \) \( (\deg(F), \deg(G) \geq 1) \).

To an affine planar differential system (1.1) we can associate a differential equation in the projective plane (see for example [30]):
\[ Q(X, Y, Z)ZdX - P(X, Y, Z)ZdY + C(X, Y, Z)dX = 0 \] (2.2)

A complex projective line \( uX + vY + wZ = 0 \) is invariant for the system \( (S) \) if either it coincides with \( Z = 0 \) or if it is the projective completion of an invariant affine line \( ux + vy + w = 0 \).

Definition 2.3. We say that an invariant straight line \( \mathcal{L}(x, y) = ux + vy + w = 0, (u, v) \neq (0, 0) \), \( (u, v, w) \in \mathbb{C}^3 \) for a quadratic vector field \( \hat{D} \) has multiplicity \( m \) if there exists a sequence of real quadratic vector fields \( \hat{D}_k \) converging to \( \hat{D} \), such that each \( \hat{D}_k \) has \( m \) distinct (complex) invariant straight lines \( \mathcal{L}_k^1 = 0, \ldots, \mathcal{L}_k^m = 0 \), converging to \( \mathcal{L} = 0 \) as \( k \to \infty \) (with the topology of their coefficients), and this does not occur for \( m + 1 \).
Definition 2.4. We say that $Z = 0$ is an invariant line of multiplicity $m$ for a quadratic system $(S)$ of the form (2.1) if and only if there exists a sequence of quadratic systems $(S_i)$ of the form (2.1) such that $(S_i)$ tend to $(S)$ when $i \to \infty$ and the systems $(S_i)$ have $m - 1$ distinct invariant affine lines $L_i = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0, 0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbb{C}^3$ $(j = 1, \ldots, m - 1)$ such that for every $j$, $\lim_{i \to \infty} (u_i^j, v_i^j, w_i^j) = (0, 0, 1)$ and they do not have $m$ invariant such lines $L_i$, $j = 1, \ldots, m$ satisfying the above mentioned conditions.

Proposition 2.1. [1] The maximum number of invariant lines (including the line at infinity and including multiplicities) which a quadratic system (2.1) with $\gcd(p, q) = 1$ could have is six.

Notation 2.4. Let $S \in \text{QSL}$. Let us denote by

$$IL(S) = \left\{ l \mid l \text{ is a line in } \mathbb{P}_2(\mathbb{C}) \text{ such that } l \text{ is invariant for } (S) \right\};$$

$M(l) =$ the multiplicity of the invariant line $l$ of $(S)$ in $\mathbb{P}_2(\mathbb{C})$.

Remark 2.5. We note that the line $l_\infty : Z = 0$ is included in $IL(S)$ for any $(S) \in \text{QSL}$. Let $l_i : f_i(x, y) = 0$, $i = 1, \ldots, k$, be all the distinct invariant affine lines (real or complex) of a system $S \in \text{QSL}$. Let $l_i' : F_i(X, Y, Z) = 0$ be the complex projective completion of $l_i$.

Notation 2.6. We denote

$$\mathcal{G} : \prod_i F_i(X, Y, Z) Z = 0; \quad \text{Sing} \mathcal{G} = \{ w \in \mathcal{G} \mid w \text{ is a singular point of } \mathcal{G} \};$$

$\nu(w) =$ the multiplicity of the point $w$, as a point of $\mathcal{G}$.

Definition 2.5.

$$D_{IL}(S) = \sum_{l \in IL(S)} M(l), \quad (S) \in \text{QSL};$$

$$\text{Supp} D_{IL}(S) = \{ l \mid l \in IL(S) \}. $$

Notation 2.7.

$$M_{IL} = \deg D_{IL}(S);$$

$$N_c = \# \text{Supp} D_{IL};$$

$$N_r = \# \{ l \in \text{Supp} D_{IL} \mid l \in \mathbb{P}_2(\mathbb{R}) \};$$

$$d_{\mathcal{G}, \sigma}^2 = \sum_{\omega \in \mathcal{G}} I_{\omega}(P, Q). \quad (2.3)$$

Definition 2.6. Consider a quadratic system possessing a singularity which is a focus or a center. It is known that such a system can be brought under the group action to the form

$$\dot{x} = cx - dy + p_2(x, y),$$

$$\dot{y} = dx + cy + q_2(x, y).$$

If $c \neq 0$ we say that the singularity at $(0, 0)$ is a strong focus. If $c = 0$ then the singularity is a weak focus.

Notation 2.8.

$$O(S) = \begin{cases} 
1 & \text{iff the system } (S) \text{ possesses at least one center;} \\
0 & \text{iff the system } (S) \text{ does not have a center.} 
\end{cases}$$
Notation 2.9. \[ \mathcal{J}_f(S) = \prod_{w \in \sigma(p, q)} i_w \]
where \( i_w \) is the Poincaré index of \( w \).

Definition 2.7. We call configuration of invariant lines of a system \( (S) \) in QSL the set of all its invariant lines (real or complex), each endowed with its own multiplicity and together with all the real isolated singular points of \( (S) \) located on these lines, each one endowed with its own multiplicity.

Notation 2.10. Let us denote by
\[ \text{QS}_{C_2=0} = \left\{ (S) \mid (S) \text{ is a system (2.1) such that } \gcd(p(x, y), q(x, y)) = 1, \max\{\deg(p(x, y)), \deg(q(x, y))\} = 2 \text{ and } C_2(a, x, y) \equiv 0 \right\}. \]

Notation 2.11. We denote by \( \text{QSL}_0 \) (respectively \( \text{QSL}_5 \)) the class of all quadratic differential systems (2.1) in QS with \( Z \upharpoonright C \), and possessing a configuration of invariant straight lines of total multiplicity \( M_\text{IL} = 6 \) (respectively \( M_\text{IL} = 5 \)) including the line at infinity and including possible multiplicities of the lines.

We also shall use the following remark:

Remark 2.12. Assume \( s, \gamma \in \mathbb{R}, \gamma > 0 \). Then the transformation \( x = \gamma^s x_1, y = \gamma^s y_1 \) and \( t = \gamma^{-s} t_1 \) does not change the coefficients of the quadratic part of a quadratic system, whereas each coefficient of the linear (respectively constant) part will be multiplied by \( \gamma^{-s} \) (respectively by \( \gamma^{-2s} \)).

2.2 The main \( T \)-comitants associated to configurations of invariant lines

Let us consider the polynomials
\[
C_i(a, x, y) = yp_i(a, x, y) - xq_i(a, x, y) \in \mathbb{R}[a, x, y], \quad i = 0, 1, 2,
\]
\[
D_i(a, x, y) = \frac{\partial}{\partial x} p_i(a, x, y) + \frac{\partial}{\partial y} q_i(a, x, y) \in \mathbb{R}[a, x, y], \quad i = 1, 2.
\]

As it was shown in [31] the polynomials
\[
\left\{ C_0(a, x, y), \ C_1(a, x, y), \ C_2(a, x, y), \ D_1(a), \ D_2(a, x, y) \right\}
\]
(2.4)
of degree one in the coefficients of systems (2.1) are \( GL \)-comitants of these systems (for detailed definitions of the polynomials which are invariant under the action of the group of affine transformations see [31], [27], [28]). In order to construct other necessary invariant polynomials let us consider the differential operator \( \mathcal{L} = x \cdot L_2 - y \cdot L_1 \) acting on \( \mathbb{R}[a, x, y] \) constructed in [2], where
\[
L_1 = 2a_{00} \frac{\partial}{\partial a_{10}} + a_{10} \frac{\partial}{\partial a_{20}} + \frac{1}{2} a_{01} \frac{\partial}{\partial a_{11}} + 2b_{00} \frac{\partial}{\partial b_{10}} + b_{10} \frac{\partial}{\partial b_{20}} + \frac{1}{2} b_{01} \frac{\partial}{\partial b_{11}},
\]
\[
L_2 = 2a_{00} \frac{\partial}{\partial a_{01}} + a_{01} \frac{\partial}{\partial a_{02}} + \frac{1}{2} a_{10} \frac{\partial}{\partial a_{11}} + 2b_{00} \frac{\partial}{\partial b_{01}} + b_{01} \frac{\partial}{\partial b_{02}} + \frac{1}{2} b_{10} \frac{\partial}{\partial b_{11}}
\]
as well as the classical differential operator \( (f, \varphi)^{(k)} \) acting on \( \mathbb{R}[x, y]^2 \) which is called the transvectant of index \( k \) (see, for example, [10, 17]):
\[
(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h {k \choose h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.
\]
Notation 2.13. Consider the polynomial $\Phi_{\alpha, \beta} = \alpha P + \beta Q \in \mathbb{R}[a, X, Y, Z, \alpha, \beta]$ where $P = Z^2 p(X/Z, Y/Z)$, $Q = Z^2 q(X/Z, Y/Z)$, $p, q \in \mathbb{R}[a, x, y]$ and $\max(\deg(x,y)p, \deg(x,y)q) = 2$. Then

$$\Phi_{\alpha, \beta} = c_{11}(a, \alpha, \beta)X^2 + 2c_{12}(a, \alpha, \beta)XY + c_{22}(a, \alpha, \beta)Y^2 + 2c_{13}(a, \alpha, \beta)XZ + 2c_{23}(a, \alpha, \beta)YZ + c_{33}(a, \alpha, \beta)Z^2,$$

$$\Delta(a, \alpha, \beta) = \det|c_{ij}(a, \alpha, \beta)|_{i,j \in \{1,2,3\}},$$

$$D(a, x, y) = 4\Delta(a, y, -x), \quad H(a, x, y) = 4\left[\det |c_{ij}(a, y, -x)|_{i,j \in \{1,2\}} \right].$$

Let us consider the following GL-comitants of systems (2.1):

Notation 2.14.

$$M(a, x, y) = 2 \text{Hessian } (C_2(x, y)), \quad \eta(a) = \text{Discriminant } (C_2(x, y)),$$

$$K(a, x, y) = \text{Jacobian } (p_2(x, y), q_2(x, y)), \quad \mu(a) = \text{Discriminant } (K(a, x, y)),$$

$$N(a, x, y) = K(a, x, y) + H(a, x, y), \quad \kappa(a) = \text{Discriminant } (N(a, x, y)).$$

Remark 2.15. We note that by the Discriminant $(C_2)$ of the cubic form $C_2(a, x, y)$ we mean the expression given in Maple via the function "discrim$(C_2, x)/y^6$".

By using the operators above and the GL-comitants $\mu(a), M(a, x, y), K(a, x, y), D_i(a, x, y)$ and $C_i(a, x, y)$ we construct the following polynomials:

$$\mu_0 = \mu/16, \quad \mu_i(a, x, y) = \frac{1}{4i!} L^{(i)}(\mu_0), \quad i = 1, ..., 4, \quad \kappa(a) = (M, K)^{(2)};$$

$$K_2 = 4 \text{Jacob}(J_2, \xi) + 3 \text{Jacob}(C_1, \xi)D_1 - \xi(16J_1 + 3J_3 + 3D_1^2),$$

where $L^{(i)}(\mu_0) = L(L^{(i-1)}(\mu_0))$ and

$$J_1 = \text{Jacob}(C_0, D_2), \quad J_2 = \text{Jacob}(C_0, C_2), \quad J_3 = \text{Discrim}(C_1), \quad J_4 = \text{Jacob}(C_1, D_2), \quad \xi = M - 2K.$$

Note. We use here the same notations as introduced in [28].

Proposition 2.2. Consider $m \leq 3$ distinct directions in the affine plane, where by direction we mean a point $[v : -u] \in \mathbb{P}_1(\mathbb{C})$. A necessary condition to have for each one of these directions an invariant line with that direction is that there exist in distinct common factors of the polynomials $C_2(a, x, y)$ and $D(a, x, y)$ over $\mathbb{C}$.

Notation 2.16.

$$B_3(a, x, y) = (C_2, D)^{(1)} = \text{Jacob}(C_2, D),$$

$$B_2(a, x, y) = (B_3, B_3)^{(2)} - 6B_3(C_2, D)^{(3)},$$

$$B_1(a) = \text{Res}_x (C_2, D) / y^9 = -2^{-9}3^{-8}(B_2, B_3)^{(4)}.$$  \quad (2.7)

Lemma 2.1 ([26]). (i) For a quadratic system $(S)$ of coefficients $a \in \mathbb{R}^{12}$ in QSL$_6$ the conditions $N(a, x, y) = 0$ and $B_3(a, x, y) = 0$ in $\mathbb{R}[x, y]$, are satisfied.

(ii) If for a quadratic system $(S)$ $M_{1L} = 5$, then for this system one of the following two conditions is satisfied:

$$N(a, x, y) = 0 = B_2(a, x, y) \text{ in } \mathbb{R}[x, y]; \quad (ii) \quad \theta(a) = 0 = B_3(a, x, y) \text{ in } \mathbb{R}[x, y].$$

Let us apply a translation $x = x' + x_0, y = y' + y_0$ to the polynomials $p(a, x, y)$ and $q(a, x, y)$. We obtain $\tilde{p}(\tilde{a}(a, x_0, y_0), x', y') = p(a, x' + x_0, y' + y_0)$, $\tilde{q}(\tilde{a}(a, x_0, y_0), x', y') = q(a, x' + x_0, y' + y_0)$, where $\tilde{a}(a, x_0, y_0)$ is the 12-tuple of coefficients of the transformed system via the transformation. Let us construct the following polynomials

$$\Gamma_i(a, x_0, y_0) \equiv \text{Res}_{x'} \left( C_i(\tilde{a}(a, x_0, y_0), x', y'), C_0(\tilde{a}(a, x_0, y_0), x', y') \right) / (y')^{i+1},$$

$$\Gamma_i(a, x_0, y_0) \in \mathbb{R}[a, x_0, y_0], \quad (i = 1, 2).$$
Notation 2.17. 
\[ \tilde{E}_i(a, x, y) = \Gamma_i(a, x_0, y_0) \mid_{x_0 = x, y_0 = y} \in \mathbb{R}[a, x, y] \quad (i = 1, 2). \] (2.8)

Observation 2.18. It can easily be checked (see [26]) that the constructed polynomials \( \tilde{E}_1(a, x, y) \) and \( \tilde{E}_2(a, x, y) \) are affine comitants of systems (2.1) and are homogeneous polynomials in coefficients \( a_{00}, \ldots, b_{02} \) and non-homogeneous in \( x, y \) and \( \deg_a \tilde{E}_1 = 3, \deg_{(x,y)} \tilde{E}_1 = 5, \deg_a \tilde{E}_2 = 4, \deg_{(x,y)} \tilde{E}_2 = 6. \)

Notation 2.19. Let \( \mathcal{E}_1(a, X, Y, Z) \) (\( i = 1, 2 \)) be the homogenization of \( \tilde{E}_1(a, x, y) \), i.e.
\[ \mathcal{E}_1(a, X, Y, Z) = \tilde{E}_1(a, X/Z, Y/Z), \quad \mathcal{E}_2(a, X, Y, Z) = \tilde{E}_2(a, X/Z, Y/Z) \]
and \( \mathcal{H}(a, X, Y, Z) = \gcd(\mathcal{E}_1(a, X, Y, Z), \mathcal{E}_2(a, X, Y, Z)) \).

