Two-Dimensional Wavelet Bases For Partial Differential Operators and Applications^{*}

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CRM-2986

December 2003

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^{*}This work was partially supported by the Natural Sciences and Engineering Research Council of Canada and the Centre de recherches mathématiques of the Université de Montréal

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Abstract

Partial differential operators are represented in non-standard form in separable two-dimensional orthonormal wavelet bases. A formal pseudodifferential approach is suggested for numerical applications. Applications are to the fifth-order quasilinear thin-film equation on an inclined plane.

1999 Mathematics Subject Classification. Primary 65T60; Secondary 35Q51

Keywords and Phrases. Periodic wavelet, differential operator, wavelet basis, non-standard representation, thin film equation

Résumé

On représente, sous forme non standard, les opérateurs aux dérivées partielles suivant une base d'ondelettes séparables bidimensionnelles et biorthogonales et l'on suggère une approche pseudo-différentielle en vue des applications numériques. On résout l'équation quasi-linéaire du 5e ordre de l'écoulement d'une couche fluide mince sur un plan incliné.

1 Introduction

Wavelets have found applications in the representation of differential operators. The representation of operators in compactly supported, one-dimensional wavelet bases, as well as the construction of their corresponding matrices, has been considered by Beylkin [3] and Beylkin, Coifman and Rokhlin [4]. The foregoing results have been employed by Beylkin and Keiser [5] in the numerical solution of evolution equations in time and one space dimension.

In this paper, we give the non-standard representation of linear partial differential operators in two-dimensional compactly supported wavelet bases. Periodized Daubechies' wavelets [6, 7] of period 1 are used in the construction of the matrices representing linear differential operators of the form $L = g(\partial_x, \partial_y)$, where g is a real-valued function coming from partial differential equations of interest [5, 8, 9]. A formal pseudodifferential approach is suggested for numerical applications.

Applications include the numerical realization of horseshoe-shaped solitons in the solution of a fifth-order partial differential equation governing the flow of a viscous fluid film on an inclined plane [10, 8].

2 Separable two-dimensional multiresolution analysis

Consider an orthogonal one-dimensional multiresolution analysis (MRA),

$$\cdots \subset V_j \subset V_{j+1} \subset \cdots, \quad j \in \mathbb{Z},$$

with scaling function $\varphi(x)$ and wavelet function $\psi(x)$ supported on $[0, L-1]$ and satisfying the two-scale equations:

$$\varphi(x) = \sqrt{2} \sum_{k=0}^{L-1} h_k \varphi(2x-k), \quad \psi(x) = \sqrt{2} \sum_{k=0}^{L-1} g_k \varphi(2x-k).$$

The scaled and shifted φ and ψ are

$$\varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k), \quad \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k).$$

The spaces

$$\cdots \subset \mathbf{V}_j = V_j \otimes V_j \subset \mathbf{V}_{j+1} \subset \cdots, \quad j \in \mathbb{Z},$$

form a separable two-dimensional MRA with scaling function $\Phi(x, y) = \varphi(x)\varphi(y)$. The orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$ is

$$\mathbf{W}_j = \mathbf{W}_j^h \oplus \mathbf{W}_j^v \oplus \mathbf{W}_j^d,$$

where

$$\mathbf{W}_j^h = V_j \otimes W_j, \quad \mathbf{W}_j^v = W_j \otimes V_j, \quad \mathbf{W}_j^d = W_j \otimes W_j,$$

and $\mathbf{W}_0$ is defined by the horizontal, vertical and diagonal wavelet functions:

$$\Psi^h(x, y) = \varphi(x)\psi(y), \quad \Psi^v(x, y) = \psi(x)\varphi(y), \quad \Psi^d(x, y) = \psi(x)\psi(y).$$

The scaled and shifted version of Φ , Ψ^h , Ψ^v and Ψ^d are

$$\begin{aligned} \Phi_{k_1, k_2}^j(x, y) &= \varphi_{j, k_1}(x) \varphi_{j, k_2}(y), & \Psi_{k_1, k_2}^{j, h}(x, y) &= \varphi_{j, k_1}(x) \psi_{j, k_2}(y), \\ \Psi_{k_1, k_2}^{j, v}(x, y) &= \psi_{j, k_1}(x) \varphi_{j, k_2}(y), & \Psi_{k_1, k_2}^{j, d}(x, y) &= \psi_{j, k_1}(x) \psi_{j, k_2}(y). \end{aligned}$$

In the following, the superscript λ will take values in the index set $\{h, v, d\}$. For each $j \in \mathbb{Z}$, bases for $\mathbf{V}_j$ and $\mathbf{W}_j$ are:

$$\{\Phi_{k_1, k_2}^j, \quad k_1, k_2 \in \mathbb{Z}\}, \quad \{\Psi_{k_1, k_2}^{j, \lambda}, \quad k_1, k_2 \in \mathbb{Z}, \quad \lambda = h, v, d\}.$$

A separable wavelet basis of $L^2(\mathbb{R}^2)$ is $\{\Psi_{k_1, k_2}^{j, \lambda}, \quad j, k_1, k_2 \in \mathbb{Z}, \quad \lambda = h, v, d\}$.

