On a discrete variant of Bernstein's
polynomial inequality

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Abstract

It is known that for any polynomial p of degree less or equal to n ,

$$\max_{|z| \leq 1} |p'(z)| \leq n \max_{0 \leq j \leq 2n-1} |p(e^{ij\pi/n})|$$

In this note we establish cases of equality and also improve the above inequality.

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1 Introduction

Let $\mathcal{P}_n$ be the class of polynomials $p(z) := \sum_{k=0}^n a_k(p)z^k$ of degree at most n with complex coefficients. We write, together with $\mathbb{D} := \{z \mid |z| < 1\}$,

$$|p|_{\mathbb{D}} := \max_{|z|=1} |p(z)|$$

and

$$|p|_{[-1,1]} := \max_{-1 \leq x \leq 1} |p(x)|.$$

According to the famous inequalities of Bernstein and Markov we have

$$|p'|_{\mathbb{D}} \leq n|p|_{\mathbb{D}} \quad (1)$$

and

$$|p'|_{[-1,1]} \leq n^2 |p|_{[-1,1]}. \quad (2)$$

We refer the reader to the book by Rahman and Schmeisser [4] for an exhaustive list of references concerning (1) and (2).

We shall be concerned in this note by the following discrete refinements of (1) and (2):

$$|p'|_{\mathbb{D}} \leq n \max_{1 \leq j \leq 2n} |p(e^{ij\pi/n})| \quad (3)$$

and

$$|p'|_{[-1,1]} \leq n^2 \max_{0 \leq j \leq n} |p(\cos(j\pi/n))|. \quad (4)$$

The inequality (4) is due to Duffin and Schaeffer [2] and it is known that equality holds in (4) only for multiples of the n th Chebyshev polynomial $T_n \in \mathcal{P}_n$ which is defined by

$$T_n(x) := \cos(n \arccos(x)), \quad x \in [-1, 1].$$

The more recent inequality (3) is due to Frappier, Rahman and Ruscheweyh [3]. However it does not seem to be known for which polynomials equality holds in (3). The main result of this note settles this question and contains an improvement of (3). We shall prove

Theorem 1 *Let $n \geq 1$ and $p(z) := \sum_{k=0}^{2n} a_k(p)z^k \in \mathcal{P}_{2n}$. Then for any real θ ,*

$$\left| e^{i\theta} p'(e^{i\theta}) - 2 \sum_{k=1}^n k a_{n+k}(p) e^{i(n+k)\theta} \right| \leq n \max_{1 \leq j \leq 2n} |p(e^{ij\pi/n})|.$$

Further, the equality

$$|p'|_{\mathbb{D}} = n \max_{1 \leq j \leq 2n} |p(e^{ij\pi/n})|$$

holds for some $p \in \mathcal{P}_n$ if and only if

$$p(z) \equiv Kz^n, \quad \text{for some complex number } K.$$

2 Proof of Theorem 1

We shall need two auxiliary results. Let $\mathcal{T}_n$ denote the class of trigonometric polynomials

$$t(\theta) := \sum_{k=-n}^n a_k e^{ik\theta}$$

with complex coefficients. Then we have

Lemma 1 *For any $t \in \mathcal{T}_{4n-1}$ and $\gamma \in \mathbb{R}$,*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} t(\theta) d\theta = \frac{1}{4n} \sum_{j=0}^{4n-1} t\left(\left(\frac{j\pi}{2} - \gamma\right)/n\right).$$

Lemma 1 is a well-known quadrature formula (see for example [1] as a ready reference). Simple computations show that the quadrature is valid for $t(\theta) := e^{ik\theta}$, $0 \leq k \leq 4n-1$. We shall also need

Lemma 2 For any $p \in \mathcal{P}_n$ and $\gamma, \varphi \in \mathbb{R}$,

$$e^{i(\gamma+\varphi)} p'(e^{i\varphi}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nv + \gamma) \frac{1 - \cos(nv)}{1 - \cos(v)} p(e^{i(\varphi-v)}) dv.$$

This last lemma is, to the best of our knowledge, new and again simple computations show that it is valid for $p(z) := z^k$, $0 \leq k \leq n$.