Theorem 2.1 ([26]). I. A straight line \( \mathcal{L}(x, y) = ax + by + c = 0, \quad a, b, c \in \mathbb{C}, \quad (a, b) \neq (0, 0) \) is an invariant line for a system (2.1) in QS corresponding to a point \( a \in \mathbb{R}^1 \) if and only if the polynomial \( \mathcal{L} \) is a common factor of the polynomials \( \tilde{E}_1(a, x, y) \) and \( \tilde{E}_2(a, x, y) \) over \( \mathbb{C} \), i.e.
\[ \tilde{E}_1(a, x, y) = (ax + by + c)^k \tilde{W}_i(x, y) \in \mathbb{C}[x, y] \quad (i = 1, 2). \]

II. If \( \mathcal{L}(x, y) = ax + by + c = 0, \quad a, b, c \in \mathbb{C}, \quad (a, b) \neq (0, 0) \) is an invariant straight line of multiplicity \( k \) for a quadratic system (2.1) then \( \mathcal{L}(x, y)^k \mid \gcd(\tilde{E}_1, \tilde{E}_2) \), i.e. there exist \( \tilde{W}_i(a, x, y) \in \mathbb{C}[x, y] \) (\( i = 1, 2 \)) such that
\[ \tilde{E}_1(a, x, y) = (ax + by + c)^k \tilde{W}_i(a, x, y), \quad i = 1, 2. \] (2.9)

III. If the line \( L \cap Z = 0 \) is of multiplicity \( k > 1 \) for a system (2.1) then \( Z^{k-1} \mid \gcd(\tilde{E}_1, \tilde{E}_2) \).

Remark 2.20. Let \( f(x, y) = 0 \) be an invariant algebraic curve for a polynomial system (1.1) where \( f(x, y) \) is an irreducible polynomial in \( x \) and \( y \) over \( \mathbb{C} \) of degree \( k \). In [6] it was proved that if the curve \( f(x, y) = 0 \) is of the multiplicity \( s \) for this system then
\[ [f(x, y)]^s \mid \mathcal{E}_k(x, y) \] (2.10)
where
\[ \mathcal{E}_k(x, y) = \begin{vmatrix} 1 & x & x^2 & \ldots & x^k & x^{k-1}y & \ldots & y^k \\ 0 & D_x & D_y & D_{x^2} & \ldots & D_{x^k} & D_{x^{k-1}}y & \ldots & D_y^k \\ 0 & D^2_x & D^2_y & D^2_{x^2} & \ldots & D^2_{x^k} & D^2_{x^{k-1}}y & \ldots & D^2_y \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & D^{k'}_x & D^{k'}_y & D^{k'}_{x^2} & \ldots & D^{k'}_{x^k} & D^{k'}_{x^{k-1}}y & \ldots & D^{k'}_y \end{vmatrix} \]
where \( k' = \frac{(k+1)(k+2)}{2} - 1 \) and \( D^i h = D^{i-1}(Dh) \). In the case of the invariant line (i.e. \( k = 1 \)) for a quadratic system (2.1) the polynomial \( \mathcal{E}_1(x, y) \) coincides with \( \mathcal{E}_1(x, y) \) defined above (see (2.8)).

We stress that the condition (2.10) in [6] is only a necessary condition. If in this particular case of an invariant line \( f = 0 \) for a quadratic system (2.1) we add the condition that \( [f(x, y)]^s \mid \mathcal{E}_2(x, y) \), these conditions jointly taken become necessary and sufficient for the line \( f = 0 \) to be of multiplicity \( s \).

In order to determine the existence of a common factor of the polynomials \( \mathcal{E}_1(a, X, Y, Z) \) and \( \mathcal{E}_2(a, X, Y, Z) \) we shall use the notion of the resultant of two polynomials with respect to a given indeterminate (see for instance, [32]).

Let us consider two polynomials \( f, g \in R[x_1, x_2, \ldots, x_r] \) where \( R \) is a unique factorization domain. Then we can regard the polynomials \( f \) and \( g \) as polynomials in \( x_r \) over the ring \( R = R[x_1, x_2, \ldots, x_{r-1}] \), i.e.
\[ f(x_1, x_2, \ldots, x_r) = a_0 + a_1 x_r + \ldots + a_n x_r^n, \]
\[ g(x_1, x_2, \ldots, x_r) = b_0 + a_1 x_r + \ldots + b_m x_r^m \quad a_i, b_i \in R. \]

Lemma 2.2. [32] Assuming \( n, m > 0 \), \( a_n b_m \neq 0 \) the resultant \( \text{Res}_{x_r}(f, g) \) of the polynomials \( f \) and \( g \) with respect to \( x_r \) is a polynomial in \( R[x_1, x_2, \ldots, x_{r-1}] \) which is zero if and only if \( f \) and \( g \) have a common factor involving \( x_r \).
3 The class of quadratic systems with all points at infinity as singularities

In this section we consider the class of all quadratic differential systems with the line at infinity filled up with singularities, i.e. the class $\text{QS}_{c_2=0}$. As for these systems the conditions $C_2(\mathbf{a}, x, y) = 0$ holds in $\mathbb{R}[x, y]$ then clearly the affine comitant $\hat{\mathcal{E}}_2(\mathbf{a}, x, y)$ vanishes (see Notation 2.17). We shall prove that in this case the number and the multiplicities of the invariant affine lines of a systems in $\text{QS}_{c_2=0}$ is governed completely by the remaining affine comitant $\hat{\mathcal{E}}_1(\mathbf{a}, x, y)$ (see Notation 2.17). To prove this we could use Theorem 2.1 and the fact that in this case $C_2(\mathbf{a}, x, y) \equiv 0$ to obtain the following

**Proposition 3.1.** A straight affine line $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant line for a system (2.1) in $\text{QS}_{c_2=0}$ corresponding to a point $\mathbf{a} \in \mathbb{R}^{12}$ if and only if the polynomial $\mathcal{L} \mid \hat{\mathcal{E}}_1$, i.e.

$$\hat{\mathcal{E}}_1(\mathbf{a}, x, y) = (ux + vy + w)W(x, y) \in \mathbb{C}[x, y].$$

For the sake of completeness we give below a direct proof of this proposition. We first prove the following lemma:

**Lemma 3.1.** A straight line $\hat{\mathcal{L}}(x, y) \equiv ux + vy = 0$ is an invariant line of a system (2.1) in $\text{QS}_{c_2=0}$ of coefficients $\mathbf{a}$ with $a^2 + b^2 \neq 0$ if and only if $C_0(\mathbf{a}, -v, u) = 0$ and $C_1(\mathbf{a}, -v, u) = 0$. These conditions are equivalent to the following one:

$$\text{Res}_x(C_1(a, x, y), C_0(a, x, y))/y^2(\mathbf{a}) = 0. \quad (3.1)$$

**Proof of the Lemma 3.1:** According to Definition 1.5 the line $\hat{\mathcal{L}}(x, y) = 0$ is invariant for a system (2.1) if and only if the identity $\hat{D}\hat{\mathcal{L}}(x, y) = \hat{\mathcal{L}}(x, y)S(x, y)$ holds for this system and this line. So in this case

$$u(p_0(\mathbf{a}) + p_1(\mathbf{a}, x, y) + p_2(\mathbf{a}, x, y)) + v(q_0(\mathbf{a}) + q_1(\mathbf{a}, x, y) + q_2(\mathbf{a}, x, y)) = (ux + vy)(S_0 + S_1(x, y)),$$

for some $S_0 \in \mathbb{C}$ and $S_1 \in \mathbb{C}[x, y]$. Herein we obtain:

(i) \quad $wp_0(\mathbf{a}) + vp_0(\mathbf{a}) = 0$;  
(ii) \quad $wp_1(\mathbf{a}, x, y) + vp_1(\mathbf{a}, x, y) = (ux + vy)S_0(\mathbf{a})$;  
(iii) \quad $wp_2(\mathbf{a}, x, y) + vp_2(\mathbf{a}, x, y) = (ux + vy)S_1(\mathbf{a}, x, y)$.

We observe that, if $x = -v$ and $y = u$ then the right-hand sides of these identities vanish. At the same time the left-hand sides of (i), (ii) and (iii) become $C_0(\mathbf{a}, -v, u)$, $C_1(\mathbf{a}, -v, u)$ and $C_2(\mathbf{a}, -v, u)$, respectively. But since the system $S(\mathbf{a})$ belongs to the class $\text{QS}_{c_2=0}$ then $C_2(\mathbf{a}, -v, u) = 0$ in $\mathbb{R}[u, v]$. Therefore the following equations are obtained:

$$C_0(\mathbf{a}, -v, u) = 0, \quad C_1(\mathbf{a}, -v, u) = 0. \quad (3.2)$$

As the degree of $C_0(\mathbf{a}, x, y)$ is one, the relation (3.1) hold.

**Proof of the Proposition 3.1:** Consider a straight line $\mathcal{L}(x, y) \equiv ux + vy = 0$. Let $(x_0, y_0) \in \mathbb{R}^2$ be any fixed non-singular point of the systems (2.1) in $\text{QS}_{c_2=0}$ (i.e. $p(x_0, y_0)^2 + q(x_0, y_0)^2 \neq 0$) which lies on the line $\mathcal{L}(x, y) = 0$, i.e. $ux_0 + vy_0 + w = 0$. Let $\tau_0$ be the translation $x = x' + x_0$, $y = y' + y_0$, $\tau_0(x', y') = (x, y)$. Then

$$\mathcal{L}(x, y) = \mathcal{L}(x' + x_0, y' + y_0) = ux' + vy' \equiv \hat{\mathcal{L}}(x', y')$$

and consider the line $ux' + vy' = 0$. By Lemma 3.1 the straight line $\hat{\mathcal{L}}(x', y') = 0$ will be an invariant line of the transformed system (2.1)$^{\tau_0}$ in $\text{QS}_{c_2=0}$ if and only if the condition (3.1) holds for this system, i.e. $\Gamma_1(\mathbf{a}, x_0, y_0)) = 0$ for each point $(x_0, y_0)$ located on the line $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, since the relation $ux_0 + vy_0 + w = 0$ is satisfied.

Thus we have $\Gamma_1(\mathbf{a}, x_0, y_0) = (ux_0 + vy_0 + w)W(a, x_0, y_0)$. Taking into account the notations (2.8) we conclude that the statement of Proposition 3.1 is true. ■
Lemma 3.2. Consider real quadratic systems in QS. Assume that a system (S) belongs to the class \( QS_{C_2=0} \). Then the system (S) can be brought via an affine transformation to the canonical form:

\[
\dot{x} = k + cx + dy + x^2, \quad \dot{y} = l + xy.
\]  

(3.3)

Proof: According to Lemma 4.7 of [26] a quadratic system with \( C_2 \equiv 0 \) can be brought via a linear transformation to the canonical form:

\[
\dot{x} = k + cx + dy + x^2, \quad \dot{y} = l + ex + fy + xy.
\]

Using the translation \( x \rightarrow x - f \) and \( y \rightarrow y - e \) in the above systems we obtain the systems (3.3).

Theorem 3.1. Consider the class \( QS_{C_2=0} \). Then:

- Every system in this class has invariant affine lines of total multiplicity three.
- The quotient space of \( QS_{C_2=0} \) under the action of the real affine group and time rescaling is formed by (i) a set of five orbits and (ii) a set of four one-parameter families.
- We give in Table 1 the nine Configurations of invariant lines corresponding to the equivalence classes in (i) and (ii). If an invariant line has multiplicity \( k > 1 \), then the number \( k \) appears next to the line, indicated in bold face in the picture. We indicate next to the finite real singularities, their multiplicities \( \{(p,q)\} \). The systems \( (C_2,i), i = 1, \ldots, 9 \) form a system of representatives of the orbits of \( QS_{C_2=0} \) under the group action. A differential system \( (S) \) in \( QS_{C_2=0} \) is in the orbit of a system belonging to \( (C_2,i) \) if and only if the corresponding invariant conditions in Table 1 are satisfied.

Proof: Let us now discuss the existence of affine invariant lines for systems in in \( QS_{C_2=0} \). According to Lemma 3.2 for such systems it is sufficient to consider the canonical form (3.3), i.e.

\[
\dot{x} = k + cx + dy + x^2, \quad \dot{y} = l + xy.
\]

(3.4)

We calculate (see Notation 2.17) for (3.4):

\[
\hat{E}_1(a, x, y) = 2[lx^3 - kx^2y - cdx^2 - dy^2 + 2cl^2 + (3dl - c^2)xy - 2dy^2 + l(c^2 + k)x + (cdl - c^2)y + l(cl + dl)].
\]

1) If \( d \neq 0 \) then and without loss of generality we may consider \( d = 1 \) via Remark 2.12 ( \( \gamma = d, \ s = 1 \)). Moreover, by replacing the parameter \( l \) by a new parameter \( u \) as follows: \( l = -u^2 + cu^2 - ku \) we obtain the following factorization of polynomial \( \hat{E}_1 \):

\[
\hat{E}_1 = 2(ux + y + cu - u^2)(c_{11}x^2 + 2c_{12}xy + c_{22}y^2 + 2c_{13}x + 2c_{23}y + c_{33}) = 2\hat{L}(x, y)F(x, y),
\]

where \( \hat{L}(x, y) \) and \( F(x, y) \) depend on 3 real parameters \( c, k, u \) and we have

\[
c_{11} = cu - u^2 - k, \quad c_{12} = \frac{u - c}{2}, \quad c_{13} = -1, \quad c_{22} = \frac{(c + u)c_{11}}{2}, \quad c_{23} = \frac{c_{11} - k}{2}, \quad c_{33} = (k + u^2)c_{11}.
\]

It can easily be verified that the determinant

\[
\begin{vmatrix}
c_{11} & c_{12} & c_{13} \\
c_{12} & c_{22} & c_{23} \\
c_{13} & c_{23} & c_{33}
\end{vmatrix}
\]

of the conic \( F(x, y) = 0 \) is identically zero. We conclude that this conic is reducible over \( \mathbb{C} \). Hence the polynomial \( \hat{E}_1 \) has three linear factors over \( \mathbb{C} \) (which can coincide) and according to Proposition 3.1 each factor of the polynomial \( \hat{E}_1 \) yields an invariant affine line. Since this polynomial contains the term \(-y^2\) we conclude that for \( d \neq 0 \) systems (3.4) possesses invariant affine lines of the total multiplicity three.
2) Assume now that $d = 0$. Then for the systems (3.4) we have

$$\dot{\xi}_1 = 2(x^2 + cx + k)(lx - ky + cl).$$

We observe that $l^2 + k^2 \neq 0$, otherwise for the systems (3.4) with $d = 0$ we have $gcd(p, q) = x$ and these systems do not belong to the class $QS_{c_2=0}$. Hence in the case $d = 0$ systems (3.4) possess invariant affine lines of the total multiplicity three, too.