Let $\langle \cdot, \cdot \rangle$ denote the inner product in $L^2(\mathbb{R}^2)$. A function $f \in L^2(\mathbb{R}^2)$ is approximated by its orthogonal projection on the space $\mathbf{V}_n$:

$$f_n = P_n f = \sum_{k_1, k_2} s_{k_1, k_2}^n \Phi_{k_1, k_2}^n, \quad \text{where} \quad s_{k_1, k_2}^n = \langle f, \Phi_{k_1, k_2}^n \rangle,$$

and represented in the wavelet basis by the series

$$f = \sum_{j \in \mathbb{Z}} \sum_{\lambda = h, v, d} \sum_{k_1, k_2 \in \mathbb{Z}} d_{k_1, k_2}^{j, \lambda} \Psi_{k_1, k_2}^{j, \lambda}, \quad \text{where} \quad d_{k_1, k_2}^{j, \lambda} = \langle f, \Psi_{k_1, k_2}^{j, \lambda} \rangle.$$

Choosing $\mathbf{V}_n$ and $\mathbf{V}_{n-J}$ as the finest and coarsest subspaces, respectively, and using $\mathbf{V}_j = \mathbf{V}_{j-1} \oplus \mathbf{W}_{j-1}$, we have

$$\mathbf{V}_n = \mathbf{V}_{n-J} \oplus \mathbf{W}_{n-J} \oplus \mathbf{W}_{n-J+1} \oplus \cdots \oplus \mathbf{W}_{n-1}.$$

Then $f_n \in \mathbf{V}_n$ can be expanded as

$$f_n = P_{n-J}f + \sum_{j=n-J}^{n-1} Q_j f, \quad \text{where} \quad Q_j f = \sum_{k_1, k_2} \sum_{\lambda=h, v, d} d_{k_1, k_2}^{j, \lambda} \Psi_{k_1, k_2}^{j, \lambda}.$$

Here, $P_{n-J}f$ is the coarsest approximation on $\mathbf{V}_{n-J}$ and the projections $Q_j = Q_j^h + Q_j^v + Q_j^d$ on $\mathbf{W}_j = \mathbf{W}_j^h \oplus \mathbf{W}_j^v \oplus \mathbf{W}_j^d$ give the detail functions $Q_j f$ on $\mathbf{W}_j$ which build up the finest approximation, f_n , of f on $\mathbf{V}_n$. Compression is achieved by setting $d_{k_1, k_2}^{j, \lambda} = 0$ below a threshold.

The two-dimensional fast wavelet transform (FWT) decomposition is obtained by lowpass and highpass filterings with downsampling:

$$\begin{aligned} s_{k_1, k_2}^j &= \sum_{m_1, m_2} H_{m_1, m_2} s_{m_1+2k_1, m_2+2k_2}^{j+1}, \\ d_{k_1, k_2}^{j, \lambda} &= \sum_{m_1, m_2} G_{m_1, m_2}^\lambda s_{m_1+2k_1, m_2+2k_2}^{j+1}, \end{aligned}$$

respectively, with real lowpass filter

$$H_{m_1, m_2} = h_{m_1} h_{m_2}$$

and real highpass filters

$$G_{m_1, m_2}^h = h_{m_1} g_{m_2}, \quad G_{m_1, m_2}^v = g_{m_1} h_{m_2}, \quad G_{m_1, m_2}^d = g_{m_1} g_{m_2}.$$

The two-dimensional inverse fast wavelet transform (IFWT) reconstruction of s^{j+1} from s^j and $d^{j, \lambda}$ is obtained by upsampling and filtering:

$$s_{2k_1, 2k_2}^{j+1} = \sum_{m_1, m_2} H_{2m_1, 2m_2} s_{k_1-m_1, k_2-m_2}^j \quad (1)$$

$$+ \sum_{\lambda=h, v, d} \sum_{m_1, m_2} G_{2m_1, 2m_2}^\lambda d_{k_1-m_1, k_2-m_2}^{\lambda, j},$$

$$s_{2k_1+1, 2k_2+1}^{j+1} = \sum_{m_1, m_2} H_{2m_1+1, 2m_2+1} s_{k_1-m_1, k_2-m_2}^j \quad (2)$$

$$+ \sum_{\lambda=h, v, d} \sum_{m_1, m_2} G_{2m_1+1, 2m_2+1}^\lambda d_{k_1-m_1, k_2-m_2}^{\lambda, j},$$

$$s_{2k_1, 2k_2+1}^{j+1} = \sum_{m_1, m_2} H_{2m_1, 2m_2+1} s_{k_1-m_1, k_2-m_2}^j \quad (3)$$

$$+ \sum_{\lambda=h, v, d} \sum_{m_1, m_2} G_{2m_1, 2m_2+1}^\lambda d_{k_1-m_1, k_2-m_2}^{\lambda, j},$$

$$s_{2k_1+1, 2k_2}^{j+1} = \sum_{m_1, m_2} H_{2m_1+1, 2m_2} s_{k_1-m_1, k_2-m_2}^j \quad (4)$$

$$+ \sum_{\lambda=h, v, d} \sum_{m_1, m_2} G_{2m_1+1, 2m_2}^\lambda d_{k_1-m_1, k_2-m_2}^{\lambda, j}.$$

