Complex Random Matrices and Applications*

T. Ratnarajah[†‡] R. Vaillancourt[†§] M. Alvo[†¶]

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†Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Ave, Ottawa ON K1N 6N5 Canada

‡tharm@mathstat.uottawa.ca

§remi@uottawa.ca

¶malvo@science.uottawa.ca

Abstract

The eigenvalue and condition number distributions of complex Wishart matrices are investigated and applied to open problems in information theory.

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Résumé

On précise la distribution des valeurs propres et du conditionnement de matrices de Wishart complexes appliquées à des problèmes ouverts en théorie de l'information.

0.1 Introduction

Let an $n \times m$ complex Gaussian random matrix A be distributed as $\mathrm{A} \sim \mathcal{C}N(M, I_n \otimes \Sigma)$ with mean $\mathcal{E}\{\mathrm{A}\} = M$ and covariance $\mathrm{cov}\{\mathrm{A}\} = I_n \otimes \Sigma$. The matrix $\mathrm{W} = \mathrm{A}^H \mathrm{A}$ is called a noncentral Wishart matrix. If $M = 0$, then W is called a central Wishart matrix. The central and noncentral Wishart distributions are denoted by $\mathcal{C}W_m(n, \Sigma)$ and $\mathcal{C}W_m(n, \Sigma, \Omega)$, respectively, where $\Omega = \Sigma^{-1} M^H M$.

The joint eigenvalue distributions of complex central Wishart matrices are represented in [4] by complex hypergeometric functions of a matrix argument, which can be expressed in terms of complex zonal polynomials. The distribution of complex noncentral Wishart matrix can also be represented by complex hypergeometric functions; however, in this case, the eigenvalue distributions cannot be solved in terms of zonal polynomials. We derive these distributions using invariant polynomials of two matrix arguments [1], [2], which extend the single matrix argument of zonal polynomials. We also derive the distributions of the largest, the smallest and the single unordered eigenvalues of central and noncentral complex Wishart matrices [7].

The condition number, $\mathrm{cond}(A)$, of a matrix A is defined as the positive square root of the ratio of the largest to the smallest eigenvalues of the positive definite Hermitian matrix $W = A^H A$. We assume that the eigenvalues of W are ordered in strictly decreasing order. $\lambda_{\max} = \lambda_1 > \cdots > \lambda_m = \lambda_{\min} > 0$ since the probability that any two eigenvalues of A be equal is zero.

The distributions of $\lambda_{\max}$ and $\lambda_{\min}$ and the condition number distribution of random matrices are studied in [3] (and the references therein) for $\Sigma = I$. In [5] the largest and the smallest eigenvalue distributions of complex Wishart matrices are studied for $\Sigma = \sigma^2 I$. In this report, we derive the eigenvalue distributions of complex central and noncentral Wishart matrices and the condition number distribution of complex random matrices for arbitrary Σ. The theory of these random matrices is used to evaluate the capacity of multiple-input, multiple-output (MIMO) wireless communication systems. Note that the capacity of the communication channel expresses the maximum rate at which information can be reliably conveyed by the channel [8].

0.2 Complex Wishart Matrices

The joint eigenvalue density of a complex central Wishart matrix is given in [4].

Theorem 1 *Let* $\mathrm{W} \sim \mathcal{C}W_m(n, \Sigma)$ *with* $n > m - 1$*. The joint density of the eigenvalues of* W *is*

$$f(\Lambda) = \frac{\pi^{m(m-1)} (\det \Sigma)^{-n}}{\mathcal{C}\Gamma_m(m) \mathcal{C}\Gamma_m(n)} \prod_{k=1}^{m} \lambda_k^{n-m} \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 {}_0F_0\left(-\Sigma^{-1}, \Lambda\right) \tag{1}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ *and the complex multivariate gamma function is*

$$\mathcal{C}\Gamma_m(n) = \pi^{m(m-1)/2} \prod_{k=1}^{m} \Gamma(n - k + 1), \quad n > m - 1.$$

The following theorems give the distributions of the largest and smallest eigenvalues of a central Wishart matrix.