In what follows we construct the orbit representatives and the respective invariant line Configurations for the systems in the class $QS_{c_2=0}$. At the same time the respective conditions for the realization of each such configuration will be determined. For this purpose we shall use the $T$-comitants constructed in [29]

$$H_9(a) = -\left((D, D(2), D_3(1)D_3\right)^{(3)},$$

$$H_{10}(a) = ((D_1, D(2), D_2)^{(1)},$$

$$H_{11}(a, x, y) = 8H [(C_2, D(2) + (D, D_2)^{(1} + 3H^2,$$

as well as a new $T$-comitant:

**Notation 3.1.** Let us denote

$$H_{12}(a, x, y) = (D, D)^{(2)} \equiv \text{Hessian} (D).$$
Consider the class of quadratic systems $\text{QS}_{c_2=0}$. According to Lemma 3.2 for such systems it is sufficient to consider the canonical form (3.3), i.e.
\[
\dot{x} = k + cx + dy + x^2, \quad \dot{y} = l + xy,
\]
for which we have $H_{10}(a) = 36d^2$. We observe that the condition $d = 0$ implies the existence of the invariant lines $k + cx + x^2 = 0$ of systems (3.5). Therefore we shall consider two cases: $H_{10}(a) \neq 0$ and $H_{10}(a) = 0$.

### 3.1 The case $H_{10}(a) \neq 0$

Then $d \neq 0$ and without loss of generality we may consider $d = 1$ via Remark 2.12 $(\gamma = d, \ s = 1)$. Moreover, by replacing the parameter $l$ by a new parameter $u$ as follows: $l = -u^3 + cu^2 - ku$ we obtain the systems
\[
\dot{x} = k + cx + y + x^2, \quad \dot{y} = u(cu - k - u^2) + xy,
\]
which possess the invariant affine line $ux + y + u(c - u) = 0$. Therefore applying the affine transformation
\[
x_1 = x + u, \quad y_1 = ux + y + u(c - u)
\]
systems (3.6) will be brought to the simpler form:
\[
\dot{x} = k_1 + c_1 x + y + x^2, \quad \dot{y} = xy.
\]
It is now convenient to use new parameters $(c, u)$ where $c = c_1/2, \ u = c_1^2 - k_1$. Thus we obtain the systems
\[
\dot{x} = (x + c)^2 - u + y, \quad \dot{y} = xy
\]
with the invariant line $y = 0$. For systems (3.7) calculations yields:
\[
D = -y[(cx + y)^2 - ux^2], \quad H_9 = -2304u(c^2 - u)^2.
\]

#### 3.1.1 The subcase $H_9(a) < 0$

In this case $u > 0$ and we may assume $u = 1$ via the transformation $x = u^{1/2}x_1, \ y = uy_1$ and $t = u^{-1/2}t_1$. This leads to the systems
\[
\dot{x} = (x + c)^2 - 1 + y, \quad \dot{y} = xy
\]
for which we have
\[
\tilde{E}_1 = -2y[(c + 1)x + y + c^2 - 1][(c - 1)x + y + c^2 - 1], \quad H_9 = -2^83^2(c^2 - 1)^2,
\]
and according to Proposition 3.1 systems (3.9) with $v = 1$ possess the following three real invariant affine lines:
\[
y = 0, \quad (c \pm 1)x + y + c^2 - 1 = 0.
\]
Since $H_9 \neq 0$ we obtain that the directions $[1 : 0], [-1 : (c - 1)]$ and $[-1 : (c + 1)]$ of these lines are distinct. Moreover, there are three distinct points of intersection of these lines and we get Config. $C_{2.1}$ (see Table 1).

#### 3.1.2 The subcase $H_9(a) > 0$.

Then $u < 0$ and in the same manner as above we may assume $u = -1$ via the transformation $x = (-u)^{1/2}x_1, \ y = -uy_1$ and $t = (-u)^{-1/2}t_1$. This leads to the systems
\[
\dot{x} = (x + c)^2 + 1 + y, \quad \dot{y} = xy
\]
which possess the following three invariant affine lines:
\[
y = 0, \quad (c \pm i)x + y + c^2 + 1 = 0.
\]
Taking into account the existence of a unique real singular point $(0, -c^2 - 1)$ we obtain Config. $C_{2.2}$. 

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3.1.3 The subcase $H_9(a) = 0$.

In this case by (3.8) the condition $H_9 = 0$ yields either (i) $u = 0$ or (ii) $c^2 - u = 0$. In both cases the T-comitant $D$ has a double factor respectively: $(cx + y)^2$ or $y^2$. We claim that the condition (i) may be reduced by an affine transformation to the condition (ii) and viceversa.

Indeed, assume that for systems (3.7) the condition (i) is fulfilled, i.e. $u = 0$. Then via the transformation

\[ x_1 = x + c, \quad y_1 = cx + y + c^2 \]

the systems (3.7) will be brought to the systems

\[ \dot{x}_1 = 2\tilde{c}x_1 + y_1 + x_1^2 \equiv (x_1 + \tilde{c})^2 - \tilde{c}^2 + y, \quad \dot{y}_1 = x_1y_1, \]

where $\tilde{c} = -c/2$. We observe that the last systems can be obtained from systems (3.7) if the condition $c^2 - u = 0$ is satisfied and this proves our claim.

So, in what follows we assume $u = 0$ and this leads to the systems

\[ \dot{x} = (x + c)^2 + y, \quad \dot{y} = xy \] (3.11)

for which calculations yield:

\[ \tilde{\xi}_1 = -2y(c^2 + cx + y)^2, \quad H_{12} = -8c^2(cx + y)^2. \]

According to Proposition 3.1 the systems (3.11) possess two real invariant affine lines: $y = 0$ and $c^2 + cx + y = 0$ and the line $y = 0$ could be double one. These line are distinct if and only if $c \neq 0$. This condition is captured by the $T$-comitant $H_{12}$ (which is the Hessian of the cubic form $D$).

3.1.3.1 Assume $H_{12} \neq 0$, i.e. $c \neq 0$ and due to the change $x \to cx, y \to c^2y$ and $t \to c^{-1}t$ we may consider $c = 1$. Hence the invariant lines of the system (3.11) are distinct. Moreover, the first line is a double one, this being confirmed by the perturbed systems in $QS_{c_2=0}$

\[ \dot{x} = 1 + \varepsilon + (\varepsilon + 2)x + y + x^2, \quad \dot{y} = xy, \] (3.12)

having the following invariant lines:

\[ y = 0, \quad y + x + 1 + \varepsilon = 0, \quad y + (\varepsilon + 1)x + \varepsilon + 1 = 0. \]

Taking into account the singular points $(-1, 0)$ (double) and $(0, -1)$ (simple) of systems (3.11) we obtain Config. $C_{2.3}$.

3.1.3.2 Assume now $H_{12} = 0$. Then $c = 0$ and we get the system

\[ \dot{x} = y + x^2, \quad \dot{y} = xy, \] (3.13)

which possesses a triple invariant line $y = 0$ and this is confirmed by the perturbed systems

\[ \dot{x} = -\varepsilon^2 + y + x^2, \quad \dot{y} = xy, \]

which have the following invariant lines:

\[ y = 0, \quad y + \varepsilon x - \varepsilon^2 = 0, \quad y - \varepsilon x - \varepsilon^2 = 0. \]

Taking into consideration the existence of the triple singular point $(0, 0)$ of systems (3.13) placed on the line $y = 0$ we get Config. $C_{2.4}$. 15
3.2 The case $H_{10}(a) = 0$

In this case $d = 0$ and the systems (3.5) become:

$$
\dot{x} = k + cx + x^2, \quad \dot{y} = l + xy,
$$

(3.14)

for which we have

$$
\tilde{E}_1(a, x, y) = 2(x^2 + cx + k)(lx - ky + cl), \quad H_{12}(a, x, y) = -8k^2x^2.
$$

3.2.1 The subcase $H_{12}(a, x, y) \neq 0$.

Then $k \neq 0$ and in this case the systems (3.14) possess the invariant line $lx - ky + cl = 0$. Therefore applying the affine transformation $x_1 = x, y_1 = lx - ky + cl$ systems (3.14) will be brought to the simpler form:

$$
\dot{x} = k_1 + cx, \quad \dot{y} = xy.
$$

(3.15)

It is now convenient to use new parameters $(c, u)$ where $c = c_1/2, u = c_1^2 - k_1$. Thus we obtain the systems

$$
\dot{x} = (x + c)^2 - u, \quad \dot{y} = xy
$$

(3.16)

These systems possess the following three invariant affine lines:

$$
y = 0, \quad x + c + 1 = 0, \quad x + c - 1 = 0
$$

and the last two parallel lines are distinct since $H_{12} = -8(c^2 - 1)^2x^2 \neq 0$. So in this case we get Config. $C_{2.5}$.

3.2.1.1 Assume firstly $H_{11} > 0$. In this case $u > 0$ and we may assume $u = 1$ via Remark 2.12 ($\gamma = u, s = 1$) and this leads to the systems

$$
\dot{x} = (x + c)^2 - 1, \quad \dot{y} = xy
$$

(3.17)

3.2.1.2 Assume now that the condition $H_{11} < 0$ holds. Then $u < 0$ and in the same manner as above we may assume $u = -1$ via Remark 2.12 ($\gamma = -u, s = 1$). Thus we obtain the systems

$$
\dot{x} = (x + c)^2 + 1, \quad \dot{y} = xy
$$

(3.18)

which possess the following three (one real and two imaginary) invariant affine lines:

$$
y = 0, \quad x + c + i = 0, \quad x + c - i = 0.
$$

This leads to Config. $C_{2.6}$.

3.2.1.3 For $H_{11} = 0$ we obtain $u = 0$ and the systems (3.15) become:

$$
\dot{x} = (x + c)^2, \quad \dot{y} = xy,
$$

(3.19)

with $c \neq 0$, otherwise we get a degenerate system, i.e. $\gcd(p, q) \neq 1$. Therefore we may assume $c = 1$ via Remark 2.12 ($\gamma = c, s = 1$) and the system (3.18) we have: $\tilde{E}_1 = -2y(x + 1)^2$, i.e. by Proposition 3.1 this system possesses two invariant lines: $y = 0$ and $x = -1$. The last line is a double one as is confirmed by the following perturbed system:

$$
\dot{x} = (x + 1)^2 - \varepsilon^2, \quad \dot{y} = xy.
$$

Thus we obtain Config. $C_{2.7}$.
3.2.2 The subcase $H_{12}(a, x, y) = 0$.
In this case for the systems (3.14) we have $k = 0$ and then $l \neq 0$ (otherwise we get degenerate systems). So we may assume $l = 1$ due to the substitution $y \rightarrow ly$ and this leads to the systems
\[ \dot{x} = cx + x^2, \quad \dot{y} = 1 + xy, \]  
(3.20)
which possess the invariant lines: $x = 0$ and $x + c = 0$. We observe that these lines are distinct for $c \neq 0$ and they coincide if $c = 0$. On the other hand for systems (3.20) we have $H_{11}(a, x, y) = 48c^2x^4$ and hence the $T$-comitant $H_{11}$ captures the condition $c = 0$.

3.2.2.1 For $H_{11} \neq 0$ we have $c \neq 0$ and we can assume $c = 1$ due to the rescaling $x \rightarrow cx$, $y \rightarrow y/c$ and $t \rightarrow t/c$. In this case the invariant line $x = 0$ is simple, whereas the line $x = -1$ is a double one. This is confirmed by the perturbed systems
\[ \dot{x} = (x + 1)(x + \varepsilon), \quad \dot{y} = 1 + xy, \]  
(3.21)
which besides the parallel lines $x = -1$ and $x = -\varepsilon$ possess the invariant line $x - \varepsilon y + 1 + \varepsilon = 0$. Thus we obtain Config. $C_{2.8}$.

3.2.2.2 Assume $H_{11} = 0$, i.e. $c = 0$. Then we obtain the system
\[ \dot{x} = x^2, \quad \dot{y} = 1 + xy, \]  
(3.22)
for which $\tilde{E}_1 = 2x^3$, i.e. the invariant line $x = 0$ of this system can be triple and the following perturbation shows this:
\[ \dot{x} = (x + \varepsilon)^2 - \varepsilon^4, \quad \dot{y} = 1 + xy. \]  
(3.23)
It could easily be checked that these systems possess the following invariant lines
\[ x + \varepsilon(1 + \varepsilon) = 0, \quad x + \varepsilon(1 - \varepsilon) = 0, \quad x + \varepsilon^2(\varepsilon^2 - 1)y + 2\varepsilon = 0. \]
Therefore we get Config. $C_{2.9}$.

**Theorem 3.2.** Every system (2.1) in $QS_{C_2=0}$ has a polynomial inverse integrating factor which is always the product of all polynomials defining the invariant lines, each one appearing in the product with its respective multiplicity (see Table 2). The system has a Darboux first integral in all the cases. Furthermore the Darboux first integral is of the form:

1. $\prod_{i=1}^{n} [f_i(x, y)]^{\lambda_i}$ if and only if the lines $f_i$ are distinct (or $f_i$'s are simple, $i = 1, 2, 3$);
2. $e^{G(x, y)}f_1^{\lambda_1}f_2^{\lambda_2}$ if and only if we have two distinct lines, one of them with multiplicity 2;
3. It is a rational function if and only if we have a single invariant affine line with multiplicity 3.