For periodic wavelets, the matrices s^j and $d^{j, \lambda}$ are $(2^j, 2^j)$ -periodic. The $[0, 1]$ -periodic scaling function and wavelets are defined by the formulae

$$\sum_{\ell_1 \in \mathbb{Z}} \sum_{\ell_2 \in \mathbb{Z}} \Phi_{k_1, k_2}^j(x - \ell_1, y - \ell_2)$$

and

$$\sum_{\ell_1 \in \mathbb{Z}} \sum_{\ell_2 \in \mathbb{Z}} \Psi_{k_1, k_2}^{j, \lambda}(x - \ell_1, y - \ell_2), \quad \lambda = h, v, d,$$

respectively.

3 Nonstandard representation of differential operators

Given an operator T , we approximate T by $T_n = P_n T P_n$ and with $\mathbf{V}_n$ and $\mathbf{V}_{n-J}$ as above, we expand $T_n = P_n T P_n$ in the telescoping sum

$$T_n = P_n T P_n = \sum_{j=0}^{J-1} [P_{n-j} T P_{n-j} - P_{n-j-1} T P_{n-j-1}] + P_{n-J} T P_{n-J},$$

and use $P_{j+1} = P_j + Q_j$ to obtain the NS-representation of T ,

$$T_n = \sum_{j=n-J}^{n-1} [A_j + B_j + C_j] + T_{n-J},$$

where $T_{n-J} = P_{n-J} T P_{n-J} : \mathbf{V}_{n-J} \rightarrow \mathbf{V}_{n-J}$ and

$$\begin{aligned} A_j &= \sum_{\lambda, \lambda' = h, v, d} A_j^{\lambda, \lambda'} = \sum_{\lambda, \lambda' = h, v, d} Q_j^\lambda T Q_j^{\lambda'} : \mathbf{W}_j^{\lambda'} \rightarrow \mathbf{W}_j^\lambda, \\ B_j &= \sum_{\lambda = h, v, d} B_j^\lambda = \sum_{\lambda = h, v, d} Q_j^\lambda T P_j : \mathbf{V}_j \rightarrow \mathbf{W}_j, \\ C_j &= \sum_{\lambda = h, v, d} C_j^\lambda = \sum_{\lambda = h, v, d} P_j T Q_j^\lambda : \mathbf{W}_j \rightarrow \mathbf{V}_j. \end{aligned}$$

The NS-representation decouples the scales by acting on $\mathbf{V}_j$ and $\mathbf{W}_j$ at each level.

Then for $f \in L^2(\mathbb{R}^2)$, we have the approximation

$$Tf \approx \sum_{j=n-J}^{n-1} \left[\sum_{\lambda, \lambda' = h, v, d} A_j^{\lambda, \lambda'} f + \sum_{\lambda = h, v, d} (B_j^\lambda + C_j^\lambda) f \right] + T_{n-J} f.$$

The steps in the application of an operator T on periodic functions $f \in L^2([0, 1]^2)$ of period 1 are as follows.

- Approximate f in $\mathbf{V}_n$ by $P_n f$ to get the $2^n \times 2^n$ coordinate matrix s^n .
- Decompose s^n down the MRA spaces using two-dimensional FWT to get $d^{j,h}$, $d^{j,v}$, $d^{j,d}$ and s^j for $j = n-1, n-2, \dots, n-J$.
- For $n-J \leq j \leq n-1$, construct the matrices $A^{j,\lambda,\lambda'}$, $B^{j,\lambda}$, $C^{j,\lambda}$ representing the operators $A_j^{\lambda,\lambda'}$, B_j^λ , C_j^λ , respectively.
- The approximation $Tf \simeq T_n f$ is then given by

$$T_n f = \sum_{j=n-J}^{n-1} \left[\sum_{\lambda = h, v, d} \left(\sum_{k_3, k_4=0}^{2^j-1} \tilde{d}_{k_3, k_4}^{j, \lambda} \Psi_{k_3, k_4}^{j, \lambda} \right) + \sum_{k_3, k_4=0}^{2^j-1} \tilde{s}^j \Phi_{k_3, k_4}^j \right],$$

where

$$\begin{aligned} \tilde{d}^{j,h} &= \sum_{\lambda = h, v, d} A^{j, h, \lambda} \odot d^{j, \lambda} + B^{j, h} \odot s^j, \\ \tilde{d}^{j,v} &= \sum_{\lambda = h, v, d} A^{j, v, \lambda} \odot d^{j, \lambda} + B^{j, v} \odot s^j, \\ \tilde{d}^{j,d} &= \sum_{\lambda = h, v, d} A^{j, d, \lambda} \odot d^{j, \lambda} + B^{j, d} \odot s^j, \\ \tilde{s}^j &= \sum_{\lambda = h, v, d} C^{j, \lambda} \odot d^{j, \lambda}, \quad \text{for } n-J+1 \leq j \leq n-1, \\ \tilde{s}^{n-J} &= \sum_{\lambda = h, v, d} C^{n-J, \lambda} \odot d^{n-J, \lambda} + T^{n-J} \odot s^{n-J}. \end{aligned}$$

The operation $\odot$ is defined as follows.