We first prove our theorem for $p \in \mathcal{P}_n$. Indeed, we have for $\gamma, \varphi \in \mathbb{R}$,

$$e^{i(\gamma+\varphi)} p'(e^{i\varphi}) = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^k \ell_k p(e^{i(\varphi-k\pi/n+\gamma/n)}) \quad (5)$$

with $\ell_k = \ell_k(\gamma) = \frac{1 - \cos(k\pi - \gamma)}{1 - \cos(\frac{k\pi}{n} - \frac{\gamma}{n})} \geq 0$ and $\frac{1}{2n} \sum_{k=0}^{2n-1} \ell_k = n$. This can be seen as follows: by Lemma 2,

$$e^{i(\gamma+\varphi)} p'(e^{i\varphi}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nv + \gamma) \left[n + \sum_{j=1}^{n-1} (n-j)(e^{ijv} + e^{-ijv}) \right] p(e^{i(\varphi-v)}) dv \quad (6)$$

and we remark that the integrand in (6) is a trigonometric polynomial $t(e^{iv})$ of degree at most $3n-1 \leq 4n$. We now use Lemma 1 and get

$$\begin{aligned} e^{i(\gamma+\varphi)} p'(e^{i\varphi}) &= \frac{1}{2n} \sum_{j=0}^{4n-1} \cos\left(\frac{j\pi}{2}\right) \frac{1 - \cos(\frac{j\pi}{2} - \gamma)}{1 - \cos(\frac{j\pi}{2n} - \frac{\gamma}{n})} p(e^{i(\varphi-j\pi/2n+\gamma/n)}) \\ &= \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^k \frac{1 - \cos(k\pi - \gamma)}{1 - \cos(\frac{k\pi}{n} - \frac{\gamma}{n})} p(e^{i(\varphi-k\pi/n+\gamma/n)}) \end{aligned} \quad (7)$$

The substitution $p(z) := z^n$ in (7) yields

$$\frac{1}{2n} \sum_{k=0}^{2n-1} \ell_k = n.$$

We remark that (5) is a special case of an identity proved in [3, Lemma 4'], although the coefficients ℓ_k were not explicitly determined there. It is now seen that (3) is a simple consequence of (7): let $p \in \mathcal{P}_n$, $\varphi \in \mathbb{R}$ and $\gamma = -n\varphi$. Then

$$\begin{aligned} |p'(e^{i\varphi})| &= |e^{i(\gamma+\varphi)} p'(e^{i\varphi})| \\ &= \left| \frac{1}{2n} \sum_{k=0}^{2n-1} \ell_k (-1)^k p(e^{-ik\pi/n}) \right| \\ &\leq \frac{1}{2n} \sum_{k=0}^{2n-1} \ell_k |p(e^{-ik\pi/n})| \\ &\leq \left(\frac{1}{2n} \sum_{k=0}^{2n-1} \ell_k \right) \max_{0 \leq j \leq 2n-1} |p(e^{ij\pi/n})| \\ &= n \max_{0 \leq j \leq 2n-1} |p(e^{ij\pi/n})|. \end{aligned} \quad (8)$$

Let now $p \in \mathcal{P}_n$ such that equality holds in (3), i.e., $\varphi \in \mathbb{R}$ and

$$|p'|_{\mathbb{D}} = |p'(e^{i\varphi})| = n \max_{0 \leq j \leq 2n-1} |p(e^{ij\pi/n})|.$$