**Proof:** From the proof of the Theorem 3.1 we get orbit representatives (placed in the first column of Table 2) and invariant lines with their respective multiplicities (placed in the second columns of Table 2). The corresponding co-factors, Darboux integrating factors and first integrals from Table 2 are obtained by straightforward computations.

In the fourth column of the Table 2 we indicate the corresponding polynomial inverse integrating factor which is the product of all polynomials defining the invariant lines, each one appearing in the product with its own multiplicity.

Except for the cases $(C_2.2)$ and $(C_2.6)$, all first integrals appearing in the last column of Table 2 are real. Also for these two cases we can compute real first integrals and they are:

\[ (C_2.2) : \quad F(x, y) = 2c \arctan \left( \frac{x}{y + cx + c^2 + 1} \right) + \ln \left[ \frac{x^2 + (y + cx + c^2 + 1)^2}{y^2} \right]; \]
\[ (C_2.6) : \quad F(x, y) = 2c \arctan \left( \frac{y}{(c + x)^2 + 1} \right). \]
We are now interested in determining the topological classification of the systems in $QS_{C_2=0}$. To obtain the phase portraits of these systems we use the information contained in Table 1, i.e. i) the canonical forms; ii) the Configurations;

For these canonical forms we study α) the finite singular points and β) the behavior of the phase curves in the vicinity of infinity. To obtain β) it suffices to use two charts of the Poincaré sphere obtained by central projection of hemispheres onto the planes $X = 1$, respectively $Y = 1$. We identify the plane $(x, y)$ with $Z = 1$ in $\mathbb{R}^3$. The coordinate changes from the chart $(x, y)$ to the chart $(u, z)$ (respectively $(v, z)$) associated to the chart $X = 1$ (respectively $Y = 1$) are given by

$$
x = \frac{1}{z}, \quad y = \frac{u}{z}, \quad \text{(respectively} \quad x = \frac{v}{z}, \quad y = \frac{1}{z}).
$$

These yield the corresponding two systems of differential equations:

$$
U \begin{cases}
\frac{du}{dt} = C(1, u, z), \\
\frac{dz}{dt} = zP(1, u, z),
\end{cases}
\quad \text{and} \quad
V \begin{cases}
\frac{dv}{dt} = C(v, 1, z), \\
\frac{dz}{dt} = -zQ(v, 1, z),
\end{cases}
$$

As in $QS_{C_2=0}$ $Z | C$, the right-hand sides of both systems contain $z$ as a factor. So to detect the behavior of the phase curves near infinity we only need to study this behavior for the systems $V^*, U^*$ in
In the neighborhood of $z = 0$, where

$$
\begin{align*}
U^* \left\{ \begin{array}{l}
\frac{du}{dt} = \frac{C(1, u, z)}{z}, \\
\frac{dz}{dt} = P(1, u, z),
\end{array} \right.
\quad \text{and} \quad
V^* \left\{ \begin{array}{l}
\frac{dv}{dt} = \frac{C(v, 1, z)}{z}, \\
\frac{dz}{dt} = -Q(v, 1, z),
\end{array} \right.
\end{align*}
$$

Theorem 3.3. I. The systems in $QS_{C^2_{c=0}}$ are classified by 5 integer-valued invariants which are collected in the sequence $(N_c, N_r, d_{G, r}, O, J_f)$. We give the full classification under the group action in Diagram 1. According to the possible values of this invariant we obtain the types of the configurations of invariant lines and all the corresponding phase portraits of such systems. The necessary and sufficient conditions in $\mathbb{R}^{12}$ for the realization of each one of the eleven phase portraits are given in Diagram 2.

II. We have exactly 11 topologically distinct phase portraits (appearing in Diagram 3) in the class $QS_{C^2_{c=0}}$:

- Among these we have exactly four phase portraits each one with exactly one real singularity and such that this singularity is a strong focus, respectively a center, a node or an elliptic point;
- There are exactly two phase portraits without real singularities and exactly two with exactly one singularity which is a saddle-node;
- The remaining three phase portraits have either two or three real singularities: two phase portraits with a saddle and either one or two nodes, and one with two nodes only.

Proof: To construct the phase portraits for the quadratic systems in $QS_{C^2_{c=0}}$ we shall examine step by step each orbit representative given in Table 1.

\begin{align*}
(C_2.1) \ \left\{ \begin{array}{l}
\dot{x} = (x + c)^2 - 1 + y, \\
\dot{y} = xy, \ c \in \mathbb{R}, \ |c| \neq 1
\end{array} \right.
\end{align*}

\(\alpha\) Finite real singular points:

$M_1(-1-c, 0)$ is a saddle for $c < -1$ and a node for $c > -1$; $M_2(1-c, 0)$ is a node for $c < 1$ and a saddle for $c > 1$; $M_3(0, 1-c^2)$ is a saddle for $|c| < 1$ and a node for $|c| > 1$.

\(\beta\) Behavior near infinity:

\begin{align*}
U^* : \left\{ \begin{array}{l}
\dot{u} = 2cu + u^2 + (c^1 - 1)uz, \\
\dot{z} = 1 + z(2c + u - z + c^2 z);
\end{array} \right.
\quad V^* : \left\{ \begin{array}{l}
\dot{i} = -1 - 2cv + z(1 - c^2), \\
\dot{z} = v.
\end{array} \right.
\end{align*}

There are no singularities on $z = 0$ and on $z = 0$ we have $\frac{dz}{du} = \frac{1}{2cu + u^2} \neq 0$ but $\frac{dz}{dv} = -\frac{v}{1 + 2cv}$. Hence we have only one contact point $N(0, 0)$ in the chart $(v, z)$. So we obtain Picture C2.1.

\begin{align*}
(C_2.2) \ \left\{ \begin{array}{l}
\dot{x} = (x + c)^2 + 1 + y, \\
\dot{y} = xy, \ c \in \mathbb{R}
\end{array} \right.
\end{align*}

\(\alpha\) Finite real singular points:

$M_1(0, -1-c^2)$: $\lambda_{1,2} = c \pm i$. Hence, $M_1$ is a strong focus for $c \neq 0$ and it is a center for $c = 0$. We need to construct an invariant polynomial which will govern the condition $c = 0$. Such a polynomial is $N_7(a)$ (see Notation 3.2 below) which for the system $(C_2.2)$ has the value: $N_7(c) = 16c(c^2 + 9)$. 

Proof: To construct the phase portraits for the quadratic systems in $QS_{C^2_{c=0}}$ we shall examine step by step each orbit representative given in Table 1.
Diagram 2 \( (Q_{S_{C_2=0}}) \)
\(\beta\) Behavior near infinity:

\[
U^* : \begin{cases}
\dot{u} = 2cu + u^2 + (c^1 + 1)uz, \\
\dot{z} = 1 + z(2c + u + z + c^2z);
\end{cases} \quad V^* : \begin{cases}
\dot{v} = -1 - 2cv + z(1 + c^2), \\
\dot{z} = v.
\end{cases}
\]

There are no singularities on \(z = 0\) and on \(z = 0\) we have \(\frac{dz}{du} = \frac{1}{2cu + u^2} \neq 0\) but \(\frac{dz}{dv} = -\frac{v}{1 + 2cv}\). Hence we have only one contact point \(N(0, 0)\) in the chart \((v, z)\). So we obtain Picture C2.2(a) for \(c \neq 0\) and Picture C2.2(b) for \(c = 0\).

Notation 3.2. We introduce

\[
N_7(a) = 2D_1^3 + 3D_1 \left[ 4(C_0, D_2)^{(1)} + 3(C_1, C_1)^{(2)} \right] + 36((C_0, C_1)^{(1)}, D_2)^{(1)}.
\]

Remark 3.3. We note that the GL-comitant \(N_7\) is a CT-comitant modulo \(\langle C_2 \rangle\) (for detailed definitions see [26]).

Indeed, if a system belongs to \(QS_{c_2=0}\) then it was shown earlier (see page 12) that this system can be brought via an affine transformation to the canonical form (3.4). Let us consider the action of the translation group \(T(2, \mathbb{R})\) on systems in \(QS_{c_2=0}\). If \(\tau \in T(2, \mathbb{R})\), i.e. \(\tau : x = \tilde{x} + x_0, y = \tilde{y} + y_0\) and \((S)\) is a system (3.4), then applying this action to \((S)\) we obtain

\[
\begin{align*}
\dot{x} &= (k + cx_0 + dy_0 + x_0^2) + (c + 2x_0)\tilde{x} + (d + y_0)\tilde{y} + \tilde{x}^2, \\
\dot{y} &= l + x_0 + y_0 + y_0\tilde{x} + y_0\tilde{y} + \tilde{x}\tilde{y},
\end{align*}
\]

for which calculation yields \(N_7(a) = 8(9ck - 2c^3 + 27dl)\). Hence the value of \(N_7\) does not depend of the vector defining the translation. Therefore we conclude that the polynomial \(N_7\) is a CT-comitant modulo \(\langle C_2 \rangle\).

\[\text{(C2.3)} \begin{cases}
\dot{x} = x + y + x^2, \\
\dot{y} = xy
\end{cases} \quad \text{Tabl.1} \quad \Rightarrow \quad \text{Config. C2.3} \quad \Rightarrow \quad \text{Picture C2.3} \]

a) Finite real singular points:
\(M_1(0, 0)\) is a saddle-node and \(M_2(-1, 0)\) is a (dicritical) node

\[\beta\) Behavior near infinity:

\[
U^* : \begin{cases}
\dot{u} = u(u + 1), \\
\dot{z} = 1 + z(1 + z);
\end{cases} \quad V^* : \begin{cases}
\dot{v} = -1 - v, \\
\dot{z} = v.
\end{cases}
\]

There are no singularities and on \(z = 0\) we have \(\frac{dz}{du} = \frac{1}{u(u + 1)} \neq 0\) but \(\frac{dz}{dv} = -\frac{v}{1 + v}\). Hence we have only one contact point \(N(0, 0)\) in the chart \((v, z)\). Hence we obtain Picture C2.3.

\[\text{(C2.4)} \begin{cases}
\dot{x} = y + x^2, \\
\dot{y} = xy
\end{cases} \quad \text{Tabl.1} \quad \Rightarrow \quad \text{Config. C2.4} \quad \Rightarrow \quad \text{Picture C2.4} \]

Finite real singular points:
\(M_1(0, 0)\) has multiplicity three and possesses exactly one hyperbolic and exactly one elliptical and exactly two parabolic sectors.

\[\beta\) Behavior near infinity:

\[
U^* : \begin{cases}
\dot{u} = u^2, \\
\dot{z} = 1 + uz;
\end{cases} \quad V^* : \begin{cases}
\dot{v} = -1, \\
\dot{v} = -1.
\end{cases}
\]

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There are no singularities on $z = 0$ and on $z = 0$ we have $\frac{dz}{du} = \frac{1}{u(u+1)} \neq 0$ but $\frac{dz}{dv} = -\frac{v}{1+v}$. Hence we have only one contact point $N(0,0)$ in the chart $(v,z)$. Therefore we obtain Picture $C_{2.4}$.

\( (C_{2.5}) \begin{cases} \dot{x} = (x+c)^2 - 1, \\ \dot{y} = xy, \ c \in \mathbb{R}, |c| \neq 1 \end{cases} \)

\( \alpha \) Finite real singular points:

$M_1(-1-c,0)$ is a saddle for $c < -1$ and a node for $c > -1$; $M_2(1-c,0)$ is a node for $c < 1$ and a saddle for $c > 1$. Hence, we have two nodes for $|c| < 1$ and one node and one saddle for $|c| > 1$. We note that these conditions are governed by the $CT$-comitant $\mu_2(a,x,y)$ as for the system $(C_{2.5})$ we have:

$\mu_2(a,x,y) = (c^2 - 1)x^2$.

\( \beta \) Behavior near infinity:

\[ U^* : \begin{cases} \dot{u} = 2cu + (c^2 - 1)uz, \\ \dot{z} = 1 + z(2c - z + c^2z); \end{cases} \quad V^* : \begin{cases} \dot{v} = -2cv + z(1-c^2), \\ \dot{z} = v. \end{cases} \]

In the chart $(v,z)$, $(0,0)$ is a singular point which is a saddle for $c^2 - 1 < 0$ and a node for $c^2 - 1 < 0$. On $z = 0$ we have $\frac{dz}{du} = \frac{1}{2cu} \neq 0$ and $\frac{dz}{dv} = -\frac{v}{2cv}$. Hence for $v \neq 0$ we have no contact point and we obtain Picture $C_{2.5}(a)$ for $\mu_2 < 0$ and Picture $C_{2.5}(b)$ for $\mu_2 > 0$.

\( (C_{2.6}) \begin{cases} \dot{x} = (x+c)^2 + 1, \\ \dot{y} = xy, \ c \in \mathbb{R} \end{cases} \)

\( \alpha \) Finite real singular points:

There are no real singular points.

\( \beta \) Behavior near infinity:

\[ U^* : \begin{cases} \dot{u} = 2cu + (c^2 + 1)uz, \\ \dot{z} = 1 + z(2c + z + c^2z); \end{cases} \quad V^* : \begin{cases} \dot{v} = -2cv - z(1+c^2), \\ \dot{z} = v. \end{cases} \]

In the chart $(v,z)$, $(0,0)$ is a singular point which is a focus for $c \neq 0$ and a center for $c = 0$. In both cases the behavior near $z = 0$ and around $(v,z) = (0,0)$ for $z > 0$ (respectively for $z < 0$) is the same. On $z = 0$ we have $\frac{dz}{du} = \frac{1}{2cu} \neq 0$ and $\frac{dz}{dv} = -\frac{v}{2cv}$. Hence for $v \neq 0$ we have no contact point. Thus we obtain Picture $C_{2.6}$.

\( (C_{2.7}) \begin{cases} \dot{x} = (x+1)^2, \\ \dot{y} = xy \end{cases} \)

\( \alpha \) Finite real singular points:

The double point $M_1(-1,0)$ is a saddle-node.
β) Behavior near infinity:

\[ U^* : \begin{cases} \dot{u} = u(2 + z), \\ \dot{z} = (1 + z)^2; \end{cases} \quad \quad \quad V^* : \begin{cases} \dot{v} = -2v - z, \\ \dot{z} = v. \end{cases} \]

In the chart \((v, z)\), \((0, 0)\) is a singular point which is a degenerate node. On \(z = 0\) we have \(\frac{dz}{du} = \frac{1}{2u} \neq 0\) and \(\frac{dz}{dv} = -\frac{v}{2v}\). Hence for \(v \neq 0\) we have no contact point. So we obtain Picture \(C_2.7\).