If P is a $k \times k$ block matrix with $k \times k$ block matrices $P^{i,j}$ and D is a $k \times k$ matrix, then

$$(P \odot D)_{i,j} = \sum_{m,n} P_{m,n}^{i,j} D_{m,n}.$$

The function $\tilde{f} = T_n f$ can be reconstructed to the finest space $\mathbf{V}_n$ as

$$\tilde{f} = \sum_{k_1=0}^{2^n-1} \sum_{k_2=0}^{2^n-1} s_{k_1,k_2}^n \Phi_{k_1,k_2}^n.$$

The coefficients s^n can be reconstructed from the coefficients $\tilde{d}^{j,\lambda}$ and $\tilde{s}^j$ of the NS-representation of f for $n - J \leq j \leq n - 1$ by the following **pseudo-inverse fast wavelet transform**.

For $j = n - J, n - J + 1, n - J + 2, \dots, n - 2$, $\hat{s}^{j+1}$ is constructed from $\tilde{s}^j$ and $\tilde{d}^{j,\lambda}$ by means of (1)–(4) and

$$s^{j+1} = \hat{s}^{j+1} + \tilde{s}^{j+1}.$$

Finally, for $j = n - 1$, s^n is constructed from s^{n-1} and $\tilde{d}^{n-1,\lambda}$ again by (1)–(4).

4 Matrix construction of the NS-representation

The operator T_n acting on a function $f \in L^2([0, 1]^2)$,

$$\begin{aligned} T_n f &:= P_n T P_n f = P_n T \left(\sum_{k_1, k_2} \langle f, \Phi_{k_1, k_2}^n \rangle \Phi_{k_1, k_2}^n \right) \\ &= \sum_{k_3, k_4} \underbrace{\left(\sum_{k_1, k_2} \underbrace{\langle T \Phi_{k_1, k_2}^n, \Phi_{k_3, k_4}^n \rangle}_{T_{k_1, k_2}^{n, k_3, k_4}} \underbrace{\langle f, \Phi_{k_1, k_2}^n \rangle}_{s_{k_1, k_2}^n} \right)}_{\tilde{s}_{k_3, k_4}^n} \Phi_{k_3, k_4}^n, \end{aligned}$$

is represented by the 4-dimensional structure:

$$T_{k_1, k_2}^{n, k_3, k_4} = \langle T \Phi_{k_1, k_2}^n, \Phi_{k_3, k_4}^n \rangle.$$

For each k_3, k_4 , the $2^n \times 2^n$ matrix

$$T_{k_1, k_2}^{n, k_3, k_4} = \langle T \Phi_{k_1, k_2}^n, \Phi_{k_3, k_4}^n \rangle, \quad 0 \leq k_1, k_2 \leq 2^n - 1,$$

is a block in the structure representation T^n of $T_n : \mathbf{V}_n \rightarrow \mathbf{V}_n$,

$$T^n = \begin{bmatrix} T^{n, 0, 0} & T^{n, 0, 1} & \dots & T^{n, 0, 2^n-1} \\ T^{n, 1, 0} & T^{n, 1, 1} & \dots & T^{n, 1, 2^n-1} \\ \vdots & \vdots & \dots & \vdots \\ T^{n, 2^n-1, 0} & T^{n, 2^n-1, 1} & \dots & T^{n, 2^n-1, 2^n-1} \end{bmatrix}.$$

Due to the structure of the MRA, the operators $A_j^{\lambda, \lambda'}$, B_j^λ , C_j^λ and T_j are represented by the structures $A^{j, \lambda, \lambda'}$, $B^{j, \lambda}$, $C^{j, \lambda}$ and T^j , respectively,

$$\begin{aligned} A_{k_1, k_2}^{j, \lambda, \lambda', k_3, k_4} &= \langle T \Psi_{k_1, k_2}^{j, \lambda'}, \Psi_{k_3, k_4}^{j, \lambda} \rangle, & B_{k_1, k_2}^{j, \lambda, k_3, k_4} &= \langle T \Phi_{k_1, k_2}^j, \Psi_{k_3, k_4}^{j, \lambda} \rangle, \\ C_{k_1, k_2}^{j, \lambda, k_3, k_4} &= \langle T \Psi_{k_1, k_2}^{j, \lambda}, \Phi_{k_3, k_4}^j \rangle, & T_{k_1, k_2}^{j, k_3, k_4} &= \langle T \Phi_{k_1, k_2}^j, \Phi_{k_3, k_4}^j \rangle. \end{aligned}$$