We assume first that $|\cos(n\varphi)| < 1$. Then

$$\ell_k = \frac{1 - (-1)^k \cos(n\varphi)}{1 - \cos(k\pi/n - \varphi)} > 0, \quad 0 \leq k \leq 2n-1$$

and equality must hold everywhere in (8). Because of the equality case in the triangle inequality, we obtain that

$$(-1)^k p(e^{-ik\pi/n}) = Me^{i\psi}, \quad 0 \leq k \leq 2n-1$$

where ψ is a fixed real number and $M := \max_{1 \leq j \leq 2n} |p(e^{ij\pi/n})|$. This amounts to the fact that the polynomial $q \in \mathcal{P}_n$ defined by

$$q(z) = p(z) - Me^{i\psi} z^n$$

has a zero at each $(2n)$ th-root of unity and therefore $q \equiv 0$, i.e., $p(z) \equiv Me^{i\psi} z^n$. We now have to discuss the case where $|\cos(n\varphi)| = 1$. We may without any loss of generality assume that $\varphi = 0$ because the set of $(2n)$ th-roots of unity is closed under multiplication by one of its members. We obtain as in (8)

$$\begin{aligned} nM = |p'(1)| &= \left| \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^k \frac{1 - \cos(k\pi)}{1 - \cos(k\pi/n)} p(e^{-ik\pi/n}) \right| \\ &= \left| \frac{n}{2} p(1) + \frac{1}{2n} \sum_{\substack{k=1 \\ k \text{ odd}}}^{2n-1} \frac{2}{1 - \cos(k\pi/n)} (-p(e^{-ik\pi/n})) \right| \\ &\leq \left(\frac{n}{2} + \frac{1}{2n} \sum_{\substack{k=1 \\ k \text{ odd}}}^{2n-1} \frac{2}{1 - \cos(k\pi/n)} \right) M \\ &= nM, \end{aligned} \tag{9}$$

i.e., equality holds everywhere in (9) and since none of the numbers $\frac{2}{1 - \cos(k\pi/n)}$ ($1 \leq k \leq 2n-1$, k odd) vanishes, there exists a real number ψ such that

$$q(z) = p(z) - Me^{i\psi} z^n \in \mathcal{P}_n$$

has a zero at each of the $n+1$ distinct points $e^{-ik\pi/n}$, ($0 \leq k \leq 2n-1$, $k = 0$ or k is odd). Just as above $p(z) \equiv Me^{i\psi} z^n$.

We now have to verify our claim concerning $\mathcal{P}_{2n}$. Let $P(z) = \sum_{k=0}^{2n} a_k(P) z^k \in \mathcal{P}_{2n}$. Applying Lemma 2 to $p(z) := P(z) - \sum_{k=n+1}^{2n} a_k(P) z^k \in \mathcal{P}_n$, we obtain after some simple computations

$$\begin{aligned} e^{i\theta} P'(e^{i\theta}) - 2 \sum_{k=1}^n k a_{n+k}(P) e^{i(n+k)\theta} \\ = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nv + \gamma) \frac{1 - \cos(nv)}{1 - \cos(v)} P(e^{i(\theta-v)}) dv. \end{aligned} \tag{10}$$

The integrand in (10) is a trigonometric polynomial of degree at most $4n-1$. Upon an application of Lemma 1 we obtain (with $\gamma = -n\theta$)

$$e^{i\theta} P'(e^{i\theta}) - 2 \sum_{k=1}^n k a_{n+k}(P) e^{i(n+k)\theta} = \frac{1}{2n} \sum_{k=0}^{2n-1} \ell_k (-1)^k P(e^{-ik\pi/n}).$$

The result follows just as above. It is however of interest to notice that the equality

$$\left| e^{i\theta} P'(e^{i\theta}) - 2 \sum_{k=1}^n k a_{n+k}(P) e^{i(n+k)\theta} \right| = n \max_{0 \leq j \leq 2n-1} |P(e^{ij\pi/n})|$$

holds for all real numbers θ and all polynomials with complex coefficient a, b of the type

$$P(z) = az^{2n} + bz^n - a, \quad z = e^{i\theta}.$$

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