Theorem 2 *If* $W\mathrm{W} \sim \mathcal{C}W_m(n, \Sigma)$ $(n \geq m)$ *and* $\lambda_{\max}$ *is the largest eigenvalue of* W*, then*

$$P(\lambda_{\max} < x) = \frac{\mathcal{C}\Gamma_m(m)}{\mathcal{C}\Gamma_m(n+m)} \frac{x^{mn}}{(\det\Sigma)^n} {}_1F_1(n, n+m, -x\Sigma^{-1}). \tag{2}$$

Theorem 3 *If* $\mathrm{W} \sim \mathcal{C}W_m(n, \Sigma)$ *and* $\lambda_{\min}$ *is the smallest eigenvalue of* W*, then*

$$P(\lambda_{\min} > x) = \operatorname{etr}\left(-x\Sigma^{-1}\right) \sum_{k=0}^{m(n-m)} \widehat{\sum_{\kappa}} \frac{C_\kappa\left(x\Sigma^{-1}\right)}{k!}, \tag{3}$$

where $\operatorname{etr}(\cdot) = \exp(\operatorname{tr}(\cdot))$ *and* $\widehat{\sum}_\kappa$ *denotes summation over the partitions* $\kappa = (k_1, \ldots, k_m)$ *of* k *with* $k_1 \leq n - m$*.*

The condition number density is given by the following theorem.

Theorem 4 *Let* $\mathrm{W} \sim \mathcal{C}W_m(n, \Sigma)$*. Then the density of* $y = 1 - 1/\operatorname{cond}(W)$ *is*

$$\begin{aligned} f(y) = {} & \frac{\pi^{m(m-1)}(\det\Sigma)^{-n}}{\mathcal{C}\Gamma_m(m)\mathcal{C}\Gamma_m(n)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\Gamma(mn+k)C_\kappa(\Sigma^{-1})}{m^{mn+k}\ k!\ C_\kappa(I)} \\ & \times \sum_{t=0}^{\infty}\sum_{\tau,\delta} \frac{(m-n)_\tau g^{\delta}_{\tau,\kappa} y^{(m-1)(m+1)+t+k-1}}{t!} [(m-1)(m+1)+k+t] \\ & \times \frac{\mathcal{C}\Gamma_{m-1}(m-1)}{\pi^{(m-1)(m-2)}} \frac{\mathcal{C}\Gamma_{m-1}(m+1,\delta)\mathcal{C}\Gamma_{m-1}(m-1)}{\mathcal{C}\Gamma_{m-1}(2m,\delta)} C_\delta(I), \end{aligned} \tag{4}$$

where κ*,* τ *and* δ *are the ordered partitions of the nonnegative integers* k*,* t*, and* $f = k + t$*, respectively, into not more than* m *parts.*

The joint eigenvalue density of a noncentral Wishart matrix is given by following theorems and lemma.

Theorem 5 *Let* $\mathrm{W} \sim \mathcal{C}W_m(n, \Sigma, \Omega)$ *with* $n > m - 1$*. The joint density of the eigenvalues of* W *is*

$$\begin{aligned} f(\Lambda) = {} & \frac{\pi^{m(m-1)}(\det\Sigma)^{-n}}{\mathcal{C}\Gamma_m(m)\mathcal{C}\Gamma_m(n)} \operatorname{etr}(-\Omega) \prod_{k=1}^{m} \lambda_k^{n-m} \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 \\ & \times \sum_{k,t=0}^{\infty} \sum_{\kappa,\tau;\phi\in\kappa.\tau} \frac{C_\phi^{\kappa,\tau}(-\Sigma^{-1}, \Omega\Sigma^{-1}) C_\phi^{\kappa,\tau}(\Lambda,\Lambda)}{k!\ t!\ [n]_\tau\ C_\phi(I_m)}, \end{aligned} \tag{5}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_m)$*,* $C_\phi^{\kappa,\tau}$ *is an invariant polynomial indexed by the ordered partitions* κ*,* τ *and* ϕ *of the nonnegative integers* k*,* t*, and* $f = k + t$*, respectively, into not more than* m *parts.*

Lemma 1 *The following inequality holds:*

$$ {}_0F_1(b;X) < {}_0F_0\left(X/b\right), \tag{6} $$

where X is an $m \times m$ complex matrix and b is an arbitrary complex number.