The entries of the lower-scale structures are obtained recursively from T^n by the formulae:

$$A_{k_1, k_2}^{j, \lambda, \lambda', k_3, k_4} = \sum_{m_1, m_2, m_3, m_4=0}^{L-1} G_{m_1, m_2}^{\lambda'} G_{m_3, m_4}^\lambda T_{m_1+2k_1, m_2+2k_2}^{j+1, m_3+2k_3, m_4+2k_4},$$

$$\begin{aligned}
B_{k_1, k_2}^{j, \lambda, k_3, k_4} &= \sum_{m_1, m_2, m_3, m_4=0}^{L-1} H_{m_1, m_2} G_{m_3, m_4}^\lambda T_{m_1+2k_1, m_2+2k_2}^{j+1, m_3+2k_3, m_4+2k_4}, \\
C_{k_1, k_2}^{j, \lambda, k_3, k_4} &= \sum_{m_1, m_2, m_3, m_4=0}^{L-1} G_{m_1, m_2}^\lambda H_{m_3, m_4} T_{m_1+2k_1, m_2+2k_2}^{j+1, m_3+2k_3, m_4+2k_4}, \\
T_{k_1, k_2}^{j, \lambda, k_3, k_4} &= \sum_{m_1, m_2, m_3, m_4=0}^{L-1} H_{m_1, m_2} H_{m_3, m_4} T_{m_1+2k_1, m_2+2k_2}^{j+1, m_3+2k_3, m_4+2k_4}.
\end{aligned}$$

For periodic wavelets, these matrices are considered as $(2^j, 2^j)$ -periodic.

5 The operator $T = \partial_x$

The operator $T = \partial_x$ has the following representation:

$$T_{k_1, k_2}^{n, k_3, k_4} = \langle \partial_x(\Phi_{k_1, k_2}^n), \Phi_{k_3, k_4}^n \rangle = 2^n r_{k_3-k_1} \delta_{k_2, k_4},$$

where

$$r_l = \int_{-\infty}^{\infty} \phi(x-l) \phi'(x) dx, \quad l \in \mathbb{Z}. \quad (5)$$

Thus, T^n is completely determined by r_l . A similar representation holds for $T = \partial_y$.

The following proposition is proved in [3].

Proposition 1 *If the integrals (5) exist, then the coefficients r_l satisfy the following system of linear equations:*

$$r_l = 2r_{2l} + \sum_{k=1}^{L/2} a_{2k-1} (r_{2l-2k+1} + r_{2l+2k-1}) \quad (6)$$

with

$$\begin{cases} \sum_{l=-(L-2)}^{L-2} l r_l = -1, \\ r_{-l} = -r_l, \\ r_l = 0 \quad \text{for } l \notin [-L+2, L-2], \end{cases} \quad (7)$$

and

$$a_n = 2 \sum_{i=0}^{L-1-n} h_i h_{i+n}, \quad n = 1, 2, \dots, L-1.$$

Moreover, if the number of vanishing moments of the wavelet $\psi(x)$ is at least two, then (6) and (7) have a unique solution with $r_i \neq 0$ for $-L+2 \leq l \leq L-2$ and $r_l = -r_l$.

With a minor change in notation, the block matrix $T_x^n (= T^n$ for $T = \partial_x$) becomes

$$T_x^n = \begin{bmatrix} \boxed{X^{1,1}} & X^{1,2} & \dots & X^{1,2^n-1} & X^{1,2^n} \\ X^{2,1} & X^{2,2} & \dots & X^{2,2^n-1} & X^{2,2^n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ X^{2^n-1,1} & X^{2^n-1,2} & \dots & X^{2^n-1,2^n-1} & X^{2^n-1,2^n} \\ X^{2^n,1} & X^{2^n,2} & \dots & X^{2^n,2^n-1} & X^{2^n,2^n} \end{bmatrix},$$

with

$$X_{k_1, k_2}^{k_3, k_4} = T_{k_1, k_2}^{n, k_3, k_4} = 2^n r_{k_3-k_1} \delta_{k_2, k_4}.$$

Each X^{k_3, k_4} contains only one nonzero column, the k_4^{th} column. The nonzero column of $X^{1,1}$ is given by

$$c = 2^n [0, r_{-1}, r_{-2}, \dots, r_{-(L-2)}, 0, \dots, 0, r_{L-2}, \dots, r_1]^T,$$

The matrices X^{k_3, k_4} are obtained by means of the ‘Forward Shift Rows Wrap aRound’ and ‘Forward Shift Columns Wrap aRound’ formulas

$$X^{k_3, k_4+1} = \text{FSRWR}(X^{k_3, k_4}), \quad X^{k_3+1, k_4} = \text{FSCWR}(X^{k_3, k_4}),$$

where, for $A = (\vec{c}_1 \ \vec{c}_2 \ \dots \ \vec{c}_{n-1} \ \vec{c}_n)$,

$$\text{FSRWR}(A) = [\vec{c}_n \ \vec{c}_1 \ \dots \ \vec{c}_{n-1}],$$

and for

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{bmatrix}, \quad \text{FSCWR}(A) = \begin{bmatrix} \vec{r}_n \\ \vec{r}_1 \\ \vdots \\ \vec{r}_{n-1} \end{bmatrix},$$

We note that only the top-left matrix, $X^{1,1}$, is calculated and stored as the other blocks, $X^{i,j}$, can be dynamically generated.