(C2.8) \{ \begin{align*} \dot{x} &= x + x^2, \\ \dot{y} &= 1 + xy \end{align*} \}

α) Finite real singular points:

\(M_1(-1, 1)\) is a (degenerate) node. At infinity after excluding the singular line \(Z = 0\) we obtain the isolated singular point \(N_1[0 : 1 : 0]\) which is saddle-node.

β) Behavior near infinity:

\[ U^* : \begin{cases} \dot{u} = u - z, \\ \dot{z} = 1 + z; \end{cases} \quad \quad \quad V^* : \begin{cases} \dot{v} = -v + vz, \\ \dot{z} = v + z^2. \end{cases} \]

In the chart \((v, z)\) the singular point \((0, 0)\) is a saddle-node with the two hyperbolic sectors separated by the invariant line \(v = 0\), which appears in Configuration \(C_2.8\) as the line which is simple. On \(z = 0\) we have \(\frac{dz}{du} = \frac{1}{u} \neq 0\) and \(\frac{dz}{dv} = -\frac{v}{v}\). Hence for \(v \neq 0\) we have no contact point and we obtain Picture \(C_2.8\).

(C2.9) \{ \begin{align*} \dot{x} &= x^2, \\ \dot{y} &= 1 + xy \end{align*} \}

α) Finite real singular points:

There are no real singular points. Considering the respective Darboux first integral in Table 2 we obtain the phase portrait given by Picture \(C_2.9\).

β) Behavior near infinity:

\[ U^* : \begin{cases} \dot{u} = -z, \\ \dot{z} = 1; \end{cases} \quad \quad \quad V^* : \begin{cases} \dot{v} = vz, \\ \dot{z} = v + z^2. \end{cases} \]

In the chart \((v, z)\) on \(z = 0\) we have only one singularity: \((0, 0)\) which has two parabolic sectors separated from the remaining hyperbolic sector by the invariant line \(v = 0\). We have \(\frac{dz}{du} = -\frac{1}{z}\) and \(\frac{dz}{dv} = \frac{v + z^2}{vz}\) and hence on \(z = 0\) we have no contact point for \(v \neq 0\). So we obtain Picture \(C_2.9\).

Theorem 3.3 is proved.
4 Bifurcation diagrams for the class $QS_{c_2=0}$ of quadratic systems

4.1 The case $H_{10} \neq 0$

The canonical form (see page 14) for this case is:

$$\dot{x} = (x + c)^2 + y - u, \quad \dot{y} = xy.$$  \hspace{1cm} (4.1)

We are now interested in drawing the bifurcation diagram of this family. For this we follow the Diagram 2 under the condition $H_{10} \neq 0$. It is clear from this Diagram that the parameter values for which $H_9 = 0$ are bifurcation points.

We have $H_9 = -2394u(c^2 - u)^2$. Hence $u(c^2 - u) = 0$ is a bifurcation curve whose components are the line $u = 0$ and the parabola $c^2 - u = 0$. From Diagram 2 we see that in case $H_9 > 0$, i.e. $u < 0$, the points on $N_7 = 0$ are bifurcation points. Since for systems \eqref{eq:4.1} $N_7 = 16(c^2 - 9u)$ and $u < 0$, this yields the bifurcation curve $c = 0$. In the case $H_9 = 0$ the points on $H_{12} = 0$ are bifurcation points. But $H_9 = 0$ means $u = 0$ or $c^2 - u = 0$. If $u = 0$, $H_{12} = -8c^2(cx + y)^2$ and if $u = c^2$, $H_{12} = -32c^2y^2$. In both cases $H_{12} = 0$ means $c = 0$, i.e. $(c, u) = (0, 0)$ is a bifurcation point on $H_9 = 0$. This discussion and Diagram 2 completes the drawing of the Bifurcation Diagram I.

![Bifurcation diagram I $(H_{10} \neq 0)$](image)

We are now interested in the quotient space $QS_{c_2=0}/\sim$ of $QS_{c_2=0}$ under the action of affine transformation and time rescaling.

We consider the canonical projection

$$QS_{c_2=0} \xrightarrow{\pi} QS_{c_2=0}/\sim = \mathcal{M}$$

$$(S) \longmapsto \text{orbit}(S)$$

We may call the topological space $\mathcal{M}$ the "moduli space" of $QS_{c_2=0}$.

We construct here the piece $\mathcal{M}_1$ of $\mathcal{M}$ obtained from the condition $H_{10} \neq 0$, i.e.

$$\mathcal{M}_1 = \left(QS_{c_2=0} \cap \{H_{10} \neq 0\}\right)/\sim$$

To do this we take orbit representatives of systems in the Bifurcation diagram I $(H_{10} \neq 0)$. For this we observe that for $u > 0$ we can take $u = 1$ via the transformation $x = u^{1/2}x_1$, $y = uy_1$ and $t = u^{-1/2}t_1$. Hence for the quotient space $\mathcal{M}_1$ from the upper part of the plane we take representatives on this line.
The set \( \{ c^2 - u = 0 \} \cup \{ u = 0 \} \setminus \{0, 0\} \) is an orbit (see subsection 3.1.3 on page 15). In particular the two points \((c, u) = (\pm 1, 1)\) are identified under \(\pi\) and hence from the line \(u = 1\) we obtain a curve with a double point in the quotient space (see Figure 1).

For \(u < 0\) we can take \(u = -1\) via the transformation \(x = (-u)^{1/2}x_1, y = -uy_1\) and \(t = (-u)^{-1/2}t_1\). On this line we have only two phase portraits, one for \(c \neq 0\) \((C_2.2(a))\) and one for \(c = 0\) \((C_2.2(b))\). So to construct the quotient space \(\mathcal{M}_1\) we take systems representatives on \(u = 1\) and \(u = -1\) and we obtain this quotient space as indicated in the Figure 2.

![Figure 1](image1.png)

**Figure 1**

In Figure 2 we use \([c, u]\) for the orbit of the system corresponding to parameter value \((c, u)\).

We obtained in Figure 2 the moduli space \(\mathcal{M}_1\) of the all systems in \(\mathcal{QS}_{C_2=0}\) for which \(H_{10} \neq 0\). We note that this is not a Hausdorff space. Indeed any open neighborhood \(\mathcal{U}\) of the point \([0, 0]\) in \(\mathcal{M}_1\) has the property that \(\pi^{-1}(\mathcal{U})\) contains points on \(u(c^2 - u) = 0\) whose projection on \(\mathcal{M}_1\) is the point \([1, 1]\). Hence \([0, 0]\) and \([1, 1]\) cannot be separated by open neighborhoods.

### 4.2 The case \(H_{10} = 0\)

The canonical form (see page 16) for this case is:

\[
\begin{align*}
\dot{x} &= k + cx + x^2, \\
\dot{y} &= l + xy.
\end{align*}
\]

We may limit ourselves to \(l = 1\). To show this we consider first \(l \neq 0\) and then \(l = 0\). If \(l \neq 0\) changing \(y\) into \(ly_1\) we obtain the system (4.2) with \(l = 1\).

If \(l = 0\) then necessarily \(k \neq 0\) as otherwise we would have degenerate systems. Using the transformation

\[
\begin{align*}
x_1 &= x, \\
y_1 &= \frac{1}{k}x + y + \frac{c}{k}
\end{align*}
\]

we obtain the systems

\[
\begin{align*}
\dot{x} &= k_1 + c_1 x + x^2, \\
\dot{y} &= 1 + xy.
\end{align*}
\]

It is now convenient to use new parameters \((c, u)\) where \(c = c_1/2, u = c_1^2 - k_1\). Thus we obtain the systems

\[
\begin{align*}
\dot{x} &= (x + c)^2 - u + y, \\
\dot{y} &= 1 + xy.
\end{align*}
\]

We are now interested in drawing the bifurcation diagram of this family. For this we follow the Diagram 2 under the condition \(H_{10} = 0\). It is clear from this Diagram that the parameter values for which \(H_{12} = 0\) are bifurcation points.

For the systems (4.3) we have

\[
\begin{align*}
H_{12} &= -8(c^2 - u)^2x^2, \\
H_{11} &= 192ux^4, \\
\mu_2 &= (c^2 - u)x^2.
\end{align*}
\]
Hence the bifurcation set is $u(c^2 - u) = 0$ whose components are the line $u = 0$ and the parabola $c^2 - u = 0$ drawn in the bold face in Bifurcation diagram II.

Following the Diagram 2 in its lower part corresponding to $H_{10} = 0$ and considering the expressions of $H_{12}$, $H_{11}$ and $\mu_2$ (see (4.4)) we have:

- For $H_{12} \neq 0$ (i.e. $c^2 - u \neq 0$ and $H_{11} > 0$ (i.e. $u > 0$):
  - if $\mu_2 < 0$ (i.e. $c^2 - u < 0$), $(c, u)$ is inside the parabola and for its associated system we have Picture C2.5(a);
  - if $\mu_2 < 0$ (i.e. $c^2 - u > 0$), $(c, u)$ is outside the parabola and for its associated system we have Picture C2.5(b);

- For $H_{12} \neq 0$ (i.e. $c^2 - u \neq 0$ and $H_{11} < 0$ (i.e. $u < 0$) we have Picture C2.6;

- For $H_{12} \neq 0$ and $H_{11} = 0$ (i.e. $u = 0 \neq c$) we have Picture C2.7;

- For $H_{12} = 0$ (i.e. $c^2 - u = 0$ and $H_{11} \neq 0$ (i.e. $u \neq 0$) we have Picture C2.8;

- For $H_{12} = 0$ and $H_{11} = 0$ (i.e. $u = 0 = c$) we have Picture C2.9.

We have obtained the following bifurcation diagram for this family:

\[ \text{Bifurcation diagram II (} H_{10} = 0 \text{)} \]

In the same way as in the case $H_{10} \neq 0$ we obtain the quotient space

\[ \mathcal{M}_2 = \left( QS_{c_2=0} \cap \{H_{10} = 0\} \right) / \sim \]

and it is given in Figure 3.

We obtained in Figure 3 the moduli space $\mathcal{M}_2$ of the all systems in $QS_{c_2=0}$ for which $H_{10} = 0$. We note that this is not a Hausdorff space. Indeed any open neighborhood $U$ of the point $[0, 0]$ in $\mathcal{M}_2$ has the property that $\pi^{-1}(U)$ contains points on $u(c^2 - u) = 0$ whose projection on $\mathcal{M}_2$ is the set $\{[1, 1], [0, 1]\}$. Hence $[0, 0]$ cannot be separated by open neighborhoods from either $[1, 1]$ or $[0, 1]$. 

27
Completion of the phase portraits in the neighborhood of infinity listed in [28] with those for which we have \( C_2(x, y) \equiv 0 \)

In [28] we gave the list of all 40 topologically distinct phase portraits in the neighborhood of infinity for all systems in QS such that \( Z \nmid C(X, Y, Z) \), i.e. \( C_2(x, y) \not\equiv 0 \). We now complete the list of the topologically distinct phase portraits in the neighborhood of infinity with those arising from the class \( QS_{C_2=0} \).

We are now interested in the behavior in the vicinity of infinity of the class \( QS_{C_2=0} \). From the eleven phase portraits denoted from \( C_2.1 \) up to \( C_2.9 \) which appear in Diagram 1 we now extract all the topologically distinct phase portraits in the vicinity of infinity of the class \( QS_{C_2=0} \). They are the ones we exhibit here below:

For each specific Figure \( i, i \in \{41, \ldots, 46\} \) we indicate below the phase portraits among the one denoted by \( C_2.1 \) up to \( C_2.9 \) which yield this particular Figure \( i \):

- Figure 41 ⇔ Pictures \( C_2.i, i = 1, 3, 4 \) and Pictures \( C_2.2(a) \) and \( C_2.2(b) \);
- Figure 42 ⇔ Picture \( C_2.5(a) \);
- Figure 43 ⇔ Picture \( C_2.6 \);
- Figure 44 ⇔ Pictures \( C_2.5(b) \) and \( C_2.7 \);
- Figure 45 ⇔ Picture \( C_2.9 \);
- Figure 46 ⇔ Picture \( C_2.8 \).

We are now interested in determining in invariant form under the group action necessary and sufficient conditions for a quadratic system (2.1) to have a specific phase portrait in the vicinity of infinity of the form Figure \( i, i \in \{41, \ldots, 46\} \). To do this we first amalgamate the various invariant conditions in Diagram 2 for which Figure \( i \) occurs, following the correspondence \((\Omega)\) for a quadratic system (2.1).
Lemma 5.1. The Picture in Figure $i$, $i \in \{41, \ldots, 46\}$ occurs if and only if $C_2 \equiv 0$ and the corresponding conditions below (jointly taken, when we have more than one condition) are satisfied:

- Figure 41 $\iff H_{10} \neq 0$;
- Figure 42 $\iff H_{10} = 0$, $H_{12} \neq 0$, $\mu_2 < 0$;
- Figure 43 $\iff H_{10} = 0$, $H_{12} \neq 0$, $\mu_2 > 0$, $H_{11} < 0$;
- Figure 44 $\iff H_{10} = 0$, $H_{12} \neq 0$, $\mu_2 > 0$, $H_{11} \geq 0$;
- Figure 45 $\iff H_{10} = 0$, $H_{12} = 0$, $H_{11} \neq 0$;
- Figure 46 $\iff H_{10} = 0$, $H_{12} = 0$, $H_{11} = 0$.

The proof follows easily from (Ω) and Diagram 2, taking into account the implication: $H_{10} = 0$ and $H_{12} \neq 0$ and $\mu_2 < 0 \Rightarrow H_{11} > 0$ (see (5.2) below).

We compare the polynomials occurring in the invariant algebraic conditions of the right hand side above with the polynomials used in the classification theorem (Theorem 7.1) from [28]. We see that the two sets of polynomials used in the two classifications are different. To obtain a uniform topological classification in the neighborhood of infinity for the whole class of quadratic differential systems in QS we need to obtain the classifications of the class $QS_{C_2=0}$ in terms of the same polynomials used in Theorem 7.1 of [28].