A numerical evaluation shows that this bound is tight. Using Lemma 1, we can express the joint eigenvalue density of a noncentral Wishart matrix as a bounded density function, which is given by the following theorem.

Theorem 6 *Let $\mathrm{W} \sim \mathcal{C}W_m(n, \Sigma_1, \Omega)$ with $n > m-1$. The joint density of the eigenvalues of W satisfies the inequality*

$$ f(\Lambda) < \frac{\pi^{m(m-1)}(\det \Sigma_1)^{-n}}{\mathcal{C}\Gamma_m(m)\mathcal{C}\Gamma_m(n)} \operatorname{etr}(-\Omega) \prod_{k=1}^{m} \lambda_k^{n-m} \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 {}_0F_0(-\Psi, \Lambda), \tag{7} $$

where $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_m)$, the diagonal elements of $\Psi = \operatorname{diag}(\psi_1, \ldots, \psi_m)$ are the eigenvalues of the matrix $\left(\Sigma_1^{-1} - \Omega\Sigma_1^{-1}/n\right)$ and $\Omega = \Sigma^{-1}M^H M$.

Due to space limitation we only give the distribution of $\lambda_{\max}$.

Theorem 7 *If $W \sim \mathcal{C}W_m(n, \Sigma, \Omega)$ and $\lambda_{\max}$ is the largest eigenvalue of W, then its distribution is given by*

$$ P(\lambda_{\max} < y) = \frac{y^{mn}\mathcal{C}\Gamma_m(m)\operatorname{etr}(-\Omega)}{\mathcal{C}\Gamma_m(n+m)(\det \Sigma)^n} \sum_{k,t=0}^{\infty} \sum_{\kappa,\tau;\phi\in\kappa.\tau} \frac{[n]_\phi \theta_\phi^{\kappa,\tau} C_\phi^{\kappa,\tau}(-y\Sigma^{-1}, y\Omega\Sigma^{-1})}{k!\ t!\ [n]_\tau\ [n+m]_\phi}, \tag{8} $$

where $C_\phi^{\kappa,\tau}$ is an invariant polynomial, indexed by the ordered partitions κ, τ and ϕ of the nonnegative integers k, t, and $f = k+t$, respectively, into not more than m parts and $\theta_\phi^{\kappa,\tau} = C_\phi^{\kappa,\tau}(I,I)/C_\phi(I)$.

0.3 Channel Capacity

An n_t-input (or transmitter) and n_r-output (or receiver) MIMO channel can be represented by an $n_r \times n_t$ complex random matrix H, as shown in Fig. 1.

The complex vector signal, $y = \mathrm{H}x + v$, received at the jth output is

$$ y_j = \sum_{i=1}^{n_t} h_{ij}x_i + v_j, \tag{9} $$

where h_{ij} is the complex channel coefficient between input i and output j, x_i is the complex signal at the ith input and v_j is complex Gaussian noise. The total power of the input is constrained to ρ,

$$ \mathcal{E}\{x^H x\} \le \rho \quad \text{or} \quad \operatorname{tr}\mathcal{E}\{xx^H\} \le \rho. $$

Assume H is a complex Gaussian random matrix and its realization is known to the receiver. If $W = \mathrm{H}^H\mathrm{H}$, then the capacity of this MIMO channel is [8]

$$ C = \mathcal{E}_{\mathrm{W}}\left\{\log\det\left(I_{n_t} + (\rho/n_t)\mathrm{W}\right)\right\}. \tag{10} $$

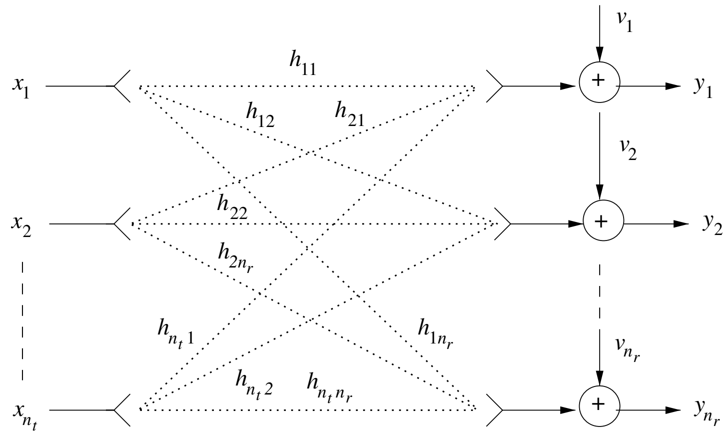

Figure 1: A MIMO communication system.