To solve for r_l we write the system for r_l as a matrix-vector equation

$$A\vec{r} = \vec{r}, \quad \vec{r} = [r_1, r_2, \dots, r_{L-2}]^T,$$

where we use $r_{-l} = -r_l$ and $r_0 = 0$. Then $\vec{r}$ is the normalized eigenvector of A corresponding to the eigenvalue $\lambda = 1$. The normalization is

$$\sum_{l=-(L-2)}^{L-2} l r_l = -1.$$

6 The operator $T = g(\partial_x, \partial_y)$

To construct the NS-forms of analytic functions of the differential operator (∂_x, ∂_y) one can introduce two approaches for computing the NS-forms of operator functions [5]:

- Compute the function of the projection of the operator on $\mathbf{V}_n$ by

$$T_n = g(P_n \partial_x P_n, P_n \partial_y P_n). \quad (8)$$

- Compute the projection of the operator function on $\mathbf{V}_n$ by

$$T_n = P_n g(\partial_x, \partial_y) P_n. \quad (9)$$

The difference between these two approaches depends on how well the absolute value square of the Fourier transform of the scaling function, $|\hat{\varphi}(\xi)\hat{\varphi}(\eta)|^2$, acts as a cutoff function and the choice may depend on the applications in hand. In this paper, representation (8) is used for the numerical application. In Subsection 6.2, representation (9) will give rise to a formal pseudodifferential formulation.

6.1 Function of projected operator

The matrix T_g^n of $g(\partial_x, \partial_y)$ on $\mathbf{V}_n$ is obtained by applying the function g to the matrices T_x^n and T_y^n representing ∂_x and ∂_y , respectively.

We use this representation to calculate $T_{k_1, k_2}^{n, k_3, k_4}$. With $T_n = g(T_x^n, T_y^n)$, we rearrange the matrices T_x^n and T_y^n as the $2^{2n} \times 2^{2n}$ matrices X and Y , so that

$$T_x^n \odot s^n \sim X \vec{s}, \quad T_y^n \odot s^n \sim Y \vec{s}.$$

where $\vec{s}$ is a 2^{2n} -long column vector rearrangement of the matrix s^n . Because of the structure of T_x^n and T_y^n , X and Y are block circulant with circulant blocks (bccb). Therefore,

$$X = (F_{2^n}^* \otimes F_{2^n}^*) \Lambda_x (F_{2^n} \otimes F_{2^n}), \quad Y = (F_{2^n}^* \otimes F_{2^n}^*) \Lambda_y (F_{2^n} \otimes F_{2^n}),$$

where F_{2^n} and $F_{2^n}^*$ are the Fourier matrix and its adjoint and Λ_x, Λ_y are $2^{2n} \times 2^{2n}$ diagonal matrices containing the eigenvalues of X and Y . Then the matrix T_g^n (in bccb form) is

$$T_g^n = g(X, Y) = (F_{2^n}^* \otimes F_{2^n}^*) g(\Lambda_x, \Lambda_y) (F_{2^n} \otimes F_{2^n}).$$

A long manipulation gives the explicit formula for the top left block $T_g^{n,1,1}$ of T_g^n in terms of the top left blocks $T_x^{n,1,1}$ and $T_y^{n,1,1}$ of T_x^n and T_y^n :

$$T_g^{n,1,1} = \frac{1}{2^n} F_{2^n} g(2^n F_{2^n}^* T_x^{n,1,1} F_{2^n}^*, 2^n F_{2^n}^* T_y^{n,1,1} F_{2^n}^*) F_{2^n}.$$

By construction, $T_g^{n,1,1}$ is real. If $g(\partial_x, \partial_y) = \partial_x$, i.e., $T = \partial_x$, then

$$T^{n,1,1} = \frac{1}{2^n} F_{2^n} \left(2^n F_{2^n}^* T_x^{n,1,1} F_{2^n}^* \right) F_{2^n} = I_n T_x^{n,1,1} I_n = T_x^{n,1,1}.$$

Similarly, if $g(\partial_x, \partial_y) = \partial_y$, i.e., $T = \partial_y$, then $T^{n,1,1} = T_y^{n,1,1}$.