Theorem 7.1 (continuation of Theorem 7.1 in [28]). A quadratic differential system with $C_2 \equiv 0$ (i.e. $Z \mid C$) has the phase portrait in the vicinity of infinity given in Figure $i$, $i \in \{41, \ldots, 46\}$ if and only if the conditions corresponding to Figure $i$ below, jointly taken, are satisfied:

- Figure 41 $\iff \mu_1 \neq 0$;
- Figure 42 $\iff \mu_1 = 0$, $\mu_2 < 0$;
- Figure 43 $\iff \mu_1 = 0$, $\mu_2 > 0$, $K_2 < 0$;
- Figure 44 $\iff \mu_1 = 0$, $\mu_2 > 0$, $K_2 \geq 0$;
- Figure 45 $\iff \mu_1 = 0$, $\mu_2 = 0$, $\mu_3 \neq 0$;
- Figure 46 $\iff \mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$. (5.1)

Proof: 1) Figure 41. According to Lemma 5.1 for Figure 41 we have $H_{10} \neq 0$. On the other hand for the systems (3.4) (see page 12) calculations yield:

$$H_{10}(a) = 36d^2, \quad \mu_0 = 0, \quad \mu_1 = dx.$$  

Hence, the condition $H_{10} \neq 0$ is equivalent to $\mu_1 \neq 0$. We note that according to Lemma 7.6 [28] the polynomial $\mu_1$ is a CT-comitant modulo $\langle \mu_0 \rangle$.

2) Figures 42-44. By Lemma 5.1 from Diagram 4 we observe that for Figures 42, 43 and 44 the conditions $H_{10} = 0$ and $H_{12} \neq 0$ are satisfied. On the other hand, for $H_{10} = 0$ a system $(S)$ in $QS_{C_2=0}$ can be brought via an affine transformation to the form (3.14) (see page 16), for which

$$H_{10} = 0, \quad H_{12} = -8k^2x^2, \quad H_{11} = 48(c^2 - 4k)x^4, \quad \mu_0 = \mu_1 = 0, \quad \mu_2 = kx^2, \quad K_2 = 48(c^2 - 4k)x^2. \quad (5.2)$$

Therefore, the condition $H_{12} \neq 0$ is equivalent to $\mu_2 \neq 0$. Furthermore, if $\mu_2 < 0$ then $H_{11} > 0$ and according Lemma 5.1 we obtain Figure 42.

Assume $\mu_2 > 0$. In this case for $H_{11} < 0$ (then $K_2 < 0$) we have Figure 43. If $H_{11} \geq 0$ (then $K_2 \geq 0$) according Lemma 5.1 we obtain Figure 44.

We note that according to Lemma 7.6 [28] the GL-comitant $\mu_2(a, x, y)$ (respectively $K_2(a, x, y)$) is a CT-comitant modulo $\langle \mu_0, \mu_1 \rangle$ (respectively modulo $\langle \eta, \mu_0, \mu_1, \kappa, \kappa_1 \rangle$), where $\eta, \kappa$ and $\kappa_1$ are the polynomials from [28] and (2.6). Since for the systems (3.14) calculations yield

$$\eta = \mu_0 = \mu_1 = \kappa = \kappa_1 = 0$$

and taking into consideration the fact that $\mu_2(a, x, y)$ and $K_2(a, x, y)$ are homogeneous polynomials of even degree in the coefficients of the systems (2.1), we conclude that the conditions for Figures 42, 43 and 44...
expressed through the polynomials \( \mu_2(a, x, y) \) and \( K_2(a, x, y) \) are invariant under the action of the affine group and time rescaling.

3) Figures 45, 46. According to Lemma 5.1 for the Figures 45 and 46 the conditions \( H_{10} = H_{12} = 0 \) hold. It was shown earlier (see page 17) that for \( H_{10} = H_{12} = 0 \) a system (S) in QS\( _{c_{2}=0} \) can be brought via an affine transformation and time rescaling to the form (3.20) for which

\[
H_{10} = H_{12} = 0, \ H_{11} = 48c^2x^4, \ \mu_0 = \mu_1 = \mu_2 = 0, \ \mu_3 = -cx^3.
\]

Hence, the condition \( H_{11} = 0 \) is equivalent to \( \mu_3 = 0 \) and by Lemma 5.1 we obtain Figure 45 for \( \mu_3 \neq 0 \) and Figure 46 for \( \mu_3 = 0 \). It remains to note that the GL-comitant \( \mu_3(a, x, y) \) is a C\( T \)-comitant modulo \( \langle \mu_0, \mu_1, \mu_2 \rangle \) (Lemma 7.6 [28]) and hence the condition \( \mu_3 = 0 \) is invariant under the group action.

**Observation 5.1.** In [16], the authors gave phase portraits at infinity for the class QS\( _{c_{2}=0} \). If we compare our result in Theorem 7.1 with the Figures obtained from Theorem 1 of [16], we see that all the figures from Fig. 41 up to Fig. 45 in this last theorem are incompletely drawn. Indeed, only some sections of the two opposite singularities on the vertical line are drawn. It is therefore necessary to complete these Figures. In this process we see that from the Fig. 43 in Theorem 1 of [16] we obtain two topologically distinct phase portraits namely Figure 44 and Figure 45 in Theorem 7.1 above appear.

6 The vicinity in QSL of the class QS\( _{c_{2}=0} \)

**Theorem 6.1.** Let (S) be an arbitrary system in QS\( _{c_{2}=0} \) and let \( \mathbf{a} \in \mathbb{R}^{12} \) be the 12-tuple of its coefficients. Then there exists a small vicinity \( \mathcal{V} \ni \mathbf{a} \), \( \mathcal{V} \subset \mathbb{R}^{12} \) such that:

i) for every point \( \mathbf{a} \in \mathcal{V} \) the corresponding system (\( \hat{S} \)) belongs to QS \( \setminus (\text{QLS}_6 \cup \text{QLS}_5) \);

ii) \( \exists \hat{\mathbf{a}} \in \mathcal{V} \) such that the corresponding system (\( \hat{S} \)) belongs to QSL\( _4 \).

**Proof:** i) It was shown before (see page 12) that any system in QS\( _{c_{2}=0} \) can be brought via an affine transformation to the form

\[
\dot{x} = k + cx + dy + x^2, \quad \dot{y} = l + xy, \quad (c, d, k, l) \in \mathbb{R}^4 \setminus \{0\}
\]

(6.1)

We claim that for any system (6.1) there could not exist perturbations in the class of quadratic systems such that the perturbed systems belong to the class QSL\( _5 \) or QSL\( _6 \). To prove this claim we examine step by step all the possible situations from Table 1.

For systems (6.1) we have \( H_{10}(\mathbf{a}) = 36d^2 \) and we shall consider two subcases: \( H_{10}(\mathbf{a}) \neq 0 \) and \( H_{10}(\mathbf{a}) = 0 \).

6.1 The case \( H_{10}(\mathbf{a}) \neq 0 \)

Suppose that there exists a system (6.1) with \( d \neq 0 \) (let us denote it by (\( S_\mathbf{a} \))) and consider a perturbation (\( S_{\mathbf{a}+\Gamma} \)) of (\( S_\mathbf{a} \)) where \( \Gamma = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{12}) \), \( 0 < |\varepsilon_i| \ll 1 \), such that the perturbed systems (\( S_{\mathbf{a}+\Gamma} \)) possess four (respectively five) distinct invariant affine lines. Hence we must have one of the Configurations 5.1 up to 5.6 (respectively Configurations 6.1 up to 6.4) from [26]. We observe that on each invariant affine line in any of these Configurations lie two singularities of the system in QSL\( _5 \) (respectively in QSL\( _6 \)).

We note that when \( \Gamma \) tends to zero the perturbed systems (\( S_{\mathbf{a}+\Gamma} \)) tend to (\( S_\mathbf{a} \)). Therefore we conclude that besides the existing three distinct invariant affine lines of systems (\( S_{\mathbf{a}+\Gamma} \)) (which are in the vicinity of the respective lines of total multiplicity three of the system (\( S_\mathbf{a} \))) there must exist at least one invariant affine line in the vicinity of the line \( Z = 0 \). Hence, two singularities of systems (\( S_{\mathbf{a}+\Gamma} \)) are in the vicinity of infinity which contradicts the fact that the systems (\( S_{\mathbf{a}+\Gamma} \)) with \( d \neq 0 \) have at most four singularities in the finite part of the phase plane.

Thus in the case \( d \neq 0 \) systems (6.1) could not belong to (QLS\( _6 \cup \text{QLS}_5 \)).
6.2 The case $H_{10}(a) = 0$

Then $d = 0$ and we shall consider two subcases: $H_{12} = -8k^2x^2 \neq 0$ and $H_{12} = 0$ (see page 16).

6.2.1 The subcase $H_{12} \neq 0$.

Then $k \neq 0$ and since in this case the systems (6.1) with $d = 0$ possess the invariant line $lx - ky + cl = 0$, without loss of generality we may assume $l = 0$ via the transformation $x_1 = x$, $y_1 = lx - ky + cl$. Hence we obtain the systems

$$\dot{x} = k + cx + x^2, \quad \dot{y} = xy$$

(6.2)

for which the line $y = 0$ is invariant and the line at infinity $Z = 0$ is filled up with singularities.

The idea in the remaining part of the proof of i) is the following:

As we are interested in the maximal number of invariant lines which can appear in perturbations from the line full of singularities at infinity, and as lines are preserved under the projective group $PGL(2, \mathbb{R})$ we shall use a projective transformation to first bring the infinite line in one affine chart, and simultaneously bring an affine line at infinity in this chart.

Following [28] we associate to a system (6.2) the differential equation in the projective plane:

$$XYZ \, dX - Z(X^2 + cXZ + kZ^2) \, dY + YZ(cX + kZ) \, dZ = 0$$

We consider the projective transformation: $X = X_1$, $Y = Z_1$, $Z = Y_1$ which takes the line $Y = 0$ to the line $Z_1 = 0$ and the line $Z = 0$ to the line $Y_1 = 0$. We obtain the following equation:

$$X_1Y_1Z_1 \, dX_1 + Y_1Z_1(cX_1 + kY_1) \, dY_1 - Y_1(X_1^2 + cX_1Y_1 + kY_1^2) \, dZ_1 = 0.$$  

This equation in the chart $Z_1 \neq 0$ leads back to the following quadratic systems (when we denote $x = X_1$, $y = Y_1$):

$$\dot{x} = y(cx + ky), \quad \dot{y} = xy,$$

(6.3)

which have the affine line $y = 0$ filled up with singularities.

We claim that there could not exist a perturbation of any system (6.3) in the class of quadratic systems such that the perturbed system belong to either the class QSL$_6$ or to QSL$_5$.

Indeed for the systems (6.3) calculations yield:

$$B_2(a, x, y) = 0, \quad N(a, x, y) = -x^2 + 2cxy - (c^2 + 2k)y^2, \quad \theta(a) = -8k.$$  

(6.4)

Since $k \neq 0$ we have $\theta(a)N(a, x, y) \neq 0$ and taking into consideration Lemma 2.1 we conclude that for any system (6.3) there could not exist such a perturbation in the class of quadratic systems such that the perturbed system belong to either the class QSL$_6$ or to QSL$_5$.

6.2.2 The subcase $H_{12} = 0$.

In this case we have $k = 0$ and since $d = 0$ in the systems (6.1) we have $l \neq 0$ (otherwise we obtain degenerate systems). Then we may assume $l = 1$ due to the substitution $y \rightarrow ly$ and this leads to the systems

$$\dot{x} = cx + x^2, \quad \dot{y} = 1 + xy.$$  

(6.5)

These systems possess the invariant affine lines $x = 0$ and $x + c = 0$ and the line at infinity $Z = 0$ is filled up with singularities.

As above we associate to a system (6.5) the differential equation on the projective plane:

$$Z(XY + Z^2) \, dX - XZ(X + cZ) \, dY + XZ(cY - Z) \, dZ = 0.$$  

We apply the projective transformation: $X = Z_1$, $Y = Y_1$, $Z = X_1$ and we obtain the equation:

$$X_1(cY_1 - X_1) \, dX_1 - X_1Z_1(cX_1 + Z_1) \, dY_1 + X_1(X_1^2 + Y_1Z_1) \, dZ_1 = 0,$$
which in the chart \( Z_1 \neq 0 \) (or \( Z_1 = 1 \)) leads back to the following quadratic systems of differential equations (when we denote \( x = X_1, y = Y_1 \)):\[
\dot{x} = x(1+cx), \quad \dot{y} = x(cy - x). \tag{6.6}
\]
These systems have the affine line \( x = 0 \) filled up with singularities. For these systems we calculate\[
C_2(a, x, y) = x^3, \quad N(a, x, y) = c^2 x^2. \tag{6.7}
\]
Our goal is to show that there cannot exists a perturbation of a system (6.6) corresponding to \( c \in \mathbb{R} \) such that perturbed system belong to either QSL\(_5\) or QSL\(_6\). Suppose the contrary, that there exists such a perturbation \( \Upsilon \) and the systems (6.6\(^T\)) possess at least four invariant affine lines:
\[
\mathcal{L}_i^\Upsilon = 0, \quad i = 1, 2, 3, 4.
\]

For the perturbed systems (6.6\(^T\)) the directions of these lines are governed by the cubic form \( C_2(\tilde{a}_\Upsilon, x, y) \). Hence the directions of the invariant lines of the perturbed systems could be either all three real or one real and two complex. We shall consider both these cases.

### 6.2.2.1 The case of real directions.

In this case considering (6.7) we conclude that for the perturbed systems we must have the following factorization over \( \mathbb{R} \):
\[
C_2(\tilde{a}_\Upsilon, x, y) = [(\gamma_1 + 1)x + \gamma_2 y][(\gamma_3 + 1)x + \gamma_4 y][(\gamma_5 + 1)x + \gamma_6 y]
\]
where \( |\gamma_i| \ll 1, i = 1, \ldots, 6 \). Clearly, via a linear transformation (for example, using the following one:
\[
x = [(1 + \gamma_1)(1 + \gamma_3)(1 + \gamma_5)]^{-1/3}[x_1 - \gamma_2(1 + \gamma_3)(1 + \gamma_5)y_1], \quad y = [(1 + \gamma_1)(1 + \gamma_3)(1 + \gamma_5)]^{2/3} y_1
\]
we can transform the cubic form \( C_2(\tilde{a}_\Upsilon, x, y) \) as follows
\[
\tilde{C}_2(\tilde{a}_\Upsilon, x_1, y_1) = x_1(x_1 + \delta_1 y_1)(x_1 + \delta_2 y_1)
\]
and in order to have distinct directions the following conditions must be satisfied:
\[
\delta_1 \delta_2 (\delta_1 - \delta_2) \neq 0. \tag{6.8}
\]
Furthermore by a translation we can move the invariant line with the direction \([0 : 1]\) to the line \( x = 0 \).