The channel is Rician distributed if $\mathrm{W} \sim \mathcal{C}W_{n_t}(n_r, \Sigma, \Omega)$ and Rayleigh distributed if $\mathrm{W} \sim \mathcal{C}W_{n_t}(n_r, \Sigma)$. These are typical satellite and fixed or mobile communication environments, respectively. The capacity (10) is computed using the complex noncentral and central Wishart distributions, respectively. It can also be computed using the joint eigenvalue density $f(\Lambda)$ or the single unordered eigenvalue density $f(\lambda_1)$, that is,

$$\begin{aligned} C &= \mathcal{E}_\Lambda \left\{ \log \left(\prod_{k=1}^{n_t} [1 + (\rho/n_t)\lambda_k] \right) \right\} = \sum_{k=1}^{n_t} \mathcal{E}_{\lambda_k} \{\log(1 + (\rho/n_t)\lambda_k)\} \\ &= n_t \mathcal{E}_{\lambda_1} \{\log(1 + (\rho/n_t)\lambda_1)\}. \end{aligned} \tag{11}$$

We only give the capacity evaluation of a correlated Rayleigh $n_r \times 2$ channel matrix. Thus, we assume that we have a two-input ($n_t = 2$), n_r-output communication system operating over a correlated Rayleigh fading environment. The joint eigenvalue density of a central Wishart matrix depends on the population covariance matrix Σ only through its eigenvalues $\upsilon_1, \ldots, \upsilon_{n_t}$, i.e.,

$${}_0F_0\left(-\Sigma^{-1}, \Lambda\right) = {}_0F_0\left(-\Upsilon^{-1}, \Lambda\right),$$

where $\Upsilon = \mathrm{diag}(\upsilon_1, \ldots, \upsilon_{n_t})$. If $n_t = 2$ and $\Upsilon^{-1} = \mathrm{diag}(a_1, a_2)$, then [6]

$$\begin{aligned} &{}_0F_0(-\Upsilon^{-1}, \Lambda) = \\ &\qquad \frac{1}{(a_2 - a_1)(\lambda_1 - \lambda_2)} \left\{ \exp\left[-(a_1\lambda_1 + a_2\lambda_2)\right] - \exp\left[-(a_1\lambda_2 + a_2\lambda_1)\right] \right\}. \end{aligned} \tag{12}$$

Substituting (12) into (1) and integrating with respect to λ_2 and dividing by 2, we obtain the single unordered eigenvalue density $f(\lambda_1)$ which can be used to evaluate (11).

Theorem 8 *Consider the two-input correlated Rayleigh channel, i.e.,* $\mathrm{H} \sim \mathcal{C}N(0, I_{n_r} \otimes \Sigma)$*, with* $n_r \geq 2$*. If the input power is constrained by* ρ*, then the capacity* C *is*

Table 1: Capacity in nats for a two-input, n_r-output communication system operating over a correlated Rayleigh fading channel, where the correlation coefficient is equal to 0.9.