6.2 A formal pseudodifferential operator approach to $T = g(\partial_x, \partial_y)$

Projecting $g(\partial_x, \partial_y)$ on $\mathbf{V}_n$, we have

$$T_n = P_n g(\partial_x, \partial_y) P_n.$$

Thus

$$T_{k_1, k_2}^{n, k_3, k_4} = \int \int \Phi_{k_3, k_4}^n g(\partial_x, \partial_y) \Phi_{k_1, k_2}^n(x, y) dx dy.$$

Using the Fourier transform, we have the formal pseudodifferential representation of the operator $g(\partial_x, \partial_y)$:

$$g(\partial_x, \partial_y) \Phi_{k_1, k_2}^n = \frac{1}{2\pi} \int \int 2^n g(-i2^n \xi, -i2^n \eta) \hat{\varphi}(\xi) \hat{\varphi}(\eta) \times e^{-i2^n x \xi} e^{-i2^n y \eta} e^{ik_1 \xi} e^{ik_2 \eta} d\xi d\eta.$$

Now, using $\Phi_{k_3, k_4}^n(x, y) = 2^n \varphi(2^n x - k_3) \varphi(2^n y - k_4)$, and after making the change of variables

$$x \rightarrow 2^{-n}(x + k_3), \quad y \rightarrow 2^{-n}(y + k_4),$$

we obtain

$$T_{k_1, k_2}^{n, k_3, k_4} = \int \int g(-i2^n \xi, -i2^n \eta) |\hat{\varphi}(\xi)|^2 |\hat{\varphi}(\eta)|^2 e^{-i(k_3 - k_1)\xi} e^{-i(k_4 - k_2)\eta} d\xi d\eta.$$

Therefore,

$$T_{k_1, k_2}^{n, k_3, k_4} = T_{l_1, l_2}^n, \quad l_1 = k_3 - k_1, \quad l_2 = k_4 - k_2.$$

To evaluate this expression, we write

$$T_{k_1, k_2}^{n, k_3, k_4} = T_{l_1, l_2}^n = \int_0^{2\pi} \int_0^{2\pi} h(\xi, \eta) e^{-il_1 \xi} e^{-il_2 \eta} d\xi d\eta,$$

where

$$h(\xi, \eta) = \sum_l \sum_m g(-i2^n(\xi + 2\pi l), -i2^n(\eta + 2\pi m)) |\hat{\varphi}(\xi + 2\pi l)|^2 |\hat{\varphi}(\eta + 2\pi m)|^2.$$

Now, for a given accuracy ε , $|\hat{\varphi}(\xi)|^2$ acts as a cutoff function in the Fourier domain, i.e., $|\hat{\varphi}(\xi)|^2 < \varepsilon$ for $|\xi| > \xi_{\text{cutoff}}$. Then we approximate $h(\xi, \eta)$ by the following sum:

$$\begin{aligned} h(\xi, \eta) &\approx \tilde{h}(\xi, \eta) \\ &= \sum_{l=-L}^L \sum_{m=-M}^M [g(-i2^n(\xi + 2\pi l), -i2^n(\eta + 2\pi m)) |\hat{\varphi}(\xi + 2\pi l)|^2 |\hat{\varphi}(\eta + 2\pi m)|^2], \end{aligned}$$

from which T_{l_1, l_2}^n is finally approximated by the integral

$$T_{l_1, l_2}^n \approx \int_0^{2\pi} \int_0^{2\pi} \tilde{h}(\xi, \eta) e^{-il_1 \xi} e^{-il_2 \eta} d\xi d\eta.$$

This integral can be further approximated using the two-dimensional FFT.

It is noted in [5] that this approach can be used for functions $g(u)$ that have rapidly converging Taylor series expansions. In this case, for a given accuracy, the remainder term of sufficiently high order will be small. If f is a polynomial of degree p , the remainder term of order larger than p will be zero. For detail on the use of the dynamic range of the variable u , the reader is referred to [5]. An interesting case would be to consider smooth wavelets or smooth frame wavelets with compact support in the Fourier domain and which are also well localized in the (x, y) domain [1, 2].

More theoretical questions, like admissible classes of symbols, are left for future research.

7 Application to PDEs

We consider the initial-value problem:

$$u_t = \mathcal{L}u + \mathcal{N}f(u), \quad u(t_0, x, y) = u_0(x, y), \quad x, y \in [0, 1]^2,$$

with the periodic boundary conditions:

$$u(t, 0, y) = u(t, 1, y), \quad u(t, x, 0) = u(t, x, 1).$$

Here $\mathcal{L}$ and $\mathcal{N}$ are constant-coefficient differential operators (independent of t) and f is typically nonlinear. For example, for $u_t = u_{xx} + uu_y$ we have $\mathcal{L} = \partial_x^2$, $\mathcal{N} = \partial_y$ and $f(u) = \frac{1}{2}u^2$.

The PDE is transformed into the integral equation

$$u(t) = e^{(t-t_0)\mathcal{L}}u(t_0) + \int_{t_0}^t e^{(t-\tau)\mathcal{L}}\mathcal{N}f(u) d\tau, \quad \text{where} \quad e^{t\mathcal{L}} = \sum_{n=0}^{\infty} \frac{(t\mathcal{L})^n}{n!}.$$

For example, for $u_t = u_{xx} + uu_y$, we obtain

$$u(t) = e^{(t-t_0)\partial_x^2}u(t_0) + \int_{t_0}^t e^{(t-\tau)\partial_x^2}u(\tau)u_y(\tau) d\tau.$$

A quadrature reduces the integral to an implicit difference equation.