At the same time after application of the above indicated affine transformation the perturbed systems (6.6\(^T\)) become
\[
\dot{x} = (\varepsilon_1 + 1) x + (c + \varepsilon_2 + \delta_1 + \delta_2) x^2 + \varepsilon_3 xy, \\
\dot{y} = \varepsilon_4 + \varepsilon_5 x + \varepsilon_6 y - x^2 + (c c + \varepsilon_2) xy + (\varepsilon - \delta_1 \delta_2) y^2. \tag{6.9}
\]
For these systems we calculate:
\[
\theta(\tilde{a}_\Upsilon) = 8 \varepsilon_3 (c \delta_1 + \varepsilon_2 \delta_1 + 2 \delta_1 \delta_2 - \varepsilon_3)(c \delta_2 + \varepsilon_2 \delta_2 + 2 \delta_1 \delta_1 - \varepsilon_3) \tag{6.10}
\]
and according to Lemma 2.1 we must have \( \theta(\tilde{a}_\Upsilon) = 0 \). Since by interchanging \( \delta_1 \) with \( \delta_2 \) the form (6.9) is maintained but the second and the third factors of the polynomial \( \theta(\tilde{a}_\Upsilon) \) are interchanged we shall consider two subcases: \( \varepsilon_3 = 0 \) and \( 0 \neq \varepsilon_3 = c \delta_1 + \varepsilon_2 \delta_1 + 2 \delta_1 \delta_2. \)
6.2.2.1.1 The subcase $\varepsilon_3 = 0$. Then as $\delta_1 \delta_2 \neq 0$ (see (6.8)) we may assume $\varepsilon_6 = 0$ via the translation $x = x_1$, $y = y_1 + \frac{\varepsilon_6}{2\delta_1 \delta_2}$. Then for the systems (6.9) with $\varepsilon_3 = \varepsilon_6 = 0$ calculations yield:

$$N(\hat{a}_\tau, x, y) = (c + 2\delta_1 + \varepsilon_2)(c + 2\delta_2 + \varepsilon_2)x^2$$

(6.11)

and we have to distinguish the case $c \neq 0$ and $c = 0$.

1) If $c \neq 0$ then for the initial systems (6.6) we have $N \neq 0$ and then the perturbed systems cannot belong to the class QSL$_6$. On the other hand, by Lemma 2.1 the perturbed systems could belong to the class QSL$_5$ only if $\theta = B_3 = 0$. However for these systems calculation yields

$$\text{Coefficient}[B_3, x^3y] = -6\delta_1 \delta_2\left[1 + 2\varepsilon_1 + \varepsilon_1^2 + ((c + \varepsilon_2)^2 - 4\delta_1 \delta_2)\varepsilon_4 - \delta_1 \delta_2 \varepsilon_5^2\right] \neq 0$$

and, hence the perturbed systems could not belong to the class QSL$_5$ in the case $c \neq 0$ (i.e. when $N \neq 0$).

2) Assume now $c = 0$. Then, considering (6.11) and the fact that interchanging $\delta_1$ with $\delta_2$ the systems (6.9) remain the same, the condition $N = 0$ yields $\varepsilon_2 = -2\delta_1$. In this case according to the Lemmas 2.1 the condition $B_2 = 0$ at least, must be satisfied. Therefore for the systems (6.9) with $\varepsilon_3 = \varepsilon_6 = c = 0$ and $\varepsilon_2 = -2\delta_1$ we calculate

$$B_2 = 648\delta_1^2 \delta_2^2 (1 + \varepsilon_1 + \delta_2 \varepsilon_5)^2 \left[4\varepsilon_4 \delta_1 (\delta_1 - \delta_2)^2 - \delta_2 (1 + \varepsilon_1 + \delta_1 \varepsilon_5)^2\right] x^4$$

Since $\delta_1 (\delta_1 - \delta_2) \neq 0$ (see the condition (6.8)) we obtain that the condition $B_2 = 0$ yields

$$\varepsilon_4 = \frac{\delta_2 (1 + \varepsilon_1 + \delta_1 \varepsilon_5)^2}{4\delta_1 (\delta_1 - \delta_2)^2}.$$

After substitution of this value of $\varepsilon_4$ we obtain that for the perturbed systems (6.9) the necessary conditions $N = 0 = B_2$ for the systems to belong to the class QSL$_5$ are satisfied. However for these systems calculations yield:

$$\mathcal{E}_1 = 8(\delta_1 - \delta_2)^2 [4\delta_1^2 (\delta_1 - \delta_2)(X + \delta_2 Y)^2 + 2\delta_2 (1 + \varepsilon_1 + \delta_2 \varepsilon_5)(X + \delta_1 Y)Z + \delta_2 (1 + \delta_1)(1 + \varepsilon_1 + \delta_1 \varepsilon_5)Z^2] \mathcal{H},$$

$$\mathcal{E}_2 = - [4\delta_1^2 (\delta_1 - \delta_2)^2 (X + \delta_2 Y)^2 - 4\delta_1 (\delta_1 - \delta_2)^2 (1 + \varepsilon_1 + \delta_2 \varepsilon_5)XZ - \delta_2 (1 + \varepsilon_1 + \delta_1 \varepsilon_5)^2 Z^2] \times$$

$$\left[2(\delta_1 - \delta_2)(2\delta_1 - \delta_2)X + 2\delta_1 \delta_2 (\delta_1 - \delta_2)Y + \delta_2 (1 + \varepsilon_1 + \delta_1 \varepsilon_5)Z\right] \mathcal{H},$$

$$\mathcal{H} = \frac{1}{16\delta_1 (\delta_1 - \delta_2)^2} [\delta_1 - \delta_2] X [\delta_1 - \delta_2] X - (1 + \varepsilon_1) Z \left[2(\delta_1 - \delta_2)(X + \delta_1 Y) - (1 + \varepsilon_1 + \delta_1 \varepsilon_5)Z\right].$$

Thus the degree of the polynomial $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2)$ is 3 and to have one more invariant line it is necessary to have a common nonconstant factor of the polynomials $\mathcal{E}_1/\mathcal{H}$ and $\mathcal{E}_2/\mathcal{H}$. Since $\delta_1 \delta_2 (\delta_1 - \delta_2) \neq 0$ the expressions of these polynomials above indicate to us that such a common factor has to depend on $Y$. Therefore we calculate:

$$\text{Res}_Y(\mathcal{E}_1/\mathcal{H}, \mathcal{E}_2/\mathcal{H}) = 2^{19} \delta_1^8 \delta_2^5 (\delta_1 - \delta_2)^{15} (1 + \varepsilon_1 + \delta_2 \varepsilon_5)^2 [(\delta_1 - \delta_2)X - (1 + \varepsilon_1)Z]^2X^2Z^2.$$

By Lemma 2.2 the condition $\text{Res}_Y(\mathcal{E}_1/\mathcal{H}, \mathcal{E}_2/\mathcal{H}) = 0$ must hold in $\mathbb{R}[X, Z]$. However this is impossible, because the condition $\delta_1 \delta_2 (\delta_1 - \delta_2) \neq 0$ holds and $|\varepsilon_i| \ll 1$ ($i = 1, 5$) and $|\delta_i| \ll 1$ ($i = 1, 2$).

6.2.2.1.2 The subcase $\varepsilon_3 \neq 0$. Then the condition

$$\varepsilon_3 = c\delta_1 + \varepsilon_2 \delta_1 + 2\delta_1 \delta_2 \neq 0$$

(6.12)

holds and in this case for the systems (6.9) with $\varepsilon_3 = \varepsilon_6 = 0$ calculations yield:

$$N(\hat{a}_\tau, x, y) = (c + \varepsilon_2)(c + 2\delta_2 + \varepsilon_2)(x + \delta_1 y)^2$$

(6.13)
and we have to distinguish the cases $c \neq 0$ and $c = 0$.

1) If $c \neq 0$ then we may assume $\epsilon_5 = 0$ in systems (6.9) via the translation $x = x_1$, $y = y_1 - \frac{\epsilon_5}{c + \epsilon_2}$. We observe that in the case $c \neq 0$ for the initial systems (6.6) we have $N \neq 0$ and then the perturbed systems cannot belong to the class QSL$_6$. On the other hand, by Lemma 2.1 the perturbed systems could belong to the class QSL$_5$ only if $\theta = B_3 = 0$. For these systems calculations yield

$$\text{Coefficient}[B_3, x^4] = 3(c + \epsilon_2)[\epsilon_4(c + \epsilon_2)(\delta_1 - \delta_2) + \epsilon_6(1 + \epsilon_1 - \epsilon_6)]$$

and hence the condition $B_3 = 0$ implies

$$\epsilon_4 = -\frac{\epsilon_6(1 + \epsilon_1 - \epsilon_6)}{(c + \epsilon_2)(\delta_1 - \delta_2)}.$$ 

Considering this relation for the whole polynomial $B_3$ calculations yield:

$$B_3 = 3\delta_1(c + \epsilon_2 + \delta_2)(1 + \epsilon_1 - \epsilon_6)^2x^2y[2x + (\delta_1 + \delta_2)y] \neq 0$$

and hence the perturbed systems could not belong to the class QSL$_5$ in the case $c \neq 0$ (i.e. when $N \neq 0$).

2) Assume now $c = 0$. Then from (6.13) the condition $N = 0$ yields $\epsilon_2(\epsilon_2 + 2\delta_2) = 0$ and condition (6.12) for $c = 0$ becomes: $\epsilon_3 = \delta_1(\epsilon_2 + 2\delta_2) \neq 0$. Hence we obtain $\epsilon_2 = 0$ and then the systems (6.9) become:

$$\begin{align*}
\dot{x} &= x[1 + \epsilon_1 + (\delta_1 + \delta_2)x + 2\delta_1\delta_2y], \\
\dot{y} &= \epsilon_4 + \epsilon_5x + \epsilon_6y - x^2 + \delta_1\delta_2y^2.
\end{align*}$$

(6.14)

Moreover we may assume $\epsilon_6 = 0$ due to the translation $x = x_1$, $y = y_1 - \frac{\epsilon_6}{2\delta_1\delta_2}$. Thus for systems (6.14) we have $N = 0$ and to belong to the class QSL$_5$ (or QSL$_4$) we need the condition $B_2 = 0$ (or $B_3 = 0$). Since the condition $B_3 = 0$ implies $B_2 = 0$ we examine the polynomial $B_2$. For these systems calculations yield

$$B_2 = -648\delta_1\delta_2x^2[\epsilon_5x - (1 + \epsilon_1)y][2(1 + \epsilon_1)x + (\delta_1 + \delta_2)(\epsilon_5x + \epsilon_1y + y) + 2\delta_1\delta_2y] \neq 0$$

and hence the perturbed systems could not belong to the class QSL$_5$ in the case $c = 0$.

### 6.2.2.2 The case of one real and two complex directions.

In this case considering (6.7) we conclude, that for the perturbed systems we must have the following factorization over $\mathbb{C}$:

$$C_2(\hat{a}, x, y) = [(1 + \gamma_1)x + \gamma_2y][(1 + \gamma_3)^2x^2 + \gamma_4^2y^2]$$

where $|\gamma_i| \ll 1$, $i = 1, \ldots, 4$. Clearly, via a linear transformation (for example, using the following one:

$$x_1 = \alpha[(1 + \gamma_1)x + \gamma_2], \quad y_1 = -(1 + \gamma_3)(1 + \gamma_3)^{-1}\gamma_2\gamma_4^{-2}x + (1 + \gamma_3)^{-1} y,$$

where $\alpha = (1 + \gamma_3)^{2/3}[(1 + \gamma_1)^2 + (1 + \gamma_3)^2\gamma_2^2\gamma_4^{-2}]^{-1/3}$) we can transform the cubic form $C_2(\hat{a}, x, y)$ as follows

$$\hat{C}_2(\hat{a}, x_1, y_1) = x_1(x_1^2 + \delta y_1^2), \quad 0 < \delta \ll 1$$

Furthermore by a translation we can move the invariant line with the direction $[0 : 1]$ to the line $x = 0$.

At the same time after application of the above indicated affine transformation the perturbed systems (6.6$^Y$) become

$$\begin{align*}
\dot{x} &= (\epsilon_1 + 1)x + (c + \epsilon_2)x^2 + \epsilon_3xy, \\
\dot{y} &= \epsilon_4 + \epsilon_5x + \epsilon_6y - x^2 + (c + \epsilon_2)xy + (\epsilon_3 - \delta)y^2.
\end{align*}$$

(6.15)

For these systems we calculate:

$$\theta(\hat{a}) = 8\epsilon_3[\delta(c + \epsilon_2)^2 + (\epsilon_3 - 2\delta)^2]$$

(6.16)

and according to Lemma 2.1 we must have $\theta(\hat{a}) = 0$ and we shall consider two subcases: $\epsilon_3 = 0$ and $\epsilon_3 \neq 0$. 

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6.2.2.2.1 The subcase $\varepsilon_3 = 0$. Then for systems (6.15) we calculate

\[
N(\hat{a}, x, y) = [(c + \varepsilon_2)^2 + 4\delta]x^2
\]

Coefficient $[B_3, x^3 y] = -6\delta[(1 + \varepsilon_1)^2 + (c + \varepsilon_2)^2\varepsilon_4 - \delta(4\varepsilon_4 + \varepsilon_5^2) - \varepsilon_6(c\varepsilon_5 + \varepsilon_2\varepsilon_5 + \varepsilon_6)]$

Since $\delta > 0$ we obtain $B_3(\hat{a}, x, y) \neq 0$ and $N(\hat{a}, x, y) \neq 0$. Therefore according to Lemma 2.1 no one of systems (6.15) with $\varepsilon_3 = 0$ could be in the class QSL$_5$ or QSL$_6$.