	SNR ρ in dB							
n_r	0 dB	5 dB	10 dB	15 dB	20 dB	25 dB	30 dB	35 dB
2	1.0326	1.9252	3.1157	4.5641	6.2023	7.9419	9.7221	11.5165
4	1.6408	2.8426	4.4118	6.3154	8.4439	10.6803	12.9577	15.2490
6	2.0685	3.4398	5.1852	7.2250	9.4266	11.6948	13.9863	16.2855
8	2.4033	3.8917	5.7454	7.8540	10.0862	12.3653	14.6604	16.9606
10	2.6804	4.2568	6.1838	8.3330	10.5817	12.8666	15.1635	17.4643
12	2.9179	4.5639	6.5437	8.7196	10.9786	13.2669	15.5650	17.8661
14	3.1265	4.8293	6.8489	9.0437	11.3096	13.6003	15.8992	18.2005
16	3.3129	5.0631	7.1139	9.3226	11.5936	13.8860	16.1853	18.4869

given by

$$
\begin{aligned}
C = {} & \frac{a_1^{n_r} a_2}{(a_2 - a_1)\Gamma(n_r)} \int_0^\infty \log[1 + (\rho/2)\lambda_1]\lambda_1^{n_r - 1}\, e^{-a_1\lambda_1}\, d\lambda_1 \\
& - \frac{a_1 a_2^{n_r}}{(a_2 - a_1)\Gamma(n_r)} \int_0^\infty \log[1 + (\rho/2)\lambda_1]\lambda_1^{n_r - 1} e^{-a_2\lambda_1} d\lambda_1 \\
& - \frac{a_1^{n_r}}{(a_2 - a_1)\Gamma(n_r - 1)} \int_0^\infty \log[1 + (\rho/2)\lambda_1]\lambda_1^{n_r - 2} e^{-a_1\lambda_1} d\lambda_1 \\
& + \frac{a_2^{n_r}}{(a_2 - a_1)\Gamma(n_r - 1)} \int_0^\infty \log[1 + (\rho/2)\lambda_1]\lambda_1^{n_r - 2} e^{-a_2\lambda_1} d\lambda_1,
\end{aligned}
\tag{13}
$$

where λ_1 is an unordered eigenvalue of $W = H^H H$ and (a_1, a_2) are the eigenvalues of Σ^{-1}.

In (13), C is measured in nats or bits if we use $\log_e$ or $\log_2$, respectively. Table 1 shows the capacity in nats for an $n_r \times 2$ correlated Rayleigh fading channel matrix with correlation coefficient 0.9. Note that each column represents different levels of input power or signal to noise ratio (SNR) in dB. Figure 2 shows the capacity vs the correlation coefficient. From the table and figure we note the following: (i) the capacity is decreasing with increasing channel correlation, (ii) the capacity is increasing with increasing n_r and SNR.

The covariance matrix is $\Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$ and its eigenvalues are 1.9 and 0.1. Hence $\Upsilon = \mathrm{diag}(1.9, 0.1)$ and $a_1 = 1/1.9, a_2 = 1/0.1$. The non-diagonal element of Σ, called correlation coefficient, gives the correlation between the channel coefficient from different transmitter antennas to a single receiver antenna.

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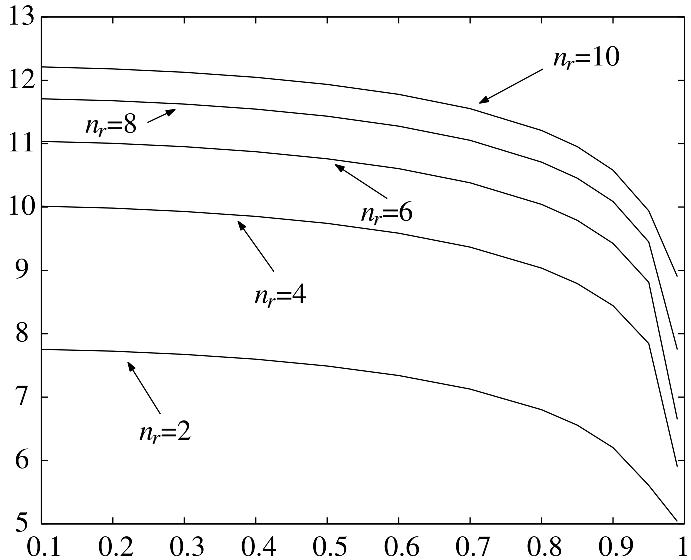

Figure 2: Capacity vs correlation coefficient for SNR=20dB and $n_t = 2$ and $n_r = 2, 4, 6, 8, 10$, i.e., H is a $n_r \times 2$ correlated Rayleigh fading channel matrix.

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