In the case $\mathcal{N}f(u) = u(\tau)v(\tau)$ with time step size $\Delta t = h$, $t_1 = t_0 + h$, we have the $O(h^2)$ approximation $\hat{I}$,

$$\begin{aligned} \int_{t_0}^{t_1} e^{(t_1-\tau)\mathcal{L}}u(\tau)v(\tau) d\tau &= (e^{h\mathcal{L}} - I)\mathcal{L}^{-1} \left(\sum_{i,j=0}^1 c_{ij}u_i v_j \right) + O(h^2) \\ &:= \hat{I}(t_1) + O(h^2), \end{aligned}$$

where $u_i = u(t_i)$, $v_i = v(t_i)$ and c_{ij} are coefficients to be determined by comparing $\hat{I}$ with, say, the approximation

$$\bar{I}(t_1) = \sum_{i,j=0}^1 f_{i,j}u_i v_j, \quad \text{where} \quad f_{i,j} = \int_{t_0}^{t_1} e^{(t_1-\tau)\mathcal{L}}L_i(\tau)L_j(\tau) d\tau.$$

Here $L_0(\tau) = 1 - \tau$ and $L_1(\tau) = \tau$ are the Lagrange linear interpolation basis at 0 and 1. Finally, the coefficients c_{ij} are determined such that $|\hat{I} - \bar{I}| = O(h^3)$.

The application **algorithm** is as follows.

- Discretize the PDE to arrive at an implicit difference equation with operator coefficients

$$u(t + \Delta t) = E(u(t)) + N(u(t), u(t + \Delta t)).$$

- Construct the matrices of the various operators appearing in the difference equation.
- Project u_0 onto the finest space $\mathbf{V}_n$ to get the coefficients s^n .
- Project s^n down on $\mathbf{V}_j$ to get $d^{j,\lambda}$, and s^j , for $j = n-1, n-2, \dots, n-J$.
- Solve the implicit difference equation by applying the operator to get the solution at the next time step given by $\tilde{d}^{j,\lambda}$, $\tilde{s}^j$, $n-J \leq j \leq n-1$.

- Use the pseudo-inverse FWT to reconstruct s^n .

This algorithm is applied to the fifth-order **thin film equation**:

$$u_t = -4\varepsilon uu_x + \mathcal{L}u,$$

where the operator $\mathcal{L}$ is

$$\mathcal{L} = -\partial_x^2 + B\partial_y^2 - C\partial_x^3 - D\partial_x\partial_y^2 - E(\partial_x^2 + \partial_y^2)^2 - F\partial_x^5 - G\partial_x^3\partial_y^2 - H\partial_x\partial_y^4$$

with coefficients

$$\begin{aligned} A &= \mu \left(\frac{8}{15}R - \frac{2}{3}\cot\alpha \right), & B &= -\frac{2}{3}\mu\cot\alpha, \\ C &= \mu^2 \left(2 + \frac{32}{63}R^2 - \frac{40}{63}R\cot\alpha \right), & D &= \mu^2 \left(2 - \frac{40}{63}R\cot\alpha \right), \\ E &= \frac{2}{3}\mu^3W, & F &= \frac{40}{63}\mu^4RW\csc\alpha, \\ G &= \frac{401}{315}\mu^4RW\csc\alpha, & H &= \frac{40}{63}\mu^4RW\csc\alpha. \end{aligned}$$

Here R is a measure of viscosity (small $R \leftrightarrow$ large viscosity), W is the surface tension, α is the plane inclination, and μ is the ratio of film thickness to streamwise lengthscale.

We have experimented with several values of the parameters of the problem with the radially symmetric initial disturbance:

$$u_0(x, y) = 0.1 e^{-60\cos^2(\pi x) - 60\cos^2(\pi y)}$$

shown in the top part of Fig. 1.

In the interesting case: $R = 10$, $W = 100$ and $\alpha = \pi/4$, we observed the formation of **horseshoe-type** waves downstream at $t = 2.5$ shown in the bottom part of Fig. 1 and at $t = 3.5$ and $t = 5.5$ shown in Fig. 2.

8 Conclusion and future work

In this paper, partial differential operators were represented in non-standard form in periodic two-dimensional separable orthonormal wavelet bases. Applications were to the thin film equation on an inclined plane. Future work can be pursued in representing partial differential operators and pseudodifferential operators in nonseparable and nonperiodic wavelet bases.

Acknowledgement

The authors thank the referee for many suggestions and corrections which greatly improved this work.

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Figure 1: Initial disturbance (top) and disturbance at $t = 2.5$ (bottom).

Figure 2: Disturbance at $t = 3.5$ (top) and at $t = 5.5$ (bottom).