6.2.2.2.2 The subcase $\varepsilon_3 \neq 0$. In this case taking into consideration the condition $\delta > 0$ from (6.16) we obtain that the condition $\theta(\hat{a}) = 0$ could be satisfied for a system (6.15) if and only if $c = 0 = \varepsilon_2$ and $\varepsilon_3 = 2\delta$. Then for the systems (6.15) calculations yield:

\[
N(\hat{a}, x, y) = 0
\]

\[
B_2(\hat{a}, x, y) = -648\delta^2[(1 + \varepsilon_1)^2 + \varepsilon_6(\varepsilon_6 - 2\varepsilon_1 - 2) + \delta\varepsilon_5^2]x^4 \neq 0
\]

and according to Lemma 2.1 no one of the systems (6.15) with $\varepsilon_3 = 0$ could be in the class QSL$_5$ or QSL$_6$ in this case.

As all the cases are examined the first part of the Theorem 6.1 is proved.

ii) To prove the existence of a system ($\hat{S}$) with the coefficients $\hat{a} \in U$ such that this system belongs to QSL$_4$ it is sufficient to construct the concrete perturbations. For each one of the canonical systems $C_{2.1}$ up to $C_{2.9}$ we give below the corresponding perturbation.

For convenience let us introduce the following notation:

\[
\varphi^\pm(c, \varepsilon) = c^2(\varepsilon + 1)^2 \pm (\varepsilon - 1)^2; \quad \lim_{\varepsilon \to 0} \varphi^\pm = c^2 \pm 1; \quad c^2 - 1 \neq 0 \iff |c| \neq 1. \tag{6.17}
\]

1) Perturbation of the systems $C_{2.1}$:

\[
\dot{x} = (x + c)^2 - 1 + y - \frac{\varepsilon}{\varphi^-(c, \varepsilon)} y^2, \quad \dot{y} = (1 + \varepsilon)xy + \frac{2\varepsilon(1 + \varepsilon)}{\varphi^-(c, \varepsilon)} y^2;
\]

Invariant lines: $y = 0$, $y + [\varepsilon(c \pm 1) + c \mp 1](x + c \pm 1) = 0$.

2) Perturbation of the systems $C_{2.2}$:

\[
\dot{x} = (x + c)^2 + 1 + y - \frac{\varepsilon}{\varphi^+(c, \varepsilon)} y^2, \quad \dot{y} = (1 + \varepsilon)xy + \frac{2\varepsilon(1 + \varepsilon)}{\varphi^+(c, \varepsilon)} y^2;
\]

Invariant lines: $y = 0$, $y + [\varepsilon(c \pm i) + c \mp 1](x + c \pm i) = 0$.

3) Perturbation of the system $C_{2.3}$:

\[
\dot{x} = x + y + (1 + \varepsilon^2)x^2 + \frac{\varepsilon(1 + \varepsilon^2)}{\varepsilon^3 - 1} y^2, \quad \dot{y} = \varepsilon y + xy + \frac{\varepsilon(2\varepsilon^3 + 2\varepsilon - 1)}{\varepsilon^3 - 1} y^2;
\]

Invariant lines: $y + (1 - \varepsilon)x = 0$, $y(\varepsilon^2 + 1)^2 - \varepsilon(1 + \varepsilon + \varepsilon^2)[1 + (\varepsilon^2 + 1)x] = 0$.

4) Perturbation of the system $C_{2.4}$:

\[
\dot{x} = 2\varepsilon x + y + x^2 + \frac{\varepsilon}{1 - 2\varepsilon^3} y^2, \quad \dot{y} = \varepsilon y + (1 + \varepsilon^3)xy - \frac{2\varepsilon^5}{1 - 2\varepsilon^3} y^2;
\]

Invariant lines: $y = 0$, $y + \varepsilon x = 0$, $y + \varepsilon(2\varepsilon^3 - 1)(x + 2\varepsilon) = 0$.

5) Perturbation of the systems $C_{2.5}$:

\[
\dot{x} = (x + c)^2 - 1 + \varepsilon y - \frac{\varepsilon^3}{\varphi^-(c, \varepsilon)} y^2, \quad \dot{y} = (1 + \varepsilon)xy + \frac{2\varepsilon(1 + \varepsilon)}{\varphi^-(c, \varepsilon)} y^2;
\]
Proposition 7.1. All systems in Table 4 are given orbit representatives with their respective Configurations of invariant lines.

6) Perturbation of the systems $C_2.6$:
\[
\dot{x} = (x + c)^2 + 1 + \varepsilon y - \frac{\varepsilon^3}{\varphi^+(c, \varepsilon)} y^2, \quad \dot{y} = (1 + \varepsilon)xy + \frac{2c\varepsilon^2(1 + \varepsilon)}{\varphi^+(c, \varepsilon)} y^2;
\]

7) Perturbation of the system $C_2.7$:
\[
\dot{x} = (x + 1)^2 + \varepsilon - \varepsilon x^2 + \varepsilon xy - \frac{\varepsilon^3(1 + \varepsilon)}{1 + \varepsilon + 4\varepsilon^3} y^2, \quad \dot{y} = x + \varepsilon(1 - \varepsilon - 2\varepsilon^2) y^2;
\]

8) Perturbation of the system $C_2.8$:
\[
\dot{x} = x + (1 - \varepsilon^2 + \varepsilon^3)x^2, \quad \dot{y} = 1 + \varepsilon y + \varepsilon x^2 + xy;
\]

9) Perturbation of the system $C_2.9$:
\[
\dot{x} = -\varepsilon x + (1 + 2\varepsilon^3)x^2, \quad \dot{y} = 1 + \varepsilon y + \varepsilon x^2 + xy;
\]

Invariant lines: $y = 0$, $\varepsilon y - (1 + 2\varepsilon)(x + 1) = 0$, $\varepsilon(\varepsilon^2 - 1)y + (1 - \varepsilon - 2\varepsilon^2)(1 + \varepsilon + x(1 - \varepsilon)) = 0$.

7 Degenerate quadratic systems "$PGL(2, \mathbb{R})$-equivalent" to systems in $QS_{C_2=0}$

The class $QS_{C_2=0}$ yields eleven topologically distinct phase portraits arising from the nine Configurations $C_2.1$ up to $C_2.9$. Each system in $QS_{C_2=0}$ yields an equation (2.2) in $\mathbb{P}_2(\mathbb{R})$. On the set of all equations
\[
A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dX = 0
\]
subject to the condition $AX + BY + CZ = 0$ ($A, B, C$ cubic homogenous polynomials in $X, Y, X$ over $\mathbb{R}$ acts the group $PGL(2, \mathbb{R})$. As all systems in $QS_{C_2=0}$ have at least one real affine invariant line, via a transformation in $PGL(2, \mathbb{R})$ we can always move our former line at infinity in the "finite plane" and this former real affine line to "infinity".

In Table 3 we give specific projective transformations which move an affine line to infinity in this way yielding degenerate systems, i.e. $\deg \gcd(p, q) > 0$.

**Notation 7.1.** $QSD = \{(S)| \max(\deg p, \deg q) = 2$ and $\deg(\gcd(p, q)) = 1\}$.

**Notation 7.2.** $LS = \{(S)| \max(\deg p, \deg q) = 1$, $\deg \gcd(p, q) = 1$ and $Z \nmid C\}$.

So any differential equation associated to a system in $QS_{C_2=0}$ is $PGL(2, \mathbb{R})$ equivalent to a differential equation associated to a system in $QSD$.

From $QS_{C_2=0}$ using the $PGL(2, \mathbb{R})$-action we obtain in this way thirteen systems in $QSD$, which become orbit representatives under the action of affine group and time rescaling.

For the sake of completeness we determine below all the configurations of systems in $QSD$ generated by systems in $LS$. For this we need the following proposition which can be easily obtained.

**Proposition 7.1.** All systems in $LS$ have invariant lines of total multiplicity three. They have either i) two distinct affine invariant lines which are either both real or both complex or ii) only one affine line of multiplicity two or one in which case $Z = 0$ has multiplicity two or iii) $Z = 0$ is of multiplicity three. In Table 4 are given orbit representatives with their respective Configurations of invariant lines.
Let \((S_a)\) be a system in \(L\).

\[
(S_a) : \quad \begin{cases} 
\dot{x} = a + cx + dy = p_0 + p_1(x, y) = p(x, y), \\
\dot{y} = b + ex + fy = q_0 + q_1(x, y) = q(x, y).
\end{cases}
\tag{7.1}
\]

Consider an affine line \(f(x, y) = ux + vy + w = 0\) which is distinct from the invariant lines of \((S_a)\) and let us form the quadratic system in \(QSD\):

\[
(S_{a, f}) : \quad \begin{cases} 
\dot{x} = f(x, y)p(x, y), \\
\dot{y} = f(x, y)q(x, y).
\end{cases}
\tag{7.2}
\]

**Proposition 7.2.** Consider the set \(\mathcal{S}\) of all systems in \(QSD\) listed in the right-hand side of Table 3.

Consider the set \(\mathcal{S}'\) of all systems \((S_{a, f})\) obtained via the above construction from the representatives of systems \(\mathcal{S}\) in \(L\). Then \(\mathcal{S} = \mathcal{S}'\).

### Table 3

| \((C_2.1)\) | \[
\begin{cases} 
\dot{x} = (x + c)^2 - 1 + y, \\
\dot{y} = xy, c \in \mathbb{R}, |c| \neq 1
\end{cases}
\] | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\] | \[
\begin{cases} 
\dot{x} = y[1+2cx+(1-c^2)y], \\
\dot{y} = xy
\end{cases}
\] |
|---|---|---|---|
| \((C_2.2)\) | \[
\begin{cases} 
\dot{x} = (x + c)^2 + 1 + y, \\
\dot{y} = xy, c \in \mathbb{R}
\end{cases}
\] | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\] | \[
\begin{cases} 
\dot{x} = y[1+2cx-(1+c^2)y], \\
\dot{y} = xy
\end{cases}
\] |
| \((C_2.3)\) | \[
\begin{cases} 
\dot{x} = x + y + x^2, \\
\dot{y} = xy
\end{cases}
\] | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\] | \[
\begin{cases} 
\dot{x} = y(1+x), \\
\dot{y} = xy
\end{cases}
\] |
| \((C_2.3)\) | \[
\begin{cases} 
\dot{x} = x + y + x^2, \\
\dot{y} = xy
\end{cases}
\] | \[
\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\] | \[
\begin{cases} 
\dot{x} = y(x-1), \\
\dot{y} = y(x+y)
\end{cases}
\] |
| \((C_2.4)\) | \[
\begin{cases} 
\dot{x} = y + x^2, \\
\dot{y} = xy
\end{cases}
\] | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\] | \[
\begin{cases} 
\dot{x} = y, \\
\dot{y} = xy
\end{cases}
\] |
| \((C_2.5)\) | \[
\begin{cases} 
\dot{x} = (x + c)^2 - 1, \\
\dot{y} = xy, c \in \mathbb{R}, |c| \neq 1
\end{cases}
\] | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\] | \[
\begin{cases} 
\dot{x} = y[2cx+(1-c^2)y], \\
\dot{y} = xy
\end{cases}
\] |
| \((C_2.5)\) | \[
\begin{cases} 
\dot{x} = (x + c)^2 - 1, \\
\dot{y} = xy, c \in \mathbb{R}, |c| \neq 1
\end{cases}
\] | \[
\begin{pmatrix} c+1 & 0 & 0 \\ 0 & c+1 & 0 \\ -1 & 0 & 1 \end{pmatrix}
\] | \[
\begin{cases} 
\dot{x} = (x-1)(1-c-2x), \\
\dot{y} = (c-1)(x-1)y
\end{cases}
\] |
Table 3 (continuation)

\[
\begin{align*}
(C_2.6) & \{ \dot{x} = (x + c)^2 + 1, \quad \dot{y} = xy, \quad c \in \mathbb{R} \} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \{ \dot{x} = y[2cx - (c^2 + 1)y], \quad \dot{y} = xy \} \\
\end{align*}
\]

\[
\begin{align*}
(C_2.7) & \{ \dot{x} = (x + 1)^2, \quad \dot{y} = xy \} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \{ \dot{x} = y(2x - y), \quad \dot{y} = xy \} \\
\end{align*}
\]

\[
\begin{align*}
(C_2.7) & \{ \dot{x} = (x + 1)^2, \quad \dot{y} = xy \} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \Rightarrow \{ \dot{x} = 1 - x, \quad \dot{y} = y(x - 1) \} \\
\end{align*}
\]

\[
\begin{align*}
(C_2.8) & \{ \dot{x} = (x + 1)^2, \quad \dot{y} = xy \} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \Rightarrow \{ \dot{x} = x(1 - x), \quad \dot{y} = (x - 1)^2 \} \\
\end{align*}
\]

\[
\begin{align*}
(C_2.8) & \{ \dot{x} = (x + 1)^2, \quad \dot{y} = xy \} \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \{ \dot{x} = x(1 + x), \quad \dot{y} = y(x - x) \} \\
\end{align*}
\]

\[
\begin{align*}
(C_2.9) & \{ \dot{x} = x^2, \quad \dot{y} = 1 + xy \} \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \{ \dot{x} = x^2, \quad \dot{y} = x \} \\
\end{align*}
\]

Table 4 (LS)

\[
\begin{align*}
(L.1) & \{ \dot{x} = cx, \quad c \in \mathbb{R}, \quad \dot{y} = y, \quad c \neq 0, 1 \} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \{ \dot{x} = x + cy, \quad c \in \mathbb{R} \} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \{ \dot{x} = x, \quad \dot{y} = y \} \\
\end{align*}
\]

\[
\begin{align*}
(L.3) & \{ \dot{x} = x + y, \quad \dot{y} = y \} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \{ \dot{x} = x + y, \quad \dot{y} = x \} \\
\end{align*}
\]

\[
\begin{align*}
(L.4) & \{ \dot{x} = 1, \quad \dot{y} = y \} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \{ \dot{x} = 1, \quad \dot{y} = x \} \\
\end{align*}
\]

\[
\begin{align*}
(L.5) & \{ \dot{x} = 1, \quad \dot{y} = x \} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \{ \dot{x} = 1, \quad \dot{y} = x \} \\
\end{align*}
\]

\[
\begin{align*}
& \Delta = (p_1 q_1)^{(1)} = Jacobian(p_1, q_1), \\
& \Theta = Discriminant(C_1) = D^2_1 - 4\Delta \\
\end{align*}
\]
References


[26] D. Schlomiuk, N. Vulpe, Planar quadratic vector fields with invariant lines of total multiplicity at least five, to appear in the journal "Qualitative Theory of Dynamical Systems".


